BOUNDED NOETHERIAN PRIME RINGS OF INJECTIVE DIMENSION ONE

By

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ABSTRACT

This thesis studies the maximal ideals and minimal overrings of the rings of the title. It is shown that then maximal ideals are segregated into projective and non-projectives with no interplay between these classes. Moreover, the projective maximal ideals behave as though the ring were hereditary.

The maximal spectrum of minimal equivalent orders is calculated in terms of that of R. This enables a comparison of their link-graphs (they are almost the same) and a characterization of when a minimal equivalent order also has the attributes of the title. This inductive property is shown to be preserved by the "cycle map" as well as passing up to the minimal equivalent order itself.

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INTRODUCTION

This thesis has grown out of an attempt to study links in the fairly tame setting of the title. Links were introduced by Mueller in [13] as obstructions to localization at a prime ideal in a fully bounded noetherian ring. It is hoped that by studying links in rings of low dimension that a better understanding of links in general will follow, shedding some light on the problem on localization in non-commutative rings in general.

Much of this thesis is inspired by the work of Eisenbud, Robson and others on the localizations of a bounded noetherian prime ring of global dimension one (all modules have injective dimension less or equal one) which are called bounded HNP rings. In this setting one of the major tools is the cycle map. It has been shown that the link directed-graph is the same as the cycle map directed-graph in a bounded HNP ring. The links are thus too well-behaved here and a more general setting is more appropriate.

The rings of the title (where only the R-modules ${}_RR$ and R_R are assumed to have injective dimension one) are precisely the bounded noetherian prime rings for which the cycle map is still well-defined (ie. is still a map). The link directed-graph is not nearly as well behaved, but one still has enough control to make their study feasible.

In Chapter I, the first three sections present known material which is necessary for the rest of the thesis. Section IV is concerned with three related types of questions: (1) Since in a bounded HNP ring all one sided ideals are projective, to what extent do projective ideals in our rings mimic ideals in bounded HNP rings? (2) Since all of the assumptions of the title are symmetric, can one-sided properties be shown to be two-sided (expecially projectivity)? (3) Since links occur exactly between non-commuting maximal ideals, what sort of commutativity results can be shown for fractional-ideals other than maximal ideals?

Chapter II is concerned with equivalent orders (especially minimal ones). The main questions are: (1) When is an equivalent order also bounded noetherian prime with injective dimension one? (2) Can one describe the spectrum of a minimal equivalent order in terms of the spectrum of R? (3) Can one describe the links and "cycle map" of a minimal equivalent order in terms of those of R?

CHAPTER I

FRACTIONAL-IDEAL THEORY

Throughout the thesis R will be a not necessarily commutative associative ring with unity denoted by 1. K will be reserved for the Goldie quotient ring of R, when it exists. S and T will be reserved for equivalent orders to R, and M, N, P, Q for R-modules or ideals. Any condition is assumed to be two-sided unless prefixed by the word right or left. $\mathcal{M}_R$ and ${}_R\mathcal{M}$ will denote the categories of right and left R-modules respectively. M-Spec(R) denotes the maximal spectrum (the set of all maximal ideals) of R. The abbreviation f.g. is used for finitely generated throughout.

In the proofs of many of the results, there are strings of equalities where each step is derived from the previous one by a substitution. To facilitate the reading of these, the quantity that is to be substituted for is bracketed on the left of the equality sign. Thus, for example, we might write $(MN)P = XP$ if we knew that $MN = X$ holds.

I. GENERALITIES ABOUT DUAL MODULES.

In this section R and R' will denote arbitrary rings.

The information in this section is all well known and hence proofs are omitted or sketched. It is, however, fundamental to all that follows so it is collected here for completeness.

1.1.1. Definition. For $M \in \mathcal{M}_R$, the right dual of M is $M^* = \hom(M,R)$. Similarly, for $N \in {}_R\mathcal{M}$, the left dual of N is ${}^*N = \hom(N,R)$.

Although, for one-sided modules the differing notations may seem redundant, for bimodules the two notations differentiate which structure is being dualized. The abelian group M^* is in a canonical way a left R-module.

1.1.2. Proposition. For right modules M and N with $\alpha \in \hom(M,N)$ α^* defined via $\alpha^*(f) = f\cdot\alpha$ is a well defined R-homomorphism from N^* to M^*. If α is epi then α^* is mono.

1.1.3. Corollary. The right R-dual of an R'-R bimodule is an R-R' bimodule.

Proof. * is a ring homomorphism from $\mathrm{End}(M)$ to $\mathrm{End}(M^*)$.q.e.d.

1.1.4. Corollary. $[M\oplus N]^* \simeq M^*\oplus N^*$ canonically.

1.1.5. Corollary. (1) $R^* \simeq {}_RR$ canonically;

(2) The dual of a f.g. projective module is f.g. projective.

1.1.6. Proposition. If R is left noetherian and M is right f.g. then M^* is left f.g.

1.1.7. Theorem. [Dual Basis Lemma] A right module M is f.g. projective if and only if $1 \in (M)(M^*)$ or equivalently, $(M)(M^*) = \mathrm{End}_R(M)$. The multiplication $(M)(M^*)$ is given by $m \cdot m^* : M \to M$ via $[m \cdot m^*](m') = m \cdot [m^*(m')]$ for $m, m' \in M$, $m^* \in M^*$.

Since the dual of a module is again a module, albeit on the other side, it is natural to iterate the process and compare successive duals.

1.1.8. Definition. For $M \in \mathcal{M}_R$, the right module $^*(M^*) = \hom(M^*, R)$ is called the second dual of M. There is a canonical R-homomorphism, $\varphi : M \to {}^*(M^*)$, called the evaluation map which is defined by $\varphi(m) : M^* \to R$ via $\varphi(m)(m^*) = m^*(m)$ for $m \in M$, $m^* \in M^*$. If φ is a monomorphism M is said to be torsionless. If φ is an isomorphism M is said to be reflexive. The submodule of M, $t(M) = \bigcap_{f \in M^*} \ker(f)$ is called the torsion part of M.

The following Proposition lists the some useful properties. The proofs are all straightforward.

1.1.9. Proposition. For $M, N \in \mathcal{M}_R$:

(1) $t(M) = \ker \varphi$.

(2) M is torsionless if and only if $t(M) = 0$.

(3) $M/t(M)$ is torsionless.

(4) M^* is torsionless or zero.

(5) If $M \subseteq N$ with N torsionless, then M is torsionless.

(6) If M is also a left S-module then φ is a bimodule homomorphism canonically.

(7) $M \otimes N$ is torsionless if and only if M and N are torsionless.

(8) $M \otimes N$ is reflexive if and only if M and N are reflexive.

(9) R is reflexive.

(10) If M is f.g. projective, it is reflexive.

II. DUALS IN A PRIME NOETHERIAN RING.

Throughout this section R is a prime Noetherian ring with Goldie quotient ring K.

Recall that K is a simple artinian ring, which is the left and right injective hull of R. It is obtained by inverting all the regular elements of R. It is an easy consequence of this fact that

for any f.g. right submodule M of K there is a regular element c, such that $cM \subseteq R$ holds.

For an essential submodule of K, there is a more concrete realization of the dual.

1.2.1. Definition. For a right submodule M of K set, $M^r = \{k \in K | kM \subseteq R\}$. Similarly for a left submodule N set, $N^\ell = \{k \in K | Nk \subseteq R\}$.

Clearly M^r is a left submodule of K and N^ℓ is a right submodule.

1.2.2. Proposition. For an essential right submodule M of K, $M^r \simeq M^*$ canonically. Further, the K-multiplication of M by M^r corresponds to homomorphism evaluation. The multiplication used in the Dual Basis Lemma is also equivalent to K-multiplication.

Proof. The proof hinges on the fact that any map from an essential submodule of K, to another essential submodule (in this case R), extends to a unique map from K to K which can be realized as an element of K. q.e.d.

Note that the proof technique can actually be used to show that for any two essential right submodules of K, M and N, $\hom(M,N) = \{k \in K | kM \subseteq N\}$. This enables us to also obtain a concrete realization of the endo-morphism ring of an essential submodule of K.

1.2.3. Definition. For M a right submodule of K set $O_\ell(M) = \{k \in K | kM \subseteq M\}$. $O_\ell(M)$ is clearly a ring. It is called the left order of M. Similarly for N a left submodule of K, $O_r(N) = \{k \in K | Nk \subseteq N\}$ is called the right order of N.

1.2.4. Proposition. For an essential right submodule M of K, $O_\ell(M) \cong \mathrm{End}_R(M)$ canonically. If M is in fact a right ideal of R, then $O_\ell(M) \subseteq M^r$.

Proof. The first statement is immediate from the above remark and the fact $\mathrm{End}_R(M) = \mathrm{hom}(M,M)$. If M is a right ideal then we have $M \subseteq R$ so for $k \in O_\ell(M)$, $kM \subseteq M \subseteq R$ holds, forcing $k \in M^r$. q.e.d.

The following lemma presents some simple but useful properties of "r".

1.2.5. Lemma. For $M, N \subseteq K$:

(1) $(M^r)M \subseteq R$.

(2) $R^r = R$.

(3) If $M \supseteq N$ then $M^r \subseteq N^r$.

(4) $(MN)^r \supseteq N^r M^r$.

(5) $(M+N)^r = M^r \cap N^r$.

(6) $(M \cap N)^r \supseteq M^r + N^r$.

Proof. Properties (1), (2), and (3)$^-$ are immediate from the definition. The proofs for (4), (5), and (6) are similar, so we present the proof of (5).

We have $k \in (M+N)^r$ if and only if $k(M+N) \subseteq R$ holds but the latter clearly gives $kM \subseteq R$ and $kN \subseteq R$, so that k is in $M^r \cap N^r$, ie. $(M+N)^r \subseteq M^r \cap N^r$. Conversely, if $k \in M^r \cap N^r$ holds, then we have $kM \subseteq R$ and $kN \subseteq R$ which yields $k(M+N) = kM + kN \subseteq R$. q.e.d.

Although the previous lemmas require no finite generation assumptions, the addition of some f.g. assumptions produces a much richer theory. The rest of this section will be concerned only with f.g. submodules of K.

1.2.6. Proposition. If M is a f.g. essential right submodule of K then:

(1) M^r is a f.g. essential left submodule of K.

(2) $M^{r\ell} \simeq {}^*(M^*)$ canonically.

(3) φ (the evaluation map) factors through the inclusion $\iota: M \to M^{r\ell}$ and the isomorphism in (2). Thus M is reflexive if and only if $M = M^{r\ell}$.

Proof. (1) Since $M^r \simeq M^*$ holds and R is noetherian, M^r is f.g. by 1.1.6. To see that M^r is essential, use the Ore condition to find a regular element c in R such that $cM \subseteq R$ holds. Thus we

have a regular element c in M^r whence M^r is essential in K.

The proofs (2) and (3) are easy exercises in element chasing. q.e.d.

Proposition 1.2.6. completes the proof of the assertion that "r" and "ℓ" may be used in place of the more abstract hom. sets when discussing duals. The next proposition gives some elementary properties of such submodules which will be needed in Section IV.

1.2.7. Proposition. For $M \subseteq N$, two f.g. essential right submodules of K:

(1) M is torsionless.

(2) M^r is reflexive.

(3) If M is reflexive and $M^r = N^r$ then $M = N$.

Proof. (1) There is a regular element c of R with cM contained in R. By 1.1.9. (5) cM and hence M is torsionless.

(2) We want to show $M^r = M^{r\ell r}$. So consider $x \in M^{r\ell r}$ ie. x is an element of K for which $xM^{r\ell} \subseteq R$ holds. But now $M^{r\ell} \supseteq M$ forces $xM \subseteq xM^{r\ell} \subseteq R$ and $x \in M^r$.

(3) Since M is reflexive, we have $M = M^{r\ell} = (N^r)^\ell \supseteq N$, with $M \subseteq N$ by assumption. q.e.d.

One technique in this thesis is the application of these tools to bisubmodules of K, especially ideals and duals of ideals. Thus it is important to know that these are essential and to have some criterion for when they are f.g. on the other side. The second part of the next proposition can be thought of as one of the main reasons for restricting to bounded rings.

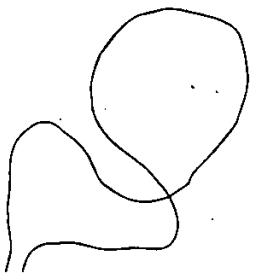
1.2.8. Proposition. For a bisubmodule M of K:

(1) M is left and right essential.

(2) If M is f.g. on one side and R is bounded then M is f.g. on the other side.

Proof. (1) $0 \neq M \cap R$ is an ideal of R which is essential in R since R is prime. But R is essential in K, so $M \cap R$ is essential in K and even moreso M is essential in K.

(2) Write $M = \sum_{\text{fin}} m_\iota R$. Now by the Ore condition there exists a regular element c with $m_\iota c \in R$ for all ι. But by boundedness, there is a c', also regular with $cR \supseteq Rc'$. Thus we have $M \cong Mc' \subseteq m_\iota cR \subseteq R$. Since R is noetherian Mc' is left f.g. so M is also left f.g. q.e.d.



III. CHARACTERIZATIONS AND PROPERTIES OF INJECTIVE DIMENSION ONE FOR (USUALLY PRIME) NOETHERIAN RINGS.

In this section R will denote a noetherian ring.

This section presents several known facts from the literature without proof. These will be used freely in Section IV.

1.3.1. Theorem. [Jans & Bass]. The following are equivalent:

(1) inj dim $(R_R) = 1$.

(2) Every f.g. torsionless left module is reflexive.

(3) Every essential left ideal is reflexive.

1.3.2. Theorem. [Zaks]. If R is prime with inj dim $(R) = 1$ then the Krull dimension of R is 1.

Note: In [17], Zaks shows that every f.g. torsion right R-module is artinian, ie. has Krull dimension zero. Since, for any essential right ideal E of R, R/E is a f.g. torsion module, we have $|R/E| = 0$. But for a noetherian prime ring R, $|R| = \sup_{E\text{ess}} (|R/E| + 1)$ holds. q.e.d.

1.3.3. Corollary. [Zaks]. If R is prime with inj dim $(R) = 1$ then a non zero prime ideal is maximal.

1.3.4. Theorem. [Mueller]. (1) If R is prime with inj dim $R = 1$ and X is an invertible ideal of R then R/X is a quasi-Frobenius.

(2) If R is prime and every ideal of R contains an invertible ideal X for which R/X is quasi-Frobenius, then inj dim $(R) = 1$.

1.3.5. Theorem. If R is prime with Goldie quotient ring K, then inj dim $(R) = 1$ if and only if K/R is an injective cogenerator.

Proof. Note that $0 \to R \to K \to K/R \to 0$ is an exact sequence with K being R-injective. Clearly inj dim $(R) = 1$ if and only if K/R is injective. Thus the content of the theorem is that K/R is actually a cogenerator when it is injective. By [1], 18.15 K/R is a cogenerator if and only if every simple R-module can be embedded into K/R which is equivalent to $EXT^1(U,R)$ being non zero for a simple R-module U. But for a simple R-module U with annihilator M, we have $0 \to R^l \to M^r \to EXT^1(U,R)$ is an exact sequence, where ι is the natural inclusion. Since M is reflexive, M^r cannot be equal to R, so $EXT^1(U,R)$ is non zero. q.e.d.

IV. [FRACTIONAL] IDEALS IN A BOUNDED NOETHERIAN PRIME RING OF INJECTIVE DIMENSION ONE.

Throughout this section R will be bounded Noetherian prime with $\text{inj dim}(R) = 1$ and Goldie quotient ring K.

This section examines mainly the maximal ideals of and the minimal equivalent orders to R. The two basic tools (besides the information in the first three sections) will be the cycle map (see Definition 1.4.1.) and links (see Definition 1.4.4.). The aim is to show "two-sidedness results" and commutativity results.

A few words about the assumptions on R are in order. The Noetherian assumption is natural for considerations of localization which was our original motivation. The primeness assumption simplifies many of the considerations, but it is to be expected that much of what is done here could be generalized to semiprime rings or perhaps even rings with artinian quotient ring. The boundedness assumption is useful both because this is the original setting for the definition of links and because it implies by 1.2.8. that all bisubmodules f.g. on one side are f.g. on the other., The injective dimension one assumption is precisely what is needed to define the cycle map (ie. the cancellation of "r" and "ℓ"). The cycle map has been used extensively in the study of bounded hereditary rings and in this case the cycles occur exactly where links occur. The object here then is to

study the rings for which the cycle map and links are both defined (with the primeness assumption for added simplicity).

1.4.1. Definition. For f.g. bisubmodules, M,N of K, write $M \to N$ read M arrow N if $M^{\ell} = N^{r}$ holds.

Since "r" and "ℓ" cancel each other $M \to N$ is equivalent to $M = N^{rr}$ and also to $M^{\ell\ell} = N$. The cycle map is clearly not only a map, it is also a bijection.

1.4.2. Proposition. r and ℓ are inverse lattice anti isomorphisms between the lattices of all right and left f.g. essential submodules of K. They respect bimodules and have fixed point R. The cycle map ($=\ell^2$) is a lattice isomorphism on the lattice of all f.g. bisubmodules of K which also has fixed point R.

The first assertion is a summary of some results in the first three sections. To see that r respects bimodules, consider M^r for M a f.g. bisubmodule of K. M^r is a left module canonically, but we also have $(M^rR)M = M^r(RM) = M^rM \subseteq R$, so that M^rR is contained in M^r and M^r is clearly thus a right module also. The second statement is a simple consequence of the first after recognizing that the cycle map is ℓ^2 (ie. the composition of two anti isomorphisms is an isomorphism) and that f.g. bisubmodules are exactly those to which ℓ can be applied twice. q.e.d.

The above proposition (and Krull dimension one which implies that the quotient module of two essential submodules of K has finite length) allows us to conclude that r, ℓ and the cycle map preserve lengths. More specifically, if M and N are two f.g. essential right submodules of K with M containing N, then the quotient modules M/N, and N^r/M^r all have the same length. If M and N are f.g. bisubmodules of K with M containing N, then the quotient bimodules M/N and M^{rr}/N^{rr} have the same length as right modules, and as bimodules.

In the rest of this section, most of the results stated have a left and a right version. Since both versions are needed in the future development, the results will be stated and proved for the right version, with the left version added in parentheses for easy reference. In all cases, the left version can be proven by a simple appeal to R^{op}.

1.4.3. Corollary. For a f.g. bisubmodule M of K, the following are equivalent:

(1) M is a maximal ideal of R.

(2) M^r is a minimal overbimodule of R.

(3) $N = M^{rr}$ (ie. $N \twoheadrightarrow M$) is a maximal ideal of R.

[Left versions: (2) M^ℓ is a minimal overbimodule of R.

(3) $N = M^{\ell\ell}$ (ie. $M \twoheadrightarrow N$) is a maximal ideal of R.]

Proof. That (1) is equivalent to (2) is an easy consequence of 1.4.2. and the remark that r preserves lengths (one sided lengths and bimodule lengths).

That (2) is equivalent to (3) follows from the left version of the equivalence of (1) and (2) using $N^{\ell} = M^{rr\ell} = M^{r}$. q.e.d.

One sees from 1.4.3. that restricting the cycle map to M-Spec(R) gives a bijection $\rightarrow$: M-Spec(R) $\rightarrow$ M-Spec(R). Alternatively, one can think of the cycle map on M-Spec(R) as imparting a directed-graph structure in which each vertex (maximal ideal) has exactly one incoming and one outgoing arrow.

There is another way of imparting a directed-graph structure on R and this is via links. Links were introduced by Mueller in [13]. They are directed arrows between prime ideals of the same Krull dimension in FBN rings. Because R has Krull dimension one, boundedness is actually fully boundedness, and non zero prime ideals are maximal ideals. Thus, we need only give the definition for a link between maximal ideals in R.

1.4.4. Definition. For maximal ideals M and N of R, write $M \rightsquigarrow N$ and read M links to N if $NM \subsetneq M \cap N$ holds.

For M = N, there is a link $M \rightsquigarrow M$ if and only if $M^2 \subsetneq M$ holds. Alternatively $M \rightsquigarrow M$ holds, if and only if the length of

R/M^2 is greater than one. Such a link is called a self link.

For $M \neq N$, the existence of at least one of the links $M \rightsquigarrow N$ or $N \rightsquigarrow M$ is equivalent to M does not commute with N since for comaximal ideals M and N, $M \cap N = MN + NM$. [ie. $M \cap N = (M \cap N)R = (M \cap N)(N + M) \subseteq MN + NM \subseteq M \cap N$]. Also $M \rightsquigarrow N$ holds if and only if the length of R/NM is greater than two. Such a link is called a non-self link.

Some remarks on links are in order. Links are important because a finite link component is precisely a minimal classically localizable set of prime ideals. What we have presented here is actually called a right link. However, the corresponding notion of left link adds no new information, since a left link $M \rightsquigarrow N$ exists if and only if the right link $N \rightsquigarrow M$ exists. This is the same as the situation for the cycle map, where there are really two cycle maps (ours and the inverse of ours), but it is sufficient to study only one.

We present two technical lemmas which will help to show the relationship between the cycle map and links.

1.4.5. Lemma. For f.g. bisubmodules M and N of K, $(MN)^r = (M^{rr}N^{rr})^{\ell}$.

[Left version: $(MN)^{\ell} = (M^{\ell\ell}N^{\ell\ell})^r$.]

Proof. We have $(MN)^r = \{x|xMN \subseteq R\} = \{x|xM \subseteq N^r\} = \{x|xM \subseteq N^{rr\ell}\} = \{x|N^{rr}xM \subseteq R\} = \{x|N^{rr}x \subseteq M^r\} = \{x|M^{rr}N^{rr}x \subseteq R\} = (M^{rr}N^{rr})^\ell$. q.e.d.

1.4.6. Corollary. For f.g. bisubmodules M and N of K, $(MN)^{rr} = M^{rr}N^{rr}$.

[Left version: $(MN)^{\ell\ell} = M^{\ell\ell}N^{\ell\ell}$.]

Proof. Apply "r" to 1.4.5. q.e.d.

The corollary says $M \twoheadrightarrow P$ and $N \twoheadrightarrow Q$ imply $MN \twoheadrightarrow PQ$, ie. the cycle map preserves products. The next lemma says that the cycle map also preserves links.

1.4.7. Lemma. For maximal ideals M and N of R, $M \rightsquigarrow N$ if and only if $M^{rr} \rightsquigarrow N^{rr}$.

[Left version: $M \rightsquigarrow N$ if and only if $M^{\ell\ell} \rightsquigarrow N^{\ell\ell}$.]

Proof. The cycle map preserves products, R and lengths of quotients. There is a link between maximal ideals exactly when the length of R/M^2 is greater than one for a self-link at $M = N$, or of R/NM is greater than two for a nonself link from M to N. q.e.d.

The above lemma, together with 1.4.3., shows that the

cycle map is an automorphism of the link-directed graph on M-Spec(R). This was noted by Mueller in [14].

The next proposition is the first in a series leading up to Theorem 1.4.20, which states that left projective is equivalent to right projective for maximal ideals. Along the way, several results concerning minimal equivalent orders and their maximal ideals must be established. These will be used not only for 1.4.20., but also in much of what comes afterwards.

1.4.8. Proposition. For a maximal ideal M, the following are equivalent.

(1) M is invertible.

(2) M is left invertible.

(3) M is left projective and $M \twoheadrightarrow M$.

(4) $M^r M \supsetneq M$.

[left versions: (2) M is right invertible.

(3) M is right projective and $M \twoheadrightarrow M$.

(4) $MM^\ell \supsetneq M$.]

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (4) Since M is left invertible we have $XM = R$ for some X. Clearly X is contained in M^r so $M^r M$ also equals R, hence properly contains M.

(2) $\Rightarrow$ (1): As just noted $M^rM = R$ holds. Since $R \subseteq M^\ell$ holds, we have $M \subseteq MM^\ell \subseteq R$, with M maximal. If we had $MM^\ell = M$, then $R = M^r(M) = (M^rM)M^\ell = RM^\ell = M^\ell$ would follow. Contradiction. Thus MM^ℓ must equal R and M is actually two-sided invertible.

(4) $\Rightarrow$ (2): This is essentially done in (2) $\Rightarrow$ (1). For if $M^rM \supsetneq M$ holds, then since M is maximal and we know $R \supseteq M^rM$, M^rM must equal R.

(1) $\Rightarrow$ (3): If M is invertible, it is projective and M^{-1} must equal both of M^r and M^ℓ so $M \Rightarrow M$ holds.

(3) $\Rightarrow$ (4): If M is left projective then we have $1 \in M^\ell M$. Since $M \Rightarrow M$ holds we have $M^r = M^\ell$ and $1 \in M^rM$. Thus M^rM cannot equal M. q.e.d.

Part of the above proposition was already noted in [14] by Mueller.

1.4.9. Lemma. For maximal ideals M and N of R; either (a) $MN^r \cap R = M$ or (b) $M = N$ is right projective with $MM^r \cap R = R$. Further, in case (b) $MM^r = R$ if M is invertible and $MM^r = M^r \supsetneq R$ if M is not invertible.

[Left version: either (a) $N^\ell M \cap R = M$ or (b) $M = N$ is left projective with $M^\ell M \cap R = R$. Further, in case (b) $M^\ell M = R$ if M is invertible and $M^\ell M = M^\ell \supsetneq R$ if M is not invertible.]

Proof. Assume that $MN^r \cap R$ does not equal M. Since $MN^r \cap R$ is an ideal which contains the maximal ideal M, $MN^r \cap R = R$ must follow. Also, by 1.4.3., R is maximal as a bisubmodule of N^r, so either $MN^r = R$ or $MN^r = N^r$ holds. If now $MN^r = R$ holds, then M is invertible with $N^r = M^{-1} = M^r$ holding, which forces $M = N$ and we are done. If, on the other hand, $MN^r = N^r$ holds, consider MN^rN. If now N is invertible, we have $N^rN = R$, whence $MN^rN = MR$, and also $MN^rN = N^rN = R$. Thus $MR = R$ holds, contradicting that M is a maximal ideal. If N is not invertible then $N^rN = N$ (by 1.4.8.), whence $MN^rN = MN$ and also $MN^rN = N^rN = N$. Thus $MN = N \subseteq M$ holds, forcing $M = N$ and $1 \in MM^r = M^r$, so that M is right projective. q.e.d.

The following lemma, which applies immediately to any non-invertible maximal ideal, characterizes when the dual of an ideal is an equivalent order.

1.4.10. Lemma. For an ideal I of R, the following are equivalent:

(1) $(I^r) = (I^r)^2$.

(2) $I^rI = I$.

(3) $I^r = O_\ell(I)$.

[Left versions: (1) $(I^\ell) = (I^\ell)^2$.

(2) $II^\ell = I$

(3) $I^{\ell} = O_r(I)$.]

Proof. (3) ⇒ (1) is clear from section II, since $O_\ell(I)$ is a ring.

(1) ⇒ (2): If $I^r = (I^r)^2$ holds, then we have $I^rI = (I^r)^2I$ and so I^r is contained in $(I^rI)^r$. Since I is an ideal, we have $I^r \supseteq R$ and $I^rI \supseteq I$ so that I^r contains $(I^rI)^r$. Thus $I^r = (I^rI)^r$ and $I = I^rI$ follow.

(2) ⇒ (3): By 1.2.4. we know $O_\ell(I) \subseteq I^r$ holds, and since $I^rI = I$ holds, we see that I^r is also contained in $O_\ell(I)$. q.e.d.

1.4.11. Corollary. A maximal ideal M is not invertible if and only if $M^r = O_\ell(M)$.

Since invertible prime ideals are always localizable, the Corollary states that for any maximal ideal which cannot be treated by this particularly nice method of localization, we have that the dual is an equivalent order.

1.4.12. Proposition. For a right projective maximal ideal P of R either P is invertible with $\cap P^n = 0$ (ie. P is not even eventually idempotent) or $P = P^2$ and $PP^r = P^r = O_r(P)$.

Proof. If P is invertible then $(P^n)^{-1} = (P^{-1})^n = P^{-n}$ and $P^{-n} P^{n+1} = P \neq P^{-n} P^n = R$ so $P^n \neq P^{n+1}$ for all n. Thus $R/(\cap P^n)$ is not artinian, and $\cap P^n$ is an ideal so $\cap P^n$ must be zero. (Since $|R| = 1$).

If P is not invertible then by the Dual Basis Lemma, 1 is in PP^r and since $P^r = (P^r)^2 = 0_r(P)$, and $PP^r = P^r$ hold, we have $P(P^rP) = P^2$, $(PP^r)P = P^rP = P$ and $P = P^2$. q.e.d.

Note that in 1.4.20. we will show that right projective implies left projective so that the right prefix can then be dropped from the hypotheses of 1.4.12. This remark also applies to the following theorem 1.4.13. which will then assert that projectivity is preserved by the cycle map on M-Spec(R).

1.4.13. Theorem. For a maximal ideal M of R, M is right projective if and only if M^{rr} is left projective.

[Left version: M is left projective if and only if $M^{\ell\ell}$ is right projective.]

Proof. Since the backwards implication is just the forward implication of the left version applied to M^{rr} (instead of M) it is only necessary to show the forward implication of the right version.

Thus, assume M is right projective. If M is invertible

then $M \rightarrow M$ holds (ie. $M = M^{rr}$) whence M is left and right projective and there is nothing to show. Thus, assume also that M is not invertible. It follows that M is distinct from both M^{rr} and $M^{\ell\ell}$, and that these are also not invertible (they do not cycle to themselves). After setting the stage with these preliminaries, we assume that M^{rr} is not left projective and establish the following three claims which will lead to a contradiction.

Claims: (1) $MM^r = M^r$; $M^rM = M$; $M^{rr}M^r = M^{rr}$

(2) $M^{rr}M \subseteq M^rM^{rr}M \subseteq MM^{rr} = M \cap M^{rr}$

(3) $MM^{rr} \supseteq M^rM^{rr}MM^r = M^rM^{rr}$

Proof. (1) Immediate from 1.4.10., 1.4.11., and the fact $M^r = M^{rr\ell}$.

(2) That $M^rM^{rr}M \supseteq M^{rr}M$ holds, follows from R being contained in M^r. Since M^{rr} and M are both (maximal) ideals of R, $M^{rr}M$ and MM^{rr} are both contained in $M \cap M^{rr}$. Now multiplying $M \cap M^{rr}$ by M^r yields $M^r(M \cap M^{rr}) \subseteq (M^rM) \cap M^rM^{rr} = M \cap M^rM^{rr} = M \cap (R \cap M^rM^{rr}) = M \cap M^{rr}$. Thus $M^rM^{rr}M$ which contains $M^{rr}M$, must also be contained in $M \cap M^{rr}$. Since M is right projective, we have $MM^r = M^r$ which forces $(M^r)M^{rr}M = M(M^rM^{rr}M) \subseteq M(M \cap M^{rr}) \subseteq MM^{rr}$. Now, since M and M^{rr} are maximal ideals, we know that $M \cap M^{rr} = MM^{rr} + M^{rr}M$ holds, whence, using $M^{rr}M \subseteq MM^{rr}$ we conclude that

$MM^{rr} = M \cap M^{rr}$ holds.

(3) Consider $M^rM^{rr}MM^r$. On the one hand we have $M^r(M^{rr}M)M^r \subseteq M^rMM^{rr}M^r = MM^{rr}$ while on the other hand we also have $M^rM^{rr}(MM^r) = M^r(M^{rr}M^r) = M^rM^{rr}$.

To finish off the proof, note that we have established that $M^rM^{rr} \subseteq MM^{rr} \subseteq R$ holds. Thus we have $M^{rr} \subset M^{r\ell} = M$, and by the maximality of M^{rr} we must have $M = M^{rr}$, which contradicts the fact that M is distinct from M^{rr}. q.e.d.

The next result, which describes explicitly most maximal ideals of a given minimal equivalent order, belongs organizationally in Chapter II. It is in fact the basis of 2.3.1. and 2.4.1. Since, however, the result is also needed in several places in the proof of 1.4.20., it is presented here. Besides describing the maximal ideals of a minimal equivalent order, the theorem is also a commutativity result: Most maximal ideals of R (all except the conductors) commute with a given minimal equivalent order.

1.4.14. Theorem. For maximal ideals M and N of R with M not invertible and N distinct from M and M^{rr}, $M^rN = NM^r$ is the unique maximal ideal of M^r which contains N.

[left version: For N distinct from M and $M^{\ell\ell}$, $M^\ell N = NM^\ell$ is the

unique maximal ideal of M^{ℓ} which contains N.]

Proof. As noted in 1.4.11., the non-invertibility of M is precisely the assumption necessary to assure that the dual of M is a minimal equivalent order. This justifies discussing the (maximal) ideals of M^r and M^{ℓ}.

Let M and N be chosen as in the hypotheses. Now N is a maximal ideal of R and R is (bimodule) maximal in M^r, so that $N \subsetneq R \subsetneq M^r$ is a bimodule composition series for M^r/N. Thus both of the series $N \subseteq M^rN \subseteq M^rNM^r \subseteq M^r$ and $N \subseteq NM^r \subseteq M^rNM^r \subseteq M^r$ must have length less or equal to 2 and we will be done after establishing that $N \neq M^rN$, $N \neq NM^r$ and $M^r \neq M^rNM^r$ all hold.

Suppose now that $N = NM^r$ holds (the case $N = M^rN$ is similar). Since we have $NM^r = N \subseteq R$, we also have $M^r \subseteq N^{\ell}$ and $M^{rr} \supseteq N$ after applying r. But, this contradicts the fact that N is a maximal ideal of R which is distinct from M^{rr}.

If now $M^r = M^rNM^r$ holds, then since $M^{rr}M^r = M^{rr}M^{rr\ell} = M^{rr}$ and $M^rM = M$ also hold, we have $M^{rr}(M) = M^{rr}(M^r)M = (M^{rr}M^r)N(M^rM) = M^{rr}NM \subseteq N$. But for N a distinct maximal ideal from M and M^{rr} we cannot have $M^{rr}M$ contained in N.

Thus we have that M^rN, NM^r and M^rNM^r are identical, and R-bimodule maximal in M^r. It is clearly an M^r-ideal, and the only

M^r ideal which contains N. q.e.d.

For a commutative domain A, with overring A', the dual of A' in A is called the conductor of A'. [See Kaplansky [9], pg. 46]. It is the largest A'-ideal contained in A. By analogy define:

1.4.15. Definition. For an equivalent order S containing R, the right conductor of S is S^r. The left conductor of S is S^l.

An equivalent order to R, thus has two conductors which may coincide in some cases (ie. if $M \rightarrow M$ when M is a not-projective maximal ideal, then M^r has left and right conductors equal to M). Theorem 1.4.14. may thus be phrased as follows: For any minimal equivalent order S, S commutes with all maximal ideals of R, except its conductors. In 2.2.1. we will see that S does not, in fact, commute with its conductors if they are distinct, whereas S does commute with its conductor if the conductors coincide.

The next lemma presents a technical condition which insures that a pair of minimal equivalent orders commute. This is immediately applied to two different situations in the following corollaries.

1.4.16. Lemma. Let M and N be non-invertible maximal ideals. If $M \cap N$ is a left ideal in both M^r and N^r then

$M^rN^r = N^rM^r = (M \cap N)^r$.

[Left version: If $M \cap N$ is a right ideal of M^ℓ and N^ℓ then $M^\ell N^\ell = N^\ell M^\ell = (M \cap N)^\ell$.]

Proof. If $M = N$ then the lemma merely reasserts 1.4.11., so we assume $M \neq N$.

Since M and N are distinct, $R/M \cap N$ has bimodule length two, and consequently so does $(M \cap N)^r/R$. Since $M \cap N$ is a left ideal in both M^r and N^r, we have $M^r(M \cap N) = N^r(M \cap N) = M^rN^r(M \cap N) = N^rM^r(M \cap N)$. Thus M^rN^r and N^rM^r are both contained in $(M \cap N)^r$ which is minimal over both M^r and N^r, which are also contained in M^rN^r and N^rM^r. Thus it is only necessary to show that M^rN^r and N^rM^r properly contain N^r and M^r. But if, for example, we had $M^r = M^rN^r$ then $N^r \subseteq M^rN^r = M^r$ would follow, forcing $M = N$ in contradiction to the fact that M and N were assumed distinct. The others are similar. q.e.d.

1.4.17. Corollary. Let M and N be distinct non-invertible maximal ideals. If $N \not\to M$ then $M \cap N$ is a left ideal in M^r. Thus if $M \not\to N$ and $N \not\to M$ we conclude $M^rN^r = N^rM^r = (M \cap N)^r$.

[Left version: If $N \not\to M$ then $M \cap N$ is a right ideal in M^ℓ.

Thus, if $M \not\to N$ and $N \not\to M$ we conclude that $M^{\ell}N^{\ell} = N^{\ell}M^{\ell} = (M \cap N)^{\ell}$.]

Proof. For $M \neq N$ and $N \not\to M$ (ie. $N \neq M^{rr}$), $(M^{r}N)M = NM^{r}M$ (by 1.4.14.) = NM (by 1.4.8. since M is not invertible). Also, $M^{r}MN = MN$. Thus $M^{r}(M \cap N) = M^{r}(MN + NM) = M^{r}MN + M^{r}NM = MN + NM = M \cap N$ which is consequently a left ideal of M^{r}. The last statement is just an application of the lemma. q.e.d.

1.4.18. Corollary. Let M and N be distinct non-invertible maximal ideals. If N is not left projective then $M \cap N$ is a left ideal in M^{r}. Thus, if M and N are both not left projective, then $M^{r}N^{r} = N^{r}M^{r} = (M \cap N)^{r}$.

[Left version: If N is not right projective then $(M \cap N)$ is a right ideal in M^{ℓ}. Thus, if M and N are both not right projective, then $M^{\ell}N^{\ell} = N^{\ell}M^{\ell} = (M \cap N)^{\ell}$.]

Proof. If $M \neq N \not\to M$ then we are done by 1.4.17. If, on the other hand, $M \neq N \to M$ (ie. $N^{\ell} = M^{r}$) and N is not left projective, then $M^{r}(M \cap N) \subseteq M^{r}M \cap M^{r}N = M \cap M^{r}N$ (by 1.4.8.) $= M \cap R \cap M^{r}N = M \cap (R \cap N^{\ell}N) = M \cap N$ (by 1.4.9. (left version)). Thus, $M \cap N$ is a left ideal of M^{r} and the final statement is just an application of the lemma. q.e.d.

The next theorem is a long promised result that right projective

is equivalent to bi-projective for maximal ideals. This result, together with the previous result, allows a much stronger statement than either 1.4.17. or 1.4.18. Thus, if the above corollaries had not been necessary for 1.4.19. it would have been neater to place them afterwards.

Let us, for a moment, consider the effect of 1.4.20. on the above. The first thing to note is that 1.4.20. makes the assumptions of the left and right versions of 1.4.18. the same. From 1.4.20. we will also conclude (in 1.4.22. and 1.4.23.) that for M invertible and N any element of $M\text{-Spec}(R)$ $MN = NM = M \cap N$. From this, one can easily derive that $M^{-1}N^r = N^rM^{-1} = (M \cap N)^r$ and $M^{-1}N^\ell = N^\ell M^{-1} = (M \cap N)^\ell$. Finally, for $M \Rightarrow N \neq M$ we will have (1.4.22.) that $M^rN^r \neq N^rM^r$. Thus:

1.4.19. Corollary. Let M and N be maximal ideals of R. Either $N^rM^r = M^rN^r$ or else $N \Rightarrow M$ or $M \Rightarrow N$ holds, where both M and N are bi-projective.

[Left version: Either $M^\ell N^\ell = N^\ell M^\ell$ or else $M \Rightarrow N$ or $N \Rightarrow M$ holds, where both M and N are bi-projective.]

It is surprising that although the link-directed graph of $M\text{-Spec}(R)$ (which joins those maximal ideals which do not commute) may be fairly arbitrary, the analogous directed-graph on the duals of $M\text{-Spec}(R)$ has no edges involving the non-projective maximal ideals, and

is the dual of the link-directed graph on the projective maximal ideals.

1.4.20. Theorem. A maximal ideal is right projective if and only if it is left projective.

Proof. In light of the symmetry of assumptions, it is sufficient to show that right projective implies left projective.

Thus, let M be a right projective maximal ideal. If $M \twoheadrightarrow M$ then M is (two-sided) invertible and hence bi-projective. Assume, therefore, that $N = M^{\ell\ell} \neq M$ and distinguish two cases: (I) $N = M^{rr}$ (II) $N \neq M^{rr}$.

Case I: Suppose $N = M^{\ell\ell} = M^{rr} \neq M$ holds and M is right but not left projective. The following observations will lead to an eventual contradiction:

(1) N is left but not right projective.

(2) $MM^r = M^r = N^\ell = N^\ell N$

(3) $N^r M^r = (MN)^r = M^\ell N^\ell = (NM)^\ell$

(4) $MN \lhd M^r$, $NM \lhd N^r$

(5) $N^r MNN^r \subseteq M \cap N \lhd N^r$

(6) $M^r NMM^r = M^r$

(7) $M^r \supsetneq (N^r M)^r \supsetneq M$; $M^r \supsetneq (NM^r)^\ell \supsetneq M$; $N^r \supsetneq (N^r M)^\ell \supsetneq N$; $N^r \supsetneq (NM^r)^r \supsetneq N$

(8) $(N^r M)^\ell = NN^r$; $(NN^r)^r = N^r M$

(9) $(N^rM)^r\,(N^rM) = M$

(10) $M^rN^rM = N^rM$

(11) $(N^rM)^r \lhd M^r$

Proof.

(1) 1.4.13.

(2) Note M and N are both not invertible and then we use 1.4.12. (and definition of $\twoheadrightarrow$).

(3) Using $M \twoheadrightarrow N \twoheadrightarrow M$, write $(MN)^r = (M^{rr}N^{rr})^\ell = (NM)^\ell$. Further, substituting $M^r = N^\ell$ and $M^\ell = N^r$ we see $N^rM^r = M^\ell N^\ell$. It only remains to show that $N^rM^r = (MN)^r$ holds. Since M is not invertible, $M^rM = M$, so that we can write $N^r(M^rM)\,N = N^rMN \subseteq N^rN \subseteq R$, and we see that N^rM^r is contained in $(MN)^r$. Conversely, for x an element of $(MN)^r$, we have $xMN \subseteq R$, and even $xM \subseteq N^r$ holding. Multiply by M^r noting that $1 \in M^r = MM^r$, to obtain $x \in xMM^r \subseteq N^rM^r$. Thus $(MN)^r \subseteq N^rM^r$ holds also and we are done.

(4) That $M^rMNM^r = MN$ holds follows immediately from $M^rM = M$ and $NM^r = NM^\ell = N$. That $N^rNMN^r = NM$ holds is similar.

(5) Since M is not left projective and N is not right projective we use 1.4.18. to conclude that $M \cap N$ is a left ideal of N^r and $M \cap N$ is also a right ideal of $M^\ell = N^r$.

(6) Since M is right projective and N is left projective, we have $MM^r = M^r$ and $N^\ell N = N^\ell$ with $N^\ell = M^r$ so that $M^r NMM^r = (M^r)^2 = M^r$ holds.

(7) We prove $M^r \supsetneq (N^r M)^r \supsetneq M$ and note that the others are similar. Since N and M are distinct, $N^r M$ cannot be contained in R whence $N^r M$ properly contains M. Further then we must have M^r properly contains $(N^r M)^r$. Also, M not left projective forces $M^\ell \supsetneq M^\ell M$ where M^ℓ is the same as N^r. Applying r now yields $M \subsetneq (M^\ell M)^r = (N^r M)^r$.

(8) From (5) we see that $(N^r M)^\ell$ contains NN^r. Thus, we conclude from (7) that $N^r \supsetneq (N^r M)^\ell = NN^r \supsetneq N$ where N^r/N has bimodule length 2. Thus $(N^r M)^\ell = NN^r$. The other equalities are all similar.

(9) Since we have $(N^r M)^r N^r M$ contained in R, we also have $(N^r M)^r N^r$ contained in M^r whence we see that $(N^r M)^r N^r M \subseteq M^r M = M$ holds. On the other hand, M is idempotent by 1.4.12. and $M \subseteq (N^r M)^r$ holds by (7) so that we have $M = M^2 = MM^\ell M = MN^r M \subseteq (N^r M)^r N^r M$.

(10) Since 1 is an element of M^r we clearly have $N^r M$ contained in $M^r N^r M$. On the other hand, multiplying $M^r N^r M$ by NN^r yields $M^r N^r MNN^r \subseteq M^r[N \cap M] \subseteq M^r M \subseteq R$ using (5) and consequently $M^r N^r M$ is contained in $(NN^r)^r$. But $(NN^r)^r = N^r M$ holds by (8) and we are done.

(11) To see that $(N^rM)^r$ is an ideal of M^r we multiply on the left and right by M^r and show that the resulting expression still multiplies N^rM into R. Thus consider $M^r(N^rM)^rM^r \cdot N^rM$. Since $M^rN^rM=N^rM$ the inside M^r is absorbed and we have $M^r(N^rM)^rM^rN^rM=M^r(N^rM)^rN^rM$. But in (9) $(N^rM)^rN^rM$ was shown to be equal to M whence $M^r[(N^rM)^rN^rM] = M^rM = M \subseteq R$ holds and $(N^rM)^r$ is indeed an ideal of M^r.

We now combine several of these observations to reach the desired contradiction. By (2) we have $M^r = M^rMM^r$ which is contained in $M^r(N^rM)^rM^r$ by (7). But $(N^rM)^r$ is a proper ideal of M^r by (7) and (11), so that $M^r \subseteq (N^rM)^r \subsetneq M^r$ holds, which is a contradiction.

Case II: Suppose now that M is right but not left projective and that $M^{\ell\ell}$, which we will denote by N, is distinct from M^{rr}. As for Case I, we present several observations, leading up to a contradiction.

(1) M, M^{rr} and N are distinct and not invertible.

(2) M^{rr} is left projective and N is not right projective.

(3) M and N are idempotent.

(4) $NM^r = M^rN \quad M^r$; $M^{rr}M^{\ell} = M^{\ell}M^{rr} \lhd M^{\ell}$

(5) $M \cap N$ is a left ideal in M^r and in M^{ℓ}

(6) $M^rM^{\ell} = M^{\ell}M^r = (M \cap N)^r$

Proof.

(1) 1.4.8.

(2) 1.4.13.

(3) M is idempotent by 1.4.12. N is them idempotent by 1.4.7.

(4) 1.4.14.

(5)' Since M is not left projective, 1.4.18. tells us that $M \cap N$ is a left ideal of $N^r = M^\ell$. On the other hand, N is distinct from M^{rr} (ie. $N \not\to M$ holds) so by 1.4.17., $M \cap N$ is a left ideal of M^r.

(6) 1.4.16. Note that $N^r = M^\ell$.

To finish off Case II, note that $MM^r = M^r$ and $MM^\ell = M$ since M is a right projective non-invertible maximal ideal. Thus $MM^rM^\ell = M^rM^\ell$ whereas $MM^\ell M^r = MM^r = M^r$ where $M^rM^\ell \supsetneq M^r$ which contradicts that $M^rM^\ell = M^\ell M^r$. q.e.d.

In light of 1.4.20. there is a partition of M-Spec(R) into $\mathcal{P}$, the biprojective maximal ideals and $\mathcal{N}$ the maximal ideals neither left nor right projective. That $\mathcal{P}$, and hence $\mathcal{N}$, is closed under both the cycle map and its inverse follows from 1.4.13. and 1.4.20. Our next aim is to show that $\mathcal{P}$ and $\mathcal{N}$ are also closed under incoming and outgoing links. This is done by completely describing all links involving an element of $\mathcal{P}$. Recall that in an HNP ring, there is a link $M \longrightarrow N$ if and only if $M \to N$ holds. The following theorem generalizes this result to elements of $\mathcal{P}$.

1.4.21. Theorem. For maximal ideals M and N of R with $M \in \mathcal{P}$:

(a) $M \rightsquigarrow N$ if and only if $M \twoheadrightarrow N$

(b) $N \rightsquigarrow M$ if and only if $N \twoheadrightarrow M$.

In either case, N is clearly also a member of $\mathcal{P}$.

Proof. The proof is immediate from the following two lemmata which describe the situation explicitly.

1.4.22. Lemma. For $M \in \mathcal{P}$ there are three possibilities:

(a) $M \twoheadrightarrow M$. Then M is invertible and $M \rightsquigarrow M$.

(b) $M \twoheadrightarrow N \twoheadrightarrow M$, $N \neq M$. Then $M^r N^r$ is incomparable with $N^r M^r$, MN is incomparable with NM and $M \rightsquigarrow N \rightsquigarrow M$.

(c) $M \twoheadrightarrow N \not\twoheadrightarrow M$, $N \neq M$. Then $M^r N^r \supsetneq N^r M^r = (M \cap N)^r$, $NM \subsetneq MN = M \cap N$ and so $M \rightsquigarrow N \not\rightsquigarrow M$.

Proof. The three possibilities are clearly exhaustive and mutually exclusive. It remains to check the further conclusions in each case.

(a) 1.4.8. and 1.4.12.

(b) Consider MN and NM. We have $M^r(MN)M^r = M^r MNN^\ell = MN$ whereas $M^r NMM^r = N^\ell NMM^r = N^\ell M^r = M^r$ holds. Thus MN cannot contain NM (else $MN \supseteq M^r \supseteq R$ #). A symmetric argument shows NM cannot contain MN, so that NM and MN are incomparable and $M \rightsquigarrow N \rightsquigarrow M$.

Essentially the same argument works for M^rN^r and N^rM^r. Thus, we have $MM^rN^rM = MM^rM^\ell M = M^rM^\ell = M^rN^r$ whereas $MN^rM^rM = M^2$ holds so that N^rM^r cannot contain M^rN^r. Similarly, multiplying both sides by N one shows that M^rN^r cannot contain N^rM^r.

(c) We have $N = M^{\ell\ell}$, distinct from M and M^{rr}. Thus we have $NM^r = M^rN$ by 1.4.14. But also $M^rN^rNM = M^rNM = NM^rM = NM \subseteq R$ holds, so that M^rN^r is contained in $(NM)^r$. Conversely, for x an element of $(NM)^r$, we have $xNM \subseteq R$, $xN \subseteq M^r$ and $x \in xN^r = xNN^r \subseteq M^rN^r$. Thus $M^rN^r = (NM)^r$ holds and a similar argument shows that $N^rM^r = (MN)^r$ holds. But, now $N^rM^rNM = N^rNM^rM = NM \subseteq R$ holds whence N^rM^r is contained in $(NM)^r$. Thus we have $N^rM^r \subseteq M^rN^r$ and $NM \subseteq MN$.

It only remains to show that these containments are proper. Thus consider the effect of multiplying N^rM^r and M^rN^r by M, noting that $MM^r = M^r$ whereas $MN^r = MM^\ell = M$. Thus we have $MN^rM^r = MM^r = M^r$ whereas $MM^rN^r = M^rN^r$. Since M^r is distinct from M^rN^r, N^rM^r is also distinct from M^rN^r. q.e.d.

The above lemma establishes 1.4.19. and also shows that there is a link following the cycle map for a pair of elements in $\mathcal{P}$. The next lemma shows that these are the only possible links involving an element of $\mathcal{P}$.

1.4.23. Lemma. For maximal ideals M and N of R, with $M \in \mathcal{P}$:

(a) If $M \rightsquigarrow N$ then $M \twoheadrightarrow N$

(b) If $N \rightsquigarrow M$ then $N \twoheadrightarrow M$.

In either case, N is also in $\mathcal{P}$.

Proof. If N is not distinct from M then the link in either (a) or (b) is a self-link. By 1.4.12., if M has a self-link then M is invertible and $M \twoheadrightarrow M$ holds by 1.4.9. Thus w.l.o.g. we assume that M is distinct from N.

Suppose now that $N \rightsquigarrow M$ holds, but $N \twoheadrightarrow M$ does not hold, with M and N distinct. By 1.4.22. (c), $M \twoheadrightarrow N$ also does not hold. Thus by 1.4.17. we have $M^r N^r = N^r M^r = (M \cap N)^r$. As in the proof of 1.4.22. (c), we can show that $(MN)^r = N^r M^r$ holds, since this only used that M is projective. But, this yields that $(MN)^r = (M \cap N)^r$ and even $MN = M \cap N$ which contradicts the existence of a link from N to M. Thus, if $N \rightsquigarrow M$ holds, $N \twoheadrightarrow M$ also holds.

A similar argument (showing that $(NM)^\ell = (M \cap N)^\ell$) shows that if $M \rightsquigarrow N$ holds then $M \twoheadrightarrow N$ also holds. Note that N is projective by 1.4.13. q.e.d.

The next result is an easy but important consequence of 1.4.21.

1.4.24. Corollary. $\mathcal{P}$ and $\mathcal{N}$ are both closed under links.

We have shown now that links and the cycle map respect the partition of M-Spec(R) into $\mathcal{G}$ and $\mathcal{N}$. In $\mathcal{G}$ we have also seen that links interact with the cycle map exactly as they do for a bounded HNP ring. The next few results show that several other properties of bounded HNP rings apply to $\mathcal{G}$.

1.4.25. Definition. For a maximal ideal M of R, the link closure of M is the smallest link closed subset of M-Spec(R) which contains M. The cycle closure of M is the smallest subset of M-Spec(R) which contains M and is closed under the cycle map and its inverse.

These are connectivity components of the corresponding directed graph structures on M-Spec(R). If the link closure of a maximal ideal is finite (ie. has a finite vertex set) then it is called a clan. The intersection of the vertices of a clan is a classically localizable semiprime ideal. It is thus of interest to know when a maximal ideal is an element of a clan. It should be noted that in a bounded HNP ring, every maximal ideal is an element of a clan, and the intersection of any clan is an invertible ideal.

1.4.26. Theorem. The link closure of an element of $\mathcal{G}$ is finite. The intersection of a clan in $\mathcal{G}$ is an invertible ideal. Thus every projective maximal ideal of R contains an invertible ideal.

Proof. Since the link graph mirrors the cycle map graph in $\mathcal{P}$, a link component in $\mathcal{P}$ is infinite if and only if there exists an infinite sequence of projective maximal ideals M_l with $M_l \rightsquigarrow M_{l+1}$, where (by 1.4.23. and 1.4.24.) these are the only links involving the M_l. In [14], Mueller shows that in any bounded prime ring of Krull dimension one, the link diagram starting from a maximal ideal cannot be a tree, (ie. it must loop back on itself at some point). Hence, such an infinite link component is impossible.

That the intersection of a clan in $\mathcal{P}$ is invertible, is proved in the same way as for a bounded HNP ring. The proof is presented here for completeness.

Thus, let $\{M_l\}_{l=1}^{n}$ be the vertices of a clan in $\mathcal{P}$, numbered so that $M_l \rightsquigarrow M_{l+1}$ ($l \bmod n$) holds for each l and also set $X = \bigcap_{l=1}^{n} M_l$. If n is 1, the ideal $M_1(=X)$ is invertible by 1.4.8. Thus, we assume $n > 1$, so that no M_l is invertible. For each k, set $S_k = (M_k)^r = (M_{k+1})^\ell$. Both $S_k M_k = M_k = M_k S_{k-1}$ and $M_k S_k = S_k = S_k M_{k+1}$ hold by 1.4.12. We can write $R \supseteq M_{k+1} = M_{k+1}S_k \supseteq XS_k \supseteq A_k M_{k+1} M_k S_k = A_k M_{k+1} S_k$ where A_k is any irredundent product of all the $\hat{M}_l$'s except M_k and M_{k+1} (if $n=2$ set $A_k=R$). Thus XS_k is contained in R, and S_k is contained in X^ℓ. Since M_k is prime,

$A_k M_{k+1}$ is not contained in M_k, and even moreso is XS_k not contained in M_k. Thus $R \supseteq XX^{\ell} \supseteq XS_k \not\subseteq M_k$ holds for all k. XX^{ℓ} is an ideal of R which is contained in no maximal ideal and consequently must equal R. A similar argument shows $X^r X = R$ also holds so that $X^r = X^{\ell} = X^{-1}$ is the inverse of X. q.e.d.

It is well known that for any ideal X in a bounded HNP ring R, R/X is a serial ring, and that this in fact characterizes HNP rings in the collection of bounded prime rings. We wish to generalize this to ideals which are in $\mathcal{P}$ (ie. ideals all of whose maximal over-ideals are in $\mathcal{P}$). We first need the following technical lemma.

1.4.27. Lemma. If $\mathcal{A}$ and $\mathcal{L}$ are finite link closed components of M-Spec(R) than setting $A = \cap\mathcal{A}$, $B = \cap\mathcal{L}$ we have $AB = BA$.

Proof. Assume first that $\mathcal{A}$ and $\mathcal{L}$ are minimal link components. Then, either $\mathcal{A} = \mathcal{L}$ or they are disjoint. If $\mathcal{A} = \mathcal{L}$ then $A = B$ and there is nothing to show, so assume that $\mathcal{A}$ and $\mathcal{L}$ are disjoint.

Claim. A and B can be written as a sum of products of maximal ideals in $\mathcal{A}$ and $\mathcal{L}$ respectively.

Proof. Write $A = \bigcap_{i=1}^{n} A_i$ where the A_i are the maximal ideals of $\mathcal{A}$. If $n = 1$, $A = A_1$, so there is nothing to show. We proceed by

induction so assume that the intersection of $\{A_1,\dots,A_k\}$ can be written as a sum of products of the A_l and consider $\bigcap_{l=1}^{k+1} A_l$. Since $\bigcap_{l=1}^{k} A_l$ and A_{k+1} are comaximal, $\bigcap_{l=1}^{k+1} A_k = A_{k+1}(\cap A_l) + (\bigcap_{l=1}^{k} A_l)A_{k+1}$. By substituting the sum of products of the A_l for $\bigcap_{l=1}^{k} A_l$ which exists by induction hypothesis we are done.

Since $\mathcal{A}$ and $\mathcal{L}$ are link closed and disjoint, $A_l B_j = B_j A_l$ for any l,j. Thus $AB = (\Sigma\ \pi A_l)(\Sigma\ \pi B_j)$ and expanding this sum we have a sum of terms of the form $\pi A_l \pi B_j$. The various A_l commute across the πB_j to give $\pi A_l \pi B_j = \pi B_j \pi A_l$ and the collection of terms then yields $(\Sigma\ \pi A_l)(\Sigma\ \pi B_j) = (\Sigma\ \pi B_j)(\Sigma\ \pi A_l)$ so that $AB = BA$ as required.

If now $\mathcal{A}$ and $\mathcal{L}$ are any finite link closed subsets in M-Spec(R), the intersection of $\mathcal{A}$ (and of $\mathcal{L}$) is precisely the product (in any order) of the intersections of the minimal link closed subsets and these all commute.

1.4.18. Theorem. For an ideal $0 \neq X$ which has every maximal overideal in $\mathcal{P}$, R/X is a serial ring.

Proof. From 1.3.4., for any invertible ideal Y of R, R/Y is quasi-Frobenius. Since $X \neq 0$, there exist only finitely many maximal ideals which contain it. Let χ = link closure of this finite set. χ is again finite by 1.4.26. Set $J = \cap\chi$, so J is a finite product of invertible ideals (ie. the intersections of the maximal link closed components) so J is invertible, as is J^n for all n. By 1.3.2. there exists an n with $X \supseteq J^n$ ie. R/X is a factor ring of R/J^n. It is thus sufficient to show that R/J^n is serial. However, J = the Jacobson radical of R/J^n and so R/J^n is a quasi-Frobenius ring for which R/J^2 is also quasi-Frobenius. By Faith [6], 25.4.3., R/J^n is serial. q.e.d.

1.4.29. Corollary. If every maximal ideal of R is projective, then R is hereditary.

Proof. Singh's Theorem [16]. q.e.d.

Although the elements of $\mathcal{N}$ cannot be so completely described, there are still some generalizations of the above that apply.

1.4.30. Proposition. For a finite cycle component $\{M_l\}_{l=1}^n$ in $\mathcal{N}$ set $J = \bigcap_{l=1}^{n} M_l$. Then $J^r = \Pi(M_l^r)$ and $J^r J = J$.

Proof. By 1.4.18., all of the M_l^r commute pairwise, so the product can be taken in any order. For each l, we have $M_l^r J \subseteq M_l^r M_l \subseteq R$, so that $M_l^r J = M_l^r(\bigcap_{j=1}^{n} M_j) \subseteq \bigcap_{j\neq l} M_l^r M_j \cap M_l^r M_l = \bigcap_{j\neq l} (M_l^r M_j \cap R) \cap M_l = \bigcap_{j=1}^{n} M_j = J$ holds. Now with each $M_l^r J$ equal to J, $(\Pi M_l^r)J$ must also equal J, since J can absorb the product factor by factor, and (ΠM_l^r) is clearly contained in J^r. Conversely, for each l, (ΠM_j^r) contains M_l^r, forcing $(\Pi M_l^r)^l$ to be contained in each $M_l^{rl} = M_l$. In other words, ΠM^r contains J^r. Since we now have established $J^r = (\Pi M_l^r)$, we also can compute $J^r J = \Pi M_l^r J = J$. q.e.d.

1.4.31. Proposition. For M_l and J as in 1.4.30., $J^r = J^l$ is an equivalent order, and even $(J^n)^r = (J^n)^l$ for all n.

Proof. An argument symmetric to the one of 1.4.30., shows that $J^l = \Pi(M_l^l)$ holds, but $M_l^l = M_{j(l)}^r$ for a permutation j of $(1,2,\ldots,n)$. Since the products can be taken in any order we have $J^r = J^l$. J^r is an equivalent order by 1.4.10. That $(J^n)^r = (J^n)^l$ for all n follows from 1.4.6., by an easy induction argument. q.e.d.

The following corollary is an interesting generalization of 1.4.8.

1.4.32. Corollary. For M_l and J as before, the following are equivalent:

(1) J is invertible.

(2) J is right invertible.

(3) J is projective

(4) $J^r J \neq J$

(5) All the M_l are in $\mathcal{P}$.

CHAPTER II

EQUIVALENT ORDERS

Chapter II examines the (especially minimal) overrings of a bounded noetherian prime ring of injective dimension one. Our aim is to compare the maximal spectrum of such a ring with the spectrum of its minimal overrings. We also want to discover when the minimal overring is again bounded noetherian and prime with injective dimension one.

In section I, some background material on bounded noetherian prime rings of Krull dimension one is presented. Section II deals with the characterization of injective dimension one for equivalent orders to a bounded noetherian prime ring of injective dimension one. Sections III and IV examine the minimal equivalent orders, separating into the cases of distinct conductors and unique conductor.

I. KRULL DIMENSION ONE.

Throughout this section R will denote a bounded noetherian prime ring with Krull dimension one, and Goldie quotient ring K.

The aim of this section is to collect some important, but well known properties of R and its overrings which can be applied to bounded noetherian prime rings of injective dimension one by 1.3.2.

2.1.1. Definition. A subring $S \supsetneq K$ which contains R is called an overring of R. An overring S of R for which there exists a regular element c of R with cS and Sc contained in R is called an equivalent order of R.

2.1.2. Theorem. [Michler]. Any overring of R has Krull dimension one. In fact, for any essential right (or left) ideal A of S, S/A has finite length even as an R-module.

Proof. [12] Hilfssatz 3.1. q.e.d.

2.1.3. Lemma. Any overring S of R is bounded.

Proof. The proof is lifted from [11] where it was used to show only that an overring of a bounded HNP ring is again bounded.

By 2.1.2., S/A has finite length as an R-module for any essential right ideal A of S. Thus, S can be written as $S = (\sum_{\text{fin}} x_1 R) + A$

Since R is bounded, there is a regular element c of R for which $(\sum x_1 R)c \subseteq R$ holds. Since A is essential in S, $R \cap A$ is essential in R, and so $R \cap A$ contains an ideal B of R. But now we have $0 \neq (S)cBS = (\sum x_1 R)cBS + AcBS \subseteq RBS + AS \subseteq AS = A$, which gives a non zero ideal of S contained in A. This shows that S is right bounded, and a similar agrument on the other side shows that S is left

bounded. q.e.d.

2.1.4. Corollary. Any overring S of R is again bounded noetherian prime with Krull dimension one.

It is occassionally necessary to be able to distinguish the equivalent orders in the class of overrings. The following proposition gives a useful characterization.

2.1.5. Proposition. For an overring S of R the following are equivalent:

(1) S is an equivalent order.

(2) S is f.g. as an R-module.

(3) $S^r \neq 0$.

(4) $S^\ell \neq 0$.

(5) R contains a non zero ideal of S.

Proof. The regular element c from the definition of equivalent order is a non zero element of both S^r and S^ℓ. Thus (1) implies (3) and (4).

(1) $\Rightarrow$ (2). Since cS is right isomorphic to S, and Sc is left isomorphic to S, and R is noetherian, (1) clearly implies that S is f.g. as R-bimodule. Conversely, if S is f.g., the Ore condition guarantees the existance of regular elements c_1 and c_2 of R with c_1S and Sc_2 contained in R. But then the regular element $c = c_2c_1$

yields cS and Sc contained in R.

(1) ➡ (5). Since S is an R-bimodule, so are S^r and S^ℓ, ie. S^r and S^ℓ are ideals of R. Since S is a ring $S^rS = S^r$ and $SS^\ell = S^\ell$ hold. Thus $S^\ell S^r$ is an ideal of S which is contained in R and non zero since R is prime.

(5) ➡ (2). For X a non zero ideal of S, X is essential since S is prime. Thus S/X has finite length as an R-module, and since X is f.g. as R-module, so is S.

(3) ➡ (2). S^r is a non zero ideal of R, hence must contain a regular element c of R. Thus cS is contained in R, so that S is right f.g. By 1.2.8., S is also left f.g.

(4) ➡ (2). is similar to (3) ➡ (2). q.e.d.

II. INJECTIVE DIMENSION ONE FOR EQUIVALENT ORDERS.

In this section, R denotes a bounded noetherian prime ring with injective dimension one and Goldie quotient ring K.

By 2.1.4., any overring of a bounded noetherian prime ring of Krull dimension one is again such. It is also true (see [11]) that any overring of a bounded HNP ring is again such. Since the class of rings we are studying is sandwiched between these two classes it is natural to

ask: When, if ever, is an overring or equivalent order to R again in this class? In light of 2.1.4., we only need to verify that the injective dimension of the overring or equivalent order is one. For general overrings of R, it seems that little can be said, for there exist minimal equivalent orders which do not have injective dimension one. We can, however, characterize when an equivalent order has injective dimension one.

Before producing the characterization, we introduce a notational device to distinguish between duals with respect to R and duals with respect to S, for a given equivalent order S of R.

2.2.1. Notation. Let S be an equivalent order to R, M a right S-submodule of K and N a left S-submodule of K. Since S contains R, M and N are also R-modules, so one can form their R-duals, as well as their S-duals. To distinguish these, we use superscripts as before for R-duals and subscripts for S-duals. Thus, we write $M_r = \{k \in K \mid kM \subseteq S\}$ and $N_\ell = \{k \in K \mid Nk \subseteq S\}$.

There is a straightforward relationship between the S-duals and the R-duals.

2.2.2. Lemma. For S an equivalent order of R with conductors $M = S^\ell$ and $N = S^r$, and A a left S-submodule of K, B a right S-submodule of K we have $B_r = (BM)^r$ and $A_\ell = (NA)^\ell$.

Proof. We have $B_r = \{k \in K | kB \subseteq S\}$. Since $M = S^\ell$ holds, we have $M^r = S$ by reflexivity so $B_r = \{k \in K | kB \subseteq M^r\} = \{k \in K | kBM \subseteq R\} = (BM)^r$ holds. The formula for A_ℓ is similar. q.e.d.

We can now give the promised characterization of when S has injective dimension one.

2.2.3. Theorem. For an equivalent order S of R with conductors $M = S^\ell$ and $N = S^r$ the following are equivalent:

(1) inj dim $(S_S) = 1$.

(2) M is left S-projective.

(3) $(NM)^\ell M = O_r(M)$.

[Left versions: (1) inj dim $({}_S S) = 1$.

(2) M is right S-projective.

(3) $N(NM)^r = O_\ell(N)$.]

Proof. As noted in 2.2.2, $S = M^r$ holds and by 1.4.10., we have $\overline{SM} = M$ so that M is in fact a left ideal of S.

Since K is also the quotient ring of S, inj dim $(S_S) = 1$ holds if and only if K/S is right S-injective. The map $\varphi: K \to \hom(M, K/R)$ defined via $\varphi(k): M \to K/R$ by $\varphi(k)(m) = \overline{km}$ is onto because K/R is injective. Its kernel is precisely $S(=M^r)$ so that K/S is isomorphic to $\hom(M, K/R)$. We can now show the equivalence of (1) and (2).

(1) ⇒ (2). Since $K/S \simeq \hom(M,K/R)$ is injective, the functor $\hom_S(\ ,\hom(M,K/R))$ is exact. By the adjoint isomorphism theorem $\hom_R(\ \otimes_S M,K/R)$ is also exact. Since K/R is a cogenerator, $-\otimes_S M$ is also exact (see eg. [1] Prop. 1.8.14). Thus M is S-flat, but S being noetherian yields M is also finitely presented, hence M is S-projective.

(2) ⇒ (1). It is standard (see eg. [10] pg. 131, Prop. 3) that for a projective S-module M and an injective R-module K/R, $\hom_R(M,K/R)$ is right S-injective.

(2) ⟺ (3). By the dual basis lemma M is left S-projective if and only if $M_\ell M = \mathrm{End}_S(M)$ holds. Now $\mathrm{End}_S(M) = O_r(M)$ and $M_\ell = (NM)^\ell$ hold by 1.2.4. and 2.2.2' respectively. q.e.d.

The special case where the conductors coincide is of sufficient interest to be stated separately.

2.2.4. Corollary. For an equivalent order S of R with unique conductor $M = S^r = S^\ell$ we have $\mathrm{inj\ dim}(S_S) = 1$ if and only if $(M^2)^\ell M = S$.

[Left version: $\mathrm{inj\ dim}({}_S S) = 1$ if and only if $M(M^2)^r = S$.]

Proof. Since S is a ring, we have $SM = MS = M$ by 1.4.10. whence S is contained in $O_r(M)$ and $O_\ell(M)$. But by 1.2.4. S

contains $O_r(M)$ and $O_\ell(M)$. q.e.d.

Since R-projectivity forces the maximal ideals of R to be very well behaved, it would be hoped that R-projectivity should guarantee S-projectivity. The next lemma shows that this is in fact true.

2.2.5. Lemma. Let S be an equivalent order to R with conductors $M = S^\ell$ and $N = S^r$. For a right R-projective right ideal X of R, XS is a right S-projective right ideal of S.
[Left version: For a left R-projective left ideal Y of R, SY is a left S-projective left ideal of S.]

Proof. X is right R-projective if and only if $1 \in XX^r$ holds. Also XS, which is certainly a right ideal of S, is right S-projective if and only if $1 \in (XS)(XS)_r$ holds. For an element k of X^r, $kXS \subseteq RS \subseteq S$ holds, so that X^r is contained in $(XS)_r$. Thus $(XS)(XS)_r \supseteq XX^r$, whence if $1 \in XX^r$ holds, $1 \in (XS)(XS)^r$ certainly holds. q.e.d.

2.2.6. Corollary. Let S be as above, and suppose that M is left R-projective. Then $\text{inj dim}(S_S) = 1$ holds.
[Left version: If N is right R-projective, then $\text{inj dim}({}_SS) = 1$ holds.]

Proof. 2.2.5. [left version] and the observation that $SM = M$ holds. q.e.d.

The foregoing has interesting applications in the commutative case. In [4], Bass investigates a commutative domain, A whose integral closure A' is a f.g. A-module, and with the property that all rings between A and A' are Gorenstein (ie. have injective dimension one). He shows that all ideals of A are projective over their endomorphism rings and conjectures that this is also characterization of these rings. Handelman [7], has verified that this condition does in fact characterize these rings. If we drop the assumption that A' is module finite over A, then by what we have just shown if A has injective dimension one, every ideal is projective over its endomorphism ring implies every equivalent order to A has injective dimension one. In fact, one only needs that the conductors of the equivalent orders are projective over their endomorphism rings.

This information (in the commutative case) suggests several interesting questions. Can one verify directly that if every ideal is projective over its endomorphism ring then the ring has injective dimension one? What can one say about a local commutative domain for which every equivalent order (or every overring) has injective dimension one without the assumption of module finiteness for the integral closure? These questions are beyond the scope of the thesis, but the author hopes to return to them later.

III. MINIMAL OVERRINGS, DISTINCT CONDUCTORS.

In this section, R is still a bounded noetherian prime ring with injective dimension one and Goldie quotient ring K. T will be used to denote a minimal overring with distinct conductors $M = T^{\ell}$ and $N = T^{r}$. Thus, we have $T = N^{\ell} = M^{r}$ and $N \neq M$ holds. Since T is a minimal overring, M and N are maximal ideals of R. After 2.3.12., M and N are assumed to be in $\mathcal{N}$ for the rest of the section.

The aim of this section is to compare the max-spectra of R and T. More specifically, we wish to describe M-Spec(T), its link directed-graph and its cycle map (when T has injective dimension one) in terms of those in R. We begin by considering the maximal ideals of T. Recall from 2.1.2. that every non zero prime ideal of T is maximal so that we are actually describing the entire prime spectrum of T.

There is a relation Θ on M-Spec(R) $\times$ M-Spec(T) given by $P \Theta Q$ if and only if P is a minimal prime ideal over $Q \cap R$. This relation is defined on the spectra whenever one has a subring of another ring. Since we are dealing with Krull dimension one and the only prime ideal (in either R or T) that is Θ-related to 0 is zero, there is no loss of generality in restricting to the max-spectra. Generally, for noetherian rings, the relation Θ is a finite to many correspondence. In our setting, however, it is almost a bijection when M and N are projective, and is a bijection when M and N are not projective.

2.3.1. Theorem. The relation Θ is an onto function from M-Spec(R) to M-Spec(T). It is a bijection if M and N are not projective.

For a maximal ideal P of R, distinct from M and N, $\Theta P = PT = TP$. If M and N are not projective, then $\Theta M = MT$ and $\Theta N = TN$. If M and N are projective then $\Theta M = \Theta N = MN$.

If M and N are projective then $\Theta^{-1}MN = \{M,N\}$. For any other maximal ideal Q of T, $\Theta^{-1}Q = Q \cap R$ holds whether or not M and N are projective.

Proof. We verify only the explicit description of Θ and Θ^{-1} given in the second and third paragraphs, since the statement of the first paragraph follows easily from them.

For a maximal ideal P of R, distinct from M and N, $PT = TP$ is the only proper T-ideal which contains $PT \cap R = P$ by 1.4.9. and 1.4.14. This shows that $\Theta(P) = PT = TP$ holds and that $\Theta^{-1}(PT) = P$.

Claim. If M and N are non-projective, then MT is the only proper T-ideal which contains $MT \cap R = M$ and TN is the only proper T-ideal which contains $TN \cap R = N$.

Proof. We only treat M, since the arguments for N are similar. Since M is a left ideal of T, MT is an ideal of T. It

is a proper ideal since M is not projective. $MT \cap R = M$ holds by 1.4.9. Since any ideal of T which contains M, must contain MT it only remains to show that MT is maximal in T. However, since M is distinct from $N(=T^r)$, MT must properly contain M. Now T/M has R-bimodule length two, so MT is even maximal as R-bimodule in T. This proves the claim.

Before verifying the rest of 2.3.1., we present some observations concerning an arbitrary ideal X of T.

Claims. (1) $X/X \cap R$ has R-bimodule length zero or one.

(2) Either $X \cap R$ is an ideal of T and $X \cap R \subseteq M \cap N$ holds, or else $X = (X \cap R)T$ or $X = T(X \cap R)$.

(3) If M and N are projective, and $X \subseteq R$ holds, then $X \subseteq MN$ also holds.

Proofs. (1) If X is contained in R then $X = X \cap R$ holds whence $X/X \cap R$ has length zero. Suppose therefore, that X is not contained in R. Since R is bimodule maximal in T, we have $R \subsetneq X + R$ forces $X + R = T$. By the classical isomorphism theorem $X/X \cap R \simeq T/R$ holds and T/R is R-bimodule simple. ok claim (1).

(2) If $X \cap R$ is an ideal of R then $T(X \cap R) = X \cap R \subseteq R$ holds forcing $X \cap R \subseteq T^{\ell} = M$. Similarly, $X \cap R$ must also be contained in N. If $X \cap R$ is not an ideal of T, then $X \cap R$ is properly contained in at least one of the bimodules $T(X \cap R)$ or

$(X \cap R)T$, both of which are contained in X. Now we are done by claim (1). ok claim (2).

(3) Since M and N are projective, we have $T = MT = TN$. Thus, writing $X = TXT$ we have $X = MTXTN = MXN \subseteq MN$. ok claim (3).

Claim. If M and N are projective, then MN is a maximal ideal of T.

Proof. Since $M = TM$ and $N = NT$ hold, MN is an ideal of T. Suppose that X is a proper ideal of T which properly contains MN. By claim (3) above, X is not contained in R. Also, by claim (3), if $X \cap R$ were a T-ideal it would be contained in MN, and then we could write $X = TX = TNX$ where $NX \subseteq NT = N \subseteq R$ holds, forcing $NX \subseteq X \cap R \subseteq MN$ and $X = TNX \subseteq TMN = MN$. Since this last part is impossible, we have that $X \cap R$ is not an ideal of T. Now by claim (2) we know that either $X = T(X \cap R)$ or $X = (X \cap R)T$ holds. We consider $X = T(X \cap R)$, noting that $X = (X \cap R)T$ is similar. Since X contains MN, $X \cap R$ also contains MN. Since X is properly contained in T, $X \cap R$ is properly contained in R, hence must be contained in some maximal ideal of R. Since $X \cap R$ contains MN the only possible such maximal ideals are M and N. If $X \cap R \subseteq M$ holds then $X = T(X \cap R) \subseteq TM = M \subseteq R$ follows which is impossible. Thus, the only maximal ideal which can contain $X \cap R$ is N. However, in a noetherian ring, any ideal contains some product of the minimal primes

over it. Thus $X \cap R$ must contain some power of N, but N being idempotent forces $X \cap R = N$. But now $X = T(X \cap R) = TN = T$ contradicts the assumption that X was proper in T. ok claim

It only remains to show now that any maximal ideal of T has one of the following forms: (1) MN if M and N are projective, (2) MT or TN if M and N are not projective, or (3) $PT = TP$ for some maximal ideal P of R distinct from M and N.

To this end let Q be a maximal ideal of T, which we assume to be different from MN if M and N are projective. We will show that Q can be written as TP or PT for some maximal ideal P of R. If M and N are not projective, and $Q \cap R$ is an ideal of T then $Q \cap R$ is contained in the T-ideal $M \cap N$ by claim (2) and 1.4.17. If Q does not contain $M \cap N$, ie. if $Q \cap R$ is properly contained in $M \cap N$, then $Q + M \cap N = T$ holds by the maximality of Q. By the modular law this forces $Q \cap R + M \cap N = R$ where we already have $Q \cap R \subseteq M \cap N$. Contradiction. Thus Q does contain $M \cap N$, so that $Q \cap R = Q \cap M = Q \cap N = M \cap N$ follows. Note that $MT \supsetneq M \cap N$ so that $M \cap N$ is not a maximal ideal of T, and further Q must properly contain $Q \cap R$. Since, by claim (1) above, $Q/Q \cap R$ is bimodule simple, if Q is not contained in either of MT or TN we also have $Q \cap R = Q \cap MT = Q \cap TN$ and $Q + MT = Q + TN = T$. Now, by the classical isomorphism theorem, we have $T/MT = Q+MT/MT \cong Q/Q \cap MT = Q/Q \cap TN \cong Q+TN/TN = T/TN$, which contradicts the fact that T/MT and

T/TN have distinct annihilators. Thus if M and N are not projective $Q \cap R$ is not an ideal of T, and we have $Q = T(Q \cap R)$ or $Q = (Q \cap R)T$. If, on the other hand M and N are projective and $Q \cap R$ is an ideal of T, then as we argued in the last claim, we have $Q = TQ = TNQ$ with $NQ \subseteq NT = N \subseteq R$, so that $Q \subseteq T(Q \cap R) \subseteq TMN = MN$ follows, contradicting our assumption that Q is different from MN. Thus for either case, we conclude that either $Q = T(Q \cap R)$ or $Q = (Q \cap R)T$.

But now $Q \cap R$ must be contained in some maximal ideal P of R, whence Q is contained in either TP or PT. If P is distinct from M and N then $TP = PT$ is T-maximal, so $Q = PT = TP$ follows and Q has the prescribed form. If M and N are non-projective and $Q \cap R$ is contained in M then $Q = T(Q \cap R)T \subseteq TMT = MT$ follows and again Q has the prescribed form. Similarly, $Q \cap R$ contained in N forces $Q = TN$. If M and N are projective and $Q \cap R$ is contained in M, then $Q = T(Q \cap R)$ forces $Q \subseteq M \subseteq R$, whence $Q \subseteq MN$ follows by claim (3) which again contradicts our assumption that Q is different from MN. If, on the other hand, $Q = (Q \cap R)T$ then $Q \cap R$ is not contained in N (else $Q = (Q \cap R)T \subseteq R$ which force $Q = MN$) and $Q \cap R$ is not contained in any other maximal ideal P (else $Q \subseteq PT$ forces $Q = PT$ and $Q \cap R = P \subsetneq M$) so $Q \cap R = M$ follows from the idempotence of M and $Q = MT = T$ contradicts that Q is a maximal ideal. Similarly, $Q \cap R \subseteq N$ leads to a contradiction. q.e.d.

In much of the rest of this section, it is necessary to consider duals with respect to both T and R. We will use the notation convention outlined in 2.2.1. throughout Section III for this situation only. The next proposition sets out several computational formulas that we will need.

2.3.2. Proposition. (1) For a right T-submodule X of K, $X_r = (XM)^r$.

(2) For a left T-submodule Y of K, $Y_\ell = (NY)^\ell$.

(3) $N_r = (NM)^r$; $M_\ell = (NM)^\ell$; $(MT)_r = (M^2)^r$; $(TN)_\ell = (N^2)^\ell$; $(MT)_\ell = (NMT)^\ell$; $(TN)_r = (TNM)^r$. All of these contain T.

(4) $(N^m)^\ell = (M^m)^r$ for all natural numbers m. In particular $(MT)_r = (TN)_\ell$ holds, and this properly contains T if and only if M (and N) has a self link.

(5) $(MT)_r MT = MT$; $TN(TN)_\ell = TN$.

(6) If $M \rightsquigarrow M$ holds, then

(a) $(MT)_r M = MT$

(b) $N(TN)_\ell = TN$

(c) $MT(MT)_\ell = MT$

(d) $(TN)_r TN = TN$

(7) $TNM = NMT$ if M and N are not projective.

Proof. (1) and (2) are just 2.2.2.

(3) Immediate from (1) and (2), and the facts that N^2 and NM and TNM are contained in N while M^2, NM and NMT are contained in M.

(4) The first statement follows from an easy induction argument on 1.4.5. The second statement follows from (3) and the fact that "r" is a lattice anti-isomorphism.

(5) We only show that $(MT)_r MT = MT$ holds, since the other assertion is similar. If M is projective, the assertion is trivial so assume M is not projective. Now $1 \in T \subseteq (MT)_r$ holds, and MT is a maximal ideal of T, so $T \supseteq (MT)_r MT$ and $(MT)_r MT \supseteq MT$ holds, and it is only necessary to show that $(MT)_r MT$ is not equal to T. But, if it were, we would have $(MT)_\ell = (T)(MT)_\ell = (MT)_r(MT(MT)_\ell) \subseteq (MT)_r T = (MT)_r$. Using (2), (3) and (4), we would then have $(NMT)^\ell \subseteq (M^2)^r = (N^2)^\ell$ and even $M \supseteq NMT \supseteq N^2$ which contradicts the assumption that M and N are distinct.

(6) We assume now that $M \longrightarrow M$ holds.

(a) Since $M \neq M^2$ holds, $(MT)_r \neq T$ holds and 1 is an element of $(MT)_r^\circ$, so $MT = (MT)_r MT \supseteq (MT)_r M \supsetneq M$ with M bimodule maximal in MT forces $(MT)_r M = MT$.

(b) Similar to (a).

(c) As in the proof of (5) we only need show that $MT(MT)_\ell$ is not

equal to T. Still following the proof of (5), if it were then $N^2 \supseteq NMT \supseteq NM$ would follow. But then $N = NR = N^2 + NM = N^2$ would follow, ie. N has no self link. However, N has a self link since M has a self link by 1.4.7.

(d) Same as (c).

(7) We consider the cases $M \rightsquigarrow M$ and $M \not\rightsquigarrow M$ separately. If $M \rightsquigarrow M$ holds then $(TN)M = N(TN)M = N((MT)M) = NMT$ by (6) and (4). If, on the other hand $M \not\rightsquigarrow M$ holds, then since $M = M^2$ we have $(M \cap N)M = (MN + NM)M = MNM + N(M^2) = MNM + NM = NM$ and similarly $N(M \cap N) = NM$ holds also. But now $NMT \subseteq N \cap N$ (by 1.4.18. forces $NM \subseteq NMT = N(NMT) \subseteq N(M \cap N) \subseteq NM$, whence $NM = NMT$. Similarly, $TNM = NM$ also holds and we are done. q.e.d.

The relation Θ outlined in 2.3.1. allows us to completely describe M-Spec(T) in terms of M-Spec(R). We turn now to the problem of describing the links in T in terms of those in R.

For M and N projective the two maximal ideals M and N of R collapse to the one maximal ideal MN of T. As we shall see in 2.3.3., modulo this collapse and the destruction of the link $N \rightsquigarrow M$, Θ and Θ^{-1} preserve the link structure.

For M and N not projective Θ is a bijection. In this case Θ^{-1} preserves all links except for $MT \rightsquigarrow TN$ which always exists in T

and Θ preserves all links except possibly the self links for M and N or links which come from the situation $M \rightsquigarrow P \rightsquigarrow N$ with P different from M and N.

2.3.3. Theorem. For M and N projective, and R-maximal ideals P and Q, we have $P \rightsquigarrow Q$ in R if and only if $\Theta P \rightsquigarrow \Theta Q$ in T except that:

(a) If $M \Rightarrow N$ holds, then $MN = \Theta M = \Theta N$ has a self link in T, but M and N do not have self links in R.

(b) If $M \not\Rightarrow N$ holds, then $N \rightsquigarrow M$ holds in R, but $\Theta N = MN = \Theta M$ does not have a self link in T.

(c) If $M \not\Rightarrow N$ holds, then $\Theta N \rightsquigarrow \Theta M^{rr}$ and $\Theta N^{ll} \rightsquigarrow \Theta M$ hold in T even though $N \not\rightsquigarrow M^{rr}$ and $N^{ll} \not\rightsquigarrow M$ hold in R.

Proof. The theorem is proved by lemmata 2.3.5., 2.3.7., 2.3.8., and 2.3.9. q.e.d.

2.3.4. Theorem. For M and N not projective and R-maximal ideals P and Q we have $P \rightsquigarrow Q$ in R if and only if we have $\Theta P \rightsquigarrow \Theta Q$ in T except that:

(a) $\Theta M \rightsquigarrow \Theta N$ holds in T whether or not $M \rightsquigarrow N$ holds in R.

(b) If P is distinct from M and N, and $M \rightsquigarrow P \rightsquigarrow N$ holds in R, it is open whether $\Theta M \rightsquigarrow \Theta P$ or $\Theta P \rightsquigarrow \Theta N$ must hold in T.

(c) If $M \rightsquigarrow N$ and M has a self link in R (and hence N has a self

link) it may happen that ΘM and ΘN have no self link in R. It is open whether exactly one of ΘM or ΘN can have a self link in this situation.

Proof. The theorem is proved by lemmata 2.3.6., 2.3.7., 2.3.8., and 2.3.9. q.e.d.

Note that there are examples for (a) and (c) in 2.3.21. and 2.3.22. respectively.

The next lemma verifies that the exceptional cases of 2.3.3. do in fact occur.

2.3.5. Lemma. For M and N projective the link directed graph in R at $\{M,N\}$ is given by $N^{ll} \rightsquigarrow N \rightsquigarrow M \rightsquigarrow M^{rr}$ whereas the link directed graph in T at MN is given by $ON^{ll} \rightsquigarrow MN \rightsquigarrow OM^{rr}$.

Proof. The statement concerning the link directed graph in R at $\{M,N\}$ follows directly from 1.4.22.

For the statement concerning the link directed graph in T, note first that by results of Section II, T is a bounded noetherian prime ring with injective dimension one.

Now, if $M \Rightarrow N$ holds, we have $M = N^{rr}$, $N = M^{ll}$ and MN is T-invertible with inverse TN^{r}. We are done now by 1.4.22.

If $M \not\rightarrow N$ holds in R, we have $MM^r = M^r$, $NN^r = N^r$ and N is distinct from M and M^{rr} so that $M^rN = NM^r$ and $MNM^rN^r = MM^rNN^r = M^rN^r \supseteq 1$ hold. By 1.4.23. we have $M^rN^r \leq N^rM^r \leq (MN)^r \leq (MN)_r$, whence we conclude that MN is T-projective. Thus MN has exactly one incoming and one outgoing link by 1.4.23. To see that ΘN^{rr} does in fact link to MN in T, we show that $MN(\Theta N^{rr})$ cannot contain $\Theta(N^{rr})MN$. Note that $\Theta N^{rr} = TN^{rr} = N^{rr}T$ by 1.4.14. and suppose that $MN\Theta(N^{rr})$ does contain $\Theta(N^{rr})MN$. This is the same as $MNN^{rr} \geq N^{rr}MN$. But now, multiplying on the right by $N^r = N^{rr\ell}$ yields $MNN^{rr}N^{rr\ell} = MNN^{rr} \geq N^{rr}MNN^r = N^{rr}MN^r = N^{rr}MM^{\ell} = N^{rr}M$. But $N \geq MNN^{rr}$ and $N \not\geq N^{rr}M$ hold. Contradiction. To show that $MN \rightsquigarrow \Theta M^{\ell\ell}$ holds is similar. q.e.d.

The lemma establishes that the exceptional cases of 2.3.3. do in fact occur. It is only left to show that all links in R that do not involve M or N are preserved by Θ (2.3.9.) and all links in T that do not involve MN are preserved by Θ^{-1} (2.3.7, and 2.3.8.).

The next lemma establishes that for M and N not projective $\Theta M \rightsquigarrow \Theta N$ always holds in T, as asserted in the exceptional case (a) of 2.3.4.

2.3.6. Lemma. For M and N not projective, $\Theta M \rightsquigarrow \Theta N$ in T.

Proof. Since $\Theta M = MT$ and $\Theta N = TN$ holds, we have the link $\Theta M \rightsquigarrow \Theta N$ in T if and only if $TNMT \subsetneq TN \cap MT$. Since NM is contained in the T-ideal $M \cap N$, so is $TNMT$. It is enough to show now that MTN, which we know is contained in $TN \cap MT$, cannot be contained in R, much less in $TNMT$. But if MTN is contained in R then MT is contained in N^r. But $N = NT$ is also contained in N^r so $N^\ell = T = RT = MT + NT$ must be contained in N^r, and minimality forces $N^r = M^\ell$ and $M = N$ which contradicts our standing assumption. q.e.d.

The next two lemmas show that Θ^{-1} preserves all links of T except as noted in 2.3.3. (a) and (c) and 2.3.4. (a).

2.3.7. Lemma. [Self Links are preserved by Θ^{-1}] If P is a maximal ideal of R which is distinct from M and N then $\Theta P \rightsquigarrow \Theta P$ in T implies $P \rightsquigarrow P$ in R. Further if M and N are non-projective then either of $\Theta M \rightsquigarrow \Theta M$ or $\Theta N \rightsquigarrow \Theta N$ in T force $M \rightsquigarrow M$ and $N \rightsquigarrow N$ in R.

Proof. For P distinct from M and N with $\Theta P \rightsquigarrow \Theta P$ in T, but not $P \rightsquigarrow P$ in R, we have $\Theta P = PT = TP$ and $P = P^2$. Thus $(\Theta P)^2 = (PT)^2 = PTPT = P^2T = PT = \Theta P$ holds contradicting the existence of the link $\Theta P \rightsquigarrow \Theta P$.

If M and N are not projective and $\Theta M \rightsquigarrow \Theta M$ holds in T,

then $(\Theta M)^2 = (MT)^2 = M^2T \subsetneq MT$ clearly forces $M^2 \neq M$ and $M \rightsquigarrow M$. Similarly, $\Theta N \rightsquigarrow \Theta N$ forces $N \rightsquigarrow N$. Finally $M \rightsquigarrow M$ holds if and only if $N \rightsquigarrow N$ holds by 1.4.7. q.e.d.

Since $M \rightsquigarrow M$ holds if and only if $N \rightsquigarrow N$ holds it would be desirable to be able to show that ΘM has a self link if and only if ΘN has one. I have been unable to show this or to find a counter-example. It is easy to see that any such counter-example would have to satisfy $M^2 = M^3$, but this does not seem to help.

2.3.8. Lemma. [Non self links are preserved by Θ^{-1}]. Let P and Q be distinct maximal ideals of R with $\Theta P \rightsquigarrow \Theta Q$ holding in T. Suppose further that P and Q are both distinct from M and N if these are projective and P is different from M when $Q = N$ holds in the case that M and N are not projective. Then $P \rightsquigarrow Q$ holds in R.

Proof. Suppose that $P \rightsquigarrow Q$ does not hold in R. Thus $QP = Q \cap P$ holds and $T \supsetneq R \supsetneq Q \supsetneq Q \cap P = QP$ is a bimodule composition series for T/QP. For Q and P as in the hypothesis, we have $\Theta P = TPT$ and $\Theta Q = TQT$. Now that $\Theta P \rightsquigarrow \Theta Q$ holds in T means precisely that $\Theta Q \Theta P = TQTPT \subsetneq TP \cap TQ$ holds. Note that since 1 is an element of T we have $QP \subseteq TQTPT$. A simple length argument shows that $QP = TQTPT$ must hold. But $QP \subseteq R$ thus forces $QTPT \subseteq T^{\ell} = N \subseteq R$ which then forces $QTP \subseteq T^{r} = M$. Reversing this shows that $QTP \subseteq N$ also holds so

we have $QP \subseteq QTP \subseteq M \cap N$. Since Q and P are distinct maximal ideals, this forces $\{Q,P\} = \{M,N\}$. By assumption Q cannot be N since then we would have $P = M$ and $Q = N$ at the same time. Thus, we must have $P = N$ and $Q = M$ so that $MTN \subseteq M \cap N \subseteq R$ holds and as in 2.3.6. this forces $MT \subseteq N^r$, $N^r = M^\ell$ and $N = M$. Contradiction. q.e.d.

The next lemma shows that Θ preserves most links by describing quite explicitly those links which might possibly not be preserved.

2.3.9. Lemma. [Most links are preserved by Θ]. Let P and Q be maximal ideals of R not necessarily distinct but both distinct from M and N when these are projective. If $P \rightsquigarrow Q$ holds in R but not $\Theta P \rightsquigarrow \Theta Q$ in T, then we have either $M \rightsquigarrow P$ and $Q = N$ or $P = M$ and $Q \rightsquigarrow N$.

Proof. Note first that the case of P or Q being an element of $\{M,N\}$ when these are projective has already been dealt with in 2.3.5.

We will prove the theorem as follows: Assuming first that P and Q are both distinct from M we show that $M \rightsquigarrow P$ and $Q = N$ hold. A dual argument, assuming that P and Q are distinct from N then

shows that $P = M$ and $Q \rightsquigarrow N$ hold. If both M and N are members of $\{P,Q\}$ then $P = N$ and $Q = M$ hold by 2.3.6. A separate argument then shows that $N \rightsquigarrow M$ implies $\Theta N \rightsquigarrow \Theta M$.

From Proposition 2 of [13] it is easy to see that $P \rightsquigarrow Q$ in R if and only if $EXT^1(V,U)$ is non zero where V and U are simple right R-modules with $ann(V) = Q$ and $ann(U) = P$. For notational purposes we write U_X for the simple right module annihilated by X. Thus the U given above would be written as U_P. Also for the simple right T-module which is annihilated by ΘP we write $U_{\Theta P}$. Recall that for R-modules A and B, $EXT^1(A,B)$ is non zero if and only if there exists a module C such that there is a non split exact sequence $0 \to B \to C \to A \to 0$. The sequence $0 \to U_P \to X \to U_Q \to 0$ which is guaranteed to exist and be non split exact since $P \rightsquigarrow Q$ will be referred to by "(*)".

Suppose now that P and Q are both distinct from M. We use (*), the exact sequence $0 \to R \to T \to T/R \to 0$ and the various hom-functors to construct the accompanying large commutative and exact diagram. Since $hom(R,\_)$ is naturally equivalent to the identity functor we identify the R-module $hom(R,Y)$ with Y for the various Y.

$$
\begin{array}{ccccccccc}
 & & & & \uparrow \delta_1 & & \uparrow & & \uparrow \\
 & & & \to & \mathrm{EXT}^1(T/R,U_P) & \to & \mathrm{EXT}^1(T/R,X) & \to & \mathrm{EXT}^1(T/R,U_Q) \to \mathrm{EXT}^2(T/R,U_P) \\
 & & & & \uparrow & & \uparrow & & \uparrow \quad \uparrow \\
0 & \to & U_P & \to & X & \to & U_Q & \to & 0 \\
 & & & & \uparrow \rho_P & & \uparrow \rho_X & & \uparrow \rho_Q \\
(*,*) & 0 & \to \hom(T,U_P) & \to & \hom(T,X) & \to & \hom(T,U_Q) & \to & \mathrm{EXT}^1(T,U_P) \\
 & & & & \uparrow & & \uparrow & & \uparrow \quad \uparrow \delta_1 \\
 & 0 & \to \hom(T/R,U_P) & \to & \hom(T/R,X) & \to & \hom(T/R,U_Q) & \to & \mathrm{EXT}^1(T/R,U_P) \\
 & & & & \uparrow & & \uparrow & & \uparrow \quad \uparrow \\
 & & & & 0 & & 0 & & 0
\end{array}
$$

Some immediate observations are in order. The row $(*,*)$ is a T-exact sequence of right T-modules. The diagram is infinitely repeating, so the δ_1 in the upper left is the same as the δ_1 in the lower right. The maps ρ are just restrictions of domain from T to R.

Our aim now is to show that, unless $M \twoheadrightarrow P$ and $Q = N$, all of the ρ maps are isomorphisms whence $(*,*)$ can be identified as an R-sequence with $(*)$. But $(*,*)$ being T-split, is certainly R-split which yields a contradiction. Note that $\operatorname{ann}_T(\hom(T,U_P)) = TP = \Theta P$ and $\operatorname{ann}_T(\hom(T,U_P)) = TQ = \Theta Q$.

Claim. ρ_P and ρ_Q are epi.

Proof. This does not even require that P and Q are distinct from M. For any maximal ideal Y (distinct from M and N when these are projective) ΘY is not contained in R. Since R is bimodule maximal in T, $\Theta Y + R = T$. The R-module $T/\Theta Y$ is right annihilated by Y, hence has right module decomposition as $T/\Theta Y \cong (U_Y)^n$ for some n. Thus there is a non zero R-map from T to U_Y, call it f. Since $f(\Theta Y) = 0$ we have $0 \neq f(T) = f(\Theta Y + R) = f(R)$. Thus $\rho_Y(f)$ is a non zero element of U_Y, and U_Y being simple forces ρ_Y to be epi.

ok claim.

Claim. ρ_P, ρ_Q and ρ_X are all mono.

Proof. We actually show that $\hom(T/R,Y) = 0$ for any R-module Y with $\operatorname{ann}(Y)$ comaximal with M. Note that $\operatorname{ann}(X) = QP$ is comaximal with M.

Consider $f:T/R \to Y$ an R-homomorphism. Now we have $f(T/R)M = f(T/R\ M) = f(0) = 0$. But also we have $f(T/R)\operatorname{ann}(Y) \subseteq Y \operatorname{ann}(Y) = 0$. Thus we have $f(T/R) = f(T/R)R = f(T/R)[M + \operatorname{ann}(Y)] = 0$. ok claim.

We have now established that ρ_P and ρ_Q are isomorphisms.

Claim. If $M \not\hookrightarrow P$ holds or $Q \neq N$ holds then ρ_X is an isomorphism.

Proof. We only need to show that ρ_X is epi. Since the diagram is exact, it is sufficient to show that $U_Q \simeq \hom(T,U_Q)$ does not embed into $EXT^1(T,U_P)$.

Consider the map δ_1. It is clear from the diagram that δ_1 is epi. Since the diagram is exact, we have $\ker(\delta_1) = \operatorname{coker}(\rho_P)$. In claim 1 we showed that ρ_P is epi, so $\ker(\delta_1) = 0$ holds and δ_1 is also mono. Thus $\delta_1 : EXT^1(T/R,U_P) \to EXT^1(T,U_P)$ is an isomorphism.

If now $M \not\hookrightarrow P$ holds then since T/R is a f.g. R-module

with right annihilator M it has a right module decomposition as $(U_M)^m$ for some m and we have $EXT^1(T/R,U_P) \simeq EXT^1(U_M^m,U_P) \simeq [EXT^1(U_M,U_P)]^m = 0$ Thus $EXT^1(T,U_P) = 0$ also holds and we cannot have $0 \neq U_Q$ embeds into $EXT^1(T,U_P)$.

If now $Q \neq N$ holds and U_Q embeds into $EXT^1(T/R,U_P) \simeq EXT^1(T/R,U_P)$ then since T/R is a bimodule with left annihilator N, $EXT^1(T/R,U_P)$ is a right module with annihilator N, whence U_Q is also annihilated by N. But Q distinct from N forces $Q + N = R$ and $U_Q = U_Q(Q + N) = 0$ which is a contradiction. ok claim.

We have thus shown that P and Q distinct from M forces $M \rightsquigarrow P$ and $Q = N$. For the other case, recall that as well as the right links which we have been using there is also a concept of left links which is exactly dual to that of right links but uses left modules and $P \rightsquigarrow Q$ holds if and only if there is a left link from Q to P. Using this notion the same argument will show that if P and Q are both distinct from N then $P = M$ and there is a left link from N to Q. But a left link from N to Q is equivalent to a (right) link from Q to N, $Q \rightsquigarrow N$.

It only remains to consider $\{P,Q\} = \{M,N\}$. As pointed out before $\Theta M \rightsquigarrow \Theta N$ always holds in T so the hypothesis that $\Theta P \rightsquigarrow \Theta Q$ does not hold in T forces $P = N$ and $Q = M$. We consider the cases

M has a self link and M does not have a self link separately. If $M \supsetneq M^2$ and $\Theta N \rightsquigarrow \Theta M$ does not hold in T, i.e. $MTN = MT \cap TN$ then $(M^2)^r M = MT$ holds by 2.3.2. and $(M^2)^r MN = MTN = MT \cap TN \supseteq NM$. We also have $N(M^2)^r = TN$, so that $MN = (T)MN \supseteq TNMN = N(M^2)^r MN \supseteq N(NM) = N^2M$ follows. Now MN also contains MNM whence we have $MN \supseteq N^2M + MNM = (M + N)NM = RNM = NM$. Thus $N \rightsquigarrow M$ holds in R. Contradiction. If, on the other hand, $M = M^2$ holds, then $N = N^2$ also holds. Now $N(M \cap N) = N(NM + MN) = N^2M + NMN = NM + NMN = NM$ holds and $(M \cap N)N = MN$ follows similarly. Thus we have $MN = (M \cap N)N \supseteq NMTN = N(MT \cap TN) \supseteq N(M \cap N) = NM$ and $N \rightsquigarrow M$ again holds in R. Contradiction. q.e.d.

2.3.10. Corollary. If Θ does not preserve the self link for either M or N then $M \rightsquigarrow N$ holds in R.

Proof. Substitute $P = Q = M$ or $P = Q = N$ into 2.3.9. q.e.d.

Analogous with U_X, we define V_X to be the simple left module annihilated by the maximal ideal X.

2.3.11. Corollary. For P distinct from M, or from both M and N when these are projective, U_P can be identified with $U_{\Theta P}$. For Q distinct from N, or from both M and N when these are projective,

V_Q can be identified with $V_{\Theta Q}$.

Proof. That ρ_P from the proof of 2.3.9. is an isomorphism depends only on P being distinct from M and N. The statement for V_Q follows from the dual argument.

The question of whether U_M can be identified with $U_{\Theta M}$ is still open for M not projective. The right R-module T/MT has the same R-length as R/M, so that $U_{\Theta M}$ has an R-decomposition $U_{\Theta M} \cong U_M^m$ for some m. In many cases, e.g. if M is actually right maximal in R, $U_{\Theta M}$ can be identified with U_M, but I have been unable to verify that m must be one in general.

We turn now to the questions of when the injective dimension of T is one and what its cycle map must be in this case. Recall from Section II that M and N projective yield that T has injective dimension one. In 2.3.5. it was established that MN is T-projective. Now by 2.3.5. and 1.4.22., we must have that $\Theta N^{\ell\ell} \to MN \to \Theta M^{rr}$ holds in T, whereas $N^{\ell\ell} \to N \to M \to M^{rr}$ holds in R. As we shall shortly see, this is the only case where the cycle map is not preserved when injective dimension of T is one.

Consider two maximal ideals P and Q of R for which $\Theta P = TPT$, $\Theta Q = TQT$ and $(\Theta P)_r = (\Theta Q)_\ell$ hold. Using 2.3.2. and 1.4.5.,

we see that $(\Theta P)_r = (TPT)_r = (TPM)^r$ and $(\Theta Q)_\ell = (TQT)_\ell = (NQT)^\ell = (MQ^{\ell\ell}T^{\ell\ell})^r$ hold. Thus $(\Theta P)_r = (\Theta Q)_\ell$ forces $TPM = MQ^{\ell\ell}T^{\ell\ell}$. Now $P = M$ forces $M^2 = TM^2 = MQ^{\ell\ell}T^{\ell\ell} \subseteq M \cap Q^{\ell\ell} \subseteq Q^{\ell\ell}$ and hence $Q^{\ell\ell} = P$ holds. Even if $P \neq M$ holds, then $MQ^{\ell\ell} \subseteq MQ^{\ell\ell}T^{\ell\ell} = TPM \supseteq P$ with P different from M forces $Q^{\ell\ell} = P$. Thus $(\Theta P)_r = (\Theta Q)_\ell$ forces $Q^{\ell\ell} = P$ and hence $Q \to P$. Thus, we have shown:

2.3.12. Theorem. If inj dim(T) = 1 holds then $P \to Q$ in R if and only if $\Theta P \to \Theta Q$ in T except that if M and N are projective $N^{rr} \to N \to M \to M^{\ell\ell}$ holds in R whereas $\Theta N^{rr} \to \Theta M = \Theta N = MN \to \Theta M^{\ell\ell}$ holds in T.

The rest of this section is devoted to characterizing and describing the siutation that inj dim(T) = 1. Since the case of M and N projective is already adequately described, we make the standing assumption for the rest of Section III:

Assumption. M and N are not projective.

Before presenting the characterization of inj dim(T) = 1, we characterize the condition that $(\Theta P)_\ell = (\Theta P^{\ell\ell})_r$.

2.3.13. Lemma. For a maximal ideal P of R distinct from M, N^{rr}

and N $(\Theta P)_\ell = (\Theta P^{\ell\ell})_r$ holds if and only if P is link independent of N.

Proof. By 2.3.2. $(\Theta P)_\ell = (TP)_\ell = (NTP)^\ell = (NP)^\ell$. Since P is distinct from N^{rr}, $P^{\ell\ell}$ is distinct from N and we have $(\Theta P^{\ell\ell})_r = (P^{\ell\ell}T)_r = (P^{\ell\ell}M)^r = (PN)^\ell$. Thus, $(\Theta P)_\ell = (\Theta P^{\ell\ell})_r$ if and only if $PN = NP$, i.e. there is no link between N and P. q.e.d.

From 2.3.2. we know that $(MT)_r = (TN)_\ell$ always holds, so the only remaining cases are $(\Theta N^{rr})_\ell = (\Theta N)_r$ and $(\Theta M)_\ell = (\Theta M^{\ell\ell})_r$. These two are similar, so we only consider $(\Theta N^{rr})_\ell = (\Theta N)_r$. If N^{rr} is distinct from M, 2.3.2. applied as above, yields $N^r N^{rr} N = NN^{rr}$ and applying 1.4.6., we obtain $TNM = MN$ with $TNM = M \cap N$ following as an easy consequence. On the other hand, if $N^{rr} = M$ holds, we obtain $TNM = MNN^r$ with $TNM = M \cap N$ again an easy consequence (i.e. TNM contains NM and MN). What is surprising, is that this condition $TNM = M \cap N$ together with $NM \subsetneq M \cap N$ as noted below, should have the full force of injective dimension one for T.

2.3.14. Lemma. If inj dim$(T_T) = 1$ holds, then $M \rightsquigarrow N$ holds in R.

Proof. We show that if M is left T-projective, but not R-projective, and $M \rightsquigarrow N$ holds in R, then M is left R-projective. Contradiction.

Thus, suppose $M \rightsquigarrow N$ holds in R, i.e. $NM = M \cap N$, and M is left T-projective, i.e. $(NM)^{\ell}M = O_r(M)$. By 1.4.18., we have $O_r(M) = (NM)^{\ell}M = (M \cap N)^{\ell}M = M^{\ell}N^{\ell}M = M^{\ell}M^{\ell}M = M^{\ell}M$. But this says precisely that M is left R-projective. q.e.d.

2.3.15. Theorem. For M and N not projective, the following are equivalent:

(1) $\operatorname{inj\,dim}(T) = 1$

(2) $\operatorname{inj\,dim}(T_T) = 1$

(3) $\operatorname{inj\,dim}({}_TT) = 1$

(4) $M \cap N = TNM \supsetneq NM$

Proof. Clearly (1) is equivalent to (2) and (3) together. We show (2) implies (4) and (3) implies (4') will follow by symmetry. That $NM \subsetneq N \cap M$ holds, follows from 2.3.14. Also, since TN is maximal in T, $(M \cap N)^r = TN^r$ and $TN^rTN = TN^r$ we have $(TN^r)_{\ell} = TN$ whence left reflexivity yields $(TNM)^r = (TN)_r = TN^r = (M \cap N)^r$ and $TNM = M \cap N$. We will be done after showing (4) $\Rightarrow$ (2) since (4) $\Rightarrow$ (3) is symmetric.

(4) ⇒ (2) We only need show that (4) yields $(NM)^{\ell}M = O_r(M)$. Note that M has a self link, for otherwise we would have $NM = TNM$ (as in the proof of 2.3.2. (7)) contrary to hypothesis.

Consider $(NM)(NM)^{\ell}$. Clearly, we have $N \supseteq NM(NM)^{\ell} \supseteq NMT = N \cap M$. If $NM(NM)^{\ell} = NMT$ holds then $TNM(NM)^{\ell} = TNMT = NMT \subseteq R$ also holds forcing $(NM)^{\ell} \subseteq (TNM)^{\ell}$ and $NM \supseteq TNM$. Contradiction. Thus $NM(NM)^{\ell} = N$ must hold. Further, we have $T \supseteq M(NM)^{\ell} \supseteq MT$ and $NM(NM)^{\ell} \neq NMT$, so $M(NM)^{\ell} = T$ must hold. But now $M^{\ell} \supseteq (NM)^{\ell}M \supseteq M^{\ell}M$ and $M(NM)^{\ell}M = TM \supsetneq M^2 = MM^{\ell}M$ follow, so $(NM)^{\ell}M$ must equal M^{ℓ} which equals $O_r(M)$. q.e.d.

The theorem has several corollaries. It is clear that condition (4) which only says that there is a link from M to N which can be bridged by multiplication with T restricts what other links which involve M or N may exist in R.

2.3.16. Corollary. If $M \cap N = TNM \supsetneq NM$ then the link directed graph at {M,N} in R is given by $M^{\ell\ell} \rightsquigarrow M \rightsquigarrow M \rightsquigarrow N \rightsquigarrow N \rightsquigarrow N^{rr}$.

Proof. $M \rightsquigarrow N$ holds since $NM \subsetneq M \cap N$ holds. That M and N have self links was established in the proof of 2.3.15. That $M^{\ell\ell} \rightsquigarrow M$ and $N \rightsquigarrow N^{rr}$ hold follows from 1.4.7. There are no other links

involving M and N by 2.3.13. and the fact that $N \not\to M$ holds unless $M \to N$ holds (whence $M^{\ell\ell} = N$ and $N^{rr} = M$). q.e.d.

2.3.17. Corollary. The cycle map preserves injective dimension one for minimal overrings.

Proof. Note first that the cycle map, which is defined on all f.g. R-bisubmodules of K, preserves minimal overrings by 1.4.3. Condition (4) of 2.3.15. involves only lengths and products with the conductors, all of which are preserved by the cycle map.

As well as passing along the cycle map, the inheritance of injective dimension one also passes up to minimal overrings of T which have conductors in the cycle component of MT. To see this it is only necessary to verify that $\text{inj dim}((MT)_r) = 1$ holds. This is essentially proved in the next lemma.

2.3.18. Lemma. If $\text{inj dim}(T) = 1$ holds, and MT is not T-projective, then $MT \cap TN = (MT)_r(TNMT) \supsetneq TNMT$.

Proof. By 2.3.15. $TNM = M \cap N = NMT$ and this is bimodule maximal in $MT \cap TN$. Since MT is not T-projective, $MT \cap TN$ is an ideal of $(MT)_r$, whence $MT \cap TN \supseteq (MT)_r(TNMT) \supseteq TNMT = M \cap N$ follows. By 2.3.16., $M \neq M^2$ holds and by 2.3.2. $(MT)_r = (M^2)^r$ holds. Since

M^2 does not contain $TNMT = N \cap M$, we cannot have $(MT)_r TNMT$ contained in R, whence $(MT)_r(TNMT) = MT \cap TN$ follows. q.e.d.

2.3.19. Corollary. If inj dim(T) = 1 holds, then inj dim$((MT)_r)$ = 1 holds.

Proof. If MT is T-projective, then $(MT)_r$ has injective dimension one by Section II. If $(MT)_r$ is not projective, then $(MT)_r$ with conductors MT and TN in T satisfies the condition (4) of 2.3.15. and again we conclude that inj dim$((MT)_r)$ = 1 holds. q.e.d.

The next corollary shows that 2.3.4. can be improved when inj dim(T) = 1 holds.

2.3.20. Corollary. If inj dim(T) = 1 holds, then either Θ and Θ^{-1} both preserve all links or else $M \Rightarrow N$ hold, MT and TN are T-projective maximal ideals and the only links which are not preserved are the self links for M and N.

Proof. Exceptional case (a) of 2.3.4. cannot occur since $M \rightsquigarrow N$ holds in T. Exceptional case (b) cannot occur by 2.3.16. Thus exceptional case (c) of 2.3.4. is the only possibility. Note that since T has injective dimension one and $TN \Rightarrow MT$ holds in T, TN has a self link if and only if MT has a self link.

Suppose now that MT does not have a self link. If MT were not T-projective then by 2.3.18., $(MT)_r$ with T-conductors MT and TN would satisfy condition (9) of 2.3.15. which would force a self link on MT by 2.3.17. Contradiction. Thus MT must be T-projective and $MT \rightsquigarrow TN$ in T (by 2.3.6.) forces $MT \twoheadrightarrow TN$ in T (by 1.4.22.) which forces $M \twoheadrightarrow N$ in R (by 2.3.12.). q.e.d.

If T has injective dimension one, then every minimal overring of T, whose conductors lie in the cycle component of MT also has injective dimension one. We can continue this construction by induction to get a tower of rings, each with injective dimension one and each one minimal over the previous one. Since the cardinality of the cycle component is preserved, this process can terminate only if the original cycle component has only two elements (i.e. if $M \twoheadrightarrow N$ holds). Thus, we have for instance, that if R were semilocal with one cycle component which had more than two elements, and with some minimal overring which also has injective dimension one then R would not have a maximal equivalent order.

2.3.21. Examples. In the following, each R_n is a bounded noetherian prime ring with injective dimension one. Each R_n is minimal over R_{n+1} and R_1 is HNP. These examples will demonstrate several things. Each R_n, $n \geq 2$ is an example of R with one cycle component of non-projective maximal ideals for which every minimal overring has injective

dimension one. The case $n = 2$ shows that exceptional case (c) of 2.3.4. can occur. Note that this example is consistent with 2.3.10. These examples also show that both possibilities of 2.3.20. can occur. Note that for $n > 2$, θ and θ^{-1} preserves all links whereas for $n = 2$, θ destroys the self links but all other links are preserved.

Let A be a discrete valuation ring with unique maximal ideal $\langle x \rangle$. For each natural number n, set $R_n = \begin{bmatrix} A & A \\ \langle x^n \rangle & A \end{bmatrix}$. R_n is clearly prime noetherian and satisfies a polynomial identity, hence is fully bounded. The element $X = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$ is a central regular element in the Jacobson radical of R_n. Thus XR_n is an invertible ideal, and since the quotient ring R_n/XR_n is quasi-Frobenius, R_n has injective dimension one by 1.3.4. The maximal ideals $M_n = \begin{bmatrix} \langle x \rangle & A \\ \langle x^n \rangle & A \end{bmatrix}$ and $N_n = \begin{bmatrix} A & A \\ \langle x^n \rangle & \langle x \rangle \end{bmatrix}$ make up M-Spec(R_n). The cycle directed graph is given by $M_n \leftrightarrows N_n$ in each R_n. We have $M_n \rightsquigarrow N_n$ and M_n and N_n have self links unless $n = 1$ holds. We have $M_n^r = N_n^\ell = R_{n-1}$ and $M_n^\ell = N_n^r = \begin{bmatrix} A & \langle x^{-1} \rangle \\ \langle x^n \rangle & A \end{bmatrix}$. It follows from Robson [14], that R_n is HNP if and only if n is one.

2.3.22. Example. The following is an example to show that exceptional case (a) of 2.3.4. can occur. Thus, we wish to find R with non projective maximal ideals M and N such that $N \Rightarrow M$ holds, but not $M \rightsquigarrow N$. This example is based on the examples of Mueller in [13].

Let A be a local noetherian integral domain, with injective dimension one, unique non projective maximal ideal $\mathfrak{m}$, and invertible ideal $\langle y \rangle \subseteq \mathfrak{m}$. Mueller shows that the ring

$$R = \begin{bmatrix} A & A & \langle y^2 \rangle & \langle y \rangle \\ \langle y \rangle & A & \langle y^3 \rangle & \langle y^2 \rangle \\ \langle y^{-1} \rangle & \langle y \rangle & A & A \\ A & \langle y^{-1} \rangle & \langle y \rangle & A \end{bmatrix}$$ is an FBN prime ring of injective dimension one, which has four maximal ideals. These are obtained by substituting $\mathfrak{m}$ for A in the $(1,1)$-position and we denote them by M_1. The cycle map is given by $M_1 \Rightarrow M_{1+1}$. The link diagram is given by: $M_1 \rightsquigarrow M_1$, $M_1 \rightsquigarrow M_{1+1}$. There are no links $M_1 \rightsquigarrow M_{1-1}$ or $M_1 \rightsquigarrow M_{1+2}$. (Subscripts are taken modulo 4)

IV. MINIMAL OVERRINGS, NON-DISTINCT CONDUCTORS.

In this section R is still a bounded noetherian prime ring with injective dimension one, and Goldie quotient ring K. T will now denote a minimal overring, with unique conductor M. Thus $M^r = M^\ell = T$ holds, and $T^r = T^\ell = M$ is a maximal ideal of R. By 2.4.8. M is

not projective.

Our aim is to provide a treatment for T with unique conductor, similar to that in Section III, where T had distinct conductors. In particular, we wish to compare M-Spec(R) with M-Spec(T), in terms of their elements and their link directed graph and their cycle directed graphs when T has injective dimension one.

As in Section III, we consider the relation Θ on M-Spec(R) $\times$ M-Spec(T) which is given by $P\Theta Q$ if and only if P is a minimal prime ideal in R over $Q \cap R$. As noted in Section III, the relation Θ is actually defined on the entire prime spectra, but in our setting there is no loss of generality in restricting to the maximal spectra.

In Section III, it was shown that for distinct conductors Θ was an onto function with exactly one collapse of maximal ideals (of R) when M and N were projective and a bijection when M and N were not projective. Here we will see that Θ^{-1} is an onto function with exactly one possible collapse of maximal ideals (of T). Also, while in Section III, all of M-Spec(T) could be completely described, here $\Theta(M)$ does not always have a natural representation.

2.4.1. Theorem. The relation Θ^{-1} is in fact an onto function from M-Spec(T) to M-Spec(R) given by $\Theta^{-1}(Q) = Q \cap R$.

For an R maximal ideal P, distinct from M, $\Theta(P) = PT = TP$.

ΘM can be M, a unique R-bimodule strictly containing M or up to two R-bimodules X and Y, containing M, with $X^r = X^\ell = Y$, $Y^r = Y^\ell = X$.

Proof. For $P \in M\text{-Spec}(R) - \{M\}$, $\Theta P = TP = PT$ is the unique proper ideal of T which properly contains P, by 1.4.14, and $TP \cap R = P$ by 1.4.9.

For Q a maximal ideal of T, $Q \cap R$ is an ideal of R, which is contained in some maximal ideal N of R. If $Q \cap R$ is not contained in M then $T(Q \cap R)$ is not contained in R. Since R is maximal in $T = Q + R$, $Q \cap R$ is maximal in Q. Thus $Q = T(Q \cap R) \subseteq TN = \Theta N$ holds, forcing $Q = \Theta N$ and $Q \cap R = N$. If, on the other hand, $Q \cap R$ is contained in M then $Q \cap R = Q \cap M$. If now, Q does not contain M, then $Q + M = T$ by the maximality of Q, forcing $Q \cap R + M = R \subseteq M$, which contradicts that M is a maximal ideal of R. Thus, if $Q \cap R \subseteq M$, then $Q \supseteq M$, whence $Q \cap R = M$ as required.

It only remains to consider ΘM. If M is not a maximal ideal of T, then there is some X which properly contains M and which is a proper T-ideal. Since $T \supsetneq R \supsetneq M$ is a bimodule composition series of T/M, X is already a maximal ideal of T. Since $T \supsetneq X \supsetneq$ [illegible] holds, we also have $T \supsetneq X^r \supsetneq M$ and $T \supsetneq X^\ell \supsetneq M$. In fact, for any other ideal Y with $T \supsetneq Y \supsetneq M$, we have $Y \cap X \supseteq M$ and $M \supseteq Y \cap X$ by the maximality

of M in X (and in Y). Thus $XY \subseteq Y \cap X \subseteq M$ and $YX \subseteq Y \cap X \subseteq M$ both hold, forcing $Y = X^{\ell} = X^{r}$. The only two possibilities given X are thus: $\Theta(M) = X = X^{r} = X^{\ell}$ or else $X \neq X^{r} = X^{\ell}$ holds and $\Theta M = \{X, X^{r} = X^{\ell}\}$ has two elements.

We now collect a few formulae for computing with duals taken with respect to T. We still use the convention introduced in 2.2.2., (i.e. subscript r or ℓ denotes T-dualization).

2.4.2. Proposition. (a) For X a right T-submodule of K, $X_{r} = (XM)^{r}$.

(b) For Y a left T-submodule of K, $Y_{\ell} = (MX)^{\ell}$.

(c) $(M^{n})^{r} = (M^{n})^{\ell}$.

(d) $M_{r} = (M^{2})^{r} = (M^{2})^{\ell} = M_{\ell}$.

(e) $\operatorname{inj\,dim}(T_{T}) = 1$ if and only if $M_{\ell}M = T$.

(f) $\operatorname{inj\,dim}({}_{T}T) = 1$ if and only if $MM_{\ell} = T$.

Proof. (a) and (b). 2.2.4.

(c). A simple induction using 1.4.5.

(d). (a), (b) and (c).

(e) and (f). 2.2.6. and (d).

The next result shows that θ and θ^{-1} preserve links away from M. The question of preservation of links involving M or an element of θM and another maximal ideal is still largely open. When T has injective dimension one, however, θM can be completely described and the question is completely settled in 2.4.8.

2.4.3. Theorem. For maximal ideals P and Q of R, both distinct from M. We have $P \rightsquigarrow Q$ if and only if we have $\theta P \rightsquigarrow \theta Q$.

Proof. Suppose $\theta P \rightsquigarrow \theta Q$ holds in T, and $P \not\rightsquigarrow Q$ holds in R. Then we have $QP = Q \cap P$ and since QP is not contained in M, $QPT \supsetneq QP$. However, $\theta P = TP = PT$ and $\theta Q = TQ = QT$ and $TP \cap TQ$ is minimal over $Q \cap P$ and contains QPT. Thus QPT, which equals $\theta Q \theta P$, also equals $\theta Q \cap \theta P$ which contradicts that $\theta P \rightsquigarrow \theta Q$ holds. Note that P and Q are not assumed to be distinct here.

The converse statement is proved by a slightly modified version of the proof of 2.3.9. One has only to substitute $N = M$ in that agrument to arrive at the contradiction that either $P = M$ or $Q = M$ holds. q.e.d.

We turn now to the questions of when T has injective dimension one, what the cycle map of T is in this case, and what restrictions this assumption puts on the link directed graph of R.

Note that by 2.4.2., if T has injective dimension one then $M \twoheadrightarrow M$ holds in T, and ΘM is closed under the cycle map, i.e. if ΘM has only one element then $\Theta M \twoheadrightarrow \Theta M$ holds in T, and if $\Theta M = \{X,Y\}$ then either we have $X \leftrightarrows Y$ or we have $X \twoheadrightarrow X$ and $Y \twoheadrightarrow Y$. On the other hand, if P is different from M and $(\Theta P)_r = (\Theta Q)_\ell$ for some maximal ideal Q of R, necessarily also distinct from M, then we have $(\Theta P)_r = (PM)^r$ and $(\Theta Q)_\ell = (MQ)^\ell = (MQ^{\ell\ell})^r$. This forces $PM = MQ^{\ell\ell} \subseteq Q^{\ell\ell}$ and $Q^{\ell\ell}$ distinct from M also forces $P = Q^{\ell\ell}$. Thus we have proved the following two results.

2.4.4. Theorem. If injective dimension of T is one, then $P \twoheadrightarrow Q$ holds in R if and only if $\Theta P \twoheadrightarrow \Theta Q$ holds in T except that if $\Theta M = \{X,Y\}$ we have $M \twoheadrightarrow M$ and either of $X \twoheadrightarrow X$ and $Y \twoheadrightarrow Y$ or $X \leftrightarrows Y$.

2.4.5. Proposition. If injective dimension of T is one and P is distinct from M then P is link independent of M.

Proof. In the above, we not only have that $P = Q^{\ell\ell}$, but also that $PM = MQ^{\ell\ell} = MP$. But this says precisely that $P \rightsquigarrow M$ and $M \rightsquigarrow P$ both hold.

Turning now to the characterizing of when $\text{inj dim}(T) = 1$ holds, note that the next theorem is analogous to 2.3.15. and also generalizes 1.4.8.

2.4.6. Theorem. The following are equivalent:

(1) inj dim(T) = 1.

(2) inj dim(T_T) = 1.

(3) inj dim($_TT$) = 1.

(4) M is left invertible in T.

(5) M is right invertible in T.

(6) M is invertible in T.

Proof. Clearly (1) is equivalent to (2) and (3) together, and also (6) is equivalent to (4) and (5) together. By 2.4.2., (2) is equivalent to (4) and (3) is equivalent to (5). Thus, the real content of this theorem is the equivalence of (2) with (3) or of (4) with (5). Note also, that by 2.4.2. if M is one-sided invertible, its inverse can be taken to be $(M^2)^r = (M^2)^\ell$.

To show the equivalence of (4) with (5), we suppose that M is left but not right invertible and arrive at a contradiction. The reverse implication is dual. With this supposition, we derive the following observations.

(1) $(M^2)^rM = T$, $M(M^2)^r$ is idempotent.

(2) For $X = [M(M^2)^r]^r$ we have $X = [M(M^2)^r]^\ell \neq M(M^2)^r$ and X is a proper ideal of T properly containing M.

(3) $XM = M^2$, $M(M^2)^rM = M(M^2)^rX = M$.

(4) $(M^2)^rX = T$.

(5) $(M^2)^r$, X and M are all not eventually idempotent; $(M^2)^r M^2 \supsetneqq M^2$, $M^2(M^2)^r \supsetneqq M^2$.

(6) $X(M^2)^r X = X = X(M^2)^r M$.

(7) $X_\ell = (MX)^\ell = (MX)^r$; $(MX)(MX)^\ell = MX$; $(MX)^\ell$ is idempotent.

(8) $(MX)^r MX \neq MX$.

Proof. (1) $(M^2)^r M = T$ holds, since M is left invertible. Thus $[M(M^2)^r]^2 = M(M^2)^r M(M^2)^r = MT(M^2)^r$ holds also.

(2) Since we have $(M^2)^r = (M)_r \neq T$ and M is not right invertible, $M \subsetneqq M(M^2)^r \subsetneqq T$ follows. Setting $X = [M(M^2)^r]^r$, $T \supsetneqq X \supsetneqq M$ is immediate, as well as X being an ideal of T. Since $M(M^2)^r$ is idempotent and not contained in R, $M(M^2)^r \neq [M(M^2)^r]^r$ holds. Similarly we have $T \supsetneqq [M(M^2)^r]^\ell \supsetneqq M$, $[M(M^2)^r]^\ell$ is an ideal of T, and $M(M^2)^r \neq [M(M^2)^r]^\ell$. Since, by 2.4.1., there are at most two T-ideals between M and T, $[M(M^2)^r]^r = [M(M^2)^r]^\ell$.

(3) By (2) we have $XM(M^2)^r \subseteq R$, whence $XM \subseteq M^2$, whereas we have $X \supseteq M$ forcing $XM \supseteq M^2$. We have $M(M^2)^r M = M$ since $(M^2)^r M = T$ holds, $M(M^2)^r X \supseteq M$ holds since $X \supseteq M$ holds. On the other hand, by (2) we know that $M(M^2)^r X \subseteq R$, so that $M(M^2)^r X$ must be contained in $M(=MM^r)$.

(4) We have $(M^2)^r X \supseteq (M^2)^r M = T$ and $(M^2)^r XM \subseteq (M^2)^r M^2 \subseteq R$ forces $(M^2)^r X \subseteq T$.

(5) Any (even one-sided) invertible ideal cannot be eventually idempotent. Since $(M^2)^r \neq ((M^2)^r)^2$ holds, we have $(M^2)^r M^2 \supsetneq M^2$ by 1.4.20. Similarly, $(M^2)^r = (M^2)^\ell$ forces $M^2(M^2)^r \supsetneq M^2$.

(6) This follows from $XT = X$ and $(M^2)^r M = (M^2)^r X = T$.

(7) We have $X_\ell = (MX)^\ell$ from 2.4.2., and $(MX)^\ell = (M^{\ell\ell}X^{\ell\ell})^r = (MX)^r$ from 1.4.5. Since X cannot be right invertible in T and we have $T \supseteq X(MX)^\ell \supseteq X$ with X maximal in T, we must have $X(MX)^\ell = X$ and $MX(MX)^\ell = MX$. That $(MX)^\ell$ is idempotent now follows from 1.4.10.

(8) Clearly, we have $M \subseteq (MX)^r M \subseteq X^r = M(M^2)^r$ with M maximal in $M(M^2)^r$. If we had $M = (MX)^r M$ then $(MX)^r \subseteq T$ would follow, forcing $MX = M$, and then even $X^2 \supseteq M$. Since $X \neq X^r$ holds, $X^2 \neq M$ also holds, and the maximality of M in X forces X to be idempotent in contradiction to (5). Thus, $(MX)^r M = M(M^2)^r$ holds, forcing $(MX)^r MX = M(M^2)^r X = M \neq MX$, ok observations.

To finish off, we have $(MX)^r$ is not idempotent by (8) and 1.4.10., while $(MX)^r = (MX)^\ell$ is idempotent by (7). Contradiction. q.e.d.

If the injective dimension of T is one, then $(M^2)^r$ which is the inverse of M in T properly contains T and hence $M^2 \neq M$. Thus M has a self-link in R and by 2.4.5., this is the only link in T which involves M. The next theorem completely describes all of the possibilities for ΘM and its links when inj dim$(T) = 1$ holds. This theorem and the above remark will then allow us to completely describe the extent of the preservation of links by Θ and Θ^{-1}.

2.4.7. Proposition. If inj dim$(T) = 1$ then exactly one of the following holds:

(1) $\Theta M = M$. In this case M is invertible in T, hence the localization T_M is a valuation ring.

(2) $\Theta M = X \supsetneq M$, and X is invertible in T. In this case T_M is again a valuation ring and $M = X^2$.

(3) $\Theta M = X \supsetneq M$ and X is not invertible in T. In this case $MX = XM$, X is a self-cycled maximal ideal in T, which is invertible in $O_r(X) = (MX)^r$.

(4) $\Theta M = \{X,Y\}$ with $X \neq Y$ and $XY = M$. In this case X and Y are invertible maximal ideals of T, (i.e. self-cycled, self-linked and projective), so that T_M is Dedekind.

(5) $\Theta M = \{X,Y\}$ with $X \neq Y$ and $XY \neq M$. In this case X and Y are T-projective maximal ideals with $X \leftrightarrow Y$ in T, so that T_M is HNP

with a two cycle.

Note that cases (1), (2) and (4) yield T locally Dedekind, so that T is the largest (only) equivalent order to R whose conductor contains a power of M. In case (5) the overrings $X_r(=Y_\ell)$ and $X_\ell(=Y_r)$ are the largest equivalent orders to R whose conductors are contained in a power of M. However, in case (3) the relationship between M and T exactly parallels that between X and $O_r(X)$ so that case (3) can conceivably lead to an infinite tower of overrings with conductors contained in M, each minimal over the last. In fact, if M does fall into case (3) then either X also falls into case (3) allowing another step in this tower, or X falls into one of the other four cases stopping the tower with at most one more step.

Proof. Clearly these possibilities are exhaustive and mutually exclusive. It only remains to verify the resulting conclusions for each case.

(1) When $\operatorname{inj\,dim}(T) = 1$ holds, M is invertible by 2.4.3. Any invertible maximal ideal is localizable with a valuation ring as its localization.

(2) It only remains to verify $X^2 = M$. But $X = X^r$ forces $X^2 \subseteq X \cap R = M$ and X an invertible maximal ideal forces X/X^2 to be T-bimodule simple.

(3) Clearly we have $M_{rr} = ((M^2)^r)_r = (M(M^2)^r)^r = T^r = M$, so that

$T \supsetneq X \supsetneq M$ forces $T = T^{rr} \supseteq X^{rr} \supseteq M^{rr} = M$, whence $X^{rr} = X$ follows. Now $X_r = (XM)^r = (X)_\ell = (MX)^\ell = (M^{\ell\ell}X^{\ell\ell})^r = (MX)^r$ forces $XM = MX$. Further, we have $(X^2)_r \supseteq (M^2)_r \supseteq (M^2)^r$ so that $(X^2)_r X \supseteq (M^2)^r X \supseteq (M^2)^r M \ni 1$ holds, and this is sufficient to guarantee $O_r(X)$ has injective dimension one by 2.2.6.

(4) Since $XY(=M)$ is an invertible ideal of T, Y is a left invertible maximal ideal of T, which must be invertible by 1.4.8. Similarly X is invertible.

(5) Since $XY \subsetneq M = X \cap Y$ holds, there is a link from Y to X, and so X and Y are both not invertible. Now $XY \supseteq M^2$ holds, since we have X and Y both contain M, and XY cannot equal M^2 since M^2 is invertible (in T). The following observations will be helpful.

(1) $X_r = (XM)^r$; $X_\ell = (MX)^\ell$; $Y_r = (YM)^r$; $Y_\ell = (MY)^\ell$.

(2) $X_r = Y_\ell = (M^2)^r Y = X(M^2)^r$; $X_\ell = Y_r = (M^2)^r X = Y(M^2)^r$.

(3) $XY = XM = MY$; $YX = MX = YM$.

(4) $Y(M^2)^r X = Y(M^2)^r = (M^2)^r X = X_\ell = Y_r$.

Proof. (1) 2.4.2.

(2) Since we have $(M^2)^r XY \subseteq (M^2)^r M = T$, $(M^2)^r X$ is contained in Y_r. Also, $(M^2)^r X \supseteq (M^2)^r M = T$ holds with $(M^2)^r X \neq T$ since X is not invertible. Now, $(M^2)^r X$ must equal Y_r by the

minimality of Y_r over T. Similarly, $X_r = Y(M^2)^r$, $X_\ell = (M^2)^r Y$ and $Y_\ell = X(M^2)^r$ all hold. Now, using $X^{rr} = X$, $Y^{rr} = Y$ and $M^{rr} = M$, we obtain $X_\ell = (MX)^\ell = (M^{\ell\ell}X^{\ell\ell})^r = (MX)^r$. Either we have $X \to Y$ in T, or $X \to X$. But, if $X \to X$ holds in T, then $X_r = X_\ell$ forces $MX = XM$. Now $MX = XM$ yields $Y = Y_\ell Y = (M^2)^r XY \geq (M^2)^r XM = (M^2)^r MX = TX = X$. But, X and Y are incomparable so MX cannot equal XM and X can not arrow to X, so that $X \to Y$ holds in T. Similarly, $Y \to X$ also holds in T.

(3) These follow immediately from (1) and (2).

(4) Both $Y(M^2)^r$ and $(M^2)^r X$ contain $T(=(M^2)^r M = M(M^2)^r)$. Thus X and Y are both contained in $Y(M^2)^r X$, whence $T = X + Y \leq Y(M^2)^r X$ also holds. Thus, we have $1 \in T \leq Y(M^2)^r X = X_\ell X$, forcing X to be left projective and then $X_\ell X = X_\ell$ holds required.

These observations, in conjunction with the results of Chapter I, finish off the remarks for case (5).

Note that case (2) is characterized by $\Theta M = X \supsetneq M$ with $X^2 = M$ while case (3) is characterized by $\Theta M = X \supsetneq M$ with $X^2 \subsetneq M$. Case (4) is characterized by $\Theta M = \{X,Y\}$ with $X \neq Y$, but $XY = YX$.

2.4.8. Examples. The following examples show that all of the possibilities of 2.3.5. can actually occur. In particular, ΘM can

have cardinality two, even when R is commutative.

The examples are numbered to correspond to the different cases of 2.3.5. The first four examples are commutative and all examples are local. We make use of Bass' criterion which states: If in a commutative domain, A, every maximal ideal is 2-generated, then A has injective dimension one. In these examples $\tilde{k}$ with denote a field, with an element $\alpha \in \tilde{k}$, which does not have a square root in $\tilde{k}$ (for Example (1)) and x and y are a pair of indeterminates.

Example (1). Set $B = (\tilde{k}[\sqrt{\alpha}])[x]_{<x>}$. B is a DVR with unique maximal ideal xB. For A, we take $xB + \tilde{k}$, which is clearly a subring of B, with unique maximal ideal, also xB. Thus A is a commutative domain which is noetherian since $B = A + \sqrt{\alpha}A$ is f.g. as an A-module. To see that xB is two generated as a A-module, write $\tilde{k}[\sqrt{\alpha}] = \tilde{k} + \tilde{k}\sqrt{\alpha}$ whence we have $B = xB + \tilde{k} + \tilde{k}\sqrt{\alpha}$ (as $\tilde{k}$-vector spaces) and $xB = x^2B + x\tilde{k} + x\sqrt{\alpha}\tilde{k} = x^2A + x\sqrt{\alpha}\tilde{k} = xA + x\sqrt{\alpha}A$. Thus A has injective dimension one, the A-dual of xB is minimal over A, hence must be B. By construction xB is also the maximal ideal of B and we have case (1).

Example (2). Let $A = [\tilde{k}[x,y]/<x^2 = y^3>]_{<x,y>}$. A is a commutative noetherian domain with unique two generated maximal ideal $<x,y>$, which we denote by m. It is easy to see that $m^* = A + Ax/y = \tilde{k}[x/y]_{<x/y>}$ is

the localization of a polynomial ring, hence a DVR. Its unique maximal ideal is $x/y(k[x/y])$, which properly contains m (i.e. $(x/y)^2 = y$, $(x/y)^3 = x$), and is invertible with inverse $y/x(k[x/y])$. Note that $[x/y(\tilde{k}[x/y])]^2$ contains both $y(=(x/y)^2)$ and $x(=(x/y)^3)$ so that $[x/y(\tilde{k}[x/y])]^2 = m$.

Example (3). For $A = [k[x,y]/\langle x^2 = y^5\rangle]_{\langle x,y\rangle}$, A is still a commutative noetherian domain with unique two generated maximal ideal $\langle x,y\rangle$ which we again denote by m. m^* is still $A + Ax/y$, but in this case, it is no longer a polynomial ring. It has unique maximal ideal $\langle y,x/y\rangle$ which properly contains m. Now also $\langle y,x/y\rangle^2 = \langle y^2,x,x^2/y^2\rangle = \langle y^2,x,y^3\rangle = \langle y^2,x\rangle$ is properly contained in m, whence we are in case (3).

Example (4). We construct the example in a manner similar to Example (1). In this case, we want B to be noetherian Dedekind domain, which is a $\tilde{k}$-algebra, with exactly two maximal ideals, $\langle x\rangle$ and $\langle y\rangle$ say, such that $B = \langle x\rangle + \tilde{k} = \langle y\rangle + \tilde{k}$ as k-vector spaces. Concretely, B might be taken as $k[x]_{\langle x^2 - x\rangle}$ with $y = x - 1$ or $k[x,y]_{\langle xy\rangle}$. Set $A = \langle xy\rangle + k$ which is clearly a commutative noetherian domain with unique maximal ideal xyB. Clearly, we have $xA + yA \subseteq B$, but also we have $xA + yA = x^2yB + xk + xy^2B + yk$ where $x^2yB + xy^2B = xy(xB + yB) =$

xyB hold, and $xyB + yk = y(xB + k) = yB$. Thus $xA + yA$ contains $xB + By$ which equals B. Thus $xyB = xy(xA + yA) = x^2yA + xy^2A$ is two generated as an A-ideal. Since xyB is an ideal of B, B is clearly the A-dual of xyB, and $\Theta = \{xB, yB\}$ both of which are invertible maximal ideals of B.

Example (5). For commutative rings, all links are self links, so case (5) can not occur. Thus, consider a DVR, A, which is a $\tilde{k}$-algebra, with unique maximal ideal $\langle x \rangle$ and $A = \langle x \rangle + \tilde{k}$ as vector space. Consider $T = \begin{bmatrix} A & A \\ \langle x \rangle & A \end{bmatrix}$ and $R = J(T) + \tilde{k} = \begin{bmatrix} \langle x \rangle & A \\ \langle x \rangle & \langle x \rangle \end{bmatrix} + \tilde{k} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

R is local with unique maximal ideal $J(T)$, and T is an HNP ring with two cycle $\begin{bmatrix} \langle x \rangle & A \\ \langle x \rangle & A \end{bmatrix} \longleftrightarrow \begin{bmatrix} A & A \\ \langle x \rangle & \langle x \rangle \end{bmatrix}$ by Example 2.3.20. To show that R is noetherian, use the fact that the simple T-modules are simple R-modules, so that any T module of finite length has finite length as an R-module. R satisfies a polynomial identity and R is prime since it contains an ideal of the prime ring T, so R is fully bounded noetherian and prime. To see that R has injective dimension one, use 1.3.4, and show that for the unique maximal ideal of R, $J(T)$, there is an invertible ideal, namely xR, contained in $J(T)$, with R/xR quasi-Frobenius. This follows from the fact that R/xR is a local, 4-dimensional k-algebra, whose socle, $J^2(R)/xR$, is simple. Clearly, T is

the left and right R dual of the unique R-maximal ideal J(T) and $\Theta(J(T)) = \left\{ \begin{bmatrix} \langle x \rangle & A \\ \langle x \rangle & A \end{bmatrix}, \begin{bmatrix} A & A \\ \langle x \rangle & \langle x \rangle \end{bmatrix} \right\}$ as required.

V. SOME OPEN PROBLEMS AND CONJECTURES.

Problem 1. Let M and N be as in Chapter II, Section III. If M is in $\mathcal{N}$ then T has injective dimension one if and only if $TNM = MN \supsetneq NM$. On the other hand $M \in \mathcal{P}$ if and only if $TNM = M \supsetneq NM \supsetneq MN$. These conditions completely determine the link directed graph locally (i.e. for the cycle component). Can this implication be reversed? In other words, can there exist cycle components with the correct link directed graph, but with injective dimension of T not one?

Problem 2. The primeness restriction is quite strong. Can these results be generalized to R semiprime? To R has a quasi-Frobenius quotient ring?

Problem 3. What can one say about non-minimal equivalent orders? For example: Is a maximal equivalent order HNP?

Problem 4. In [7], Handelman shows that if every ring between a commutative ring A and its module finite integral closure has injec-

ative dimension one then every module between A and A' is a ring.

Can this be generalized to the non-commutative setting?

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