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DISCRETE LOT STREAMING IN THREE-MACHINE FLOW SHOPS*

By

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Abstract

Lot streaming is the process of splitting a job or lot to allow overlapping between successive operations in a multistage production system. This use of transfer lots usually results in a substantially shorter makespan for the corresponding schedule. In this paper, we study the discrete version of schedules which minimize the makespan for a single job in a three-machine flow shop. We present new insights into the structure of optimal schedules, which lead to the first polynomial time solution of the problem.

1 Introduction

Lot streaming is the process of using transfer batches to move the processed portion of a production lot to downstream machines so that the makespan of the schedule can be shortened and the work-in-process inventory levels can be lowered. The term was introduced by Reiter [16], but the idea has been considered many times under different names. The increased interest in its applications over the last few years is probably due to the fact that it is consistent with the Just-In-Time (JIT) philosophy of making small sublots and it also agrees with the basic idea of the OPT scheduling package [7], [9].

Szendrovits[17] analyzes the lot streaming problem in a flow shop for a single job with equal sublot sizes. Goyal [10] finds the optimal sublot sizes in Szendrovits'

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model. Moily[14], Jacobs and Bragg [12], Kulonda [13] and Graves and Kostreva [11] also demonstrate reductions in production time and cost by using transfer lots. Steiner and Truscott [18] find the optimal lot streaming schedules in an open shop with equal size transfer lots and no idling on the machines. Cetinkaya and Gupta [3] analyze the lot streaming problem for a single job in a flow shop with the total flow time criterion.

Most papers on lot streaming consider the objective of minimizing the makespan in an *m-machine flow shop* where each item is processed on the m machines in the order $1, \ldots, m$. Trietsch, in [19] and [20], and Baker [1] independently develop a conceptual framework for the problem. They present a classification scheme and review the most important results in [21]. Vickson [22] solves the lot streaming problem for multiple jobs in a two-machine flow shop with job setup times and sublot transfer times.

In this paper, we consider the problem of minimizing the makespan by splitting a single job of U items into s *discrete* (integer valued) sublots in an three-machine flow shop. The three machines are denoted by M_1, M_2, M_3, and each item of the job has positive processing times p_1, p_2, p_3 on M_1, M_2, M_3, respectively. If $x_{i,j}$ $(i = 1, 2, 3$ and $j = 1, \ldots, s)$ is the size of the jth sublot on M_i, then our objective is to find the integer $x_{i,j}$ values which minimize the makespan. If the integrality requirement for the $x_{i,j}$ is relaxed, we have the *continuous version* of the problem.

Under the assumption of *item availability* individual items become available for processing at the next machine as soon as they are finished on the current machine (unit size transfer lots). Under the more restrictive assumption of *batch availability* items become available for processing at the next machine after the current machine finished processing the last item in their sublot (batch). For $m = 2$ there is no difference between the two assumptions, and Vickson and Alfredsson [23] solve the continuous makespan minimization problem for this case. The same problem is solved with detached setups in [4] and with attached setups in [2]. Another, frequently used model further restricts the batch availability assumption by considering only *consistent sublots*, i.e., $x_{i,j} = x_{i+1,j}$ for $i = 1, \ldots, m-1,\ j = 1, \ldots, s$. In this case, we can write x_j instead of $x_{i,j}$.

Most analytical results assume batch availability and apply to the continuous version of flow shop problems. Baker [1] shows that linear programming can be used to find the consistent sublot sizes which minimize the makespan. Potts and Baker [15] show that for a single job, it is sufficient to consider identical sublot sizes on the first two machines, and on the last two machines. The $m = 2$ case is solved in [15] and in [19]. Glass et. al. [8] develop the solution to minimize the makespan for a single job in a three stage production process. Their algorithms compute the minimum makespan in $O(log\ s)$ time for both the flow shop and job shop problem.

Although we need integer valued solutions for most practical applications, it seems to be much more difficult to find these, and much less is known about these discrete problems. Trietsch and Baker [21] give dynamic programming algorithms which solve the two- and three-machine problem in $O(s^2U)$ time. Vickson [22] uses bisection search to find the optimal integer solution in $O(s \log U)$ time for $m = 2$. Chen and Steiner give [5] a strongly polynomial solution, requiring $O(s)$ time, for the same problem. As Baker [1] points out, the best consistent sublot solution for the discrete version of the m-machine flow shop problem can be found by integer linear programming, however, this is not a satisfactory solution method in general. Chen and Steiner [6] found quickly obtainable approximations for this general case.

In this paper, we present the first known polynomial time algorithms for finding the discrete optimal solution with minimum makespan in a three-machine flow shop. We can find the optimal sublot sizes in $O(s)$ time if $p_2^2 \leq p_1p_3$ and $p_1 \neq p_3$. If $p_2^2 \leq p_1p_3$ and $p_1 = p_3$, then the optimal sublot sizes can be found in $O(1)$ time. If $p_2^2 > p_1p_3$, it takes $O(s^4 \log^2 U)$ time to identify the optimal solution.

The paper is organized as follows. Section 2 briefly reviews a network representation of the problem. Section 3 contains the detailed network analysis. Section 4 presents the optimization algorithms which are based on the structural insights obtained in the previous section. Summary and conclusions are presented in Section 5.

2 Network representation

It is known that there is always a consistent sublot optimal solution for the three-machine problem [15]. Thus we can restrict our considerations to such solutions. Let $C_{i,j}$ denote the completion time of sublot j on machine i ($i = 1,2,3 \quad j = 1,2,...,s$). The following constraints must be satisfied by any consistent sublot feasible solution.

1) Machine capacity constraints :

$$C_{i,j} \geq C_{i,j-1} + p_i\, x_j \qquad (\ i = 1,2,3 \quad j = 2,...,s);$$

2) Production constraints :

$$C_{i,j} \geq C_{i-1,j} + p_i\, x_j \qquad (i = 2,3 \quad j = 1,2,...,s);$$

3) Lot sizing constraints:

$$\textstyle\sum_{j=1}^{s} x_j = U, \ x_j \geq 0 \ \text{ integer} \qquad (j = 1,2,...,s).$$

Such a solution can be represented by a network $N(x)$ which contains a vertex for each sublot on every machine (see Fig. 1). In the network, x_j ($j = 1,2,...,s$) is the jth sublot size. The directed arc from vertex (i,j) to vertex $(i+1,j)$ ($i = 1,2$) represents the production constraint that sublot j can be processed on machine $(i+1)$

only after it is completed on machine i. The directed arc from vertex (i,j) to vertex $(i,j+1)$ $(1 \le j < s)$ represents the machine capacity constraint that sublot $(j+1)$ can start on M_i only when the jth sublot is finished on it. The vertex (i,j) has weight $p_i x_j, 1 \le i \le 3, 1 \le j \le s$.

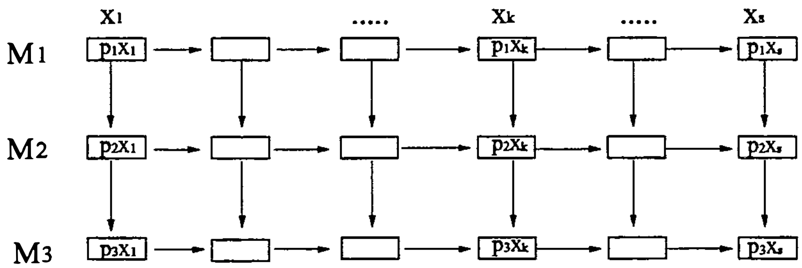

Figure 1: Network representation for a solution

Using the network representation, the objective becomes to determine the sublot sizes which minimize the length of the longest path in the network, where the *length* of any path is the sum of the weights of the vertices on it. Any longest path is referred to as a *critical path.* A subpath of a (critical) path is called a *(critical) segment.* Sublot i is a *critical sublot* if there is a critical vertical segment through it, i.e., $(l,i)-(l+1,i)$ is critical for some $l \epsilon \{1,2\}$. A path from $(1,1)$ to $(3,s)$ is called *maximal.* $M(x)$ denotes the length of a critical path in $N(x)$.

3 Structure of optimal solutions

If $U \le s$, then it is clear that the best strategy is to move each item to the downstream machine once it finished on the current machine. So for the remainder of the paper we assume that $U > s$.

We introduce some additional notation which will make it easier to refer to different parts of the network. A segment (path) $P_{i,j}[(l_1,k_1),(l_2,k_2)]$ is unambiguously defined by specifying its *starting node,* (l_1,k_1), its *last node,* (l_2,k_2), and the index of the one or two sublots, i and j, for which it contains the *vertical subsegments* $(l_1,i)-(l_1+1,i)$ and $(l_2-1,j)-(l_2,j)$. For example, $P_{i,j}[(1,2),(3,s-1)]$ is short notation for the segment $(1,2)-(1,3)-...-(1,i)-(2,i)-(2,i+1)-...-(2,j)-(3,j)-(3,j+1)-...-(3,s-1)$. If a segment contains only one sublot on which it has a vertical subsegment, then we naturally omit the index j. We use $L_{i,j}$ $[(l_1,k_1),(l_2,k_2)]$ to denote *the length* of $P_{i,j}[(l_1,k_1),(l_2,k_2)]$. If $(l_1,k_1) = (1,1)$ and $(l_2,k_2) = (3,s)$, i.e.,

$P_{i,j}[(l_1,k_1),(l_2,k_2)]$ is a maximal path in the network, then we will omit its first and last node, and refer to it simply by $P_{i,j}$.

To distinguish the *length of maximal paths* from others and to show the network where they are considered, we use $M_{i,j}(x)$ and $M_i(x)$ to denote the length of $P_{i,j}$ and P_i, respectively, in $N(x)$.

Theorem 1 *If $p_2^2 \leq p_1p_3$, then for* any *feasible solution x there exists a k such that segment $(1,k)-(2,k)-(3,k)$ is critical in $N(x)$.*

Proof. Assume that no such sublot exists, then there must be i,j $(i<j)$ such that $P_{i,j}[(1,i),(3,j)]$ is critical but $P_i[(2,i),(3,j)]$ and $P_j[(1,i),(2,j)]$ are not. Therefore, $P_i[(1,i),(2,j)]$ should be longer than $P_j[(1,i),(2,j)]$, i.e.,

$$p_1 \sum_{l=i+1}^{j} x_l < p_2 \sum_{l=i}^{j-1} x_l, \tag{1}$$

and $P_j[(2,i),(3,j)]$ should be longer than $P_i[(2,i),(3,j)]$, i.e.,

$$p_2 \sum_{l=i+1}^{j} x_l > p_3 \sum_{l=i}^{j-1} x_l. \tag{2}$$

Multiplying (1) and (2) yields a contradiction with the assumption of $p_2^2 \leq p_1p_3$. The optimal network is demonstrated in Fig. 2, with the heavy arcs highlighting the critical path.□

We must distinguish three cases, depending on whether $p_2^2 \leq p_1p_3$ and $p_1 \neq p_3$, $p_2^2 \leq p_1p_3$ and $p_1 = p_3$,or $p_2^2 > p_1p_3$.

Case 1 $p_2^2 \leq p_1p_3$ and $p_1 \neq p_3$

For convenience, we assume that $p_1 < p_3$. We can similarly solve the reverse problem if $p_1 > p_3$, i.e., p_3, p_2, p_1 can be treated as the unit processing time on M_1, M_2, M_3, respectively.

Theorem 2 *If $p_2^2 \leq p_1p_3$, $p_1 < p_3$ and the first sublot is not critical when $x_1 = 1$, then all sublot sizes are positive in* every *optimal network $N(x)$, i.e., $x_i > 0$ for $1 \leq i \leq s$.*

Proof. Suppose there was an optimal solution in which some sublot sizes are zero. It can be written as $x = (x_1, ..., x_r, 0, ..., 0)$, where $x_i > 0$ for $1 \leq i \leq r < s$.

By Theorem 1, there must be at least one sublot i with critical segment $(1,i)-(2,i)-(3,i)$. Let k be the last one with critical segment $(1,k)-(2,k)-(3,k)$. We show that $x_k > 1$.

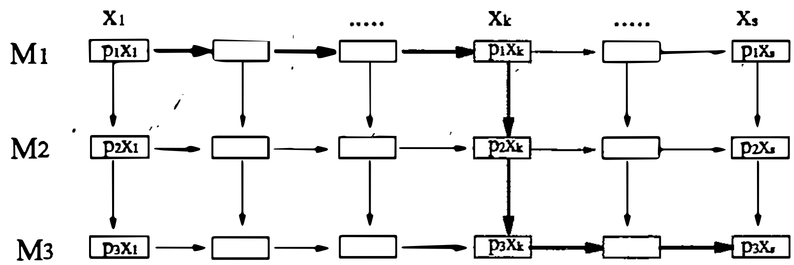

Figure 2: Network for the three-machine problem when $p_2^2 \le p_1 p_3$

Suppose $x_k = 1$, then $k > 1$ by assumption. We know that $P_k[(1, k-1), (3, k)]$ should be at least as long as $P_{k-1}[(1, k-1), (3, k)]$. Therefore, we have

$$(p_1 + p_2)x_k = p_1 + p_2 \ge (p_2 + p_3)x_{k-1}, \tag{3}$$

which yields a contradiction with $x_{k-1} > 0$ and $p_1 < p_3$. So $x_k > 1$ indeed.

Suppose $k < s$. We can construct another solution $x' = (x_1', ..., x_s')$ with a shorter critical path, which yields a contradiction with the optimality of x.

Let $x_1' = 1$, $x_{k+1}' = x_k - 1$ and $x_i' = x_{i-1}$ for $i \in \{2, ..., s\} - \{k\}$.

By Theorem 1, we only need to consider the paths containing $(1,i) - (2,i) - (3,i)$ $(1 \le i \le s)$ for critical path.

If the first sublot is not critical in $N(x)$, then

$$\begin{aligned} M_1(x') &= (p_1 + p_2)x_1' + p_3 \sum_{l=1}^{s} x_l' \\ &\le (p_1 + p_2)x_1 + p_3 \sum_{l=1}^{s} x_l \\ &< M(x). \end{aligned} \tag{4}$$

If the first sublot is critical in $N(x)$, then $x_1 > 1$ by assumption. Thus

$$\begin{aligned} M_1(x') &= (p_1 + p_2)x_1' + p_3 \sum_{l=1}^{s} x_l' \\ &< (p_1 + p_2)x_1 + p_3 \sum_{l=1}^{s} x_l \\ &= M(x). \end{aligned}$$

For $1 < i < k + 1$, we have

$$M_i(x') = p_1 \sum_{l=1}^{i} x_l' + p_2 x_i' + p_3 \sum_{l=i}^{s} x_l'$$

$$
\begin{aligned}
&\leq p_1 \sum_{l=1}^{i-1} x_l + p_2 x_{i-1} + p_3 \sum_{l=i-1}^{s} x_l + p_1 - p_3 \\
&= M_{i-1}(x) + p_1 - p_3 \\
&< M(x). \qquad (5)
\end{aligned}
$$

We also have

$$
\begin{aligned}
M_{k+1}(x') &= p_1 \sum_{l=1}^{k+1} x_l' + p_2 x_{k+1}' + p_3 \sum_{l=k+1}^{s} x_l' \\
&\leq p_1 \sum_{l=1}^{k} x_l + p_2 x_k + p_3 \sum_{l=k}^{s} x_l - p_2 - p_3 \\
&= M_k(x) - p_2 - p_3 \\
&< M(x). \qquad (6)
\end{aligned}
$$

For $k+1 < j \leq s$, we have

$$
\begin{aligned}
M_j(x') &= p_1 \sum_{l=1}^{j} x_l' + p_2 x_j' + p_3 \sum_{l=j}^{s} x_l' \\
&= p_1 \sum_{l=1}^{j-1} x_l + p_2 x_{j-1} + p_3 \sum_{l=j-1}^{s} x_l \\
&= M_{j-1}(x) \\
&< M(x), \qquad (7)
\end{aligned}
$$

where the last inequality follows from the fact that P_{j-1} cannot be critical by the definition of k. Combining (4), (5), (6) and (7) yields a contradiction with the optimality of x, so we cannot have $k < s$. □

Corollary 3 *If $p_2^2 \leq p_1 p_3$, $p_1 < p_3$ and the first sublot is not critical when $x_1 = 1$, then increasing s, the number of allowed sublots, results in a shorter minimum makespan.*

Proof. Let x and x' be optimal solutions with maximum s and $s+1$ sublots, respectively. By defining $x_{s+1} = 0$, $(x_1, x_2, ..., x_{s+1})$ is a feasible solution for the problem with maximum $s+1$ sublots. By Theorem 2, however, this solution cannot be optimal when $s+1$ sublots are allowed, so we must have $M(x) > M(x')$.□

Glass *et al* [8] have proved that when $p_2^2 \leq p_1 p_3$, the optimal network for the continuous problem is perfectly balanced in the sense that each path P_i $(1 \leq i \leq s)$ is critical. The following crucial theorem shows that there is also an optimal solution for the discrete problem for which the network remains *almost perfectly balanced.*

Theorem 4 *If $p_2^2 \leq p_1 p_3$, $p_1 < p_3$ and the first sublot is not critical when $x_1 = 1$, then there must be an optimal solution x in which every path P_i $(1 \leq i \leq s)$ has a length which is within $p_2 + p_3$ of the length of a critical path, i.e., $M_i(x) > M(x) - p_2 - p_3$.*

Proof. Suppose we have an optimal solution $x = (x_1, ..., x_s)$ for which the theorem is not true. We can construct alternative optimal sublot sizes $x' = (x'_1, ..., x'_s)$ for which the theorem holds.

Suppose the theorem did not hold for the path P_s, i.e., $M_s(x) \leq M(x) - p_2 - p_3$.

Let $x'_i = x_i$ for $1 \leq i < s-1$, $x'_{s-1} = x_{s-1} - 1$ and $x'_s = x_s + 1$. We note that $x'_{s-1} \geq 0$ by Theorem 2.

$$\begin{aligned} M_s(x') &= p_1 \sum_{l=1}^{s} x'_l + (p_2 + p_3)x'_s \\ &= M_s(x) + p_2 + p_3 \\ &\leq M(x), \end{aligned} \tag{8}$$

where the inequality holds by our assumption that the theorem is false for $M_s(x)$.

$$\begin{aligned} M_{s-1}(x') &= p_1 \sum_{l=1}^{s-1} x'_l + p_2 x'_{s-1} + p_3 \sum_{l=s-1}^{s} x'_l \\ &= M_{s-1}(x) - p_1 - p_2 \\ &< M(x). \end{aligned} \tag{9}$$

For $1 \leq i < s-1$, we have

$$\begin{aligned} M_i(x') &= p_1 \sum_{l=1}^{i} x'_l + p_2 x'_i + p_3 \sum_{l=i}^{s} x'_l \\ &= M_i(x) \\ &\leq M(x). \end{aligned} \tag{10}$$

From (8), (9) and (10), it is clear that x' is still optimal. By Theorem 2, we must have $x'_i > 0$ for $i = 1, 2, ..., s$. So we can repeat the preceding transformation until the theorem holds for the path containing $(1, s) - (2, s) - (3, s)$, i.e., $M(x') < M_s(x') + p_2 + p_3$.

We can similarly prove that the theorem holds for $i = s-1, ..., 2$. So suppose now that x is an optimal solution for which $M(x) < M_i(x) + p_2 + p_3$ for $2 \leq i \leq s$. By Theorem 2, $x_i > 0$ for $i = 2, ..., s$. Further suppose that x did not satisfy the theorem for $i = 1$, i.e., $M_1(x) \leq M(x) - p_2 - p_3$.

We construct alternative sublot sizes $x' = (x'_1, ..., x'_s)$ with a shorter critical path, which contradicts the assumption that x is optimal.

Consider $x'_1 = x_1 + 1, x'_i = x_i$ for $2 < i < s$ and $x'_s = x_s - 1$. Then

$$M_1(x') = (p_1 + p_2)x'_1 + p_3 \sum_{l=1}^{s} x'_l$$

$$
\begin{aligned}
&= M_1(x) + p_1 + p_2 \\
&< M_1(x) + p_2 + p_3 \\
&\leq M(x).
\end{aligned} \tag{11}
$$

For $1 < i < s$, we have

$$
\begin{aligned}
M_i(x^{'}) &= p_1 \sum_{l=1}^{i} x_l^{'} + p_2 x_i^{'} + p_3 \sum_{l=i}^{s} x_l^{'} \\
&= M_i(x) + p_1 - p_3 \\
&< M(x).
\end{aligned} \tag{12}
$$

We also have

$$
\begin{aligned}
M_s(x^{'}) &= p_1 \sum_{l=1}^{s} x_l^{'} + (p_2 + p_3) x_s^{'} \\
&= M_s(x) - p_2 - p_3 \\
&< M(x).
\end{aligned} \tag{13}
$$

From (11), (12) and (13), it is clear that $x^{'}$ yields a shorter critical path, which contradicts the optimality of x.□

Case 2 $p_2^2 \leq p_1 p_3$ and $p_1 = p_3$

Lemma 5 *If $p_2^2 \leq p_1 p_3$ and $p_1 = p_3$, then a lower bound of the makespan is $(p_1 + p_2) \lceil U/s \rceil + p_3 U \leq M(x)$.*

Proof. Let k be a sublot with the largest sublot size, i.e., $x_k \geq x_i$ for $i \in \{1, ..., s\} - \{k\}$. Then $\sum_{l=1}^{s} x_l \leq s x_k$ and $\sum_{l=1}^{s} x_l = U$ implies

$$
x_k \geq U/s. \tag{14}
$$

By looking at P_k, we get

$$
\begin{aligned}
M(x) &\geq p_1 \sum_{l=1}^{k} x_l + p_2 x_k + p_3 \sum_{l=k}^{s} x_l \\
&= (p_1 + p_2) x_k + p_3 \sum_{l=1}^{s} x_l \\
&\geq (p_1 + p_2) \lceil U/s \rceil + p_3 U,
\end{aligned}
$$

where the last inequality holds by (14). □

Theorem 6 *If $p_2^2 \leq p_1 p_3$ and $p_1 = p_3$, then there exists a k such that*

i) each segment $(1, i) - (2, i) - (3, i)$ is critical for $1 \leq i \leq k$;

ii) each segment $(1, j) - (2, j) - (3, j)$ is non-critical for $k < j \leq s$.

Proof. By Theorem 1, there exists a sublot i for which $(1,i)-(2,i)-(3,i)$ is critical in any feasible solution. Let $x=(x_1,...,x_s)$ be the optimal solution in which the index k of the last sublot with critical segment $(1,k)-(2,k)-(3,k)$ is as small as possible among all the optimal solutions.

Suppose there was a sublot $i<k$, such that $(1,i)-(2,i)-(3,i)$ was a non-critical segment in $N(x)$. By the definition of k, segment $P_k[(1,i),(3,k)]$ should be critical. Therefore, it is longer than the segment $P_i[(1,i),(3,k)]$, i.e., $p_1\sum_{l=i+1}^{k}x_l+p_2x_k > p_2x_i+p_3\sum_{l=i}^{k-1}x_l$, implying $(p_1+p_2)x_k>(p_2+p_3)x_i$. We have $p_1=p_3$, however, so $x_k>x_i$. This means that segment $P_k[(1,i),(3,k)]$ is longer by at least p_1+p_2 than segment $P_i[(1,i),(3,k)]$, i.e.,

$$M_i(x)+p_1+p_2\leq M_k(x)=M(x). \tag{15}$$

We construct alternative optimal sublot sizes $x'=(x'_1,...,x'_s)$ with non-critical $(1,j)-(2,j)-(3,j)$ for $k\leq j\leq s$, which contradicts the definition of x.

Let $x'_i=x_i+1, x'_k=x_k-1$ and $x'_j=x_j$ for $i\in\{1,...s\}-\{i,k\}$.

By Theorem 1, we only need to consider P_j, $1\leq j\leq s$, for critical paths.

For $1\leq j<i$, we have

$$\begin{aligned} M_j(x') &= p_1\sum_{l=1}^{j}x'_l+p_2x'_j+p_3\sum_{l=j}^{s}x'_l \\ &= p_1\sum_{l=1}^{j}x_l+p_2x_j+p_3\sum_{l=j}^{s}x_l \\ &\leq M(x). \end{aligned} \tag{16}$$

We also have

$$\begin{aligned} M_i(x') &= p_1\sum_{l=1}^{i}x'_l+p_2x'_i+p_3\sum_{l=i}^{s}x'_l \\ &= p_1\sum_{l=1}^{i}x_l+p_2x_i+p_3\sum_{l=i}^{s}x_l+p_1+p_2 \\ &= M_i(x)+p_1+p_2 \\ &\leq M(x), \end{aligned} \tag{17}$$

where the inequality follows from (15).

For $i<j<k$, we have

$$\begin{aligned} M_j(x') &= p_1\sum_{l=1}^{j}x'_l+p_2x'_j+p_3\sum_{l=j}^{s}x'_l \\ &= M_j(x)+p_1-p_3 \\ &\leq M(x). \end{aligned} \tag{18}$$

For k, we have

$$\begin{aligned} M_k(x') &= p_1 \sum_{l=1}^{k} x_l' + p_2 x_k' + p_3 \sum_{l=k}^{s} x_l' \\ &= p_1 \sum_{l=1}^{k} x_l + p_2 x_k + p_3 \sum_{l=k}^{s} x_l - p_2 - p_3 \\ &= M_k(x) - p_2 - p_3 \\ &< M(x). \end{aligned} \tag{19}$$

For $k < j \le s$, we have

$$\begin{aligned} M_j(x') &= p_1 \sum_{l=1}^{j} x_l' + p_2 x_j' + p_3 \sum_{l=j}^{s} x_l' \\ &= M_j(x) \\ &< M(x). \end{aligned} \tag{20}$$

From (16), (17), (18), (19) and (20), we see that x' is still optimal and $(1,j)-(2,j)-(3,j)$ is non-critical for $k \le j \le s$ in $N(x')$, so the index k' of the last critical segment $(1,k')-(2,k')-(3,k')$ in $N(x')$ must satisfy $k' < k$ which contradicts the definition of x.$\square$

Case 3 $p_2^2 > p_1 p_3$

In this case, we *assume* that $p_1 \le p_3$. If $p_1 > p_3$, we can similarly solve the reverse problem.

Lemma 7 *If* $p_2^2 > p_1 p_3$, g *denotes the last sublot with critical segment* $(1,g)-(2,g)$ *and* h' *the first sublot with critical segment* $(2,h')-(3,h')$, *then* $g \le h'$.

Proof. Suppose the lemma was not true, i.e., $g > h'$.

From the definition of sublots g and h', we know that $P_g[(1,h'),(2,g)]$ should be critical. So, it should be at least as long as $P_{h'}[(1,h'),(2,g)]$, i.e.,

$$p_1 \sum_{l=h'+1}^{g} x_l \ge p_2 \sum_{l=h'}^{g-1} x_l. \tag{21}$$

$P_{h'}[(2,h'),(3,g)]$ should be not shorter than $P_g[(2,h'),(3,g)]$, i.e.,

$$p_3 \sum_{l=h'}^{g-1} x_l \ge p_2 \sum_{l=h'+1}^{g} x_l. \tag{22}$$

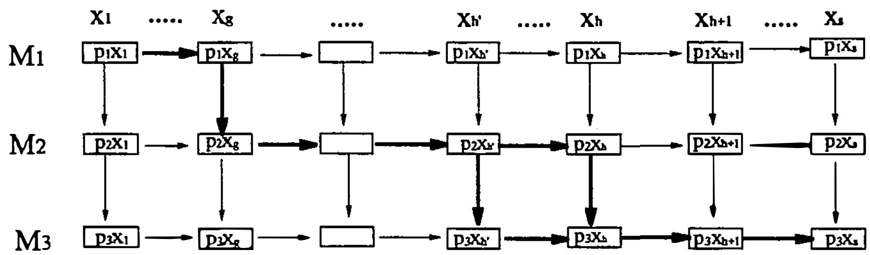

Figure 3: The network structure when $p_2^2 > p_1p_3$

Multiplying (21) with (22) yields a contradiction with the assumption $p_2^2 > p_1p_3$. □

Without loss of generality, we can write the optimal sublot sizes $x = (0, ..., 0, x_r, ..., x_s)$ with $x_i > 0$ for $r \leq i \leq s$, where r is the first sublot with positive size. Lemma 7 implies that the critical segment $(1,1) - ... - (1,g) - (2,g)$ is continued in a critical path of the form $P_{g,h'}$ when $p_2^2 > p_1p_3$. It may be possible to have a "block" of critical segments of the form $(2,j) - (3,j)$ $(j \geq h')$, so let us denote by h the index of the sublot for which $P_{g,j}$ is critical for $j \in [h', h]$, but $P_{g,h+1}$ is not (if it exists). The network is demonstrated in Figure 3.

Glass *et al* [8] have shown that when $p_2^2 > p_1p_3$, the optimal network for the continuous problem is perfectly balanced in the sense that there is a sublot k such that every $P_{i,j}$ is critical for $1 \leq i \leq k \leq j \leq s$. Although the situation is much more complicated in the discrete case, we show in the following series of lemmas that there is an *almost perfectly balanced* optimal solution in this case too.

Lemma 8 *If* $p_2^2 > p_1p_3$, *then there is an optimal solution for which*

a) $L_i[(1,i),(2,g)] + p_2 > L_g[(1,i),(2,g)]$ *for* $r < i < g$, *and*

b) $L_r[(1,r),(2,g)] + p_2 > L_g[(1,r),(2,g)]$ *if* $g \neq s$ *and* $p_1 < p_3$.

Proof. Let $i > r$ be the first sublot in an optimal solution $x = (x_1, ..., x_s)$, which does not satisfy the lemma. We can construct alternative optimal sublot sizes $x' = (x'_1, ..., x'_s)$ so that sublot i satisfies the lemma in $N(x')$.

Let $x'_{i-1} = x_{i-1} - 1$, $x'_i = x_i + 1$ and $x'_l = x_l$ for $l \in \{1, ...s,\} - \{i-1, i\}$.

For any path containing a segment $(l, i-2) - (l, i-1) - (l, i) - (l, i+1)$, its length remains unchanged for $l = 1, 2, 3$.

Any path containing the segment $(1, i-1) - (2, i-1) - (2, i) - (2, i+1)$ is shorter in $N(x')$ than in $N(x)$.

Any path containing the segment $(2,i-1)-(3,i-1)-(3,i)$ has a shorter length in $N(x')$ than in $N(x)$.

Since sublot i does not satisfy the lemma, we know that $p_2 \sum_{l=i}^{g-1} x_l + p_2 \leq p_1 \sum_{l=i+1}^{g} x_l$, implying

$$p_2 \sum_{l=i}^{g-1} x_l' \leq p_1 \sum_{l=i+1}^{g} x_l'. \tag{23}$$

This means that $P_i[(1,i),(2,g)]$ is not longer than $P_g[(1,i),(2,g)]$ in $N(x')$.

No path containing the segment $(2,i)-(3,i)$ can be critical in $N(x')$: otherwise, $P_i[(2,i),(3,g)]$ would not be shorter than $P_g[(2,i),(3,g)]$, i.e.,

$$p_3 \sum_{l=i}^{g-1} x_l' \geq p_2 \sum_{l=i+1}^{g} x_l', \tag{24}$$

and multiplying (23) and (24) would yield a contradiction with the assumption $p_2^2 > p_1 p_3$.

It follows from the above that $P_{g,h'}$ is still critical in $N(x')$ with length

$$\begin{aligned} M(x') &= p_1 \sum_{l=1}^{g} x_l' + p_2 \sum_{l=g}^{h} x_l' + p_3 \sum_{l=h}^{s} x_l' \\ &= M(x). \end{aligned}$$

Thus x' is optimal and we can keep repeating the transformation until the lemma holds for sublot i.

We can apply similarly the transformation of shifting an item to the right for all the i values to $g-1$ (in increasing order), and finally we obtain an optimal solution x' satisfying the lemma for $r < i < g$.

Assume that the lemma does not hold when $i = r$ and $g \neq s$. We construct alternative sublot sizes x' with a shorter critical path in $N(x')$, which yields a contradiction with the optimality of x.

Consider $x_r' = x_r + 1$, $x_l' = x_l$ for $r < l < s$ and $x_s' = x_s - 1$.

By assumption, we know that $p_2 \sum_{l=r}^{g-1} x_l + p_2 \leq p_1 \sum_{l=r+1}^{g} x_l$, implying

$$p_2 \sum_{l=r}^{g-1} x_l' \leq p_1 \sum_{l=r+1}^{g} x_l'. \tag{25}$$

This means that $P_g[(1,1),(2,g)]$ is not shorter than $P_r[(1,1),(2,g)]$ in $N(x')$.

$P_r[(2,r),(3,g)]$ cannot be critical in $N(x')$: otherwise, it would not be shorter than $P_g[(2,r),(3,g)]$, i.e.,

$$p_3 \sum_{l=r}^{g-1} x_l' \geq p_2 \sum_{l=r+1}^{g} x_l', \tag{26}$$

and multiplying (25) and (26) would yield a contradiction with the assumption of $p_2^2 > p_1 p_3$.

We can show similarly that $P_i[(2,i),(3,g)]$ cannot be critical in $N(x')$ for $r < i < g$.

The length of every path $P_i[(1,1),(2,g)]$, for $r < i \leq g$, is greater by the same amount (p_1) in $N(x')$ than in $N(x)$. Thus $L_i[(1,1),(2,g)] \leq L_g[(1,1),(2,g)]$ for $r < i < g$ both in $N(x)$ and $N(x')$.

In summary, $P_g[(1,1),(2,g)]$ is still critical in $N(x')$. The length of any path $P_j[(2,g),(3,s)]$, for $g \leq j \leq s$, is shorter by at least p_3 in $N(x')$ than in $N(x)$. Therefore,

$$M(x') \leq M(x) + p_1 - p_3 < M(x),$$

which yields a contradiction with the optimality of x, so b) must hold in $N(x)$.□

Lemma 9 *If $p_2^2 > p_1 p_3$, then there is an optimal solution with a sublot q, $g \leq q \leq h$, such that*

a) $L_l[(1,g),(2,l)] + p_1 > L_g[(1,g),(2,l)]$ for $g < l \leq q$;

b) $L_l[(2,l),(3,h)] + p_2 > L_h[(2,l),(3,h)]$ for $q < l < h$.

Proof. Before proving the lemma, we show that there is an optimal solution x' in which every sublot j, $g < j < h$, satisfies at least one of the following two properties in $N(x')$:

$$L_j[(1,g),(2,j)] > L_g[(1,g),(2,j)] - p_1, \tag{27}$$

$$L_j[(2,j),(3,h)] > L_h[(2,j),(3,h)] - p_2. \tag{28}$$

Suppose there was a sublot i, $g < i < h$, which satisfies neither of (27) and (28). We can construct alternative sublot sizes $x' = (x_1', ..., x_s')$ such that x' is still optimal and satisfies at least one of the two properties.

Case a. $i < h - 1$

Let $x_l' = x_l$ for $l \in \{1, ..., s\} - \{i, i+1\}$, $x_i' = x_i + 1$ and $x_{i+1}' = x_{i+1} - 1$.

By assumption, $L_i[(1,g),(2,i)] \leq L_g[(1,g),(2,i)] - p_1$ in $N(x)$, i.e., $p_1 \sum_{l=g+1}^{i} x_l \leq p_2 \sum_{l=g}^{i-1} x_l - p_1$, implying

$$p_1 \sum_{l=g+1}^{i} x_l' \leq p_2 \sum_{l=g}^{i-1} x_l'. \tag{29}$$

This means that $L_g[(1,g),(2,i)] \geq L_i[(1,g),(2,i)]$ in $N(x')$.

By assumption, $L_i[(2,i),(3,h)] \leq L_h[(2,i),(3,h)] - p_2$ in $N(x)$, i.e., $p_3 \sum_{l=i}^{h-1} x_l \leq p_2 \sum_{l=i+1}^{h} x_l - p_2$, implying

$$p_3 \sum_{l=i}^{h-1} x_l' \leq p_2 \sum_{l=i+1}^{h} x_l'. \tag{30}$$

This means that $L_h[(2,i),(3,h)] \geq L_i[(2,i),(3,h)]$ in $N(x')$.

It is easy to see that all the other paths – not containing $(1,i)-(2,i)$ or $(2,i)-(3,i)$ – are not longer in $N(x')$ than before in $N(x)$. So x' is still an optimal solution and $M_{g,h}(x')$ is the length of a critical path.

We can keep repeating the transformation of shifting one unit to the left until at least one of (27) or (28) is satisfied by sublot i, for $i = g+1, ..., h-2$.

Case b. $i = h-1$

W.l.o.g. we assume that x is an optimal solution for which h is as small as possible among all optimal solutions. Suppose $i = h-1$ does not satisfy either of (27) or (28) in $N(x)$. Consider $x_l' = x_l \; for \; l \in \{1,...,s\} - \{h-1,h\}$, $x_{h-1}' = x_{h-1} + 1$ and $x_h' = x_h - 1$.

The path $P_{g,h-1}$ has length

$$\begin{aligned} M_{g,h-1}(x') &= p_1 \sum_{l=1}^{g} x_l' + p_2 \sum_{l=g}^{h-1} x_l' + p_3 \sum_{l=h-1}^{s} x_l' \\ &= M_{g,h-1}(x) + p_2 \\ &\leq M(x), \end{aligned} \tag{31}$$

where the inequality follows since (28) does not hold for $j = h-1$, by assumption.

The path $P_{g,h}$ has length

$$\begin{aligned} M_{g,h}(x') &= p_1 \sum_{l=1}^{g} x_l' + p_2 \sum_{l=g}^{h} x_l' + p_3 \sum_{l=h}^{s} x_l' \\ &= M_{g,h}(x) - p_3 \\ &< M(x). \end{aligned} \tag{32}$$

By assumption, $L_{h-1}[(1,g),(2,h-1)] \leq L_g[(1,g),(2,h-1)] - p_1$ in $N(x)$, i.e., $p_1 \sum_{l=g+1}^{h-1} x_l \leq p_2 \sum_{l=g}^{h-2} x_l - p_1$, implying

$$p_1 \sum_{l=g+1}^{h-1} x_l' \leq p_2 \sum_{l=g}^{h-2} x_l'. \tag{33}$$

This means that $L_g[(1,g),(2,h-1)] \geq L_{h-1}[(1,g),(2,h-1)]$ in $N(x')$, so $P_g[(1,g),(2,h-1)]$ must be critical in $N(x')$.

$M_{g,h+1}(x') = M_{g,h+1}(x) < M(x)$ by the definition of h, and $P_{g,h}$ cannot be critical by (32), so we must have $M_{g,h-1}(x') = M(x)$ in (31). This, however, would make $h-1$ the last critical sublot from the interval $[g,h]$ in $N(x')$, contradicting the assumption that x was such that h is the lowest possible value for this last critical sublot.

Now, we are ready to prove the lemma. W.l.o.g let us denote by x the original (possibly transformed) solution for which every sublot j, $g < j < h$, satisfies at least one (27) or (28).

Suppose the lemma did not hold. There must be sublots i and j, $i < j$, such that j satisfies (27) but not (28) and i satisfies (28) but not (27).

Since sublot j satisfies (27), we have

$$p_1 \sum_{l=g+1}^{j} x_l > p_2 \sum_{l=g}^{j-1} x_l - p_1. \tag{34}$$

Since sublot i does not satisfy (27),

$$p_2 \sum_{l=g}^{i-1} x_l - p_1 \geq p_1 \sum_{l=g+1}^{i} x_l. \tag{35}$$

Combining (34) and (35), we get

$$p_2 \sum_{l=i}^{j-1} x_l < p_1 \sum_{l=i+1}^{j} x_l. \tag{36}$$

Since sublot j does not satisfy (28),

$$p_2 \sum_{l=j+1}^{h} x_l - p_2 \geq p_3 \sum_{l=j}^{h-1} x_l. \tag{37}$$

Since sublot i satisfies (28), we have

$$p_2 \sum_{l=i+1}^{h} x_l - p_2 < p_3 \sum_{l=i}^{h-1} x_l. \tag{38}$$

Combining (37) and (38), we obtain

$$p_3 \sum_{l=i}^{j-1} x_l > p_2 \sum_{l=i+1}^{j} x_l. \tag{39}$$

Multiplying (36) and (39), however, yields a contradiction with the assumption of $p_2^2 > p_1 p_3$. □

We note that if $g = h$, then Lemma 9 holds vacuously, and the structure of the optimal network is similar to the simpler structure of Case 1.

Lemma 10 *If $p_2^2 > p_1p_3$ and $p_2 > p_3$, then there is an optimal solution for which $L_j[(2,h),(3,j)] + p_2 > L_h[(2,h),(3,j)]$, for $h < j \leq s$.*

Proof. Suppose the lemma does not hold for some sublots. Let $j \in [h+1, s]$ be the smallest such index. We construct sublot sizes x' such that x' is still optimal and satisfies the lemma for sublot j.

Let $x'_l = x_l$ for $l \in \{1, ..., s\} - \{j, j+1\}$, $x'_j = x_j + 1$ and $x'_{j+1} = x_{j+1} - 1$.

First we prove that $M_{g,h}(x')$ is still the length of the longest path in $N(x')$.

By assumption, $p_2 \sum_{l=h+1}^{j} x_l \leq p_3 \sum_{l=h}^{j-1} x_l - p_2$, implying

$$p_2 \sum_{l=h+1}^{j} x'_l \leq p_3 \sum_{l=h}^{j-1} x'_l. \tag{40}$$

This means that we have in $N(x')$

$$L_h[(2,h),(3,s)] \geq L_j[(2,h),(3,s)]. \tag{41}$$

We claim that $P_j[(1,h),(2,j)]$ cannot be critical in $N(x')$: If it was, it would be at least as long as $P_h[(1,h),(2,j)]$, i.e., $p_1 \sum_{l=h+1}^{j} x'_l \geq p_2 \sum_{l=h}^{j-1} x'_l$ and multiplying this with (40) would yield a contradiction with the assumption of $p_2^2 > p_1p_3$.

Since $x'_l = x_l$ for $l < j$ and $(2,h) - (3,h)$ is critical in $N(x)$,

$$p_2 \sum_{l=h+1}^{f} x'_l \leq p_3 \sum_{l=h}^{f-1} x'_l \quad for\ h < f < j. \tag{42}$$

Similarly, $(1,f) - (2,f)$ cannot be critical in $N(x')$, since if it was, we would have

$$p_1 \sum_{l=h+1}^{f} x'_l \leq p_2 \sum_{l=h}^{f-1} x'_l, \tag{43}$$

and multiplying (42) and (43) would again yield a contradiction with $p_2^2 > p_1p_3$.

Since the length of every other path (not covered by the above claims) is the same in $N(x)$ and $N(x')$, we have shown that the longest paths from $(1,1)$ to $(3,j)$ must be the same in $N(x')$ as in $N(x)$, and a longest path must go through $(2,h)$. Combining this fact with (41) implies that $P_{g,h}$ is still critical in $N(x')$ with $M(x') = M_{g,h}(x') = p_1 \sum_{l=1}^{g} x'_l + p_2 \sum_{l=g}^{h} x'_l + p_3 \sum_{l=h}^{s} x'_l = M(x)$.

If sublot j still does not satisfy the lemma in $N(x')$, then we can repeat the procedure by transforming further units from sublot $j+1$ to j, until the lemma is satisfied. This constructive proof can be applied to every sublot j for $h < j < s$.

Suppose sublot s did not satisfy the lemma. Consider $x'_{h-1} = x_{h-1} - 1$, $x'_s = x_s + 1$ and $x'_j = x_j$ for $j = \{1, ..., s-1\} - \{h-1\}$.

By assumption, $p_2 \sum_{l=h+1}^{s} x_l \leq p_3 \sum_{l=h}^{s-1} x_l - p_2$, implying

$$p_3 \sum_{l=h}^{s-1} x'_l \geq p_2 \sum_{l=h+1}^{s} x'_l, \tag{44}$$

which means $L_h[(2,h),(3,s)] \geq L_s[(2,h),(3,s)]$ in $N(x')$.

We claim that $P_s[(1,h),(2,s)]$ cannot be critical in $N(x')$: If it was, it would be at least as long as $P_h[(1,h),(2,s)]$, i.e.,

$$p_1 \sum_{l=h+1}^{s} x'_l \geq p_2 \sum_{l=h}^{s-1} x'_l, \tag{45}$$

and multiplying (44) and (45) yields a contradiction with $p_2^2 > p_1 p_3$.

Using a similar argument, it can be shown that the segment $P_f[(1,h),(2,f)]$ cannot be critical in $N(x')$ for $h < f < s$.

In summary, $P_h[(2,h),(3,s)]$ is still critical in $N(x')$.

The length of every path $P_i[(1,1),(2,h-1)]$ for $r \leq i \leq h-1$ is shorter by at least p_2 in $N(x')$ than in $N(x)$. Thus $L_i[(1,1),(2,h-1)] \leq L_g[(1,1),(2,h-1)]$ both in $N(x)$ and $N(x')$, and $P_{g,h}$ is still critical in $N(x')$. Therefore,

$$\begin{aligned} M(x') &= p_1 \sum_{l=1}^{g} x'_l + p_2 \sum_{l=g}^{h} x'_l + p_3 \sum_{l=h}^{s} x'_l \\ &= M(x) + p_3 - p_2 \\ &< M(x), \end{aligned}$$

which contradicts the assumption that x is optimal.□

Lemma 11 *If $p_2^2 > p_1 p_3$ and $p_2 \leq p_3$, then there is an optimal solution for which*

a) $L_j[(2,h),(3,j)] + p_3 > L_h[(2,h),(3,j)]$ for $h+1 < j \leq s$;

b) $L_{h+1}[(2,h),(3,h+1)] + p_3 + p_2 > L_h[(2,h),(3,h+1)]$.

Proof. Suppose part a) of the lemma does not hold for some sublots. Let $j \in (h+1, s]$ be the biggest such index. We construct sublot sizes x' such that x' is still optimal and satisfies the lemma for sublot j.

Let $x'_l = x_l$ for $l \in \{1, ..., s\} - \{j-1, j\}$, $x'_{j-1} = x_{j-1} - 1$ and $x'_j = x_j + 1$.

First we prove that $M_{g,h}(x')$ is still the length of the longest path in $N(x')$.

By assumption, $p_2 \sum_{l=h+1}^{j} x_l + p_3 \leq p_3 \sum_{l=h}^{j-1} x_l$, implying

$$p_2 \sum_{l=h+1}^{j} x_l^{'} \leq p_3 \sum_{l=h}^{j-1} x_l^{'}. \tag{46}$$

This means that we have in $N(x^{'})$

$$L_h[(2,h),(3,j)] \geq L_j[(2,h),(3,j)]. \tag{47}$$

We claim that $P_j[(1,h),(2,j)]$ cannot be critical in $N(x^{'})$: If it was, it would be at least as long as $P_h[(1,h),(2,j)]$, i.e.,

$$p_1 \sum_{l=h+1}^{j} x_l^{'} \geq p_2 \sum_{l=h}^{j-1} x_l^{'}, \tag{48}$$

and multiplying (46) and (48) would yield a contradiction with $p_2^2 > p_1 p_3$.

$L_f[(1,h),(2,j)]$ cannot be longer in $N(x^{'})$ than in $N(x)$ for $h < f < j$, since we shifted a unit from sublot $j-1$ to j.

Since the length of every other path (not covered by the above claims) is the same in $N(x)$ and $N(x^{'})$, we have shown that the longest path from $(1,1)$ to $(3,j)$ must be the same in $N(x^{'})$ as in $N(x)$, and this longest path must go through $(2,h)$. Combining this fact with (47) implies that $P_{g,h}$ is still critical in $N(x^{'})$ with $M(x^{'}) = M_{g,h}(x^{'}) = p_1 \sum_{l=1}^{g} x_l^{'} + p_2 \sum_{l=g}^{h} x_l^{'} + p_3 \sum_{l=h}^{s} x_l^{'} = M(x)$.

If sublot j still does not satisfy the lemma in $N(x^{'})$, then we can repeat the procedure by shifting further units from sublot $j-1$ to j, until the lemma is satisfied. This constructive proof can be applied to every sublot j for $h+1 < j \leq s$ (in decreasing order of j).

W.l.o.g. we assume that x is now an optimal solution which satisfies part a) of the lemma and for which h is as small as possible among all the optimal solutions.

For part b), suppose sublot $h+1$ does not satisfy the lemma in $N(x)$. Consider $x_h^{'} = x_h - 1$, $x_{h+1}^{'} = x_{h+1} + 1$ and $x_l^{'} = x_l$ for $l = \{1, ..., s\} - \{h, h+1\}$.

By assumption, $p_2 x_{h+1} \leq p_3 x_h - p_2 - p_3$, which implies

$$p_3 x_h^{'} \geq p_2 x_{h+1}^{'}. \tag{49}$$

We claim that segment $(1,h+1)-(2,h+1)$ cannot be critical in $N(x^{'})$: If it was, segment $(1,h)-(1,h+1)-(2,h+1)$ would be at least as long as segment $(1,h)-(2,h)-(2,h+1)$, i.e.,

$$p_1 x_{h+1}^{'} \geq p_2 x_h^{'}, \tag{50}$$

and multiplying (49) and (50) yields a contradiction with $p_2^2 > p_1p_3$.

It is clear that

$$\begin{aligned} M_{g,h}(x') &= p_1 \sum_{l=1}^{g} x_l' + p_2 \sum_{l=g}^{h} x_l' + p_3 \sum_{l=h}^{s} x_l' \\ &= M(x) - p_2 \\ &< M(x). \end{aligned}$$

Furthermore, $M_{g,h+1}(x') = M_{g,h+1}(x) + p_3 \leq M_{g,h}(x) - p_2 < M(x)$, where the first inequality follows from the assumption that b) does not hold.

Thus, we cannot have $M(x') = M_{g,h}(x')$ or $M_{g,h+1}(x')$, as these would contradict the optimality of x. The only possibility left is to have $M_{g,h-1}(x') = M(x') = M(x)$, but this contradicts the definition of x, i.e., that it is a solution with the smallest possible h. □

Theorem 12 *If* $p_2^2 > p_1p_3$ *and* $p_2 > p_3$, *then there is an optimal solution for which Lemma 8, 9 and 10 simultaneously hold. If* $p_2^2 > p_1p_3$ *and* $p_2 \leq p_3$, *then there is an optimal solution for which Lemma 8, 9 and 11 simultaneously hold.*

Proof. We observe that the transformations applied in the proof of the three applicable lemmas always changed *different* parts of the solution. (The transformations considered for Lemma 8.b, Lemma 11.b, and for the $j = s$ case of Lemma 10 were not implemented. They were only considered to derive the necessary contradictions.) Therefore, the actually applied transformations were never in conflict, i.e., they can be used sequentially to arrive at a solution satisfying all three applicable lemmas. □

4 Problem solution

Case 1 $p_2^2 \leq p_1p_3$ and $p_1 \neq p_3$

Theorem 13 *If* $p_2^2 \leq p_1p_3$ *and* $p_1 < p_3$, *we can determine in* $O(s)$ *time whether there exist lot sizes* $x = (x_1, ..., x_s)$ *satisfying the following additional conditions:*

i) the segment $(1,1) - (2,1) - (3,1)$ *is critical;*

ii) $x_1 = 1$.

If such a solution exists, then it is optimal and we can find it in $O(s)$ *time.*

Proof. By Theorem 1, there exists a k such that $(1,k)-(2,k)-(3,k)$ is critical in $N(x)$ for any feasible x. Therefore, $(1,1)-(2,1)-(3,1)$ is critical iff $L_1[(1,1),(3,j)] \geq L_j[(1,1),(3,j)]$ in $N(x)$, i.e.,

$$p_1 \sum_{l=2}^{j-1} x_l + (p_1+p_2)x_j \leq p_3 \sum_{l=1}^{j-1} x_l + p_2 x_1 \text{ for } 2 \leq j \leq s. \tag{51}$$

Substituting $x_1 = 1$ into (51) for $j = 2, ..., s$, we can recursively determine the largest $x_2, ..., x_s$ values allowed by (51).

If for the resulting x values, we have $\sum_{l=1}^{s} x_l > U$, then optimal sublot sizes $x^* = (x_1^*, ..., x_s^*)$ can be obtained in the following way: Let $x_i^* = x_i$ for $1 \leq i \leq r$, where the r is such that $\sum_{l=1}^{r} x_l \leq U$ and $\sum_{l=1}^{r+1} x_l > U$, $x_{r+1}^* = U - \sum_{l=1}^{r} x_l$ and $x_j^* = 0$ for $r+1 < j \leq s$.

It is clear that P_1 is a critical path in $N(x^*)$ with length

$$\begin{aligned} M(x^*) &= (p_1+p_2)x_1^* + p_3 \textstyle\sum_{l=1}^{s} x_l^* \\ &= p_1 + p_2 + p_3 U, \end{aligned}$$

which is also a lower bound in any network. Therefore, x^* is optimal.

If for the resulting x values, we have $\sum_{l=1}^{s} x_l = U$, then the preceding argument can be repeated to show that $x^* = x$ is an optimal solution.

If $\sum_{l=1}^{s} x_l < U$, then we show that the segment $(1,1)-(2,1)-(3,1)$ cannot be critical when $x_1 = 1$.

Suppose there is an optimal solution $x^* = (x_1^*, ..., x_s^*)$, in which $x_1^* = 1$ and $(1,1)-(2,1)-(3,1)$ is critical. This implies that x^* must also satisfy (51). Since we chose each x_j to be as large as possible, when recursively solving (51), we have $x_j^* \leq x_j$ for $j = 1, 2, ..., s$. Therefore, $\sum_{l=1}^{s} x_l^* \leq \sum_{l=1}^{s} x_l < U$, which yields a contradiction with the feasibility of x^*.

It is clear that the total time required for the recursive solution of (51) is $O(s)$. □

When the "extreme" solution of Theorem 13 does not exist, we look for the balanced optimal solution of Theorem 4. The following bound was proved in [6].

Theorem 14 *If $p_2^2 \leq p_1 p_3$, M_c denotes the optimal makespan for the continuous version of the problem, then the optimal solution x^* of the discrete problem satisfies $M(x^*) < M_c + p_1 + p_2$.*

Theorem 15 *If $p_2^2 \leq p_1 p_3$, $p_1 < p_3$ and no optimal solution satisfying the conditions of Theorem 13 exists, then the sublot sizes of a balanced optimal solution can be found in $O(s)$ time.*

Proof. Let M_0 be the trial length of the critical path in the optimal network. If M_0 is the optimal value for the makespan, then there is also a balanced optimal solution $x^0 = (x_1^0, ..., x_s^0)$ which satisfies Theorem 4. This solution must satisfy $M_0 \geq M_s(x^0) = p_1 \sum_{l=1}^{s} x_l^0 + (p_2+p_3)x_s^0 > M_0 - (p_2+p_3)$, implying

$$(M_0 - p_1U)/(p_2+p_3) \geq x_s^0 > (M_0 - p_1U)/(p_2+p_3) - 1. \tag{52}$$

Note, however, that there is only a *unique* integer value x_s^0 which satisfies (52).

Similarly, every other path P_i $(1 \leq i < s)$ must satisfy $M_0 \geq M_i(x^0) = p_1 \sum_{l=1}^{i} x_l^0 + (p_2+p_3)x_i^0 + p_3 \sum_{l=i+1}^{s} x_l^0 > M_0 - (p_2+p_3)$. So, we must have

$$\frac{M_0 - p_1(U - \sum\limits_{l=i+1}^{s} x_l^0) + p_2 \sum\limits_{l=i+1}^{s} x_l^0}{p_2+p_3} \geq \sum_{l=i}^{s} x_l^0 > \frac{M_0 - p_1(U - \sum\limits_{l=i+1}^{s} x_l^0) + p_2 \sum\limits_{l=i+1}^{s} x_l^0}{p_2+p_3} - 1 \tag{53}$$

Note again that, given $x_s^0, ..., x_{i+1}^0$, there is a *unique* integer value for x_i^0 which satisfies (53). Thus we can obtain recursively $x_{s-1}^0, ..., x_1^0$.

If $\sum_{l=1}^{s} x_s^0 > U$, then this implies $M(x^*) < M_0$: Assume that $M(x^*) \geq M_0$ and w.l.o.g. we can assume that x^* is a balanced solution satisfying Theorem 4. Therefore, x^* must satisfy (52) and (53) if we replace everywhere M_0 by $M(x^*)$. $M(x^*) \geq M_0$, however, implies that $\sum_{l=i}^{s} x_l^* \geq \sum_{l=i}^{s} x_l^0$ for $i = s, s-1, ..., 1$. Therefore, $\sum_{l=1}^{s} x_l^* \geq \sum_{l=1}^{s} x_l^0 > U$, which yields a contradiction with the feasibility of x^*.

If $\sum_{l=1}^{s} x_s^0 < U$, then we can show similarly that $M(x^*) > M_0$.

If $\sum_{l=1}^{s} x_s^0 = U$, then we show that $M(x^*) \leq M_0$: Since x^0 satisfies all inequalities in (53), this implies that $M_i(x^0) \leq M_0$ for $1 \leq i \leq s$. Since $M(x^0) = \max\limits_{1 \leq i \leq s} M_i(x^0)$, we have $M(x^*) \leq M(x^0) \leq M_0$.

In summary, we can decide the feasibility of an M_0 value in $O(s)$ time. Theorem 14 establishes the narrow feasible range of $[M_c, M_c+p_1+p_2]$ for M_0. Since $M_s(x^0)\epsilon(M_0 - p_2 - p_3, M_0]$ by (52), we must have $M_s(x^0)\ \epsilon(M_c - p_2 - p_3, M_c + p_1 + p_2]$. Notice, however, that increasing x_s^0 by 1 unit increases $M_s(x^0)$ by $p_2+p_3 > p_1+p_2$. Thus, there are at most two feasible x_s^0 values for which $M_s(x^0)$ falls in the right range. So we need to test only at most three M_0 values. Thus, the total time required is $O(s)$ indeed.□

Case 2 $p_2^2 \leq p_1 p_3$ and $p_1 = p_3$

Theorem 16 *For* $p_2 \leq p_1 p_3$ *and* $p_1 = p_3$,

i) if U *is divisible by* s, *then the optimal solution is* $x_i = U/s$ *for* $1 \leq i \leq s$;

ii) if U *is not divisible by* s, *then the optimal solution is*

$$\begin{array}{lll} x_i = \lceil U/s \rceil & \text{for } 1 \le i \le k \\ x_j = \lceil U/s \rceil - 1 & \text{for } k < j \le s \end{array}, \quad \text{where } k = U - s\lfloor U/s \rfloor .$$

Proof. By Theorem 1, there is always a critical path containing $(1,k)-(2,k)-(3,k)$ for some $k \epsilon \{1,...,s\}$. If U is divisible by s, it is clear that all paths P_i, $1 \le i \le s$, have the same length in the network when $x_i = U/s$ for $1 \le i \le s$, i.e.,

$$\begin{aligned} M_i(x) &= p_1 \sum_{l=1}^{i} x_l + p_2 x_i + p_3 \sum_{l=i}^{s} x_l \\ &= (p_1 + p_2)U/s + p_3 U, \end{aligned}$$

which is a lower bound by Lemma 5. Therefore, x is optimal indeed.

If U is not divisible by s and x is as defined in ii), it is clear that P_i is critical when $1 \le i \le k$ and is non-critical when $k < i \le s$. Thus, for $1 \le i \le k$,

$$\begin{aligned} M(x) &= p_1 \sum_{l=1}^{i} x_l + p_2 x_i + p_3 \sum_{l=i}^{s} x_l \\ &= p_1 \lceil U/s \rceil + p_2 U, \end{aligned}$$

which is also a lower bound by Lemma 5. Therefore, x is optimal again.□

Case 3 $p_2^2 > p_1 p_3$

Theorem 17 *If $p_2^2 > p_1 p_3$, then a balanced optimal solution can be obtained in $O(s \log U)$ time if $g = s$.*

Proof. Let us assume $g = s$ and let $x^0 = (x_1^0, ..., x_s^0)$ be trial values for an optimal solution $x^* = (x_1^*, ..., x_s^*)$ satisfying Lemma 8. Since we do not know the index r in the lemma, we assume that $r = 1$ for the time being. Accordingly, x^0 must satisfy

$$\frac{p_1}{p_2} \sum_{l=i+1}^{s} x_l^0 \ge \sum_{l=i}^{s-1} x_l^0 > \frac{p_1}{p_2} \sum_{l=i+1}^{s} x_l^0 - 1 \text{ for } 1 < i < s. \tag{54}$$

Notice that if we assign x_s^0 an arbitrary trial value, then there is only a *unique* integer between the lower and upper bounds of (54), so a recursive calculation determines x_i^0 for $i = s-1, ..., 2$. Since $g = s$, $L_s[(1,1),(2,s)] \ge L_1[(1,1),(2,s)]$, i.e.,

$$p_1 \sum_{l=2}^{s} x_l^0 \ge p_2 \sum_{l=1}^{s-1} x_l^0. \tag{55}$$

Thus, x_1^0 can be determined by making it the largest integer allowed by (55).

We claim that if $\sum_{l=1}^{s} x_l^0 \geq U$, then $x_s^* \leq x_s^0$.

Define now r as the unique index for which $\sum_{l=r}^{s} x_l^0 \leq U$ and $\sum_{l=r-1}^{s} x_l^0 > U$. Letting $x_i^{'} = x_i^0$ for $r \leq i \leq s$, $x_{r-1}^{'} = U - \sum_{l=r}^{s} x_l^0$, and $x_j^{'} = 0$ for $1 \leq j < r-1$, it is clear that $x^{'}$ is feasible ($\sum_{l=1}^{s} x_l^{'} = U$), so $M(x^*) \leq M(x^{'})$. Since no path can be longer in $N(x^{'})$ than in $N(x)$, P_s is still critical in $N(x^{'})$, and $M(x^{'}) = p_1 \sum_{l=1}^{s} x_l + (p_2+p_3)x_s^{'} = p_1 U + (p_2+p_3)x_s^0$. By assumption $g = s$, so $M(x^*) = p_1 U + (p_2+p_3)x_s^*$. Therefore, we have $x_s^* \leq x_s^0$ indeed.

We show that if $\sum_{l=1}^{s} x_l^0 < U$, then $x_s^* > x_s^0$.

W.l.o.g. we can assume that x^* is an optimal solution satisfying Lemma 8 a) with some r^* value, i.e., $\sum_{l=r^*}^{s} x_l^* = U$ and

$$\frac{p_1}{p_2} \sum_{l=i+1}^{s} x_l^* \geq \sum_{l=i}^{s-1} x_l^* > \frac{p_1}{p_2} \sum_{l=i+1}^{s} x_l^* - 1 \text{ for } r^* < i < s. \tag{56}$$

If we had $x_s^* \leq x_s^0$, then by recursively comparing (54) and (56), we would have $\sum_{l=i}^{s-1} x_l^* \leq \sum_{l=i}^{s-1} x_l^0$ for $r^* < i \leq s-1$. Similarly to (55), we would also have

$$p_1 \sum_{l=\max\{2,r^*+1\}}^{s} x_l^* \geq p_2 \sum_{l=\max\{1,r^*\}}^{s-1} x_l^*, \tag{57}$$

implying that $x_{r^*}^* \leq x_{r^*}^0$. Thus, $\sum_{l=1}^{s} x_l^* \leq \sum_{l=1}^{s} x_l^0 < U$, which contradicts the feasibility of x^*.

In summary, we can do bisection search for x_s^* over $[1, U]$. The time required is clearly $O(s \log U)$. □

Theorem 18 *If $p_2^2 > p_1 p_3$ and $g < s$, then a balanced optimal solution can be found in $O(s^4 \log^2 U)$ time.*

Proof. Let g^0, h^0 and q^0 be trial values for the indexes of sublots g, h and q, respectively, and let $x_{g^0}^0, x_{h^0}^0$ be trial values for the sizes of sublots g^0 and h^0 in an optimal solution $x^* = (x_1^*, ..., x_s^*)$ satisfying Theorem 12.

By Lemma 8, we have $p_1 \sum_{l=i+1}^{g^0} x_l^0 \geq p_2 \sum_{l=i}^{g^0-1} x_l^0 > p_1 \sum_{l=i+1}^{g^0} x_l^0 - p_2$, implying

$$\frac{p_1}{p_2} \sum_{l=i+1}^{g^0} x_l^0 \geq \sum_{l=i}^{g^0-1} x_l^0 > \frac{p_1}{p_2} \sum_{l=i+1}^{g^0} x_l^0 - 1 \text{ for } r \leq i < g^0. \tag{58}$$

For the time being, assume that $r = 1$ and recursively find the *unique* value of x_i^0, for $i = g^0 - 1, ..., 1$, which satisfies (58). Define r as the unique index for which $x_r^0 > 0$ and $x_{r-1}^0 = 0$ if it exists, otherwise r remains at 1.

By Lemma 9, we have $p_2 \sum_{l=g^0}^{i-1} x_l^0 \geq p_1 \sum_{l=g^0+1}^{i} x_l^0 > p_2 \sum_{l=g^0}^{i-1} x_l^0 - p_1$, implying

$$\frac{p_2}{p_1} \sum_{l=g^0}^{i-1} x_l^0 \geq \sum_{l=g^0+1}^{i} x_l^0 > \frac{p_2}{p_1} \sum_{l=g^0}^{i-1} x_l^0 - 1 \text{ for } g^0 < i \leq q^0. \tag{59}$$

Recursively, we can determine the *unique* value of x_i^0 which satisfies (59) for $i = g^0+1, \dots, q^0$.

Case i $p_2 > p_3$

By Lemma 10, we have $p_3 \sum_{l=h^0}^{j-1} x_l^0 \geq p_2 \sum_{l=h^0+1}^{j} x_l^0 > p_3 \sum_{l=h^0}^{j-1} x_l^0 - p_2$, implying

$$\frac{p_3}{p_2} \sum_{l=h^0}^{j-1} x_l^0 \geq \sum_{l=h^0+1}^{j} x_l^0 > \frac{p_3}{p_2} \sum_{l=h^0}^{j-1} x_l^0 - 1 \text{ for } h^0 < j \leq s. \tag{60}$$

Recursively, we can find the *unique* value of x_i^0 which satisfies (60) for $j = h^0+1, \dots, s$.

By Lemma 9, we have $p_2 \sum_{l=i+1}^{h^0} x_l^0 \geq p_3 \sum_{l=i}^{h^0-1} x_l^0 > p_2 \sum_{l=i+1}^{h^0} x_l^0 - p_2$, implying

$$\frac{p_2}{p_3} \sum_{l=i+1}^{h^0} x_l^0 \geq \sum_{l=i}^{h^0-1} x_l^0 > \frac{p_2}{p_3} \sum_{l=i+1}^{h^0} x_l^0 - \frac{p_2}{p_3} \text{ for } q^0 < i < h^0. \tag{61}$$

We can recursively determine the *unique largest* values y_l^0 $(q^0 < l < h^0)$ satisfying (61). Let $Upper = \sum_{l=q^0+1}^{h^0-1} y_l^0$. By recursively calculating the *unique smallest* values z_l^0 $(q^0 < l < h^0)$ which satisfy (61), we obtain $Lower = \sum_{l=q^0+1}^{h^0-1} z_l^0$.

We show that if $\sum_{l=1}^{q^0} x_l^0 + Lower + \sum_{l=h^0}^{s} x_l^0 > U$, then either $x_{g^0}^* < x_{g^0}^0$ or $x_{h^0}^* < x_{h^0}^0$.

Suppose there was an optimal solution x^*, satisfying Lemma 8, 9 and 10, with $g = g^0, h = h^0$ and $q = q^0$, in which $x_g^* \geq x_g^0$ and $x_h^* \geq x_h^0$. Since x^* satisfies both (58) and (59), x_l^*, for $l \epsilon \{1, \dots, g_0 - 1\} \cup \{g_0+1, \dots, q^0\}$, must be the unique value satisfying (58) and (59) when we substitute x_g^* into these inequalities. Therefore, $x_{g^0}^* \geq x_{g^0}^0$ implies $\sum_{l=1}^{g^0} x_l^* \geq \sum_{l=1}^{g^0} x_l^0$ and $\sum_{l=g^0+1}^{q^0} x_l^* \geq \sum_{l=g^0+1}^{q^0} x_l^0$. Similarly, x^* must satisfy (60) and (61), so $x_{h^0}^* \geq x_{h^0}^0$ implies $\sum_{l=h^0+1}^{s} x_l^* \geq \sum_{l=h^0+1}^{s} x_l^0$ and $\sum_{l=g^0+1}^{h^0-1} x_l^* \geq \sum_{l=g^0+1}^{h^0-1} z_l^0 = Lower$. Thus, $\sum_{l=1}^{s} x_l^* \geq \sum_{l=1}^{q^0} x_l^0 + Lower + \sum_{l=h^0}^{s} x_l^0 > U$, contradicting the feasibility of x^*.

We can show similarly that if $\sum_{l=1}^{q^0} x_l^0 + Upper + \sum_{l=h^0}^{s} x_l^0 < U$, then either $x_{g^0}^* > x_{g^0}^0$ or $x_{h^0}^* > x_{h^0}^0$.

Now we study the case when $\sum_{l=1}^{q^0} x_l^0 + Lower + \sum_{l=h^0}^{s} x_l^0 \leq U \leq \sum_{l=1}^{q^0} x_l^0 + Upper + \sum_{l=h^0}^{s} x_l^0$. We show how to find the yet undefined x_l^0 $(q^0 < l < h^0)$ values, so that x^0 becomes a feasible solution satisfying Theorem 12.

Letting $A = U - \sum_{l=1}^{q^0} x_l^0 - \sum_{l=h^0}^{s} x_l^0$, we can rewrite (61) as follows.

$$(1-\frac{p_3}{p_2})(A-\sum_{l=q^0+1}^{i-1} x_l^0)+x_h^0 \geq x_i^0 > (1-\frac{p_3}{p_2})(A-\sum_{l=q^0+1}^{i-1} x_l^0)+x_h^0-1 \text{ for } q^0 < i < h^0. \quad (62)$$

We can recursively find the *unique* $x_{q^0+1}, ..., x_{h^0-2}$ integer values satisfying (62). In order to ensure the feasibility of x^0, let $x_{h^0-1}^0 = A - \sum_{l=q^0+1}^{h^0-2} x_l^0$. We claim that $z_{h^0-1}^0 \leq x_{h^0-1}^0 \leq y_{h^0-1}^0$, which implies that (61) and (62) are satisfied by x^0 for $i = h^0 - 1$ too.

Suppose $x_{h^0-1}^0 < z_{h^0-1}^0$. Since (61) is satisfied by both x^0 and z^0 for $q^0 < i < h^0-1$,

$$p_2 \sum_{l=i+1}^{h^0} x_l^0 \geq p_3 \sum_{l=i}^{h^0} x_l^0 > p_2 \sum_{l=i+1}^{h^0} x_l^0 - p_2 \text{ for } q^0 < i < h^0 - 1 \quad (63)$$

and

$$p_2 \sum_{l=i+1}^{h^0} z_l^0 \geq p_3 \sum_{l=i}^{h^0} z_l^0 > p_2 \sum_{l=i+1}^{h^0} z_l^0 - p_2 \text{ for } q^0 < i < h^0 - 1. \quad (64)$$

As $x_{h^0-1}^0 < z_{h^0-1}^0$ by assumption, it follows recursively from (63) and (64) that $p_2 \sum_{l=i+1}^{h^0} x_l^0 \leq p_2 \sum_{l=i+1}^{h^0} z_l^0$ for $i = h^0 - 2, ..., q^0 + 1$. Therefore, $\sum_{l=i}^{h^0-1} x_l^0 \leq \sum_{l=i}^{h^0-1} z_l^0$ for $i = h^0 - 2, ..., q^0 + 1$. We have $U = \sum_{l=1}^{s} x_l^0 < \sum_{l=1}^{q^0} x_l^0 + \sum_{l=q^0+1}^{h^0-1} z_l^0 + \sum_{l=h^0}^{s} x_l^0$, which contradicts the assumption of $\sum_{l=1}^{q^0} x_l^0 + Lower + \sum_{l=h^0}^{s} x_l^0 \leq U$.

We can show similarly that $x_{h^0-1}^0$ cannot be bigger than $y_{h^0-1}^0$.

In summary, we can do bisection search on the values of x_g and x_h over $[1, U]$ for assumed positions of sublots q, g and h. This takes $O(s \log^2 U)$ time. The total time needed to search over all possible g, q and h is at most $O(s^4 \log^2 U)$.

Case ii $p_2 \leq p_3$

In addition to (58) and (59), we have by Lemma 9, $p_2 \sum_{l=i+1}^{h^0} x_l^0 \geq p_3 \sum_{l=i}^{h^0-1} x_l^0 > p_2 \sum_{l=i+1}^{h^0} x_l^0 - p_2$, implying, as $p_2/p_3 \leq 1$,

$$\frac{p_2}{p_3} \sum_{l=i+1}^{h^0} x_l^0 \geq \sum_{l=i}^{h^0-1} x_l^0 > \frac{p_2}{p_3} \sum_{l=i+1}^{h^0} x_l^0 - 1 \text{ for } q^0 < i < h^0. \quad (65)$$

Recursively, we find the *unique* value of x_i^0 which satisfies (65) for $q^0 < i < h^0$.

By Lemma 11, $p_3 x_{h^0} \geq p_2 x_{h^0+1} > p_3 x_{h^0} - p_2 - p_3$, implying

$$(p_3/p_2) x_{h^0} \geq x_{h^0+1} > (p_3/p_2) x_{h^0} - 1 - p_3/p_2. \quad (66)$$

Also by Lemma 11, we have $p_3 \sum_{l=h^0}^{j-1} x_l^0 \geq p_2 \sum_{l=h^0+1}^{j} x_l^0 > p_3 \sum_{l=h^0}^{j-1} x_l^0 - p_3$, implying

$$\frac{p_3}{p_2} \sum_{l=h^0}^{j-1} x_l^0 \geq \sum_{l=h^0+1}^{j} x_l^0 > \frac{p_3}{p_2} \sum_{l=h^0}^{j-1} x_l^0 - \frac{p_3}{p_2} \text{ for } h^0 + 1 < j \leq s. \quad (67)$$

We can recursively determine the *unique largest* values Y_l^0 ($h^0 < l \le s$) satisfying (66) and (67). Let $UPPER = \sum_{l=h^0+1}^{s} Y_l^0$. By recursively calculating the *unique smallest* values Z_l^0 ($h^0 < l \le s$) which satisfy (66) and (67), we obtain $LOWER = \sum_{l=h^0+1}^{s} Z_l^0$.

We show that if $\sum_{l=1}^{h^0} x_l^0 + LOWER > U$, then either $x_{g^0}^* < x_{g^0}^0$ or $x_{h^0}^* < x_{h^0}^0$.

Suppose there was an optimal solution x^*, satisfying Lemma 8, 9 and 11, with $g = g^0, h = h^0$ and $g = g^0$, in which $x_g^* \ge x_g^0$ and $x_h^* \ge x_h^0$. Since x^* satisfies both (58) and (59), x_l^*, for $l \epsilon \{1, ..., g_0 - 1\} \cup \{g_0 + 1, ..., q^0\}$, must be the unique value satisfying (58) and (59) when we substitute x_g^* into these inequalities. Therefore, $x_{g^0}^* \ge x_{g^0}^0$ implies $\sum_{l=1}^{g^0} x_l^* \ge \sum_{l=1}^{g^0} x_l^0$ and $\sum_{l=g^0+1}^{q^0} x_l^* \ge \sum_{l=g^0+1}^{q^0} x_l^0$. Similarly, x^* must satisfy (65), (66) and (67), so $x_{h^0}^* \ge x_{h^0}^0$ implies $\sum_{l=q^0+1}^{h^0-1} x_l^* \ge \sum_{l=q^0+1}^{h^0-1} x_l^0$, and $\sum_{l=h^0+1}^{s} x_l^* \ge \sum_{l=h^0+1}^{s} Z_l^0 = LOWER$. Thus, $\sum_{l=1}^{s} x_l^* \ge \sum_{l=1}^{h^0} x_l^0 + LOWER > U$, contradicting the feasibility of x^*.

We can show similarly that if $\sum_{l=1}^{h^0} x_l^0 + UPPER < U$, then either $x_{g^0}^* > x_{g^0}^0$ or $x_{h^0}^* > x_{h^0}^0$.

Now we study the case when $\sum_{l=1}^{h^0} x_l^0 + LOWER \le U \le \sum_{l=1}^{h^0} x_l^0 + UPPER$. We show how to find the yet undefined x_l^0 ($h^0 < l \le s$) values so that x^0 becomes a feasible solution satisfying Theorem 12.

Letting $B = U - \sum_{l=1}^{h^0} x_l^0$, we can rewrite (67) as follows.

$$(1 - \frac{p_2}{p_3})(B - \sum_{l=j+1}^{s} x_l^0) + x_h^0 \ge x_j^0 > (1 - \frac{p_2}{p_3})(B - \sum_{l=j^0+1}^{s} x_l^0) + x_h^0 - 1 \text{ for } h^0 + 1 < j \le s. \tag{68}$$

We can recursively find the *unique* $x_s^0, ..., x_{h^0+2}^0$ integer values satisfying (68). In order to ensure the feasibility of x^0, let $x_{h^0+1}^0 = B - \sum_{l=h^0+2}^{s} x_l^0$. We claim that $Z_{h^0+1}^0 \le x_{h^0+1}^0 \le Y_{h^0+1}^0$, which implies that (66) is also satisfied by x^0. Suppose $x_{h^0+1}^0 < Z_{h^0+1}^0$. Since (67) is satisfied by both x^0 and Z^0,

$$p_3 \sum_{l=h^0}^{j-1} x_l^0 \ge p_2 \sum_{l=h^0+1}^{j} x_l^0 > p_3 \sum_{l=h^0}^{j-1} x_l^0 - p_3 \text{ for } h^0 + 1 < j \le s \tag{69}$$

and

$$p_3 x_{h^0}^0 + p_3 \sum_{l=h^0+1}^{j-1} Z_l^0 \ge p_2 \sum_{l=h^0+1}^{j} Z_l^0 > p_3 x_{h^0}^0 + p_3 \sum_{l=h^0+1}^{j-1} Z_l^0 - p_3 \text{ for } h^0 + 1 < j \le s. \tag{70}$$

As by assumption $x_{h^0+1}^0 < Z_{h^0+1}^0$, it follows recursively from (69) and (70) that $p_3 \sum_{l=h^0+1}^{j-1} x_l^0 < p_3 \sum_{l=h^0+1}^{j-1} Z_l^0$ for $j = h^0 + 2, ..., s$. We have $U = \sum_{l=1}^{s} x_l^0 < \sum_{l=1}^{h^0} x_l^0 + \sum_{l=h^0+1}^{s} Z_l^0$, which contradicts the assumption of $\sum_{l=1}^{h^0} x_l^0 + LOWER \le U$.

We can show similarly that $x_{h^0+1}^0$ cannot be bigger than $Y_{h^0+1}^0$.

In summary, we can do bisection search on the values of x_g and x_h over $[1, U]$ for assumed positions of sublots q, g and h. This takes $O(s \log^2 U)$ time. The total time needed to search over all possible g, q and h is at most $O(s^4 \log^2 U)$.□

5 Summary and concluding remarks

We have analyzed the structural properties of discrete lot streaming schedules which minimize the makespan for a single job in a three-machine flow shop. The structure of the optimal network depends on the relative size of the processing times on the three machines. We have proved the existence of almost perfectly balanced optimal schedules, in which the longest paths all have almost the same length. This fact was used in designing the first polynomial time algorithms for the optimal solution of the problem. The makespan of these optimal solutions is surprisingly close to the minimum makespan of the continuous version. The sublots for the continuous problem follow various geometric progressions, depending on the relative sizes of the processing times on the different machines. Although these types of patterns cannot be followed exactly by the integer valued solutions, our results imply that the discrete sublot sizes deviate only little from these patterns.

In certain situations it is desirable to have *no-wait* schedules, i.e., to be able to start the processing of each sublot on every machine immediately after it is finished on the preceding machine. It can be easily checked from the structure of the optimal networks for the continuous case [8] that the no-wait requirement can be satisfied *without* increasing the length of the optimal schedule. Since the length of our optimal discrete schedules is always within $max\{p_1 + p_2, p_2 + p_3\}$ of the minimum makespan for the continuous relaxation [6], the balanced optimal schedules are also very close to satisfying the no-wait requirement.

There are many related topics for future investigation. It seems to be natural to try to extend the results in this paper to the case of job shops and open shops and to the case of more than three machines in flow shops. Some of these problems will be studied in the future.

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