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OPTIMAL ORDERING DECISION AND INCENTIVES FOR YIELD IMPROVEMENT UNDER RANDOM DEMAND

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Abstract

In this thesis, we focus on the applications of incentive mechanism design in operations and supply chain management (OSCM). Most significant—and interesting—topics arising in OSCM are concerned with the management of relationships among supply chain members under asymmetric information. Since the incentive mechanism design based on the principal-agent model deals with asymmetric information in a satisfactory way, it has become an important tool in investigating OSCM-related asymmetric information problems.

We start with an introduction in Chapter 1. In this chapter, we briefly describe the theory of incentive mechanism design and its applications to OSCM, and the organization structure of this thesis. In Chapter 2, we study the optimal wage scheme and effort level in a contracting problem where both the principal and the agent are risk-averse. This chapter is a starting point for analyzing the buyer's optimal ordering decision and incentives for yield improvement in Chapters 3 and 4. Chapter 3 investigates the buyer's optimal ordering decision and incentives for yield improvement in the setting of random yield for the critical component and uncertain demand for the finished product. In Chapter 4, we assume the supplier's effort and yield become continuous and study a continuous optimization problem where the buyer decides the optimal order quantities and incentives for yield improvement under random demand. Our thesis ends with a conclusion and addresses the future research in Chapter 5.

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Chapter 1 Introduction

This chapter describes the theory of incentive mechanism design and its applications to operations and supply chain management (OSCM), and also indicates the organizational structure of this thesis. We explain the theory of incentive mechanism design and address two important concepts related with asymmetric information. One is adverse selection and the other is moral hazard. The theory of incentive mechanism design has been applied to diverse fields and we focus our attention on its applications to OSCM-related asymmetric information problems. Specifically, we study how a buyer designs an incentive mechanism (or a contract) to induce his supplier's effort in improving supply yield and what's the buyer's optimal ordering decision in a setting of random yield for the critical component and uncertain demand for the finished product.

1.1 Incentive Mechanism Design and OSCM

In the last two decades incentive mechanism design theory has found useful applications in economics and related disciplines including finance (e.g., Bolton and Scharfstein [6]), accounting (e.g., Demski and Feltham [12]), organizational behavior (e.g., Eisenhardt [14]), management science (e.g., Ha [19] and van Ackere [43]), and marketing (e.g., Basu, et al. [3] and Lal and Srinivasan [29]). The importance of this theory was recognized by awarding the 2007 Nobel prize in Economic Sciences to Leonid Hurwicz who initiated the theory, and to Eric Maskin and Roger Myerson who refined and applied it.

In its simplest form, incentive mechanism design is concerned with a bilateral relationship where one individual (i.e., principal) contracts another (i.e., agent) to carry out an action, or exert effort to complete a task. In such problems information shared between the parties is usually asymmetric. Before the contract is signed, the agent may hold private information related to the action to be taken giving rise to an *adverse selection* problem. For example, the agent's marginal cost of production may be private and the principal may only have a prior distribution of this cost.

If, on the other hand, the informational asymmetry arises after the contract is signed and the principal cannot verify the agent's effort in completing the task, then a *moral hazard* problem arises. In such cases, the outcome of the agent's effort is a random variable which creates an additional level of complexity. For example, if the agent is contracted to complete a project, her actions may not be observable to the principal and a given level of agent's effort results in possibly different outcomes.

Both the adverse selection and moral hazard problems described above are formulated as mathematical programming problems with the objective of maximizing the principal's expected utility, subject to the agent's participation and incentive compatibility constraints. The first constraint assures that, if she accepts the contract, the agent's expected utility will be at least as high as what she can obtain from an alternative arrangement. The second constraint takes into account the fact that the agent will choose the optimal effort level to maximize her expected utility. For an excellent treatment of these problems, see Bolton and Dewatripont [5], Laffont and Martimort [28], Macho-Stadler and Pérez-Castrillo [33] and Salanié [40].

As we have mentioned at the beginning, the theory of incentive mechanism design has been applied to diverse fields and we put our focus on its applications to OSCM-related asymmetric information problems. Specifically, we are interested in its applications to random yield literature, which studies the decision-making problem when the received amount from an order is uncertain. Some random yield papers are concerned with hidden information problems, which lead to adverse selection problems. For example, Yang, Aydin, Babich and Beil [44] analyze a problem where a risk-neutral manufacturer designs an adverse selection model to induce the supplier to reveal her true reliability type. The supplier's reliability with two types (high and low) is hidden information to the manufacturer. Related work by Corbett, Zhou and Tang [10], Ha [19], Hu, Duenyas and Beil [24], Lee and Yang [31], Özer and Raz [35] provide further glimpses of incentive mechanism design related to random yield, and their focus is the adverse selection models in the presence of private cost and/or demand information.

Some other random yield papers are concerned with hidden actions rather than hidden information. For example, van Ackere [43] analyzes a batch-size problem under random yield and demand through a principal-agent model under moral hazard. In the paper of van Ackere [43], the worker as the agent exerts costly effort that affects the yield of good units and the manager as the principal decides the batch-size as well as the worker's wage scheme. Similar to van Ackere [43], we study a buyer's decision-making problem when his supplier has some hidden effort in improving her yield performance. However, different from van Ackere [43] that considers a specific yield function of the effort, our model assumes a more general yield function where the effort improves yield in the sense of first-order stochastic dominance (FSD), see Bunn [8, Ch. 4]. Furthermore, we develop an algorithm based on Lagrangian duality to solve the buyer's problem under multiple effort choices and yield outcomes. Note that Zhu, Zhang and Tsung [48] investigate a contracting problem where both the buyer and the supplier exert effort in the quality improvement in a setting with deterministic demand. Our research work differs from their paper as we study the contracting problem where the supplier is rewarded to improve the yield performance in a setting with random yield and uncertain demand.

1.2 Organization and Overview

This thesis is organized as follows. In Chapter 2, we study a principal-agent model under a symmetric information case where both principal and agent are risk-averse. Assuming that both participants may possess either a power or exponential utility function, we obtain the optimal wage scheme explicitly as the solution of the resulting nonlinear ordinary differential equation. We also provide an analysis of the agent's level of effort that is optimal for the principal and show that the highest effort level is not always optimal. This chapter is a starting point for analyzing the buyer's optimal ordering decision and incentives for yield improvement in Chapters 3 and 4. Chapter 3 investigates how a buyer designs an incentive mechanism (or a contract) to induce his supplier's effort in improving supply yield and what's the buyer's optimal ordering decision under random yield and demand. We first build a model with binary effort choices and yield outcomes, and obtain the optimal solutions under both symmetric (first-best) and asymmetric information (second-best) cases. We then extend it to a model with multiple effort choices and yield outcomes, and develop an algorithm based on Lagrangian duality to solve the buyer's problem. In Chapter 4, we assume the supplier's effort and yield become continuous and study a continuous optimization problem for the buyer. Under the first-best case, we find that the optimal transfer function is increasing and convex in the effort level but independent of the order quantity. We present the first-best optimal order quantity and effort level through the buyer's first-order conditions. Under the second-best case, we find the buyer's second-best optimal solution through the first-order approach of Holmström [21], where the supplier's maximization problem is concave with respect to the effort. In Chapter 5, we summarize this thesis and address our future research work. We present a two-period discrete model and we plan to extend it to a N-period problem as well as provide practitioners some managerial insights as our future work.

Chapter 2
Optimal Reward and Effort Level in a Contracting Problem with Risk-Averse Principal and Agent

This chapter considers a contracting problem between a risk-averse principal and a risk-averse agent. The principal designs an outcome-based wage scheme (i.e., a reward function) and enforces the agent to exert the demanded effort level if the agent accepts the offer. Under the "first-best" case, the agent's effort and outcome are observable and verifiable, but the outcome is a random function of the effort. For a given effort level, the optimal wage scheme is obtained as the solution of a nonlinear ordinary differential equation (ODE). In order to solve explicitly the resulting ODE we assume that the risk-averse principal and agent may possess either a power or exponential utility function and provide the exact solutions for the resulting four combinations. As it is well-known, the exponential utility function has constant risk aversion which makes it an unrealistic choice for decision analytic studies. We show that when the principal has the exponential utility, unless a minimum liability constraint is imposed, the optimal wage function obtained from the solution of the ODE may be unacceptable, thus revealing the unrealistic nature of this utility function in the principal-agent model. In all cases we present results for the concavity or convexity properties of the reward function. The chapter concludes with an analysis of the agent's level of effort that is optimal for the principal and shows that the highest effort level is not always optimal.

2.1 Introduction

In order to understand the effects of information asymmetry inherent in mechanism design problems, it is necessary to analyze first the symmetric information (or, "first-best") case where both participants share the same information both before the contract is signed, and during the contract execution. Once the first-best solution is found, one can then compare the results of the "second-best" case where the principal cannot observe the agent's actions (or, the effort level) which results in the adverse selection or moral hazard problem. In this chapter we provide a thorough analysis of the first-best problem when both participants are risk averse and find the optimal wage scheme and the optimal effort level. That is, our first-best analysis involves only the principal's objective function and the agent's participation constraint. The second-best case which involves the incentive compatibility constraint giving rise to the moral hazard problem is not considered in this chapter.

It is well-known that in the first-best case when the principal is risk-neutral and the agent is risk-averse, then the principal accepts all the risk by completely insuring the agent who receives a fixed wage; see, Macho-Stadler and Pérez-Castrillo (hereafter MS/PC) [33, p. 25]. For the same case, when the principal is risk-averse and the agent is risk-neutral, then the opposite result is found and the agent insures the principal against randomness in the results; see, MS/PC [33, p. 27]. In this case the agent keeps the outcome and pays the principal a fixed amount, resulting in a franchise agreement. For example, Crama, Reyck and DeGraeve [11] consider a model under symmetric information in a licensing contract design problem for multi-phase R&D projects where the licensor (principal) is risk averse and the licensee (agent) is risk neutral.

When at most one participant is risk-averse, as noted above, the optimal contract can be found relatively easily. When they are both risk-averse, the principal and the agent must accept part of the risk depending on their level of risk-aversion. In this case the actual wage scheme is the solution of a (possibly complicated) nonlinear ordinary differential equation (ODE), which can be quite difficult to solve explicitly depending on the form of the utility functions. However, when both participants have the exponential utility function with constant risk-aversion (assumed for analytical convenience), the ODE is solved easily which gives rise to a linear contract. (See, Hosseinian and Carmichael [23] and MS/PC [33, p. 27] for the symmetric; and Holmström and Milgrom [22] for the asymmetric information case). But since few decision makers would have constant risk-aversion, linear contracts would be an unlikely outcome in most practical situations. For example, the executive compensation schemes such as stock-options commonly used in practice are convex in nature; see Chaigneau [9], Dittmand and Maug [13], and Hall and Murphy [20].

Principal	Agent	Optimal wage scheme
Risk-neutral	Risk-averse	Fixed wage (MS/PC 2001, p. 25)
Risk-averse	Risk-neutral	Franchise agreement (MS/PC 2001, p. 27)
Risk-averse	Risk-averse	**Linear contract or others?**

Table 1. Optimal wage scheme when Principal or/and Agent are risk-averse.

To our knowledge, the problem of finding the optimal wage scheme and effort level when both participants possess non-exponential utility functions has not been analyzed in a satisfactory manner in the available literature. The exception appears to be the paper by Chaigneau [9] who considers the case where both participants have the quadratic utility function, which has the undesirable property of exhibiting *increasing* absolute and relative risk aversion; however, Chaigneau [9] does not provide an analysis of the optimal effort level for the agent. We believe that our study provides some insights into the evaluation of the wage function and the computation of the optimal effort level when both principal and the agent are risk averse. As shown in Table 1, we are interested in the optimal wage scheme when both participants are risk-averse. Will the optimal wage scheme always give a linear contract or other forms? Moreover, the ODE given in MS/PC [33, p. 27, Eqn. (2.7)] is written in terms of the risk aversion measures of the participants which involve the first and second (partial) derivatives. But the steps leading to the ODE are not clearly delineated as the variables of differentiation (principal's net income? agents' outcome?) are not specified when the second partial derivative of the principal's utility is computed.

In this chapter we hope to provide a clear and unambiguous set of steps leading to the development of the ODE for the wage (reward) function given a fixed effort level. We explicitly solve the resulting nonlinear ODE assuming the principal and the agent are both risk averse and possess power and exponential utility functions. We also show that depending on the form of the utility functions, the optimal first-best reward function may be concave or convex in the outcome.

For the special case where both participants have power utility functions with the same degree of risk aversion, the optimal contract is linear in outcome. In the case where the principal has exponential utility while the agent has power utility, an unrealistic result may be obtained where, for low outcomes, the principal pays the agent more than the outcome itself. This happens when the principal is "generous" with the agent for high outcome levels. When both participants possess exponential utility functions, further unrealistic results are obtained where the agent may be *penalized* for poor outcomes unless a minimum liability constraint is introduced (see, e.g., Laffont and Martimort [28, p. 118]). Our results thus reveal that it may be advisable to avoid the use of the exponential utility not just in first-best principal-agent problems, but also in the more general adverse selection and moral hazard problems.

The remainder of the chapter is organized as follows. In Section 2.2 we derive the nonlinear ODE for the optimal wage function in terms of the principal's and agent's Arrow-Pratt measures of absolute risk-aversion. In Sections 2.3 to 2.6 we solve the ODE assuming the principal and the agent are risk averse and possess power and exponential utility functions. In these sections we also examine the convexity/concavity properties of the wage functions under different utility function combinations and point out the practical difficulties arising when the principal has the exponential utility function. In Section 2.7 we present an easy-to-use method for finding the agent's optimal effort level with two illustrative examples. Finally, conclusions are presented in Section 2.8.

2.2 First-Best Contracting Problem with Risk-Averse Participants

In this section we consider the first-best case with risk-averse principal and agent. We find that the principal's optimal wage scheme is the solution of a nonlinear ODE, which is consistent with the result in MS/PC [33, p. 27, Eqn. (2.7)]. However, the ODE given in MS/PC [33, p. 27, Eqn. (2.7)] is written in terms of the risk aversion measures of the participants, but the steps leading to the ODE appear ambiguous. Thus, we first present a clarification of the steps that lead to the development of the nonlinear ODE.

The first-best case is described as follows: The principal offers the agent a take-it-or-leave-it contract under the assumption that both participants share the same information on the agent's effort level and the outcome. The agent is not willing to accept this contract unless her expected utility is as much as what she can obtain from another, alternate, contract, i.e., her reservation utility U_0. If she accepts the contract, then she will choose an effort level $e \in [e_0, e_1]$, which affects the (random) outcome X taking values in the interval $[x_0, x_1]$. The random variable X has a density function $f(x; e) > 0$ which depends on the effort level e. It is reasonable to assume that a higher level of effort is associated with a higher probability of achieving a specific outcome in the sense of first-order stochastic dominance (FSD), that is, $\Pr(X > \hat{x})$ is increasing in e for any given outcome $\hat{x}$ (Bunn [8, Ch. 4]). Both the effort e and the outcome x are observable and verifiable under the first-best case.

The principal aims to design a wage scheme $w(x)$ contingent on the outcome x to reward the agent. Thus, given the outcome x, the principal's net income can be written as $y = x - w(x)$ while the agent's net income is $w = w(x)$. We assume that both the principal and the agent are risk averse with *increasing and concave* utility functions $B(y)$ and $u(w)$, respectively; that is, $B'(y) > 0$, $B''(y) < 0$, $u'(w) > 0$ and $u''(w) < 0$. The principal prefers the agent to exert a higher level of effort, which will cost the agent greater disutility $v(e)$, where $v(e)$ is assumed to be increasing convex, i.e., $v'(e) > 0$ and $v''(e) > 0$. Under the first-best case, the principal can directly control the agent's effort, and thus, the principal's problem under the symmetric information case can be formulated as,

$$\begin{array}{cc} \max_{e,w(x)} & \int_{x_0}^{x_1} B(x - w(x)) f(x; e)\, dx \\ \text{s.t.} & \int_{x_0}^{x_1} u(w(x)) f(x; e)\, dx - v(e) \geq U_0, \end{array}$$

where the constraint is the agent's *participation constraint* (PC).

The development of the ODE for the wage function will involve partial derivatives of the principal's and agent's utility functions. Note that the principal receives x and pays $w(x)$, while the agent receives $w(x)$ for a given x. Thus, B is a function of the outcome x and wage w, and u is a function of w. MS/PC [33, p. 27, Eqn. (2.7)] use only the notation B'' in the development of their arguments which may lead to some ambiguity. In order to avoid potential ambiguities, where applicable, we explicitly indicate the partial and mixed partial of $B(x-w(x))$ as either $B'_w(x-w(x))$, or $B''_{ww}(x-w(x))$ or $B''_{wx}(x-w(x))$ in the analyses to follow.

Remark 1 We had assumed above that the principal's utility $B(y)$ was increasing concave in his net income $y = x - w\,(x)$. It thus follows that the principal's utility is a decreasing and concave function of the wage paid to the agent $w(x)$ for a given x. That is, using chain rule, we have the first derivative as

$$B'_w(x-w(x)) = B'(y)\frac{dy}{dw} = -B'(y) < 0, \tag{2.1}$$

since $B'(y) > 0$ and $dy/dw = -1$. Similarly, the second derivative is found as,

$$B''_{ww}(x-w(x)) = -\frac{dB'(y)}{dy}\frac{dy}{dw} = B''(y) < 0. \qquad \blacktriangleleft \tag{2.2}$$

The above result shows that the principal's problem possesses desirable properties with respect to the wage paid to the agent; that is, given an effort level e, the principal's problem is a well-behaved optimization problem in the wage scheme $w\,(x)$ for any outcome $x \in [x_0, x_1]$. This follows since $B(x-w(x))$ is concave in $w(x)$, the principal's objective function $\int_{x_0}^{x_1} B(x-w(x))f(x;e)\;dx$ is concave in $w(x)$. Moreover, since $\int_{x_0}^{x_1} u(w(x))f(x;e)\;dx$ is concave and $U_0+v(e)$ is a constant, the feasible region corresponding to the agent's PC is convex. Thus, we can use Karush–Kuhn–Tucker conditions [4, Ch.4] to find the principal's global optimal wage scheme.

To solve the principal's problem, we associate a multiplier λ with the agent's participation constraint, and write the Lagrangian function as,

$$\hat{\mathcal{L}}\,(w(x),\lambda) = \int_{x_0}^{x_1} B(x-w(x))f(x;e)\;dx - \lambda\left[U_0 - \int_{x_0}^{x_1} u(w(x))f(x;e)\;dx + v\,(e)\right].$$

As shown by Laffont and Martimort [28, p. 185] the problem can also be stated in terms of pointwise optimization and written as,

$$\mathcal{L}\,(w(x),\lambda) = B(x-w(x))f(x;e) - \lambda\left[U_0 - u(w(x))f(x;e) + v(e)\right]. \tag{2.3}$$

For any given outcome x, the first-order condition of (2.3) w.r.t. $w\,(x)$ is written as,

$$\frac{\partial\mathcal{L}}{\partial w\,(x)} = f(x;e)B'_w(x-w(x)) + \lambda f(x;e)u'_w(w(x)) = 0,$$

which implies

$$B'_w(x - w(x)) + \lambda u'_w(w(x)) = 0. \tag{2.4}$$

Thus we have that,

$$\lambda = -\frac{B'_w(x - w(x))}{u'_w(w(x))}, \quad \text{for } x \in (x_0, x_1). \tag{2.5}$$

It's straightforward to show that the multiplier λ is always positive for $x \in [x_0, x_1]$ since $B'_w(x - w(x)) < 0$ and $u'_w(w(x)) > 0$; which is known as the Borch rule [7]. This implies that the associated participation constraint is always binding at the optimal solution from Karush–Kuhn–Tucker theory [4, Ch.4].

Given an effort level e, what is the wage $w(x)$ that should be offered to the agent? To answer this question we develop an ordinary differential equation for the unknown function $w(x)$. First, we differentiate (2.4) w.r.t. x and get

$$B''_{wx}(x - w(x)) + \lambda u''_{wx}(w(x)) = 0. \tag{2.6}$$

Substituting (2.5) into (2.6) and simplifying we find

$$\frac{B''_{wx}(x - w(x))}{B'_w(x - w(x))} = \frac{u''_{wx}(w(x))}{u'_w(w(x))}. \tag{2.7}$$

The next theorem derives an ordinary differential equation for $w(x)$ which is given in terms of the principal's and agent's Arrow-Pratt measures of absolute risk-aversion. Letting $u'(w(x)) \equiv u'_w(w(x))$ and $u''(w(x)) \equiv u''_{ww}(w(x))$, these measures for the principal and the agent are respectively given as,

$$r_P(y) = -\frac{B''(y)}{B'(y)}, \quad \text{and} \quad r_A(w(x)) = -\frac{u''(w(x))}{u'(w(x))}, \tag{2.8}$$

see Arrow [2, Ch. 3] and Pratt [39]. Note that both measures are positive, i.e., $r_P(y) > 0$ and $r_A(w(x)) > 0$ since both the principal and the agent are assumed to be risk averse with increasing and concave utility functions $B(y)$ and $u(w)$, respectively.

Theorem 1 Suppose we fix the agent's effort at some level e and consider an arbitrary outcome $a \in [x_0, x_1]$. The wage function $w(x) \equiv w(x; e)$ is found as the solution of the ODE

$$\frac{dw(x)}{dx} = \frac{r_P(x - w(x))}{r_P(x - w(x)) + r_A(w(x))}, \tag{2.9}$$

and it increases in the outcome x with a slope not exceeding 1. The initial condition for the ODE is $w(a) = b$ for $b < a$ where the unknown b is determined from the equality part of the participation constraint

$$\int_{x_0}^{x_1} u(w(x))f(x;e)\ dx = v(e) + U_0. \tag{2.10}$$

Proof. Differentiating (2.1) w.r.t. x and using the chain rule, we obtain the mixed partial derivative of the principal's utility function as,

$$\begin{aligned} B''_{wx}\left(x - w\left(x\right)\right) &= -\frac{dB'(y)}{dy}\frac{dy}{dx} \\ &= -B''(y)\frac{dy}{dx} = -B''(y)\left[1 - \frac{dw\left(x\right)}{dx}\right], \end{aligned} \tag{2.11}$$

since from (2.1) we have $B'_w(y) = -B'(y)$ and $dy/dx = 1 - dw\left(x\right)/dx$. Similarly, the mixed partial derivative of the agent's utility function is found as,

$$u''_{wx}\left(w(x)\right) = \frac{du'\left(w(x)\right)}{dw}\frac{dw(x)}{dx} = \left[u''\left(w(x)\right)\right]\frac{dw\left(x\right)}{dx}. \tag{2.12}$$

Using (2.11) and (2.12), equation (2.7) can be written as,

$$\left[\frac{B''(y)}{B'(y)}\right]\left[1 - \frac{dw\left(x\right)}{dx}\right] = \frac{u''\left(w(x)\right)}{u'\left(w(x)\right)}\frac{dw\left(x\right)}{dx}. \tag{2.13}$$

Since (2.13) involves the Arrow-Pratt measures of absolute risk-aversion, it can be written equivalently as,

$$-r_P\left(y\right)\left[1 - \frac{dw\left(x\right)}{dx}\right] = -r_A\left(w(x)\right)\frac{dw\left(x\right)}{dx},$$

where, as before, $y = x - w(x)$. Finally, isolating the derivative $dw(x)/dx$, we obtain the nonlinear ODE for the unknown function $w(x)$ as,

$$\frac{dw\left(x\right)}{dx} = \frac{r_P\left(x - w\left(x\right)\right)}{r_P\left(x - w\left(x\right)\right) + r_A\left(w\left(x\right)\right)},$$

which clearly has the property that $dw(x)/dx < 1$ since $r_A\left(w\left(x\right)\right) > 0$. To solve the ODE we need an initial condition $w(a) = b$. Since a is given, using the fixed effort level e, we find b by solving $\int_{x_0}^{x_1} u(w(x))f(x;e)\ dx = v(e) + U_0$. This proves the theorem. ■

2.3 Principal and Agent both have the Power Utility Functions

In this section, we assume that both the principal (hereafter, "P") and the agent (hereafter, "A") are risk-averse with power utility functions $B\left(y\right)$ and $u\left(w\left(x\right)\right)$ respectively, given as,

$$B(y) = \frac{y^{1-c_1}}{1-c_1}, \quad \text{and} \quad u(w\left(x\right)) = \frac{w\left(x\right)^{1-c_2}}{1-c_2},$$

with c_1, $c_2 > 0$ and c_1, $c_2 \neq 1$, see Larsen [30]. Then both participants' Arrow-Pratt risk aversion functions can be calculated as,

$$\begin{aligned} r_P(x - w\left(x\right)) &= -\frac{B''(y)}{B'(y)} = \frac{c_1}{x - w\left(x\right)}, \\ r_A(w\left(x\right)) &= -\frac{u''\left(w\left(x\right)\right)}{u'\left(w\left(x\right)\right)} = \frac{c_2}{w\left(x\right)}. \end{aligned}$$

Using (2.9) in Theorem 1, the ODE corresponding to the optimal wage scheme $w\left(x\right)$ for outcome x is,

$$\frac{dw\left(x\right)}{dx} = \frac{c_1 w(x)}{c_1 w(x) + c_2[x - w(x)]}. \tag{2.14}$$

Remark 2 Note that since $B'' < 0$, $B' > 0$, (i.e., the principal is risk averse with an increasing and concave utility function) in this case we must have $r_P > 0$, implying that $w(x) < x$ for all x. The fact that $w(x) < x$ also follows by noting that $dw/dx < 1$, from which we obtain the inequality. Similarly, as $u'' < 0$, $u' > 0$, we have $r_A > 0$, implying that $w(x) > 0$. Thus, the principal would not be willing to pay more than x for *any* given outcome x. ◀

We set the initial condition for the ODE (2.14) as $w(a) = b$ with the property that $b/a < 1$. Then the solution to the ODE (2.14) is found implicitly as

$$w(x) + K_1[w(x)]^{c_2/c_1} = x,$$

where the constant K_1 is determined using the condition $w(a) = b$.

Proposition 2 If the principal is less risk-averse than the agent, i.e., $c_1 < c_2$, then the wage function $w(x)$ which solves (2.14) is concave in x; otherwise i.e., if $c_1 > c_2$, it is convex.

Proof. Differentiating (2.14), we obtain

$$w''(x) = \frac{c_1 c_2 [w'(x)\, x - w(x)]}{\{c_1 w(x) + c_2 [x - w(x)]\}^2}. \tag{2.15}$$

Substituting the RHS of (2.14) in (2.15) and simplifying gives

$$w''(x) = \frac{c_1 c_2 w(x) [(c_1 - c_2)(x - w(x))]}{\{c_1 w(x) + c_2 [x - w(x)]\}^3}.$$

Since the denominator is always positive, we find that $w''(x) < 0$, i.e., $w(x)$ is concave if $c_1 < c_2$; and convex otherwise. ■

Next, we discuss three cases where the principal and the agent may have different degrees of risk aversion. First, we obtain an explicit solution to the special case $c_1 = c_2$ where the optimal wage function $w(x)$ is a linear contract in x. For $c_1 \neq c_2$, we provide (graphical) examples and show that if $c_1 < c_2$ the wage function is concave in x; and if $c_1 > c_2$ then it is convex in x.

Example 1 For the special case $c_1 = c_2$ (where both P and A have the same degree of risk aversion), we have $w(x) + K_1 w(x) = x$ with K_1 as the constant of integration. Using the initial condition $w(a) = b$ with $b/a < 1$ gives $K_1 = a/b - 1$, from which we obtain the explicit solution as

$$w(x) = \left(\frac{b}{a}\right) x,$$

which is a linear contract. As expected, this solution has the property that $w(x) < x$, since $b/a < 1$. However, note that even though a can be fixed arbitrarily since $a \in [x_0, x_1]$, the exact value of b depends on the assumed value of the effort $e \in [e_0, e_1]$ and the solution of the agent's participation constraint (2.10). Figure 1 presents a graphical description of a family of solutions (i.e., direction fields) for $w(x)$ for different values of b when $c_1 = c_2 = \frac{1}{5}$. It should be noted that a unique $w(x)$ will correspond to the optimal effort e^* and the constant b which will maximize the principal's objective function.

Let's now consider the case where P is less risk-averse than A, i.e., $c_1 < c_2$. In this case one would expect the outcomes to have less effect on A's wages. When $c_1 \neq c_2$, the solution of the differential equation must be obtained numerically (except in simple cases where c_2/c_1 is a small integer). For example, when $c_1 = \frac{1}{5}$, $c_2 = \frac{1}{2}$, the family of solutions is found as in Figure 2 where the $w(x)$ curve is below x and it is always concave (as was shown in Proposition 2).

When $c_1 > c_2$, P is more risk-averse than A, and the opposite happens, i.e., A's wages should be more sensitive to the outcomes. For example, when $c_1 = \frac{1}{2}$ and

$c_2 = \frac{1}{5}$, the family of solutions $w(x)$ is shown in Figure 3 with the property that $\lim_{x\to\infty} x/w(x) = 1$, i.e., the asymptotic solution for the wages $w(x)$ is approximated in the ratio as the result of A's outcome x. In this case the $w(x)$ function is always convex (as was shown in Proposition 2). ♦

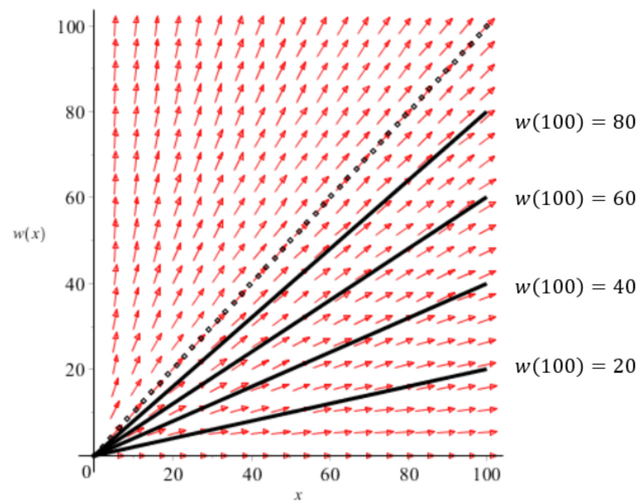

Figure 1. Direction fields for the ODE (2.14) when both participants have the power utility with $c_1 = c_2 = \frac{1}{5}$. In this case the wage function is always linear in the outcome x.

2.4 Principal has Power and Agent has Exponential Utility Function

In this section, the principal's utility is the same as that in Section 2.3, i.e., $B(y) = y^{1-c_1}/(1-c_1)$ with $c_1 > 0$ and $c_1 \neq 1$, and the agent's utility follows the exponential utility function given as,

$$u\left(w\right) = \frac{1}{r_A}\left[1 - \exp(-r_A w)\right]$$

with $r_A > 0$. The principal's and the agent's risk aversion measures are given respectively as,

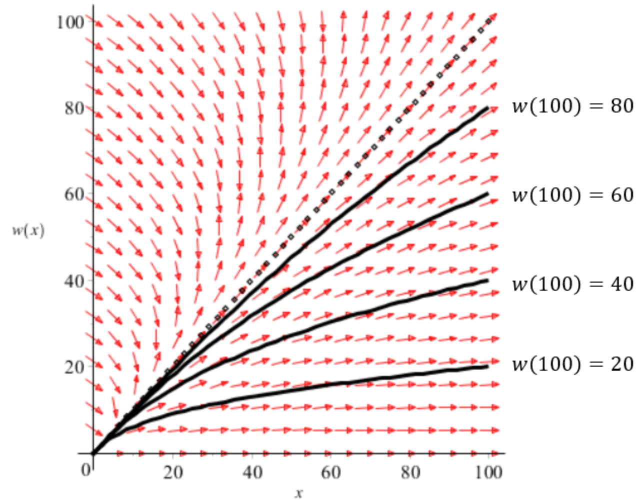

Figure 2. Direction fields for the ODE (2.14) when both participants have the power utility with $c_1 = \frac{1}{5}$ and $c_2 = \frac{1}{2}$. In this case the wage function is always concave in the outcome x.

$$\begin{aligned} r_P\left(x - w\left(x\right)\right) &= \frac{c_1}{x - w\left(x\right)}, \\ r_A(w\left(x\right)) &= r_A. \end{aligned}$$

Substituting the risk aversion functions into (2.9), we find the ODE for this case as

$$\frac{dw\left(x\right)}{dx} = \frac{c_1}{c_1 + r_A[x - w(x)]}. \tag{2.16}$$

Remark 3 As in Remark 2, we note that since $r_P > 0$, we have $w\left(x\right) < x$ for every $x \in [x_0, x_1]$, i.e., the principal would not be willing to pay more than x for a given outcome x. This result also follows by solving the inequality $dw/dx < 1$. ◄

With the initial condition $w(a) = b$, (where $b < a$) the solution to this ODE is

$$w(x) = x - \frac{c_1}{r_A}\mathcal{W}\left(K_1 \frac{r_A}{c_1} \exp\left(\frac{r_A}{c_1}x\right)\right),$$

where K_1 is the constant of integration. Here, $\mathcal{W}(x)$ is Lambert's $\mathcal{W}$-function which is the solution of the equation $\mathcal{W}(x)\exp(\mathcal{W}(x)) = x$. Lambert's $\mathcal{W}$-function is implemented in Maple (`www.maplesoft.com`) and it is a concave increasing function in its

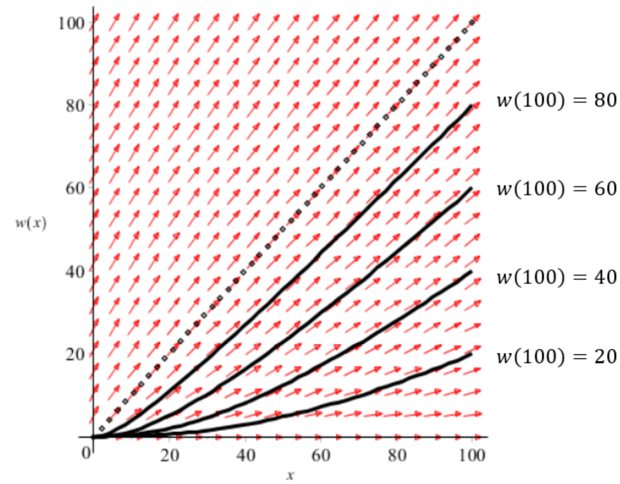

Figure 3. Direction fields for the ODE (2.14) when both participants have the power utility with $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{5}$. In this case the wage function is always convex in the outcome x.

argument with $\mathcal{W}(0) = 0$, as shown in Figure 4.

Proposition 3 The wage function $w(x)$ as the solution of (2.16) is always concave in x.

Proof. Differentiating (2.16), we obtain,

$$w''(x) = -\frac{c_1 r_A [1 - w'(x)]}{\{c_1 + r_A[x - w(x)]\}^2} < 0,$$

since $w'(x) < 1$ from Theorem 1, thus proving the Proposition. ■

Example 2 For the special case of $c_1 = r_A$, and with, say, $w(100) = 60$, the solution is found explicitly as

$$w(x) = x - \mathcal{W}\left(40 \exp\left(x - 60\right)\right).$$

It can be shown that, as before, and as expected, $w(x) < x$ for all x. This function is concave, as shown in Proposition 3.

Now consider the case where $c_1 < r_A$, i.e., in a "relative" sense P is less risk-averse than A. When $(c_1, r_A) = (\frac{1}{5}, \frac{1}{2})$, and $w(100) = 60$, we find the solution exactly as

$$w(x) = x - \frac{2}{5}\mathcal{W}\left(100 \exp\left(\frac{5}{2}x - 150\right)\right).$$

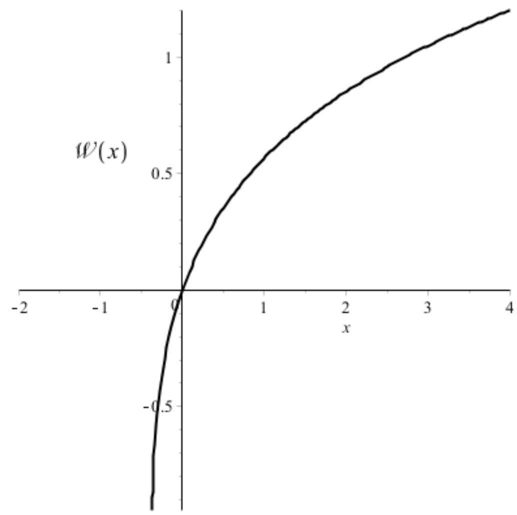

Figure 4. The shape of Lambert's $\mathcal{W}$-function as the solution of $\mathcal{W}(x)\exp(\mathcal{W}(x)) = x$.

In this solution, as the one in Example 1 with $c_1 < c_2$, the wage function $w(x)$ is concave.

Finally, for the case where $c_1 > r_A$, we set $(c_1, r_A) = (\frac{1}{2}, \frac{1}{5})$, and for $w(100) = 60$ we find the solution as,

$$w(x) = x - \frac{5}{2}\mathcal{W}\left(16\exp\left(\frac{2}{5}x - 24\right)\right).$$

Since from Proposition 3 we know that $w(x)$ is always concave in x, we plot the solution curves for the present case with $(c_1, r_A) = (\frac{1}{2}, \frac{1}{5})$ in Figure 5. ♦

2.5 Principal has Exponential and Agent has Power Utility Function

Now consider the case where the principal has an exponential utility function given as

$$B(y) = \frac{1}{r_P}\left[1 - \exp(-r_P y)\right]$$

with $r_P > 0$, and the agent has the power utility function as in Section 2.3, i.e., $u(w(x)) = w(x)^{1-c_2}/(1-c_2)$ with $c_2 > 0$ and $c_2 \neq 1$. Then both participants'

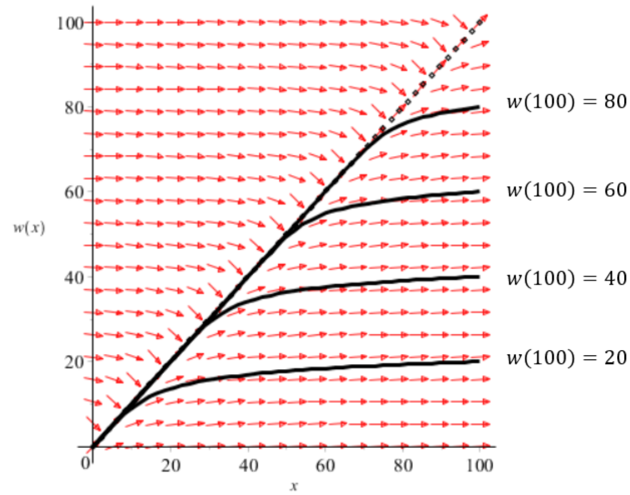

Figure 5. Direction fields for the ODE (2.16) when principal has power and agent has exponential utility with $c_1 = \frac{1}{2}$ and $r_A = \frac{1}{5}$. In this case the wage function is always concave in the outcome x.

Arrow-Pratt risk aversion functions can be calculated as,

$$\begin{aligned} r_P(x - w(x)) &= -\frac{B''(y)}{B'(y)} = r_P, \\ r_A(w(x)) &= -\frac{u''(w(x))}{u'(w(x))} = \frac{c_2}{w(x)}. \end{aligned}$$

Substituting the above risk aversion functions into (2.9), we find the ODE for this case as,

$$\frac{dw(x)}{dx} = \frac{r_P w(x)}{r_P w(x) + c_2}. \tag{2.17}$$

Remark 4 In Remark 2 we had shown that when both participants have the power utility we have $0 < w(x) < x$ for all x. The fact that $w(x) < x$ also followed from the observation $dw/dx < 1$. In the present case when the principal has the exponential utility and agent has the power utility, it is not possible to show that $w(x) < x$ for all x since $r_P(x - w(x)) = r_P$, a constant which does not involve x. The condition $dw/dx < 1$ in the ODE (2.17) also gives $c_2 > 0$ which does not give any information on the wage function $w(x)$. Thus, in this case it is possible that for some values of x, the principal may pay the agent more than the outcome she produces. Note, however, that the principal cannot punish the agent for any outcome since $r_A(w(x)) = c_2/w(x) > 0$ implies that the wage is always positive. ◀

The solution of the ODE (2.17) is found as,

$$w(x) = \exp\left(\frac{r_P}{c_2}(K_1 + x) - \mathcal{W}\left(\frac{r_P}{c_2}\exp\left(\frac{r_P}{c_2}(K_1 + x)\right)\right)\right),$$

where K_1 is the constant of integration to be determined from the initial condition $w(a) = b$ and $W(x)$ is Lambert's W-function as described in Section 2.4.

Proposition 4 The wage function $w(x)$ as the solution of (2.17) is always convex in x.

Proof. Differentiating (2.17), substituting the RHS of (2.17) into the result, we obtain,

$$w''(x) = \frac{r_P^2 c_2 w(x)}{[r_P w(x) + c_2]^3}.$$

Since $w(x) > 0$, we always have $w''(x) > 0$, thus proving the proposition. ■

Example 3 Consider the case where $r_P < c_2$, i.e., in a "relative" sense P is less risk-averse than A. When $(r_P, c_2) = (\frac{1}{5}, \frac{1}{2})$, and $w(100) = 60$, we find the solution exactly as

$$w(x) = \frac{5}{2}\mathcal{W}\left(24\exp\left(\frac{2}{5}x - 16\right)\right).$$

In this solution $w(0) \simeq 0$, and the wages are sensitive to the outcome since $w(x)$ is convex as shown in Figure 6 with $\lim_{x\to\infty} x/w(x) = 1$. Note however that in the extreme case when $w(100) = b$ for a large $b \simeq 100$, (say, $b = 99$) we find that the principal still pays the agent a small but positive amount, i.e., $w(0) = 6$. This outcome is unlikely to occur in practice since the principal would not be willing to reward the agent to such an extent.

Now consider $r_P > c_2$ with $(r_P, c_2) = (\frac{1}{2}, \frac{1}{5})$, which indicates that P is more risk-averse than A. With the initial condition $w(100) = 60$, the solution to the ODE (2.17) is found as,

$$w(x) = \frac{2}{5}\mathcal{W}\left(150\exp\left(\frac{5}{2}x - 100\right)\right).$$

In this case the direction fields are quite similar to the one in Figure 6, thus we do not reproduce it here. ⧫

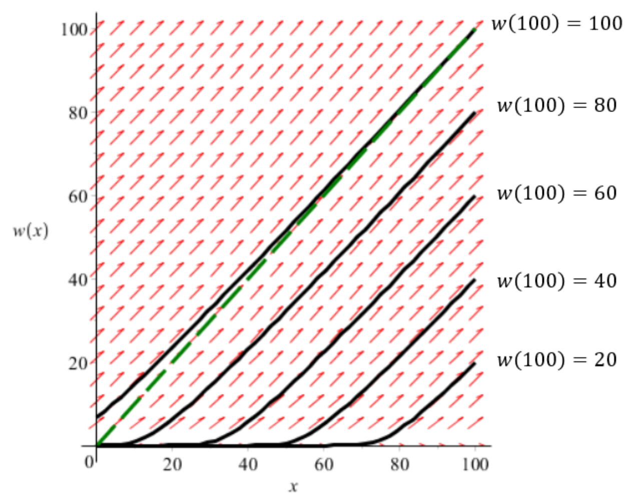

Figure 6. Direction fields for the ODE (2.17) when principal has exponential and agent has power utility with $r_P = \frac{1}{5}$ and $c_2 = \frac{1}{2}$. In this case the wage function is always convex in the outcome x.

2.6 Principal and Agent both have Exponential Utility Functions

For the case where both participants' utility functions are exponential, the risk aversion measures are, respectively, $r_P\left(x - w\left(x\right)\right) = r_P$ and $r_A\left(w\left(x\right)\right) = r_A$, with constant $r_P > 0$ and $r_A > 0$. In this case the ODE (2.9) reduces to one with a constant RHS,

$$\frac{dw\left(x\right)}{dx} = \frac{r_P}{r_P + r_A}, \tag{2.18}$$

which suggests a linear contract in x.

Remark 5 We had shown in Remark 2 that when both participants have the power utility we obtain $0 < w(x) < x$ for all x. In Remark 3 we found that $w(x) < x$, but $w(x)$ could possibly be negative. Remark 4 showed that either $w(x) < x$ or $w(x) > x$ could be possible, but we always had $w(x) > 0$. In the current section we again have either $w(x) < x$ or $w(x) > x$ since $r_P\left(x - w\left(x\right)\right)$ is a positive constant not involving either x or $w(x)$. Moreover, since $r_A\left(w\left(x\right)\right)$ is also a positive constant, $w(x)$ can be either negative or positive, meaning that the principal can punish or reward the agent depending on the outcome. ◀

The solution of the ODE (2.18), with the initial condition $w\left(a\right) = b$ is obtained as,

$$w(x) = \frac{r_P}{r_P + r_A}(x - a) + b. \tag{2.19}$$

The linear contract solution (2.19) indicates that if r_A is very large, then $w(x) \approx b$, a constant; that is, the principal must insure the agent against any uncertainty in the outcomes and pay her a fixed wage. On the other hand, if r_P is very large, then $w(x) \approx x + (b - a) < x$, then the wage payment to the agent depends linearly on the actual outcome.

Example 4 Consider now the case where the participants have the exponential utility with the same risk aversion constant, i.e., $r_P = r_A = \frac{1}{2}$. Figure 7 displays the direction fields where the wage function is linear with a slope of $\frac{1}{2}$. Clearly, in this case it is possible to have either $w(x) < x$, or $w(x) > x$. It is also possible to have a negative $w(x)$ meaning that the principal can punish the agent for some outcomes, unless a minimum liability constraint is introduced. ♦

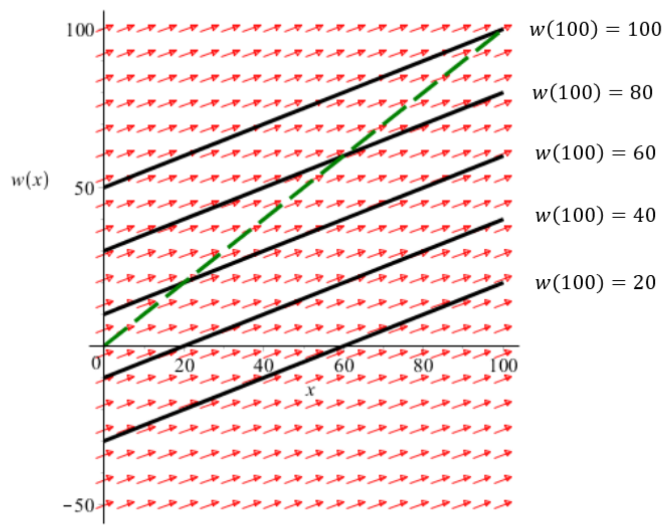

Figure 7. Direction fields for the ODE (2.18) when principal and agent has exponential utility with $r_P = r_A = \frac{1}{2}$. In this case the wage function is always linear in the outcome x.

Table 2 summarizes the properties of the wage function $w\,(x)$ under different utility combinations for the principal and the agent. These findings demonstrate that the exponential utility function with constant risk aversion makes it an unrealistic choice for decision analytic studies in the first-best problem.

Combination for P and A	**ODE**	**Properties of $w(x)$**
Power - Power	$\frac{dw(x)}{dx} = \frac{c_1 w(x)}{c_1 w(x) + c_2 x - c_2 w(x)}$	$\begin{cases} \text{linear if } c_1 = c_2, \\ \text{concave if } c_1 < c_2, \\ \text{convex if } c_1 > c_2, \end{cases}$
Power - Exponential	$\frac{dw(x)}{dx} = \frac{c_1}{c_1 + r_A x - r_A w(x)}$	concave in x,
Exponential - Power	$\frac{dw(x)}{dx} = \frac{r_P w(x)}{r_P w(x) + c_2}$	convex in x,
Exponential - Exponential	$\frac{dw(x)}{dx} = \frac{r_P}{r_P + r_A}$	linear in x.

Table 2. Four utility combinations for the principal (P) and the agent (A) and the corresponding wage function properties.

2.7 The Agent's Optimal Level of Effort

Our discussion so far focussed on the explicit solution of the nonlinear ODE (2.9) to find the wage function $w(x)$ for the agent. Since an initial condition for the ODE was not (yet) available, we demonstrated the solution of the ODE as a family of curves determined by the direction fields. In this section, we consider the problem of finding the *optimal* effort level the principal demands from the agent to maximize the former's expected utility.

As noted by MS/PC [33, p. 28], in general, finding the optimal effort level is a difficult problem as the principal's objective may not be concave with respect to the agent's effort since the density function of the random outcome depends on the effort level. But given an effort level $e \in [e_0, e_1]$, in principle it is not too difficult to evaluate the principal's expected utility. Since the participation constraint is always satisfied as an equality, we can determine the wage function and evaluate the principal's objective $J(e)$ as follows:

(1) Fix the effort level $e \in [e_0, e_1]$,
(2) select an arbitrary outcome, say, the highest one as $a = x_1$ to anchor the initial condition $w(a) = b$ of the ODE (2.9) where b is still an unknown,
(3) with e and a given, find $b = b(e)$ for which the unique solution $w(x; e)$ of the ODE (2.9) satisfies the agent's participation constraint $\int_{x_0}^{x_1} u(w(x; e)) f(x; e)\, dx =$

$v(e) + U_0$,

(4) use $w(x;e)$ in the principal's objective to calculate $J(e) = \int_{x_0}^{x_1} B(x - w(x;e)) f(x;e)\, dx$, and

(5) using an optimization routine, maximize $J(e)$ over $e \in [e_0, e_1]$.

We now consider two cases where both the principal and the agent possess the power utility function. In the first case, they have the same degree of risk aversion, i.e., $c_1 = c_2 = c$ which gives rise to a closed-form solution for the wage function $w(x;e)$ in terms of the effort level e and simplifies the optimization analysis considerably to find the agent's optimal effort level. In the second case where $c_1 \neq c_2$, the closed-form solution for the wage function becomes difficult (and sometimes impossible) to obtain and thus we use numerical methods to find the agent's optimal effort level.

2.7.1 Principal and Agent have Power Utility Functions with the Same Degree of Risk Aversion

Suppose both participants possess power utility functions with the same degree of risk aversion, i.e., $c_1 = c_2 = c$, where, without loss of generality, we assume that $c \in (0,1)$. That is, the utility functions are given as

$$B(y) = \frac{y^{1-c}}{1-c}, \quad \text{and} \quad u(w(x)) = \frac{w(x)^{1-c}}{1-c},$$

with $y = x - w(x)$. For this case, as shown in Section 2.3, the wage function $w(x)$ is linear in x for a given effort level e, i.e., $w(x) = (b/a)\,x$ where $0 < b < a$ for an arbitrary outcome $a \in [x_0, x_1]$. As discussed above, for a given effort level e, the agent's participation constraint (2.10) is always binding at the optimal solution. In order to find the correct value of b corresponding to a fixed a, we solve the agent's binding participation constraint $\int_{x_0}^{x_1} u(w(x;e)) f(x;e)\, dx = v(e) + U_0$. This gives,

$$b \equiv b(e) = a \left[\frac{(1-c)(v(e) + U_0)}{\int_{x_0}^{x_1} x^{1-c} f(x;e)\, dx} \right]^{\frac{1}{1-c}}. \tag{2.20}$$

Since $c \in (0,1)$, we have $b(e) > 0$. In order to assure that $b(e) < a$ for all effort levels $e \in [e_0, e_1]$, we impose the condition that

$$v(e) + U_0 < \frac{1}{(1-c)} \int_{x_0}^{x_1} x^{1-c} f(x;e)\, dx. \tag{2.21}$$

Clearly, if condition (2.21) is not satisfied, then the principal pays the agent more than what she (the agent) is producing; an undesirable outcome for the principal!

Now, writing the wage function as $w(x;e) = b(e)\,x$ and substituting it in the principal's objective, we have,

$$\begin{aligned}
\max_{e\in[e_0,e_1]} J(e) &\equiv \max_{e\in[e_0,e_1]} \int_{x_0}^{x_1} \left[\frac{(x-b(e)\,x)^{1-c}}{1-c}\right] f(x;e)\;dx \\
&= \max_{e\in[e_0,e_1]} \frac{(1-b(e))^{1-c}}{1-c} \int_{x_0}^{x_1} x^{1-c} f(x;e)\;dx. \qquad (2.22)
\end{aligned}$$

Thus, the principal's problem is to find the optimal effort level $e \in [e_0, e_1]$ which maximizes the expected utility $J(e)$. To that end, let's define the functions $G(e)$ and $H(e)$ as,

$$G(e) = \frac{[1-b(e)]^{1-c}}{1-c}, \quad \text{and} \quad H(e) = \int_{x_0}^{x_1} x^{1-c} f(x;e)\;dx,$$

so that $J(e) = G(e)\,H(e)$. The first two derivatives of $J(e)$ w.r.t. e are $J'(e) = G'(e)\,H(e) + G(e)\,H'(e)$ and $J''(e) = G(e)\,H''(e) + 2G'(e)\,H'(e) + G''(e)\,H(e)$. The principal's problem (2.22) is concave in e if and only if the second-order condition $J''(e) < 0$ is satisfied. If it does, then we can use the first-order condition to find the optimal effort level e^*. Of course, if $J'(e) > 0$ for all $e \in [e_0, e_1]$, then $e^* = e_1$. Otherwise, e^* is obtained by solving $J'(e^*) = 0$ for e^*.

Example 5 In this example we assume that the agent's effort level $e \in (0, 1]$, that is, we exclude $e = 0$ as the agent is expected to provide some positive level of effort. We also assume that the random outcome X takes values in the unit interval, i.e., $x \in [0, 1]$. The density function for X is chosen as $f(x;e) = 1 + (x - \frac{1}{2})e$ with the expected value $E(X;e) = \frac{1}{2} + \frac{1}{12}e$, and the survivor function $\bar{F}(x) = \Pr(X > x) = 1 - \frac{1}{2}e(x^2 - x) - x$. Since $\bar{F}'_e(x) = -\frac{1}{2}x(x-1) > 0$, we see that $\bar{F}(x)$ is increasing in e for any given outcome x; that is, the first-order stochastic dominance condition is satisfied.

The agent's disutility is a convex increasing function $v(e) = \gamma e^2$, for $\gamma > 0$. Solving the participation constraint we find that, for the initial condition $a = 1$, we have,

$$b(e) = \left[\frac{2(3-c)(2-c)(1-c)\left(\gamma e^2 + U_0\right)}{6 + e(1-c) - 2c}\right]^{1/(1-c)}.$$

To gain further insights into the problem, in our numerical example we assume $\gamma = \frac{1}{10}$, $U_0 = \frac{3}{10}$. Then the wage scheme is obtained as

$$w(x;e) = \left[\frac{1}{5}\frac{(3-c)(2-c)(1-c)\left(e^2+3\right)}{6 + e(1-c) - 2c}\right]^{1/(1-c)} x.$$

$\gamma \backslash c$	0.1	0.3	0.5	0.7	0.9
0.1	$(0.488, 0.350)$	$(0.812, 0.749)$	$(1.000, 1.411)$	$(1.000, 2.730)$	$(1.000, 9.307)$
0.3	$(0.163, 0.335)$	$(0.278, 0.717)$	$(0.706, 1.355)$	$(1.000, 2.726)$	$(1.000, 9.307)$
0.5	$(0.098, 0.331)$	$(0.168, 0.710)$	$(0.459, 1.334)$	$(1.000, 2.718)$	$(1.000, 9.307)$
0.7	$(0.070, 0.330)$	$(0.120, 0.707)$	$(0.343, 1.325)$	$(1.000, 2.702)$	$(1.000, 9.307)$
0.9	$(0.054, 0.329)$	$(0.094, 0.706)$	$(0.276, 1.320)$	$(0.898, 2.682)$	$(1.000, 9.307)$

Table 3. The joint sensitivity analyis of γ and c on the optimal solution $(e^*, J(e^*))$.

We substitute the above wage scheme $w(x; e)$ into the principal's objective function $J(e)$ in (2.22) and provide graphs involving the derivatives of the objective $J(e)$.

As shown in Figure 8, for some fixed values of $c \in (0, 1)$, the first derivative $J'(e)$ decreases in e and falls below zero, indicating that $J(e)$ has a unique interior solution, i.e., $e^* < 1$. [Figure 9 shows that the second derivative $J''(e)$ is always negative for any $c \in (0, 1)$, implying that $J(e)$ is concave.] For some other values of c, $J'(e)$ decreases but always stays above zero, indicating that the global optimal solution is at the boundary, i.e., $e^* = 1$. Performing the numerical optimization, we indeed find that for $c \in (0, 0.36)$ we have $e^* < 1$ and for $c \in (0.36, 1)$, the optimal solution is at $e^* = 1$. For example, when $c = 0.2$, we find $e^* = 0.60$ and $J(e^*) = 0.52$, but when $c = 0.6$, we have $e^* = 1$, and $J(e^*) = 1.92$. This implies that when the principal's risk aversion increases, he requires the agent to work harder.

A natural question to ask is the following: How does the optimal solution [in terms of e^* and $J(e^*)$] vary as the agent's disutility shifts upwards with a higher γ (so she is less willing to put in the effort) and both participants become more risk averse with a higher c? To answer this question we fix $U_0 = \frac{3}{10}$ and re-solve the problem for different combination of (γ, c). The results are summarized in Table 3 where both γ and c vary from 0.1 to 0.9 in increments of 0.2 resulting in $(e^*, J(e^*))$ as the optimal solution.

We note that for a fixed value of γ, as both participants become more risk-averse, i.e., as c increases, the optimal effort level increases until the highest possible effort level $e = 1$. With higher levels of effort, the principal's expected utility also increases. Although the agent with higher degree of risk aversion is not willing to exercise higher effort level since it will bring her more disutility, the principal who has the power of deciding the agent's effort level can still demand higher effort level from the agent.

For fixed values of c, as γ increases, the optimal effort level decreases since the increasing value of γ brings higher disutility for the agent. When c assumes a very high level ($c = 0.9$), we note that the optimal effort reaches its maximum possible level of $e^* = 1$ for all γ values. This is the consequence of the principal being highly risk-averse who can force the agent to exert the maximum effort. ♦

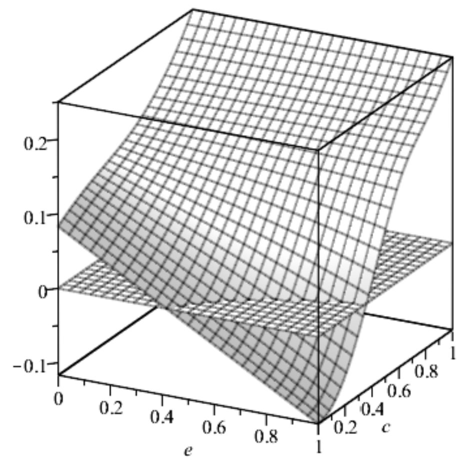

Figure 8. The first derivative $J'(e)$ of the principal's expected utility drawn as a function of c and e.

2.7.2 Principal and Agent have Power Utility Functions with Different Degrees of Risk Aversion

Suppose the principal and the agent both possess power utility functions

$$B(y) = \frac{y^{1-c_1}}{1-c_1}, \quad \text{and} \quad u(w\,(x)) = \frac{w\,(x)^{1-c_2}}{1-c_2},$$

with different degrees of risk aversion, i.e., $c_1 \neq c_2$ where, as we assumed above, $c_1, c_2 \in (0,1)$ and $y = x - w\,(x)$. For this case, and for a fixed effort level $e \in [e_0, e_1]$, the wage function $w\,(x) = w(x;e)$ is the solution of the nonlinear ODE given in (2.14).

However, unlike the previous case discussed in Section 2.7.1, when $c_1 \neq c_2$, the $b(e)$ function cannot be obtained in closed-form which makes it impossible to find the wage function $w(x;e)$ explicitly. For a given e, and fixing a as before, the correct $w(x;e)$ function is found numerically as the one which satisfies the participation constraint $\int_{x_0}^{x_1} u(w(x;e))f(x;e)\,dx = v(e) + U_0$ for that value of e. Once $w(x;e)$ is determined, it can be substituted into the principal's objective $J(e) = \int_{x_0}^{x_1} B(x - w(x;e))f(x;e)\,dx$. With this procedure, one can evaluate $J(e)$ for different (discrete) values of e and find the optimal e^* that maximizes $J(e)$.

We now provide an example that uses the above numerical procedure to find the optimal effort level when the principal is less risk-averse than the agent, i.e., $c_1 < c_2$. The optimal effort level for $c_1 > c_2$ follows a similar procedure, which is omitted here.

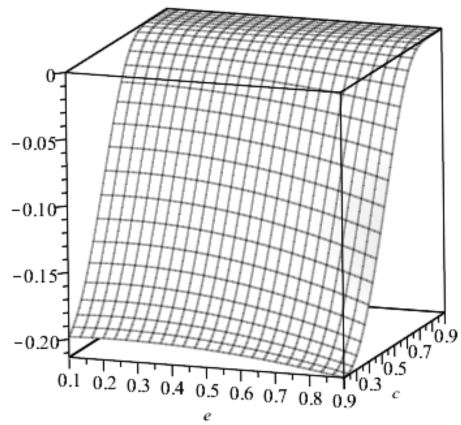

Figure 9. The second derivative $J''(e)$ is always negative indicating that the function $J(e)$ is concave with a unique maximum.

Example 6 In this example we assume that the agent's effort level e takes 10 discrete values between 0.1 and 1.0, i.e., $e \in \{0.1, 0.2, \ldots, 1.0\}$, and as before, the random variable X takes values in the unit interval, i.e., $x \in [0, 1]$. The density function for X, the agent's disutility function and reservation utility are the same as those in Example 5, i.e., $f(x; e) = 1 + (x - 1/2)e$, $v(e) = \gamma e^2$ for $\gamma = \frac{1}{10}$ and $U_0 = \frac{3}{10}$.

Letting $c_1 = \frac{1}{4}$ and $c_2 = \frac{1}{3}$, the nonlinear ODE (2.14) is found as, $dw(x)/dx = 3w(x)/(4x - w(x))$ with the general solution $w(x) + K_1[w(x)]^{4/3} = x$. Now, fixing $a = 1$, and $e = 0.1$, we find that the correct value of $b(e)$ which solves the ODE and satisfies the PC is $b(0.1) = 0.167$; the corresponding $w(x; e)$ is concave in the outcome x; see, Figure 10. Now, evaluating the principal's objective at $e = 0.1$ by substituting the corresponding $w(x; e)$ gives $J(0.1) = 0.638$. For the other effort levels, i.e., $e = 0.2, \ldots, 1.0$, we summarize the results for $b(e)$ and $J(e)$ in Table 4 which shows that the agent's optimal effort level is $e^* = 0.7$, which results in the principal's maximum expected utility as $J(e^*) = 0.691$.

The nonlinear contracts may appear too complex for some real-life implementations. One possible way to make the contracts easier to implement is to provide a look-up table that lists the optimal wage for different discrete outcomes. For example, Table 5 lists the optimal wage for the outcome between 0.1 and 1.0, given the optimal effort level $e^* = 0.7$. ♦

e	$b(e)$	$J(e)$
0.1	0.167	0.638
0.2	0.167	0.647
0.3	0.169	0.656
0.4	0.172	0.664
0.5	0.177	0.673
0.6	0.183	0.682
0.7	**0.191**	**0.691**
0.8	0.201	0.626
0.9	0.213	0.635
1.0	0.226	0.643

Table 4. The optimal solution with $c_1 = \frac{1}{4} < c_2 = \frac{1}{3}$ is $e^* = 0.7$ and $J(e^*) = 0.691$ indicated in **bold** font.

x	$w(x)$
0.1	0.031
0.2	0.053
0.3	0.075
0.4	0.093
0.5	0.112
0.6	0.129
0.7	0.146
0.8	0.161
0.9	0.176
1.0	0.191

Table 5. The optimal wage scheme $w(x)$ for different discrete outcomes given the optimal effort $e^* = 0.7$.

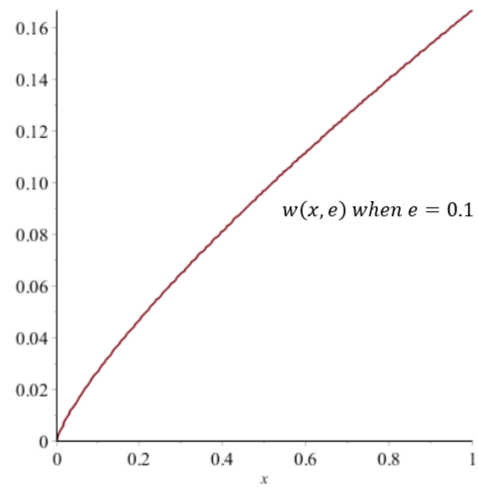

Figure 10. The optimal wage function $w(x, e)$ is concave when $e = 0.1$ and $c_1 = \frac{1}{4} < c_2 = \frac{1}{3}$.

2.8 Summary and Concluding Remarks

In this chapter we consider a first-best contracting problem where the principal and agent are both risk-averse, characterized by power and/or exponential utility functions. Even though the agent's effort is observable and thus it can be enforced by the principal, the outcome is a random variable depending on the effort level. The principal's problem is to determine the optimal wage as a function of the outcome and the effort level which will maximize his expected utility subject to the agent's participation constraint. For a given effort level, we present a step-by-step development of the ordinary differential equation (ODE) whose solution yields the wage function.

We examine combinations of power and exponential utilities for the principal and the agent and find the wage functions for each combination by solving the resulting ODE. The exponential utility function is well-known for its unrealistic property of having a constant risk-aversion. When the principal has the exponential utility, the wage function assumes a form where the agent may either be paid more than what she produces (in terms of the outcome), or be "punished" for low outcomes, unless a minimum liability constraint is included. These results show that the exponential utility assumption may not be realistic even in the first-best case and thus it may be advisable *not* to use it in more general adverse selection and moral hazard problems.

We also present a discussion of computing the optimal level of effort for the agent by using a five-step procedure. We illustrate this method in two numerical examples when both participants possess the power utility functions. When they have the same degree of risk-aversion, we find the closed-form optimal wage scheme that depends on both the effort level and the outcome. We also find that as both participants become more risk-averse, the principal may demand the highest effort level but the agent may not be willing to do so.

For different degrees of risk aversion we use a numerical procedure to find optimal wage for each effort level and evaluate the principal's objective function. This analysis then yields the optimal effort level corresponding to the maximum value of the objective function. Assuming a specific density function that satisfies the first-order stochastic dominance, our results show that it is not always optimal for the principal to demand the highest effort level even under the first-best case.

Complex nonlinear contracts may be difficult to implement in real-life applications. One possible approach that makes a nonlinear contract easier to implement is to list the optimal wage for different discrete outcomes, as we have shown in Table 5. The alternative approach is to use a piece-wise linear contract in the principal's original problem so that it can be converted to another maximization problem. We then solve the new problem without developing the ODE. For example, given an effort level e, the piece-wise linear contract takes the form of

$$w(x;e) = \begin{cases} m_1 x, & \text{for } 0 \leq x < x_0, \\ m_2 x + (m_1 - m_2)\, x, & \text{for } x_0 \leq x \leq 1, \end{cases}$$

where $0 \leq m_1, m_2, x_0 \leq 1$. Substituting $w(x;e)$ into the principal's original problem, we only need to choose the optimal values of m_1, m_2 and x_0 that maximize the principal's expected utility subject to participation constraint, given the effort level e. Depending on both participants' utility functions, the optimal piece-wise linear contract may possess concave, convex, or linear properties. Once the optimal piece-wise linear contract given the effort level is found, the principal can determine the optimal effort level that maximizes his expected utility. In the future, we will investigate such piece-wise linear contracts numerically in terms of efficiency with respect to the optimal wage scheme obtained from the corresponding ODE.

Chapter 3 Optimal Ordering Decision and Incentives for Yield Improvement under Random Demand: A Discrete Model

This chapter studies how a buyer designs an incentive mechanism (or a contract) to induce his supplier's effort in improving supply yield and what's the buyer's optimal ordering decision under random yield and demand. We embed the random yield model into the principal-agent framework under moral hazard, in which the buyer and the supplier behave in a self-interested manner. In our model, the *risk-neutral* buyer (principal) faces random yield for a critical component and random demand for a finished product, which is assembled by the critical component. The buyer offers the *risk-averse* supplier (agent) an incentive contract where the unit transfer price solely depends on yield outcomes. We first develop a model with binary effort choices and yield outcomes. When the supplier's effort cannot be observed (moral hazard arises), our results indicate that the buyer should schedule a higher transfer price for the high yield but place a smaller order. We then build a general model with multiple effort choices and yield outcomes under moral hazard. We develop an algorithm based on Lagrangian duality to solve the buyer's problem. Given the order quantity, the resulting dual problem may become linear, quadratic, or other nonlinear programing problem, depending on the form of the supplier's utility function. Finally, we present the numerical results for the model with three effort levels and three yield levels. We find that it is optimal for the buyer to induce the second-highest effort level that maximizes the buyer's expected profit function.

3.1 Introduction

Recently, supply uncertainty has received increasing research attention due to its significant impact on a buyer's overall performance. If defective components are supplied to the buyer who assembles them in the final products, then serious quality problems may arise. For example, in the auto industry, the defective airbag supplied by Takata Corporation caused four deaths on US roads and one in Malaysia, which resulted in Honda's huge profit loss and degraded this company's brand equity (The New York Times [42], Nov. 13, 2014). Having realized that supply uncertainty and the related quality problems affect the buyer's business, the buyer intends to improve his main supplier's performance through an incentive contract for purchasing a critical component (Kim, Cohen and Netessine [25]). Such incentive contract can boost the supplier's effort to deliver more desirable units and thus improve the supplier's performance (Li, Zhang and Fine [32]). However, the supplier's effort is not easily observed nor verifiable by the buyer, which results in the supplier's moral hazard issue. This issue can be solved if the buyer has direct control of the supplier's effort and writes it in the incentive contract (Laffont and Martimort [28]). The deployment of such detailed contracts, nevertheless, may destroy the long-term relationship between the buyer and his main supplier (Zhou et al. [47]). Therefore, the question arises: how to design an effective incentive contract to maintain the relationship between the buyer and his main supplier under the moral hazard issue?

The incentive mechanism design based on the principal-agent model under moral hazard provides an important tool to analyze how a principal should optimally design an incentive mechanism (or a contract) for the agent who exerts some hidden actions (Bolton and Dewatripont [5], Laffont and Martimort [28], Salanié [40], etc). We apply this important tool to study the buyer's incentive contract design problem under random yield and demand. In our study, the risk-neutral buyer (principal, "he") offers an incentive contract of transfer price to the risk-averse supplier (agent, "she"), who exerts costly effort in improving her performance. The unit transfer price received by the supplier is linked with the supplier's observable performance, which is measured by the supplier's yield of good units. Note that the supplier's yield is random and the probabilities of different yield levels depend explicitly on the supplier's effort. That is, the realized yield is only a noisy signal of the supplier's effort. Assuming that the supplier's effort improves yield in the sense of first-order stochastic dominance (see Bunn [8, Ch. 4]), then the buyer will prefer the stochastic distribution of yield induced by a high effort level to that induced by a low effort level. Thus, through an incentive contract based on the supplier's observable performance, the buyer wants to induce, at a reasonable cost, a high effort level from the supplier although it's impossible for him directly to contract the transfer price on the supplier's effort level.

This chapter also studies the buyer's optimal ordering decision given the effort level he wants to induce in a setting of random yield for the critical component and random demand for the finished product. Since a high level of effort brings the buyer a high expected yield of good units, it's better for the buyer to order fewer units. However, the total yield of good units may not be enough to meet the market demand, which results in supply shortage. Therefore, the buyer has to consider the trade-off carefully between the benefit brought from a high level of effort and the risk associated with supply shortage. Then the questions arise: what's the buyer's optimal ordering decision given the effort level he wants to induce? By comparing the expected profit given each effort level, which level of effort is finally optimal for the buyer to induce?

To answer the above questions, we first build a model with binary effort choices and yield outcomes and investigate the optimal incentive contract and order quantity for inducing the high level of effort under both the symmetric information (first-best) and asymmetric information (second-best) cases. Next, we construct a general model with multiple effort choices and yield outcomes, when the supplier's effort cannot be observed. We develop an algorithm based on Lagrangian duality to determine the second-best incentive contract given an effort level. Then for each possible effort level, we evaluate the buyer's expected profit that is maximized to obtain the second-best optimal effort level and order quantity.

Our study contributes to the random yield literature by considering incentives in the setting of random yield and random demand. It also contributes to applications of the principal-agent model in supply chain management. Moreover, this chapter presents an algorithm based on Lagrangian duality to solve the buyer's incentive mechanism design problem under multiple effort choices and yield outcomes. Our algorithm helps obtain the optimal solution to the original problem by solving its associated dual problem, without the need to explicitly involve the complicated participation and incentive constraints. It also provides guidelines for the buyer to decide which level of effort to induce by using the proper incentive contract. Furthermore, it addresses the buyer's optimal ordering decisions under random yield, random demand and moral hazard issues.

The rest of the chapter is organized as follows. After a brief review of related literature in Section 3.2, we develop a model for binary effort choices and yield outcomes in Section 3.3, and review the solutions to the random yield model that only involves the buyer's ordering decision. In Section 3.3.3, we analyze the optimal incentive mechanism design when the supplier's effort can be observed and then obtain the first-best solutions. Then in Section 3.3.4, we analyze the optimal incentive mechanism design when the supplier's effort cannot be observed and obtain the second-best solutions. In Section 3.4, we provide a general model with discrete effort choices and discrete yield outcomes, and present an algorithm based on Lagrangian duality to solve the buyer's problem. Finally, we summarize our findings and provide some concluding remarks in Section 3.5.

3.2 Literature Review

Our research is closely related to two streams of existing literature. The first stream is random yield literature. Yano and Lee [46] provides a comprehensive review of lot sizing problems with random yield. Typically in the random yield literature, the total yield is assumed to be proportional to the order quantity and the supplier's capacity is assumed to be infinite. Under such assumptions, the inventory manager simply decides how many units to order so as to maximize the expected profit in a centralized setting. Examples from this literature include Gerchak, Vickson and Parlar [16] and Gurnani, Akella and Lehoczky [18]. In a decentralized setting, supply diversification is used to mitigate supply uncertainty, see Parlar and Wang [37], Federgruen and Yang [15], and more recently, Mardan, Amalnik and Rabbani [34]. However, one implicit assumption in this literature is that incentives do not play a role. This assumption will be less reasonable if the supplier is incentivized to improve the yield performance. Hence, our study contributes to the random yield literature by considering incentives in the setting of random yield and demand.

In this study, we assume that the buyer incentivizes the supplier to produce a critical component in good quality, which leads to the second related stream of literature—incentive mechanism design. Based on the principal-agent model (for examples, see Bolton and Dewatripont [5], Laffont and Martimort [28], Macho-Stadler and Pérez-Castrillo [33] and Salanié [40]), this literature analyzes how a principal should optimally design an incentive mechanism (or a contract) for the agent who possesses private information or exerts hidden actions. Before the contract is signed, the agent may hold private information of the action to be taken, giving rise to an adverse selection problem. If the informational asymmetry arises after the contract is signed and the principal cannot verify the agent's effort in completing the task, then a moral hazard problem arises. Our research problem belongs to the latter one as we study the buyer's incentive contract design when the supplier's effort cannot be easily observed in the setting of random yield and demand. Hence, this study is an application of the principal-agent model under moral hazard in random yield literature.

For applications of the principal-agent model in random yield literature, van Ackere [43] provides a good example, where a batch-size problem with random yield and random demand is analyzed through a principal-agent model under moral hazard. Similar to van Ackere [43], this study investigates the buyer's decision-making problem when his supplier has some hidden effort in improving her yield performance. However, different from van Ackere [43] that considers a specific yield function of the effort, our model assumes a more general yield function where the effort improves yield in the sense of first-order stochastic dominance (FSD), see Bunn [8, Ch. 4]. Furthermore, our study develops an algorithm based on Lagrangian duality to solve the buyer's problem under multiple effort choices and yield outcomes. Note that Zhu, Zhang and Tsung [48] investigate a contracting problem where both the buyer and the supplier exert effort in the quality improvement in a setting with deterministic demand. Our study differs from their paper as we study the contracting problem where the supplier is rewarded to improve the yield performance in a setting with random yield and random demand.

Some other papers also investigate the asymmetric information problem under random yield; however, they are concerned with hidden information rather than hidden actions. Such problems lead to adverse selection, rather than moral hazard models. For example, Yang, Aydin, Babich and Beil [44] analyze a problem where a risk-neutral manufacturer designs an adverse selection model to induce the supplier's true reliability type. The supplier's reliability with two types (high and low) is hidden information to the manufacturer. Related work by Corbett, Zhou and Tang [10], Ha [19], Hu, Duenyas and Beil [24], Lee and Yang [31], Özer and Raz [35] provide further glimpses of incentive mechanism design related to random yield, but they again focus on adverse selection models in the presence of private cost and/or demand information.

3.3 Binary Effort Choices and Yield Outcomes

We start a simple model with binary effort choices and yield outcomes. The supplier's effort takes two values, the low level of effort e_1 and the high level of effort e_2. The supplier's yield outcome U is stochastic and only takes two values u_1 and u_2 with $0 < u_1 < u_2 < 1$ and $u_2 - u_1 \equiv \Delta u > 0$. We define $p_2 = \Pr(U = u_2 \mid e_2)$ as the probability that the yield of good units will be u_2 given the high effort level e_2, where $0 < p_2 < 1$. We also define $p_1 = \Pr(U = u_2 \mid e_1)$ as the probability that the yield will be u_2 given the low effort level e_1, where $0 < p_1 < 1$. The supplier's effort improves her yield in the sense of first-order stochastic dominance (FSD), i.e., $\Pr(U > \hat{u})$ is increasing in e for any given yield level $\hat{u}$, (see, Bunn [8, Ch. 4]). Thus, we have $p_2 > p_1$ with $p_2 - p_1 \equiv \Delta p > 0$. Moreover, the expected yield is higher given the high effort than given the low effort, i.e., $0 < E\left(U \mid e_1\right) < E\left(U \mid e_2\right) < 1$. Depending on the level of yield u_1 or u_2, the buyer pays the supplier a transfer of $t(u_1)$ or $t(u_2)$ per good unit delivered, respectively. We write $t_j \equiv t(u_j)$ as the transfer to the supplier if u_j is observed for $j = 1, 2$. The buyer must determine not only the scalar decision variable Q but also the transfer menu $\{t_1, t_2\}$ in the incentive mechanism he proposes to the supplier. If the supplier accepts the buyer's offer, then the buyer will receive a total of Qu_j good units for $j = 1, 2$ and process them into final products, which are then sold at the market. We assume that the buyer faces a random market demand X with p.d.f. $f(x)$ and c.d.f. $F(x)$. The buyer's per unit sales revenue is r, but if there is a shortage then this results in a penalty of s per unit; the salvage value for any unsold units is v per unit. It is assumed that $r + s > v$.

The sequence of events is summarized as follows:

(1) The buyer offers the supplier a contract menu $\{t_1, t_2\}$, i.e., the supplier will receive

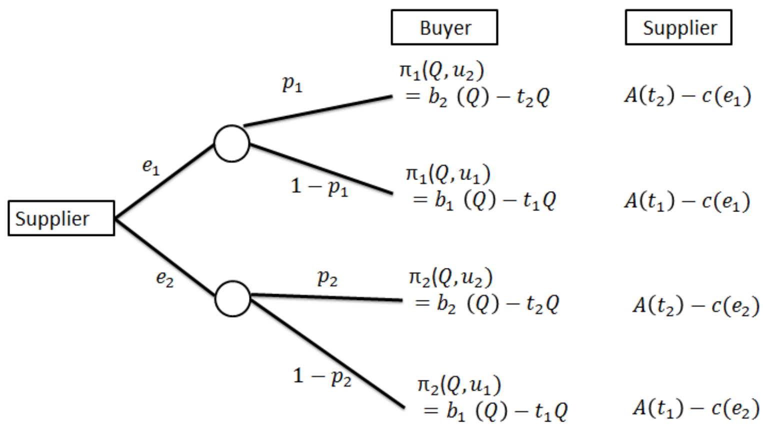

Figure 11. The supplier's decision tree after she accepts the buyer's offer.

a transfer price of t_1 per unit if her yield's realization is u_1; otherwise, she will receive a transfer price of t_2 per unit.

(2) The supplier decides whether to accept the buyer's offer or not. If she accepts, she will choose a contract from the menu.

(3) Once the supplier accepts, the buyer places an order of Q units. The supplier makes the production with her effort level. The yield rate of the critical component is realized to be u. The supplier delivers a total of Qu good units to the buyer. See Figure 11 for the supplier's decision tree where the buyer's net profit and the supplier's net utility are presented under each outcome.

(4) The buyer processes the critical component into the final product. The demand for the finished good is realized. The buyer receives his profit and makes a transfer to the supplier according to the terms of the contract, i.e., the transfer contract is executed.

Note that in stage 1 the buyer offers a menu of contracts to the supplier, with each contract intended for a possible effort level. For example, the contract $\{t_1 = \$4, t_2 = \$6\}$ is intended for the low effort level e_1 and the contract $\{t_1 = \$3, t_2 = \$7\}$ is intended for the high effort level e_2. In stage 3, the total yield is contingent on the order quantity in the manner of QU. Such multiplicative form of random yield indicates that a total of $(1-U)\,Q$ defective units must be discarded due to poor quality, see Agrawal and Nahmias [1]. We assume the cost of discarding defective items to be zero in our study.

Given the realized values of the random variables $X = x$ and $U = u_j$ for $j = 1, 2$, if demand falls short of available units, i.e., $x \le Qu_j$, then the buyer's profit is $\Pi = rx + v(Qu_j - x) - t_j Q$. Otherwise, if $x > Qu_j$, then the profit is

$\Pi = rQu_j - s(x - Qu_j) - t_jQ$. As in most published works in this area, (e.g., Hu et al. [24] and Yang et al. [45]), we assume the buyer to be risk-neutral in this basic model, and thus find his expected profit given the effort level e as,

$$\Pi\left(Q,t_1,t_2\right) = \Pr\left(U = u_2 \mid e\right)\left[b_2(Q) - t_2Q\right] + \left(1 - \Pr\left(U = u_2 \mid e\right)\right)\left[b_1(Q) - t_1Q\right], \tag{3.1}$$

where

$$b_j\left(Q\right) = \int_0^{Qu_j} [(r-v)x + vQu_j]f(x)\ dx + \int_{Qu_j}^{\infty} [(r+s)Qu_j - sx]f(x)\ dx, \text{ for } j = 1,2. \tag{3.2}$$

The buyer' problem is to decide whether to induce the supplier's high effort level or not. If he decides to do so, which incentive mechanism should be used? Once the supplier accepts the offer, what's the buyer's optimal order quantity?

3.3.1 The Buyer's Ordering Decision Problem

Before studying the buyer's problem that involves both incentive mechanism design and ordering issues, we review the properties of the random yield model that only involves the order decision variable Q. Given an effort level e and a transfer agreement $\{t_1, t_2\}$, we denote T as the expected transfer price, i.e.,

$$T = \Pr\left(U = u_2 \mid e\right) t_2 + \left(1 - \Pr\left(U = u_2 \mid e\right)\right) t_1, \tag{3.3}$$

and then write the buyer's expected profit $\Pi\left(Q,t_1,t_2\right)$ in (3.1) as,

$$\Pi\left(Q\right) = \Pr\left(U = u_2 \mid e\right) b_2(Q) + \left(1 - \Pr\left(U = u_2 \mid e\right)\right) b_1(Q) - TQ, \tag{3.4}$$

where $b_2\left(Q\right)$ and $b_1\left(Q\right)$ are defined in (3.2). Next, we present the concavity of $\Pi\left(Q\right)$ in the following lemma.

Lemma 5 The buyer's expected profit $\Pi\left(Q\right)$ in (3.4) is concave in Q.

Proof. Taking first- and second-order derivatives of $\Pi\left(Q\right)$ in (3.4) w.r.t. Q gives

$$\Pi'\left(Q\right) = \Pr\left(U = u_2 \mid e\right) b_2'(Q) + \left(1 - \Pr\left(U = u_2 \mid e\right)\right) b_1'(Q) - T, \tag{3.5}$$

and

$$\Pi''\left(Q\right) = \Pr\left(U = u_2 \mid e\right) b_2''(Q) + \left(1 - \Pr\left(U = u_2 \mid e\right)\right) b_1''(Q),$$

where

$$\begin{aligned} b_j'\left(Q\right) &= u_j\left[(r+s) - (r+s-v)\,F\left(Qu_j\right)\right], \\ b_j''\left(Q\right) &= -\left(r+s-v\right)u_j^2 f\left(Qu_j\right), \end{aligned} \tag{3.6}$$

for $j = 1, 2$. Since $r + s > v$, and $f(Qu_j) \geq 0$ for $j = 1, 2$, we have $b_1''(Q) \leq 0$ and $b_2''(Q) \leq 0$, which suggest that

$$\Pi''(Q) = \Pr(U = u_2 \mid e)\, b_2''(Q) + (1 - \Pr(U = u_2 \mid e))\, b_1''(Q) \leq 0$$

for the probability value between 0 and 1. Therefore, we proved that the buyer's expected profit $\Pi(Q)$ in (3.4) is concave in Q. Thus, we proved the lemma. ■

Since $\Pi(Q)$ is concave, the buyer's problem has a unique optimal solution. We denote the buyer's unique optimal order quantity as $\bar{Q}$, which is given as,

$$u_2 F\left(\bar{Q}u_2\right) \Pr(U = u_2 \mid e) + u_1 F\left(\bar{Q}u_1\right)(1 - \Pr(U = u_2 \mid e)) = \frac{(r+s)\, E(U \mid e) - T}{r + s - v}. \tag{3.7}$$

To guarantee that $\bar{Q}$ is finite and positive, the expected transfer price T should satisfy the condition in Proposition 6.

Proposition 6 Given any effort levels $e \in \{e_1, e_2\}$, the optimal order quantity $\bar{Q}$ is finite and positive under the condition that $vE(U \mid e_2) < T < (r+s)\, E(U \mid e_1)$.

Proof. The unique optimal order quantity $\bar{Q}$ in (3.7) is finite and positive if and only if $\Pi'(Q) > 0$ at $Q = 0$ and $\Pi'(Q) < 0$ at $Q = +\infty$. So we investigate the sign of $\Pi'(Q)$ in (3.5) at $Q = 0$ and $Q = +\infty$ respectively. At $Q = 0$, we have $F(Qu_j) = 0$ and $b_j'(Q) = u_j(r+s)$ for $j = 1, 2$. The first-order derivative is computed as,

$$\begin{aligned}\Pi'(Q) &= u_2(r+s)\Pr(U = u_2 \mid e) + u_1(r+s)(1 - \Pr(U = u_2 \mid e)) - T \\ &= (r+s)\, E(U \mid e) - T,\end{aligned}$$

which is positive if and only if $T < (r+s)\, E(U \mid e)$ given any effort levels $e \in \{e_1, e_2\}$. Since $E(U \mid e_1) < E(U \mid e_2)$, we only need $T < (r+s)\, E(U \mid e_1)$ to ensure that $\Pi'(Q) > 0$ for any effort levels. Next, at $Q = +\infty$, we have $F(Qu_j) = 1$ and $b_j'(Q) = vu_j$ for $j = 1, 2$. The first-order derivative is computed as,

$$\begin{aligned}\Pi'(Q) &= vu_2 \Pr(U = u_2 \mid e) + vu_1(1 - \Pr(U = u_2 \mid e)) - T \\ &= vE(U \mid e) - T,\end{aligned}$$

which is negative if and only if $T > vE(U \mid e)$ given any effort levels $e \in \{e_1, e_2\}$. Since $vE(U \mid e_2) > vE(U \mid e_1)$, we only need $T > vE(U \mid e_2)$ to ensure that $\Pi'(Q) < 0$ for any effort levels. Therefore, under the condition $vE(U \mid e_2) < T < (r+s)\, E(U \mid e_1)$, the unique optimal solution $\bar{Q}$ is finite and positive for any effort levels $e \in \{e_1, e_2\}$. Thus, the proposition is proved. ■

Remark 6 The expected transfer price T defined in (3.3) is actually the buyer's expected unit purchasing cost, which is usually assumed to be greater than v and less than $r+s$, see Gerchak et al. [16] and Parlar and Wang [37]. However, in our model, we have a more restricted condition $vE(U \mid e_2) < T < (r+s)\,E(U \mid e_1)$ as shown in Proposition 6. Such condition guarantees that the buyer will always place a positive order even if the supplier exerts the low effort level e_1, so that we can analyze whether it's worthwhile for the buyer to induce the supplier's high effort or not. ◀

The optimal order quantity $\bar{Q}$ in (3.7) depends on r, s and v as described in the following proposition.

Proposition 7 $\bar{Q}$ is increasing in r, s and v, but decreasing in T.

Proof. Taking implicit differentiation of $\bar{Q}$ in (3.7) w.r.t. v gives

$$\begin{aligned} & \Pr(U=u_2 \mid e)\, u_2^2 f\left(\bar{Q}u_2\right)\frac{d\bar{Q}}{dv} + \left(1-\Pr(U=u_2 \mid e)\right) u_1^2 f\left(\bar{Q}u_1\right)\frac{d\bar{Q}}{dv} \\ = \; & \frac{(r+s)\,E(U \mid e) - T}{(r+s-v)^2}. \end{aligned}$$

Isolating $\frac{d\bar{Q}}{dv}$ yields,

$$\frac{d\bar{Q}}{dv} = \frac{1}{\hat{U}\left(\bar{Q},e\right)}\frac{(r+s)\,E(U \mid e) - T}{(r+s-v)^2},$$

where

$$\hat{U}\left(\bar{Q},e\right) = u_2^2 f\left(\bar{Q}u_2\right)\Pr(U=u_2 \mid e) + u_1^2 f\left(\bar{Q}u_1\right)\left(1-\Pr(U=u_2 \mid e)\right) > 0$$

when $\bar{Q} > 0$. Since $E(U \mid e_1) < E(U \mid e_2)$ and $T < (r+s)\,E(U \mid e_1)$ from Proposition **6**, we have $T < (r+s)\,E(U \mid e_1) < (r+s)\,E(U \mid e)$, which indicates that $\frac{d\bar{Q}}{dv} > 0$ for $e \in \{e_1, e_2\}$. The optimal order quantity $\bar{Q}$ is increasing in v. Similarly, we take implicit differentiation of $\bar{Q}$ w.r.t. r, s, and T respectively, and we have,

$$\begin{aligned} \frac{d\bar{Q}}{dr} &= \frac{1}{\hat{U}\left(\bar{Q},e\right)}\frac{T - vE(U \mid e)}{(r+s-v)^2} > 0, \\ \frac{d\bar{Q}}{ds} &= \frac{1}{\hat{U}\left(\bar{Q},e\right)}\frac{T - vE(U \mid e)}{(r+s-v)^2} > 0, \\ \frac{d\bar{Q}}{dT} &= -\frac{1}{\hat{U}\left(\bar{Q},e\right)}\frac{1}{r+s-v} < 0, \end{aligned}$$

since $E(U \mid e_1) < E(U \mid e_2)$, $vE(U \mid e_2) < T$ from Proposition **6** and $r+s > v$. The optimal order quantity $\bar{Q}$ is indeed increasing in r and s, but decreasing in T. Thus, we proved the proposition. ■

3.3.2 The Buyer's Incentive Mechanism Design Problem

Different from the random yield model discussed above, our model also incorporates the supplier's effort decision e and the buyer's incentive decision t_j for rewarding the supplier's yield level u_j ($j = 1, 2$). We assume that the supplier is risk-averse with an increasing and concave utility function $A(t)$ and $A(0) = 0$, i.e., $A' > 0$ and $A'' < 0$. The buyer's order quantity Q is assumed to be a positive scalar for the supplier. Thus, given any order quantity $Q > 0$, the supplier will be reluctant to accept the buyer's offer unless her expected utility is as much as what she can obtain from another, alternate, contract, i.e., her reservation utility $a_0 > 0$. If the supplier accepts the offer and exerts effort e, then this will result in a disutility level of $c(e)$, which is increasing and convex in e, i.e., $c' > 0$ and $c'' > 0$. For $e \in \{e_1, e_2\}$, we define the disutility as $c(e_1) = c_1$ and $c(e_2) = c_2$ with $c_2 - c_1 \equiv \Delta c > 0$. Therefore, the supplier's participation constraint (PC) given an effort level e is written as,

$$\Pr(U = u_2 \mid e) A(t_2) + (1 - \Pr(U = u_2 \mid e)) A(t_1) - c(e) \geq a_0. \tag{3.8}$$

If the supplier's effort cannot be observed by the buyer, then there arises an optimization problem for the supplier, i.e.,

$$\max_{e} \left[\Pr(U = u_2 \mid e) A(t_2) + (1 - \Pr(U = u_2 \mid e)) A(t_1) - c(e)\right].$$

If the buyer intends to induce the supplier's high effort level e_2, then the supplier's optimization problem can be reduced to one incentive constraint (IC) since there are only two effort choices, i.e.,

$$p_2 A(t_2) + (1 - p_2) A(t_1) - c_2 \geq p_1 A(t_2) + (1 - p_1) A(t_1) - c_1. \tag{3.9}$$

In the next two sections, we will present the buyer's optimal incentive mechanism and order quantity under both symmetric and asymmetric information cases.

3.3.3 First-Best Optimal Solution

As a benchmark, we first consider the symmetric information ("first-best") case where the supplier's effort is verifiable and can be included in a contract enforced by the court of law. We denote the buyer's problem for inducing the supplier's high effort e_2 as (FB-H) problem, which is formulated as,

$$\text{(FB-H):} \quad \begin{array}{ll} \max_{\{Q, t_1, t_2\}} & \Pi_2 = p_2 \left[b_2(Q) - t_2 Q\right] + (1 - p_2) \left[b_1(Q) - t_1 Q\right] \\ \text{s. t.} & p_2 A(t_2) + (1 - p_2) A(t_1) - c_2 \geq a_0, \end{array}$$

where the constraint is the supplier's participation constraint (PC) and $b_j(Q)$ is defined in (3.2). Indeed, since the supplier can be forced to exert any effort level, the buyer only needs to consider the supplier's PC in (3.8). If the supplier were not choosing the effort level that the buyer induced her to exert, then the supplier could be heavily punished, and the court could commit to enforcing such a punishment.

On the other hand, if the buyer lets the supplier choose the low effort level, then the buyer solves another maximization problem denoted as (FB-L) given by,

$$\text{(FB-L):} \qquad \begin{array}{ll} \max_{\{Q,t_1,t_2\}} & \Pi_1 = p_1\left[b_2(Q) - t_2 Q\right] + (1-p_1)\left[b_1(Q) - t_1 Q\right] \\ \text{s. t.} & p_1 A(t_2) + (1-p_1) A(t_1) - c_1 \geq a_0. \end{array}$$

Theorem 8 Given $Q > 0$, the first-best transfer price for inducing e_i is found as,

$$t_1^* = t_2^* \equiv t^*(e_i) = h(a_0 + c_i), \qquad \text{for } i = 1, 2, \tag{3.10}$$

where h is the inverse function of $A(\cdot)$.

Proof. Given $Q > 0$, both (FB-H) and (FB-L) problems are indeed standard first-best models in the principal-agent literature: The risk-neutral principal will accept all the risk by completely insuring the risk-averse agent a fixed wage (see, Macho-Stadler and Pérez-Castrillo [33, p. 25]). Thus, given $Q > 0$, the first-best transfer price for the low yield is the same as that for the high yield, i.e., $t_1^* = t_2^*$ for inducing any effort level. Note that when $Q = 0$, the buyer will not do any business with the supplier and the contractual relationship between the buyer and the supplier ends.

Under the (FB-H) problem, the supplier's PC is binding at optimality, i.e., $p_2 A(t_2^*) + (1-p_2) A(t_1^*) - c_2 = a_0$. Together with $t_1^* = t_2^*$, we explicitly find the first-best transfer price for inducing e_2 as

$$t_1^* = t_2^* \equiv t^*(e_2) = A^{-1}(a_0 + c_2) = h(a_0 + c_2),$$

where h is the inverse function of A with $A' > 0$ and $A'' > 0$. Similarly, under the (FB-L) problem, the supplier's PC is binding at optimality, i.e., $p_1 A(t_2^*) + (1-p_1) A(t_1^*) - c_1 = a_0$. Together with $t_1^* = t_2^*$, the first-best transfer price for inducing e_1 is found as,

$$t_1^* = t_2^* \equiv t^*(e_1) = A^{-1}(a_0 + c_1) = h(a_0 + c_1).$$

Thus, given any effort level e_i for $i = 1, 2$, the supplier is fully insured by the buyer with a fixed transfer price, i.e., $t^*(e_i) = h(a_0 + c_i)$. Moreover, the transfer price $t^*(e_i)$ doesn't depend on the buyer's order quantity Q for $i = 1, 2$. The theorem is proved. ■

Next, replacing T in (3.7) with $t^*(e_i)$, we find the first-best optimal order quantity Q_i^* for inducing e_i as,

$$\begin{aligned} & u_2 F\left(Q_i^* u_2\right) \operatorname{Pr}\left(U=u_2 \mid e_i\right)+u_1 F\left(Q_i^* u_1\right)\left(1-\operatorname{Pr}\left(U=u_2 \mid e_i\right)\right) \\ = & \frac{(r+s) E\left(U \mid e_i\right)-t^*\left(e_i\right)}{r+s-v}, \end{aligned} \tag{3.11}$$

for $i=1,2$. Proposition 6 suggests that Q_i^* is finite and positive under the condition that $vE\left(U \mid e_i\right)<t^*(e_i)<(r+s) E\left(U \mid e_i\right)$, for $i=1,2$. Substituting Q_i^* into $\Pi_i\left(Q_i\right)$ for $i=1,2$, we write the buyer's expected profit for inducing e_1 and e_2 respectively as,

$$\Pi_1\left(Q_1^*\right)=p_1 b_2\left(Q_1^*\right)+\left(1-p_1\right) b_1\left(Q_1^*\right)-h\left(a_0+c_1\right) Q_1^*, \tag{3.12}$$

and

$$\Pi_2\left(Q_2^*\right)=p_2 b_2\left(Q_2^*\right)+\left(1-p_2\right) b_1\left(Q_2^*\right)-h\left(a_0+c_2\right) Q_2^*. \tag{3.13}$$

Remark 7 Under the first-best case, it is optimal for the buyer to induce the supplier's high effort level when $\Pi_2\left(Q_2^*\right) \geq \Pi_1\left(Q_1^*\right)$, i.e.,

$$\begin{aligned} & p_2 b_2\left(Q_2^*\right)+\left(1-p_2\right) b_1\left(Q_2^*\right)-\left[p_1 b_2\left(Q_1^*\right)+\left(1-p_1\right) b_1\left(Q_1^*\right)\right] \\ \geq\; & h\left(a_0+c_2\right) Q_2^*-h\left(a_0+c_1\right) Q_1^*, \end{aligned} \tag{3.14}$$

where $b_j\left(Q\right)$ for $j=1,2$ is defined in (3.2), Q_1^* and Q_2^* are solved through (3.11) for $i=1,2$, respectively. ◀

Below we provide a numerical example to illustrate the solution to the buyer's problem under the first-best case.

Example 7 Assume that the risk-averse supplier's utility equals the square root of the transfer received, i.e., $A(t)=\sqrt{t}$, and her disutility of effort takes a quadratic term to capture the decreasing marginal returns of effort, i.e., $c(e)=e^2$. Given $Q>0$, solving (3.10) for $i=1,2$ gives the first-best transfer price for inducing e_1 and e_2 respectively as

$$t^*\left(e_1\right)=\left(a_0+e_1^2\right)^2, \qquad t^*\left(e_2\right)=\left(a_0+e_2^2\right)^2,$$

which indicates that inducing the high effort level is more costly than inducing the low effort level under the first-best case. We assume that the random variable X follows an Erlang distribution with parameters $\alpha>0$ and $\beta>0$, i.e.,

$$f\left(x\right)=\frac{\alpha^\beta x^{\beta-1} \exp\left(-\alpha x\right)}{(\beta-1)!}, \qquad x \geq 0, \qquad E\left(X\right)=\beta / \alpha, \qquad Var\left(X\right)=\beta / \alpha^2,$$

where α is called rate parameter with $\alpha>0$ and β is the shape parameter, which

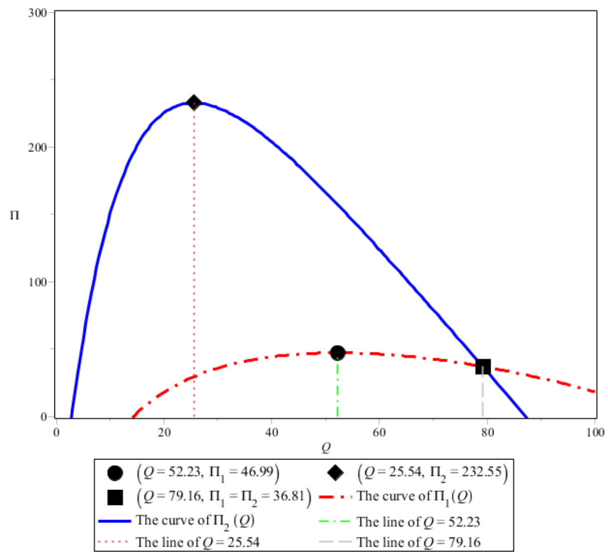

Figure 12. The buyer's expected profit under the first-best case.

takes positive integers. The following values for the parameters are used: $\alpha = 0.1$; $\beta = 1$ $(E(X) = 10$ units$)$; $e_1 = 0$; $e_2 = 1$; $a_0 = 2$; $u_1 = 0.1$; $u_2 = 0.9$; $p_1 = 0.2$, $E(U \mid e_1) = 0.26$; $p_2 = 0.9$, $E(U \mid e_1) = 0.82$; $r = \$50$; $v = \$5$; $s = \$10$. Then we have

$$t^*(e_1) = \$4, \qquad t^*(e_2) = \$9.$$

Solving (3.11) for $i = 1, 2$ with these parameter values and substituting the results in (3.12) and (3.13) gives

$$Q_1^* = 52.23 \text{ units}, \qquad Q_2^* = 25.54 \text{ units}, \qquad \Pi_1(Q_1^*) = \$46.99, \qquad \Pi_2(Q_2^*) = \$232.55.$$

Obviously we have $\Pi_2(Q_2^*) > \Pi_1(Q_1^*)$, i.e., it's optimal for the buyer to induce the high effort level e_2. As shown in Figure 12, the curve of $\Pi_2(Q)$ is much higher than the curve of $\Pi_1(Q)$ before the interaction $(Q = 79.16, \Pi_1 = \Pi_2 = 36.81)$. However, if the buyer needs to place a large order, i.e., $Q > 79.16$ units, then the buyer is not willing to induce the supplier's high effort level since $\Pi_2(Q) < \Pi_1(Q)$ for $Q > 79.16$ units. ♦

3.3.4 Second-Best Optimal Solution

In this section, we assume it's optimal for the buyer to induce the supplier's high effort level under the first-best case, i.e., the condition (3.14) holds. Then the buyer has motivation to induce the supplier's effort when it becomes unobservable. In this case, the buyer can only design a contract based on the supplier's observable yield outcomes to induce the supplier's high effort level at a reasonable cost. Such a contract must be an incentive feasible contract (Laffont and Martimort [28, p.147]), which satisfies both a participation constraint (3.8) and an incentive constraint (3.9), for inducing the high effort level. We denote the buyer's problem under the second-best case as (SB-H), which is written as,

$$\text{(SB-H):}\quad \begin{array}{ll} \max_{\{Q,t_1,t_2\}} & \Pi_2 = p_2\left[b_2(Q) - t_2 Q\right] + (1-p_2)\left[b_1(Q) - t_1 Q\right] \\ \text{s. t.} & p_2 A(t_2) + (1-p_2) A(t_1) - c_2 \geq a_0, \\ & p_2 A(t_2) + (1-p_2) A(t_1) - c_2 \geq p_1 A(t_2) + (1-p_1) A(t_1) - c_1, \end{array}$$

where the first constraint is the supplier's PC and the second one is the IC. Given $Q > 0$, it's not obvious that (SB-H) is a concave program w.r.t. t_1 and t_2 since the concave function $A(t)$ appears on both sides of the IC. To ensure the concavity of the program, we change the variables: define $A_1 = A(t_1)$ and $A_2 = A(t_2)$, or equivalently let $t_1 = h(A_1)$ and $t_2 = h(A_2)$, where h is the inverse function of A. Then the PC becomes,

$$p_2 A_2 + (1-p_2) A_1 - c_2 \geq a_0, \tag{3.15}$$

and the IC becomes,

$$p_2 A_2 + (1-p_2) A_1 - c_2 \geq p_1 A_2 + (1-p_1) A_1 - c_1. \tag{3.16}$$

The buyer's (SB-H) problem can now be replaced by the new program (SB-H)′ as,

$$\text{(SB-H)}': \quad \begin{array}{ll} \max_{\{Q,A_1,A_2\}} & \Pi_2 = p_2\left[b_2(Q) - h(A_2) Q\right] + (1-p_2)\left[b_1(Q) - h(A_1) Q\right] \\ \text{s. t.} & (3.15) \text{ and } (3.16). \end{array}$$

It turns out that the new program (SB-H)′ given $Q > 0$ is indeed a concave program: the objective function is strictly concave in A_1 and A_2 since h is strictly convex in A_1 and A_2; the constraints are now linear and the interior of the constrained set is nonempty. Therefore, the Karush–Kuhn–Tucker conditions are sufficient and necessary to characterize the program's optimality (Bazaraa, Sherali and Shetty [4, Ch. 4]).

Theorem 9 Given $Q > 0$, the second-best transfer price $\hat{t}_1$ and $\hat{t}_2$ are found as,

$$\hat{t}_1 = h\left(a_0 + c_2 - \frac{p_2}{\Delta p}\Delta c\right), \qquad \hat{t}_2 = h\left(a_0 + c_2 + \frac{(1-p_2)}{\Delta p}\Delta c\right), \tag{3.17}$$

where $\hat{t}_1 < \hat{t}_2$.

Proof. Given $Q > 0$, associate a multiplier $\lambda \geq 0$ with (3.15) and $\mu \geq 0$ with

(3.16), and write the Lagrangian function as,

$$\begin{aligned}\mathcal{L}\left(A_1, A_2, \lambda, \mu\right) &= p_2\left[b_2(Q) - h(A_2)Q\right] + \left(1-p_2\right)\left[b_1\left(Q\right) - h\left(A_1\right)Q\right] \\ &\quad -\lambda\left[a_0 + c_2 - p_2A_2 - \left(1-p_2\right)A_1\right] \\ &\quad -\mu\left[p_1A_2 + \left(1-p_1\right)A_1 - c_1 + c_2 - p_2A_2 - \left(1-p_2\right)A_1\right].\end{aligned}$$

Optimizing w.r.t. A_1 and A_2 respectively, we get the following first-order conditions,

$$\begin{aligned}-\left(1-p_2\right)h'(\hat{A}_1)Q + \lambda\left(1-p_2\right) - \mu\left(p_2-p_1\right) &= 0, \\ -\left(1-p_2\right)h'(\hat{A}_2)Q + \lambda\left(1-p_2\right) - \mu\Delta p &= 0,\end{aligned}$$

where $\hat{A}_1$ and $\hat{A}_2$ are denoted as the second-best utility for receiving the second-best transfer price $\hat{t}_1$ and $\hat{t}_2$, respectively. Simplifying gives

$$h'(\hat{A}_1) = \left[\lambda - \mu\frac{\Delta p}{\left(1-p_2\right)}\right]\frac{1}{Q}, \qquad h'(\hat{A}_2) = \left[\lambda + \mu\frac{\Delta p}{p_2}\right]\frac{1}{Q}.$$

Solving the above two equations for λ and μ gives,

$$\lambda = [\left(1-p_2\right)h'(\hat{A}_1) + p_2h'(\hat{A}_2)]Q, \tag{3.18}$$

$$\mu = \frac{p_2\left(1-p_2\right)}{\Delta p}[h'(\hat{A}_2) - h'(\hat{A}_1)]Q, \tag{3.19}$$

where $\lambda > 0$ since $Q > 0$, $p_2 < 1$, $h'(\hat{A}_1) > 0$ and $h'(\hat{A}_2) > 0$. The IC in (3.16) implies that $(\hat{A}_2 - \hat{A}_1) \geq \left(c_2 - c_1\right)/\Delta p > 0$ since $c_2 > c_1$ and thus $\hat{A}_2 > \hat{A}_1$. The inverse function h is an increasing convex function, i.e., $h'(\hat{A}_2) > h'(\hat{A}_1)$, which indicates that the right-hand side of (3.19) is strictly positive and thus $\mu > 0$. Therefore, both PC and IC are binding given $Q > 0$. Solving the binding PC and IC gives,

$$\hat{A}_1 = a_0 + c_2 - \frac{p_2}{\Delta p}\Delta c, \qquad \hat{A}_2 = a_0 + c_2 + \frac{\left(1-p_2\right)}{\Delta p}\Delta c,$$

where $\hat{A}_1 < \hat{A}_2$ for $0 < p_2 < 1$. Since h is the inverse function of A, the second-best transfer price $\hat{t}_1$ and $\hat{t}_2$ are computed as,

$$\begin{aligned}\hat{t}_1 &= h\left(a_0 + c_2 - \frac{p_2}{\Delta p}\Delta c\right), \\ \hat{t}_2 &= h\left(a_0 + c_2 + \frac{\left(1-p_2\right)}{\Delta p}\Delta c\right),\end{aligned}$$

where $\hat{t}_1 < \hat{t}_2$ since h is an increasing function of A. Therefore, we proved the theorem. ■

Remark 8 The second-best transfer prices $\hat{t}_1$ and $\hat{t}_2$ are independent of the order quantity Q, but depends on the supplier's reservation utility, the disutility of high effort level, and the yield distribution. Moreover, the second-best transfer price for the high yield level is higher than that for the low yield level, i.e., $\hat{t}_2 > \hat{t}_1$. ◀

Replacing T in (3.7) with $p_2\hat{t}_2 + (1-p_2)\,\hat{t}_1$, we find the second-best optimal order quantity $\hat{Q}$ for inducing e_2 as,

$$p_2 u_2 F(\hat{Q}u_2) + (1-p_2)\,u_1 F(\hat{Q}u_1) = \frac{(r+s)\,E\,(U \mid e_2) - [p_2\hat{t}_2 + (1-p_2)\,\hat{t}_1]}{r+s-v}. \quad (3.20)$$

Proposition 6 suggests that $\hat{Q}$ is finite and positive under the condition that

$$vE\,(U \mid e_2) < p_2\hat{t}_2 + (1-p_2)\,\hat{t}_1 < (r+s)\,E\,(U \mid e_2)\,.$$

Proposition 10 When moral hazard issue arises, inducing the supplier's high effort level costs the buyer a higher expected transfer price, i.e., $p_2\hat{t}_2 + (1-p_2)\,\hat{t}_1 > t^*\,(e_2)$, however, the buyer should order fewer units of the critical component, i.e., $\hat{Q} < Q_2^*$.

Proof. The participation constraint is binding under both first-best and second-best case when inducing the high effort level, which gives,

$$A(t^*\,(e_2)) = c_2 + a_0 = p_2 A(\hat{t}_2) + (1-p_2)\,A(\hat{t}_1) < A(p_2\hat{t}_2 + (1-p_2)\,\hat{t}_1),$$

following from Jensen's inequality. Thus we have $p_2\hat{t}_2 + (1-p_2)\,\hat{t}_1 > t^*\,(e_2)$ since $A\,(t)$ is increasing in t. Moreover, Proposition 7 shows that given an effort level, the optimal order quantity is decreasing in the expected transfer price T, then we have $\hat{Q} < Q_2^*$ where $\hat{Q}$ is given by (3.20) and Q_2^* is given by (3.11) for $i = 2$. Hence, we proved the proposition. ■

The buyer's second-best expected profit for inducing e_2 is then written as,

$$\Pi_2(\hat{Q}) = p_2 b_2(\hat{Q}) + (1-p_2)\,b_1(\hat{Q}) - \left[p_2\hat{t}_2 + (1-p_2)\,\hat{t}_1\right]\hat{Q}, \quad (3.21)$$

where $\hat{Q}$ is given by 3.20 and $\hat{t}_1$ and $\hat{t}_2$ are given by (3.17).

Next, we consider the model with moral hazard and minimum liability issues. As we have discussed in Remark 6, the expected transfer price is actually the buyer's unit purchasing cost, which is usually assumed to be greater than the salvage value. To guarantee that the buyer will place a positive order even when the supplier exerts the low effort level, we now assume that the transfer price must be at least the salvage value, i.e., $t_1 \geq v$ and $t_2 \geq v$. Therefore, we impose the minimum liability and denote the buyer's problem under both moral hazard and minimum liability as (SB-ML), which is given by,

$$
\text{(SB-ML):}\quad
\begin{array}{ll}
\max_{\{Q,t_1,t_2\}} & \Pi_2 = p_2\left[b_2(Q) - t_2 Q\right] + (1-p_2)\left[b_1(Q) - t_1 Q\right] \\
\text{s. t.} & p_2 A(t_2) + (1-p_2) A(t_1) - c_2 \geq a_0, \\
 & p_2 A(t_2) + (1-p_2) A(t_1) - c_2 \geq p_1 A(t_2) + (1-p_1) A(t_1) - c_1, \\
 & t_1 \geq v, \\
 & t_2 \geq v,
\end{array}
$$

where the first constraint is the supplier's PC, the second one is the IC associated with e_1, and the remaining are the minimum liability constraints on t_1 and t_2, respectively. Given $Q > 0$, we provide the solution to the (SB-ML) problem in the following theorem.

Theorem 11 With minimum liability imposed, the second-best transfer price for inducing the supplier's high effort level e_2 entails:

- For $0 < v < h(a_0 + c_2 - p_2\Delta c/\Delta p)$, only PC and IC are binding. Then $\hat{t}_1$ and $\hat{t}_2$ are given by (3.17).
- For $v \geq h(a_0 + c_2 - p_2\Delta c/\Delta p)$, then PC and $t_1 \geq v$ are binding. The solution is then given by

$$
\hat{t}_1 = v, \tag{3.22}
$$

$$
\hat{t}_2 = h\left(A(v) + \frac{\Delta c}{\Delta p}\right). \tag{3.23}
$$

Proof. The buyer's program (SB-ML) is replaced by a new program (SB-ML)$'$ as,

$$
\text{(SB-ML)}':\quad
\begin{array}{ll}
\max_{\{Q,A_1,A_2\}} & \Pi_2 = p_2\left[b_2(Q) - h(A_2) Q\right] + (1-p_2)\left[b_1(Q) - h(A_1) Q\right] \\
\text{s. t.} & (3.15),\ (3.16),\ h(A_1) \geq v \text{ and } h(A_2) \geq v.
\end{array}
$$

where $A_1 = A(t_1)$, $A_2 = A(t_2)$, $t_1 = h(A_1)$ and $t_2 = h(A_2)$. From (3.16), we have $A_2 - A_1 \geq (c_2 - c_1)/\Delta p > 0$ since $c_2 > c_1$ and thus $A_2 > A_1$. The inverse function h is an increasing convex function, so that we must have $h(A_2) \geq v$ if $h(A_1) \geq v$ is satisfied. Therefore, we can drop $h(A_2) \geq v$ from the (SB-ML)$'$ program. Moreover, the supplier's participation constraint (3.15) is always binding to ensure that the supplier accepts the buyer's offer, which leaves either (3.16) or $h(A_1) \geq v$ is binding (otherwise, no moral hazard problem exists). Thus, we discuss the following two cases: (i) when (3.15) and (3.16) are binding, then the solution to the program (SB-ML)$'$ are the same as those without minimum liability, if and only if $\hat{t}_1 > v$, i.e., $h(a_0 + c_2 - p_2\Delta c/\Delta p) > v$, and (ii) when $v \geq h(a_0 + c_2 - p_2\Delta c/\Delta p)$, then (3.15) and

$h(A_1) \geq v$ are binding, and the solution are obtained as, $\hat{t}_1 = v$ and

$$\begin{aligned} \hat{t}_1 &= v, \\ \hat{t}_2 &= h\left(\frac{a_0 + c_2 - (1-p_2)\,A(v)}{p_2}\right). \end{aligned}$$

Thus, we proved the theorem. ■

Remark 9 When the salvage value v becomes larger, the buyer should schedule a better transfer price menu, i.e., the second-best transfer prices for both high and low yields should be higher when minimum liability take effects. ◀

Next we provide a numerical example to illustrate the buyer's optimal solution under moral hazard and minimum liability issues.

Example 8 We keep the same assumptions and parameter values as those in Example 7. The second-best solution for inducing the supplier's high effort level entails: (i) No minimum liability imposed, then solving (3.17) gives the second-best transfer price for u_1 and u_2, respectively, as,

$$\hat{t}_1 = \$2.94 \qquad \hat{t}_2 = \$9.88.$$

Substituting $\hat{t}_1$ and $\hat{t}_2$ into (3.21), we calculate the buyer's second-best optimal order quantity and expected profit as,

$$\hat{Q} = 25.09 \text{ units} \qquad \Pi_2(\hat{Q}) = \$227.90.$$

(ii) With the minimum liability imposed as $t_1 \geq 5$ and $t_2 \geq 5$, we find the salvage value is larger than the threshold value, i.e., $v > h(a_0 + c_2 - p_2\Delta c/\Delta p)$. Then PC and $t_1 \geq 5$ are binding. The second-best transfers are then found as

$$\hat{t}_1 = \$5 \qquad \hat{t}_2 = \$13.43$$

by solving (3.22) and (3.23). Substituting into (3.21), the buyer's second-best order quantity and expected profit are calculated as

$$\hat{Q} = 19.04 \text{ units} \qquad \Pi_2(\hat{Q}) = \$153.74. \qquad \blacklozenge$$

For the above example, we perform a sensitivity analysis of the salvage value v on the second-best order quantity $\hat{Q}$, and put the result in Figure 13. As shown in Figure 13, without minimum liability constraints, the second-best order quantity $\hat{Q}$ is increasing in v, which indicates that the buyer should order more units as the salvage value increases. However, with the minimum liability $t_1 \geq v$ and $t_2 \geq v$ imposed, the buyer should order fewer units when v is larger than a threshold value, i.e., $v > h(a_0 + c_2 - p_2\Delta c/\Delta p) = \$2.94$. Such result implies that the buyer has to sacrifice the benefits brought from the increasing salvage value and provide more incentives to the supplier, when it's worthwhile for the buyer to induce the supplier's high effort level.

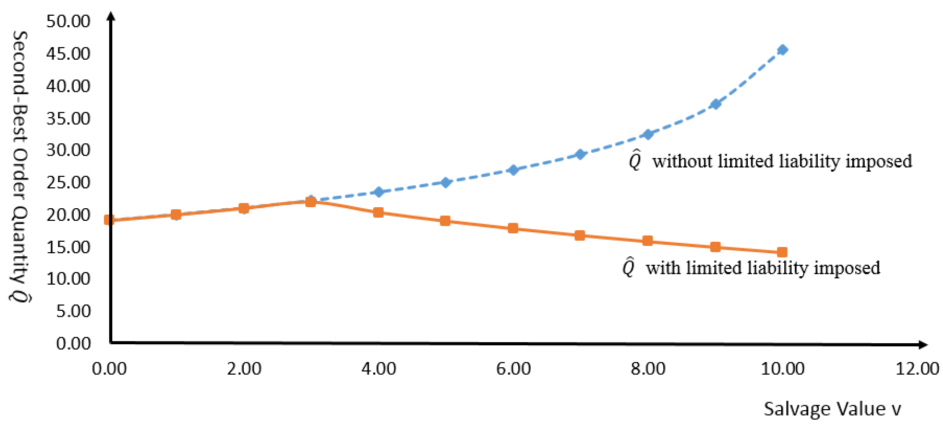

Figure 13. The buyer's second-best order quantity affected by the salvage value v.

3.4 General Discrete Effort Choices and Discrete Yield Outcomes

In this section, we consider a general model where the supplier can exert more than two effort levels, i.e., $e_i \in \{e_1, \ldots, e_m\}$ for $m \geq 2$, where $e_1 < e_2 < \cdots < e_m$, and more than two possible yield levels can happen, i.e., $u_j \in \{u_1, \ldots, u_n\}$ for $n \geq 2$, where $u_1 < u_2 < \cdots < u_n$. For a fixed value of the effort e_i ($i = 1, \ldots, m$), we define $p_{ij} = \Pr(U = u_j \mid e = e_i)$ as the probability that the outcome will be u_j ($j = 1, \ldots, n$). We also write $t_j \equiv t(u_j)$ as the unit transfer price to the agent if u_j is observed and define $\mathbf{t} = (t_1, \ldots, t_n)$. In this general model, if the supplier accepts the offer and chooses the effort level e_i, then her participation constraint (PC) is written as,

$$\sum_{j=1}^{n} A(t_j)p_{ij} - c(e_i) \geq a_0, \tag{3.24}$$

and the $m-1$ incentive constraints (ICs)

$$\sum_{j=1}^{n} A(t_j)p_{ij} - c(e_i) \geq \sum_{j=1}^{n} A(t_j)p_{kj} - c(e_k) \tag{3.25}$$

must hold for $k = 1, \ldots, m$ and $k \neq i$. For a fixed e_i, the buyer's expected profit is given as,

$$\Pi_i(Q, \mathbf{t}) = \sum_{j=1}^{n} [b_j(Q) - t_j Q] p_{ij}, \tag{3.26}$$

where $b_j(Q) = \int_0^{Qu_j} [(r-v)x + vQu_j] f(x)\ dx + \int_{Qu_j}^{\infty} [(r+s)Qu_j - sx] f(x)\ dx$ for $j = 1, \ldots, n$. Finally, we denote the buyer's problem for inducing the supplier's effort e_i under moral hazard as (SB-e_i), which is given as,

$$\text{(SB-}e_i\text{):} \qquad \begin{array}{ll} \max_{\{Q,\mathbf{t}\}} & \Pi_i(Q,\mathbf{t}) = \sum_{j=1}^{n} [b_j(Q) - t_j Q] p_{ij} \\ \text{s. t.} & \text{(3.24) and (3.25).} \end{array}$$

As in Section 3.3.4, we first change the variables in the way of $A_j = A(t_j)$ or equivalently $t_j = h(A_j)$ for $j = 1, \ldots, n$. Then the supplier's PC in (3.24) is changed to

$$\sum_{j=1}^{n} A_j p_{ij} - c(e_i) \geq a_0, \tag{3.27}$$

and the $m-1$ ICs in (3.25) are changed to

$$\sum_{j=1}^{n} A_j p_{ij} - c(e_i) \geq \sum_{j=1}^{n} A_j p_{kj} - c(e_k), \tag{3.28}$$

for $k = 1, \ldots, m$ and $k \neq i$. Thus, the buyer's (SB-e_i) problem is transformed to the (SB-e_i)$'$ problem given by

$$\text{(SB-}e_i\text{)}': \qquad \begin{array}{ll} \max_{\{Q,\mathbf{A}\}} & \Pi_i(Q,\mathbf{A}) = \sum_{j=1}^{n} [b_j(Q) - h(A_j)\,Q] p_{ij} \\ \text{s. t.} & \text{(3.27) and (3.28),} \end{array} \tag{3.29}$$

where $\mathbf{A} \equiv \{A_1, A_2, \ldots A_n\}$, $j = 1, \ldots, n$, $k = 1, \ldots, m$ and $k \neq i$. Given $Q > 0$, we find that the buyer's new program (SB-e_i)$'$ is indeed a concave program w.r.t. $\mathbf{A}$. Thus, the Karush–Kuhn–Tucker conditions are sufficient and necessary to characterize the program's optimality.

Given $Q > 0$, for a fixed effort level e_i, associate the multiplier $\lambda \geq 0$ with PC in (3.27), and the multipliers $\mu_k \geq 0$ ($k = 1, \ldots, m$, $k \neq i$) with the $m-1$ ICs in

(3.28) and then form the Lagrangian for the buyer's problem (SB-e_i)$'$ in (3.29) as,

$$\begin{aligned}\mathcal{L}\left(\mathbf{A},\lambda;\mu_1,\mu_2,\ldots,\mu_m\right) &= \sum_{j=1}^{n}[b_j(Q)-h\left(A_j\right)Q]p_{ij}-\lambda\left(a_0+c(e_i)-\sum_{j=1}^{n}A_jp_{ij}\right)\\ &\quad -\sum_{k=1;k\neq i}^{m}\mu_k\left(\sum_{j=1}^{n}A_jp_{kj}-c(e_k)+c(e_i)-\sum_{j=1}^{n}A_jp_{ij}\right).\end{aligned}$$

Differentiating the above Lagrangian function w.r.t. A_j and regrouping terms gives,

$$h'(A_j)=\left[\lambda+\sum_{k=1;k\neq i}^{m}\mu_k\left(1-\frac{p_{kj}}{p_{ij}}\right)\right]\frac{1}{Q}, \tag{3.30}$$

for $j=1,\ldots,n$. To guarantee the transfer t_j to be nondecreasing in the yield u_j, or equivalently, A_j is nondecreasing in u_j, we assume that the yield probabilities p_{kj}/p_{ij} satisfy the monotone likelihood ratio property (MLRP), see Laffont and Martimort [28, p. 164] and Salanié [40, p. 127]. That is, $1-p_{kj}/p_{ij}$ is nondecreasing in j for $j=1,\ldots,n$. Under the MLRP assumption, we find both first-best and second-best transfer prices in the following theorem.

Theorem 12 When the supplier's effort can be observed, then only PC (3.27) is binding, which gives the first-best transfer prices as $t^*\left(e_i\right)=h\left(a_0+c\left(e_i\right)\right)$. When the supplier's effort cannot be observed, then PC and at least one of ICs must be binding, which gives the second-best transfer prices $\hat{t}_1\leq\hat{t}_2\leq\cdots\leq\hat{t}_n$ and $\hat{\mathbf{t}}=\left\{\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_n\right\}$ is independent of the order quantity Q.

Proof. The function h is the inverse function of A with $h'>0$ and $h''>0$. Since $h'>0$, the left-hand side of (3.30) must be positive. To be equal, the right-hand side of (3.30) must also be positive given $Q>0$. That is, we must have $\lambda>0$ to guarantee that the supplier accepts the offer, which indicates that PC must be binding. Under the first-best case where the supplier's effort can be observed, no incentive constraints are considered, i.e., all $\mu_k=0$ for $k=1,\ldots,m$ and $k\neq i$. The equation (3.30) thus becomes

$$\lambda=h'(A_1^*)Q=h'(A_2^*)Q=\cdots=h'(A_n^*)Q,$$

where $\lambda>0$ given $Q>0$ and thus $A_1^*=A_2^*=\cdots=A_n^*>0$. Note that when $Q=0$, the buyer will not do any business with the supplier and the contractual relationship between the buyer and the supplier ends. Together with the supplier's binding PC, we explicitly find the first-best utility as $A_1^*=A_2^*=\cdots=A_n^*=a_0+c\left(e_i\right)$ and thus the risk-averse supplier receives a fixed transfer price as

$$t_1^* = t_2^* = \cdots = t_n^* \equiv t^*(e_i) = A^{-1}\left(a_0 + c\left(e_i\right)\right) = h\left(a_0 + c\left(e_i\right)\right),$$

Therefore, the supplier is fully insured by the buyer.

Under the second-best case where the supplier's effort cannot be observed, at least one of the μ_k in (3.30) must be positive (otherwise, no moral hazard problem exists). We have assumed that the supplier's yield probabilities satisfy the monotone likelihood ratio property, i.e., $1 - p_{kj}/p_{ij}$ is nondecreasing in j for $j = 1, \ldots, n$. Then the right-hand of (3.30) is nondecreasing in j. To be equal, the left-hand side of (3.30) has to be also nondecreasing in j. Since h is an increasing convex function with $h' > 0$ and $h'' > 0$, the nondecreasing of $h'\left(A_j\right)$ in j implies that A_j has to be nondecreasing in j, or equivalently t_j is nondecreasing in j, for $j = 1, \ldots, n$. Then we have $0 \le \hat{A}_1 \le \hat{A}_2 \le \cdots \le \hat{A}_n$, or equivalently, $0 \le \hat{t}_1 \le \hat{t}_2 \le \cdots \le \hat{t}_n$ at optimality. Since PC and at least one of ICs are binding at optimality, then the following equations,

$$\sum_{j=1}^{n} \hat{A}_j p_{ij} = a_0 + c(e_i),$$

and

$$\sum_{j=1}^{n} \hat{A}_j p_{ij} - \sum_{j=1}^{n} \hat{A}_j p_{kj} = c(e_i) - c(e_k),$$

must hold for some k and $k \neq i$. Note that these two equations don't involve the order quantity Q, which indicates that the second-best utility $\hat{\mathbf{A}}$, or equivalently $\hat{\mathbf{t}}$ that satisfy these equations must be independent of the order quantity Q. Therefore, we proved the theorem. ■

3.4.1 Large-Scale Nonlinear Program

Theorem 12 shows that PC must be binding and one of the ICs must be binding at optimality. However, we don't know which of ICs is/are binding and thus it is difficult to obtain the closed-form solution for the second-best transfer prices. Moreover, the buyer's problem in (3.29) becomes a large-scale nonlinear program with $n+1$ decision variables and m constraints, when m and n are large numbers, for example, $m = 200$ and $n = 100$. For such problem, we can use Lagrangian duality to find the optimal solution.

The original nonlinear programming problem is called the primal problem, which is associated with another nonlinear programming problem called the Lagrangian dual problem. The primal and dual problem will have equal optimal objective values under certain convexity assumptions and suitable constraint qualification, see Bazaraa, Sherali and Shetty [4, Ch. 6]. Thus, instead of solving the primal problem with m complicated constraints, we obtain the optimal solution by solving the dual problem which only has nonnegative constraints. The method of Lagrangian duality is used by many researchers such as Parlar and Vickson [36] who solved a discrete-time optimal control problem by converting it to a large-scale nonlinear program. Note that given $Q > 0$, the buyer's (SB-e_i)$'$ program can be turned into a convex optimization program with respect to $\mathbf{A}$ with (3.27) and (3.28) all linear constraints. Thus, we use the following steps to form the Lagrangian for the buyer's minimization problem with respect to $\mathbf{A}$.

(1) Given $Q > 0$ and e_i, simplify ICs in (3.28) to

$$a_0 + c(e_k) - \sum_{j=1}^{n} A_j p_{kj} \geq 0$$

for $k = 1, \ldots, m$ and $k \neq i$, since PC in (3.27) is binding at optimality (see Theorem 12).

(2) Let

$$G^1(\mathbf{A}) = a_0 + c(e_i) - \sum_{j=1}^{n} A_j p_{ij}, \qquad G^2(\mathbf{A}) = a_0 + c(e_k) - \sum_{j=1}^{n} A_j p_{kj}$$

where $G^1(\mathbf{A}) = 0$ is associated with the binding PC and $G^2(\mathbf{A}) \geq 0$ is associated with the $m-1$ ICs, for $k = 1, \ldots, m$, $k \neq i$, and $j = 1, \ldots, n$.

(3) Convert the buyer's maximization problem to a minimization problem, i.e., given $Q > 0$, the buyer's (SB-e_i)$'$ problem in (3.29) is equivalent to

$$\begin{array}{ll} \min_{\mathbf{A}} & \sum_{j=1}^{n} p_{ij} h(A_j) Q \\ \text{s. t.} & G^1(\mathbf{A}) = 0 \text{ and } G^2(\mathbf{A}) \geq 0. \end{array}$$

(4) Let $Q = 1$, the solution of $\mathbf{A}$ will stay the same since it's independent of Q at optimality (see Theorem 12). Then we define the buyer's problem given $Q = 1$ as,

$$\begin{array}{ll} \min_{\mathbf{A}} & J(\mathbf{A}) = \sum_{j=1}^{n} p_{ij} h(A_j) \\ \text{s. t.} & G^1(\mathbf{A}) = 0 \text{ and } G^2(\mathbf{A}) \geq 0. \end{array} \tag{3.31}$$

(5) Associate the multiplier γ with $G^1(\mathbf{A}) = 0$ and the multipliers $\delta_k \geq 0$ ($k = 1, \ldots, m$, $k \neq i$) with $G^2(\mathbf{A}) \geq 0$ and form the Lagrangian for the buyer's problem in (3.31) as,

$$\mathcal{L}(\mathbf{A}, \gamma, \mathbf{\Psi}) = J(\mathbf{A}) + \gamma G^1(\mathbf{A}) - \mathbf{\Psi}^{\mathbf{T}} G^2(\mathbf{A}), \tag{3.32}$$

where $\mathbf{\Psi} = (\delta_1, \delta_2, \ldots, \delta_k, \ldots, \delta_m)$ for $k = 1, \ldots, m$, $k \neq i$.

Then we use the following theorem to obtain the necessary conditions for optimality, see Bazaraa, Sherali and Shetty [4, pp. 75–77].

Theorem 13 (Kuhn-Tucker) If $\hat{\mathbf{A}}$ minimizes $J(\mathbf{A})$ such that $G^1(\mathbf{A}) = 0$ and $G^2(\mathbf{A}) \geq 0$, it's necessary that there exists some $\hat{\gamma}$ and $\hat{\mathbf{\Psi}}$, so that the followings conditions are satisfied:

$$
\begin{aligned}
\mathcal{L}_{\hat{\mathbf{A}}} &= 0 \Longrightarrow \mathcal{L}_{\hat{\mathbf{A}}_j} = 0, && (j = 1, 2, \ldots, n) \\
\mathcal{L}_{\hat{\gamma}} &= 0, && \\
\mathcal{L}_{\hat{\mathbf{\Psi}}} &\leq 0 \Longrightarrow \mathcal{L}_{\hat{\delta}_k} \leq 0, && (k = 1, \ldots, m, k \neq i) \\
(\mathcal{L}_{\hat{\mathbf{\Psi}}})^T \hat{\mathbf{\Psi}} &= 0 \Longrightarrow (\mathcal{L}_{\hat{\delta}_k})^T \hat{\delta}_k = 0, && (k = 1, \ldots, m, k \neq i) \\
\hat{\mathbf{\Psi}} &\geq 0 \Longrightarrow \hat{\delta}_k \geq 0, && (k = 1, \ldots, m, k \neq i).
\end{aligned}
$$

Applying the necessary conditions, we get

$$
\begin{aligned}
p_{ij} h'(\hat{A}_j) - \hat{\gamma} p_{ij} + \sum_{k=1; k \neq i}^{m} \hat{\delta}_k p_{kj} &= 0, \qquad (j = 1, 2, \ldots, n) && (3.33) \\
a_0 + c(e_i) - \sum_{j=1}^{n} \hat{A}_j p_{ij} &= 0, && (3.34) \\
a_0 + c(e_k) - \sum_{j=1}^{n} \hat{A}_j p_{kj} &\geq 0, \qquad (k = 1, \ldots, m, k \neq i) && \\
\hat{\delta}_k \left[a_0 + c(e_k) - \sum_{j=1}^{n} \hat{A}_j p_{kj} \right] &= 0, \qquad (k = 1, \ldots, m, k \neq i) && \\
\hat{\delta}_k &\geq 0, \qquad (k = 1, \ldots, m, k \neq i). &&
\end{aligned}
$$

The necessary conditions for the optimality of a solution to the transformed problem are also sufficient because of the convexity of the objective function in (3.31).

Theorem 14 (Duality Theorem) The point of $(\hat{\mathbf{A}}, \hat{\gamma}, \hat{\mathbf{\Psi}})$ is a saddle point of the Lagrangian $\mathcal{L}(\mathbf{A}, \gamma, \mathbf{\Psi})$ defined by (3.32) if and only if:

(1) $(\hat{\mathbf{A}}, \hat{\gamma})$ solves the primal problem: $\min \left\{ \bar{J}(A) = J(A) + \gamma G^1(A), \text{ such that } G^2(A) \geq 0 \right\}$.

(2) $\hat{\mathbf{\Psi}}$ solves the dual problem: $\max_{\mathbf{\Psi}\in D}$, $D = \{\mathbf{\Psi}|\omega(\mathbf{\Psi})$ exists, $\mathbf{\Psi} \geq \mathbf{0}\}$, where $\omega(\mathbf{\Psi}) = \min_{\mathbf{A},\gamma} \mathcal{L}(\mathbf{A},\gamma,\mathbf{\Psi})$ is the dual function.

(3) $\bar{J}(\hat{\mathbf{A}}) = \omega(\hat{\mathbf{\Psi}})$.

The next theorem gives insight into the nature of the dual function.

Theorem 15 The dual function $\omega(\mathbf{\Psi})$ is concave over any convex subset of its domain D.

Next, we obtain the dual function by first expressing $\hat{A}_j$ and $\hat{\gamma}$ in terms of the dual variables δ_k which constitutes $\mathbf{\Psi}$. Once this is done, $\hat{A}_j$ and $\hat{\gamma}$ will be substituted in the Lagrangian $\mathcal{L}(\mathbf{A},\gamma,\mathbf{\Psi})$ in (3.32), and the resulting function of $\mathbf{\Psi}$ will be the dual function.

Using the necessary conditions (3.33) and (3.34), $\hat{A}_j$ and $\hat{\gamma}$ are characterized by the following equations,

$$\begin{aligned}
\hat{A}_j &= (h')^{-1}\left(\hat{\gamma} - \sum_{k=1;k\neq i}^{m} \delta_k \frac{p_{kj}}{p_{ij}}\right), \\
a_0 + c(e_i) &= \sum_{j=1}^{n} p_{ij}\,(h')^{-1}\left(\hat{\gamma} - \sum_{k=1;k\neq i}^{m} \delta_k \frac{p_{kj}}{p_{ij}}\right).
\end{aligned}$$

Substituting $\hat{A}_j$ and $\hat{\gamma}$ into the Lagrangian $\mathcal{L}(\mathbf{A},\gamma,\mathbf{\Psi})$ in (3.32) gives the dual problem as,

$$\begin{aligned}
\max_{\mathbf{\Psi}\geq\mathbf{0}} \omega(\mathbf{\Psi}) &= \sum_{j=1}^{n} p_{ij} h(\hat{A}_j) + \hat{\gamma}\left[a_0 + c(e_i) - \sum_{j=1}^{n} \hat{A}_j p_{ij}\right] \\
&\quad - \sum_{k=1;k\neq i}^{m} \delta_k \left[a_0 + c(e_k) - \sum_{j=1}^{n} \hat{A}_j p_{kj}\right]
\end{aligned} \tag{3.35}$$

Remark 10 If the supplier is risk-neutral, for example, $A_j = t_j$, or equivalently, $h(A_j) = A_j$ for $j = 1,\ldots,n$, then the dual problem (3.35) becomes a linear programming w.r.t. $\mathbf{\Psi} \geq \mathbf{0}$. If the supplier is risk-averse with $A(t_j) = \sqrt{t_j}$, or equivalently, $h(A_j) = A_j^2$ for $j = 1,\ldots,n$, then the dual problem (3.35) will become a quadratic programming w.r.t. $\mathbf{\Psi} \geq \mathbf{0}$. If the supplier is risk-averse with the other utility functions, then the dual problem (3.35) will become the other nonlinear programming w.r.t. $\mathbf{\Psi} \geq \mathbf{0}$. ◀

For tractability, we assume that the risk-averse supplier possesses the utility function with $A(t_j) = \sqrt{t_j}$, or equivalently, $h(A_j) = A_j^2$ for $j = 1, \ldots, n$. Then we obtain the following closed-form solution for $\hat{A}_j$ and $\hat{\gamma}$ as,

$$\hat{A}_j = a_0 + c(e_i) + \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{k=1;k\neq i}^{m}\delta_k p_{kj} - \sum_{k=1;k\neq i}^{m}\delta_k \frac{p_{kj}}{p_{ij}}\right), \tag{3.36}$$

$$\hat{\gamma} = 2(a_0 + c(e_i)) + \sum_{j=1}^{n}\sum_{k=1;k\neq i}^{m}\delta_k p_{kj}. \tag{3.37}$$

Note that $\hat{A}_j$ in (3.36) and $\hat{\gamma}$ in (3.37) are obtained as *linear functions* of $\mathbf{\Psi}$. We then use the duality theorem to find the dual function $\omega(\mathbf{\Psi})$, which is obtained directly by substituting $\hat{A}_j$ and $\hat{\gamma}$ into (3.35). In fact $\omega(\mathbf{\Psi})$ is the minimum of $\mathcal{L}$ over the variables $\hat{\gamma}$ and $\hat{A}_j$ since these are obtained from the necessary and sufficient conditions for optimality. Therefore, the dual problem is written as,

$$\begin{aligned}\max_{\mathbf{\Psi}\geq\mathbf{0}} \omega(\mathbf{\Psi}) = {} & \frac{1}{4}\sum_{j=1}^{n} p_{ij}\left(\hat{\gamma} - \sum_{k=1;k\neq i}^{m}\delta_k\frac{p_{kj}}{p_{ij}}\right)^2 \\ & -\frac{1}{2}\sum_{j=1}^{n}\left(\hat{\gamma}^2 p_{ij} - \sum_{k=1;k\neq i}^{m}\hat{\gamma}\delta_k p_{kj}\right) \\ & +\frac{1}{2}\sum_{k=1;k\neq i}^{m}\sum_{j=1}^{n}\hat{\gamma}\delta_k p_{kj} - \frac{1}{2}\sum_{k=1;k\neq i}^{m}\frac{(\delta_k p_{kj})^2}{p_{ij}} \\ & +\hat{\gamma}(a_0 + c(e_i)) - \sum_{k=1;k\neq i}^{m}(a_0 + c(e_k))\delta_k.\end{aligned} \tag{3.38}$$

where $\hat{A}_j \equiv \frac{1}{2}\left[\hat{\gamma} - \sum_{k=1;k\neq i}^{m}\delta_k p_{kj}/p_{ij}\right]$ and $\hat{\gamma}$ is given in (3.37), both linear with $\mathbf{\Psi}$.

We note that $\omega(\mathbf{\Psi})$ in (3.38) is quadratic in $\mathbf{\Psi} = (\delta_1, \delta_2, \ldots, \delta_k, \ldots, \delta_m)$ for $k = 1, \ldots, m$, $k \neq i$ so it can be expressed as

$$\omega(\mathbf{\Psi}) = \mathbf{\Psi}^T\mathbf{M}\mathbf{\Psi} + \mathbf{q}^T\mathbf{\Psi}.$$

We are faced with the task of solving a quadratic programing problem. But now our task is relatively straightforward because the only constraints in the problem are the nonnegativity restrictions on the variables $\mathbf{\Psi}$. We can use Maple's optimization package to obtain the solution denoted as $\hat{\mathbf{\Psi}} = (\hat{\delta}_1, \ldots, \hat{\delta}_k, \ldots, \hat{\delta}_m)$ to the dual problem (3.38). Next, substituting $\hat{\mathbf{\Psi}}$ into (3.36) and (3.37), we obtain $\hat{A}_j$ and $\hat{\gamma}$, i.e., the solution to the primal problem given Q and e_i.

Since there are m levels of effort, the buyer has to consider each level of effort so that he can decide which effort level is optimal for him to induce. We provide the following four-step procedure to help the buyer find second-best optimal transfer prices and order quantity for inducing the optimal effort level:

(1) Given $Q = 1$, let $i = m$, the highest effort level e_m with the yield probabilities p_{mj} for $j = 1, \ldots, n$, generate the $\mathbf{M}$ matrix and the $\mathbf{q}$ vector to be used in the solution of the quadratic programming problem $\max_{\mathbf{\Psi} \geq \mathbf{0}} \omega(\mathbf{\Psi})$.

(2) After obtaining the solution to the dual problem (3.38) as $\hat{\mathbf{\Psi}} = (\hat{\delta}_1, \ldots, \hat{\delta}_k, \ldots, \hat{\delta}_m)$ values for $k = 1, \ldots, m$, $k \neq i$, solve (3.36) and (3.37) to obtain $\hat{A}_j$ and $\hat{\gamma}$ for $j = 1, \ldots, n$.

(3) Use $\hat{\mathbf{A}} = (\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_n)$ to compute $\hat{\mathbf{t}} = (\hat{t}_1, \ldots, \hat{t}_n)$ through $A(t_j) = \sqrt{t_j}$ and substitute $\hat{\mathbf{t}}$ into the buyer's expected profit function (3.26), which is maximized to obtain the second-best order quantity $\hat{Q}_m$ and expected profit $\Pi_m(\hat{Q}_m)$ for inducing the highest effort level e_m.

(4) Repeat the above three steps for the other effort levels in the set $\{e_1, \ldots, e_{m-1}\}$ to obtain the correspondingly second-best order quantity $\{\hat{Q}_1, \ldots, \hat{Q}_{m-1}\}$ and expected profit $\{\Pi_1(\hat{Q}_1), \ldots, \Pi_{m-1}(\hat{Q}_{m-1})\}$, respectively. The second-best optimal solution is obtained by solving the problem: $\max\Big(\Pi_1(\hat{Q}_1), \ldots, \Pi_m(\hat{Q}_m)\Big)$.

3.4.2 A Numerical Example with $m = 3$ and $n = 3$

To illustrate our solution procedure to the general case, we provide a simple example with $m = 3$ and $n = 3$. The matrix $\mathbf{P}$ is used to capture the probability $p_{ij} = \mathbf{P}[i, j]$ that the outcome will be u_j given the effort level e_i for $i, j = 1, 2, 3$, as

$$\mathbf{P} = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.1 & 0.3 & 0.6 \\ 0.025 & 0.075 & 0.9 \end{bmatrix}.$$

Observing that the highest effort level e_3 is the maximum likelihood estimator of the effort given the highest yield u_3 since $p_{33} > p_{23} > p_{13}$ and that the lowest effort level e_1 is the maximum likelihood estimator of the effort given the lowest yield u_1 since $p_{11} > p_{21} > p_{31}$, the buyer first considers to induce the supplier's highest effort level e_3. Moreover, given the highest effort level e_3, the yield probabilities satisfy the monotone likelihood ratio property, i.e., $1 - p_{11}/p_{31} \leq 1 - p_{12}/p_{32} \leq 1 - p_{13}/p_{33}$ and $1 - p_{21}/p_{31} \leq 1 - p_{22}/p_{32} \leq 1 - p_{23}/p_{33}$. Thus, the realization of a high yield is good news about the supplier's effort choice. The second-best problem for inducing e_3 is written as,

$$
\text{(SB-}e_3\text{):} \qquad \begin{array}{ll}
\max_{\{Q,\mathbf{t}\}} & \Pi_3(Q,\mathbf{t}) = \sum_{j=1}^{3}[b_j(Q) - t_j Q]p_{3j} \\
\text{s. t.} & \sum_{j=1}^{3} A(t_j)p_{3j} - c(e_3) \geq a_0, \\
& \sum_{j=1}^{3} A(t_j)p_{3j} - c(e_3) \geq \sum_{j=1}^{3} A(t_j)p_{1j} - c(e_1), \\
& \sum_{j=1}^{3} A(t_j)p_{3j} - c(e_3) \geq \sum_{j=1}^{3} A(t_j)p_{2j} - c(e_2).
\end{array}
$$

Given $Q = 1$, the closed-form solution for $\hat{A}_j$ and $\hat{\gamma}$ are obtained by letting $i = 3$, $m = 3$ and $n = 3$ in (3.36) and (3.37), i.e.,

$$
\begin{aligned}
\hat{A}_j &= a_0 + c(e_3) + \frac{1}{2}\left[\sum_{j=1}^{3}(\delta_1 p_{1j} + \delta_2 p_{2j}) - \left(\delta_1 \frac{p_{1j}}{p_{3j}} + \delta_2 \frac{p_{2j}}{p_{3j}}\right)\right], \\
\hat{\gamma} &= 2\left(a_0 + c(e_3)\right) + \sum_{j=1}^{3}(\delta_1 p_{1j} + \delta_2 p_{2j}).
\end{aligned}
$$

Assume that the agent's effort level takes three possible values, $e_1 = 0$, $e_2 = 0.5$, and $e_3 = 1$, and the yield outcome also takes three possible values, $u_1 = 0.1$, $u_2 = 0.5$, and $u_3 = 0.9$. The values for the other parameters are assumed to be the same as Example 7. Then substituting $\hat{A}_j$ and $\hat{\gamma}$ in (3.38) with these values, we calculate the dual problem for e_3 as,

$$
\max_{\mathbf{\Psi}\geq\mathbf{0}} \omega\left(\mathbf{\Psi}\right) = \max_{\mathbf{\Psi}\geq\mathbf{0}} \left[\delta_1 - 3.653(\delta_1)^2 - 1.333\delta_1\delta_2 + 0.750\delta_2 - 0.250(\delta_2)^2 + 9.000\right],
$$

Solving the above quadratic programming problem gives

$$
\hat{\delta}_1 = 0, \qquad \hat{\delta}_2 = 1.500,
$$

which indicates that the IC associated with e_1 is non-binding but the IC associated with e_2 is binding. The second-best solution for $\mathbf{A}$ are then found as

$$
\hat{A}_1 = 0.750, \qquad \hat{A}_2 = 0.750, \qquad \hat{A}_3 = 3.250,
$$

or equivalently,

$$
\hat{t}_1 = (\hat{A}_1)^2 = \$0.562, \qquad \hat{t}_2 = (\hat{A}_2)^2 = \$0.562, \qquad \hat{t}_3 = (\hat{A}_3)^2 = \$10.563.
$$

Substituting $\hat{\mathbf{t}} = \left(\hat{t}_1, \hat{t}_2, \hat{t}_3\right)$ into the buyer's expected profit function $\Pi_3(Q)$, which is maximized to obtain the second-best order quantity $\hat{Q}_3$ and expected profit $\Pi_3(\hat{Q}_3)$ for inducing the highest effort level, as

$$
\hat{Q}_3 = 25.180 \text{ units}, \qquad \Pi_3(\hat{Q}_3) = \$242.495.
$$

Next, we follow the same solution procedure to finding the solutions given e_1 and e_2 respectively, and summarize all results in Table 6. Note that in the table, we use "=" to denote the binding constraint and ">" to denote the non-binding constraint. The notation "IC-e_i" is used for the IC associated with the effort level e_i for $i = 1, 2, 3$. Therefore, from Table 6, it's straightforward to see that $\Pi_2(\hat{Q}_2) > \Pi_3(\hat{Q}_3) > \Pi_1(\hat{Q}_1)$. That is, it's optimal for the buyer to induce the second-highest effort level. We also note that the order size is decreasing as inducing a higher level of effort, i.e., $\hat{Q}_3 < \hat{Q}_2 < \hat{Q}_1$.

e_i	PC	IC–e_1	IC–e_2	IC–e_3	$\hat{\mathbf{t}} = (\hat{t}_1, \hat{t}_2, \hat{t}_3)$ (\$)	$\hat{Q}_i$ (units)	$\Pi_i(\hat{Q}_i)$ (\$)
e_1	=	N/A	>	>	$(4, 4, 4)$	54.317	111.180
e_2	=	=	N/A	>	$(3.318, 5.063, 5.389)$	46.743	320.645
e_3	=	>	=	N/A	$(0.562, 0.562, 10.563)$	25.180	242.495

Table 6. The buyer's second-best optimal transfer price $\hat{\mathbf{t}}$, order quantity $\hat{Q}_i$ and expected profit $\Pi_i(\hat{Q}_i)$ when the supplier is incentivized to exert effort e_i for $i = 1, 2, 3$, respectively.

3.5 Summary and Concluding Remarks

In this chapter, we study how a buyer designs an incentive mechanism to induce his supplier's effort in improving supply performance and what's the buyer's optimal order decision under random yield and demand. By embedding the random yield model into the principal-agent framework under moral hazard, we first develop a model with binary effort choices and yield outcomes and then extend to a general model with multiple effort choices and yield outcomes. The results of the binary model show that when moral hazard issue arises, the buyer should schedule a higher transfer price for the high yield level but place a smaller order. Moreover, with the minimum liability imposed, the results suggest the buyer to consider a trade-off between risk and incentives when the salvage value is larger than a threshold value. As for the general model, the algorithm based on Lagrangian duality is quite useful to determine which incentive constraints are binding. Furthermore, if the supplier is risk-averse with the utility equaling the square root of any transfer price received, then the dual program will become a quadratic programming problem w.r.t. nonnegative multipliers. We then provide a four-step solution procedure to help the buyer decide which level of effort is optimal to induce and how many units are optimal to order. Finally, we present the numerical results for the model with three effort levels and three yield levels. We find that it's optimal for the buyer to induce the second-highest effort level that maximizes the buyer's expected profit function.

Our study explores the incentive issues in supply management where the buyer doesn't directly involve in improving supplier's performance. An interesting follow-up question is how to design the incentive mechanism if the buyer also exerts effort in improving the supplier's performance. The buyer's any activities related to improving his supplier's performance or capability is called "supplier development" (Krause [26] and Krause, Handfield and Tyler [27]). The buyer's direct involvement provides his commitment to involving actively in supplier development. In such case, both the buyer and supplier exert effort in improving supplier's performance, which results in double moral hazard when their effort cannot be observed. It would be worthwhile to investigate the optimal incentive contract design and the buyer's optimal ordering decision under random yield, random demand and double moral hazard issues.

Chapter 4 Optimal Ordering Decision and Incentives for Yield Improvement under Random Demand: A Continuous Model

In this chapter, we study the buyer's incentive mechanism design and ordering decisions under the assumption that the supplier's effort and yield are both continuous variables. Under the first-best case, we find that the optimal transfer function is increasing and convex in the effort level but independent of the order quantity. We present the first-best optimal order quantity and effort level through the buyer's first-order conditions. Under the second-best case, we find the buyer's second-best optimal solution through the first-order approach of Holmström [21], where the supplier's maximization problem is concave with respect to the effort. We run one numerical example to show that the buyer cannot implement the first-best effort level when the supplier' effort becomes unobservable. The buyer's second-best optimal order quantity and expected profit are smaller than those under the first-best case. We also perform the sensitivity and our results imply that when the supplier's disutility of effort increases, it's optimal for the buyer to schedule the transfer function with a higher coefficient for inducing a lower effort level.

4.1 Introduction

Recall that in a single-period random yield model, the buyer (as the principal) faces a random demand X with p.d.f. $f(x)$ and c.d.f. $F(x)$ and places an order of Q units (a decision variable) with the supplier (as the agent) for a critical component. However, due to the randomness of the yield, the amount received is $Y = QU$ where U is the yield random variable with p.d.f. $g(u; e)$, $(0 \leq u \leq 1)$, depending on the supplier's effort level $e \in [e^-, e^+]$ with $0 < e^- < e^+ \leq 1$. The effort improves yield in the sense of first-order stochastic dominance (FSD), i.e., $\Pr(U_e > \hat{u})$ is increasing in e for any given yield level $\hat{u}$, (see, Bunn [8, Ch. 4]). Depending on the level of yield u, the buyer pays the supplier a transfer of $t(u)$ per unit ordered. The buyer must determine not only the scalar decision variable Q but also the function $t(u)$ in the incentive mechanism he proposes to the supplier. The buyer processes the received component to his final product and sells it to the market. The buyer's per unit sales revenue is r, but if there is a shortage then this results in a penalty of s per unit; the salvage value for any unsold units is v per unit. We define the buyer's random profit as π that is a function of Q and $t(u)$ and his expected profit as $\Pi\left(Q, t\left(u\right)\right)$.

As in Chapter 3, we assume that the supplier is risk-averse with an increasing and concave utility function $A(t(\cdot))$ and $A(0) = 0$, i.e., $A' > 0$ and $A'' < 0$. She will accept the buyer's incentive offer if her expected utility exceeds her reservation utility a_0. If the supplier accepts and exerts effort e, then this will result in a disutility level of $c(e)$, which is increasing and convex in e, i.e., $c' > 0$ and $c'' > 0$. Given the realized values of the random variables $X = x$ and $U = u$, if demand falls short of available units, i.e., $x \leq Qu$, then the buyer's profit is $\pi = rx + v(Qu - x) - t(u)Q$. Otherwise, if $x > Qu$, then the profit is $\pi = rQu - s(x - Qu) - t(u)Q$. Thus, the risk-neutral buyer finds his expected profit as,

$$\Pi\left(Q, t\left(u\right)\right) = \int_0^1 [b(Q, u) - t(u)Q]g(u; e)\ du, \tag{4.1}$$

where

$$b(Q, u) = \int_0^{Qu} [(r - v)x + vQu]f(x)\ dx + \int_{Qu}^{\infty} [(r + s)Qu - sx]f(x)\ dx.$$

The supplier's participation constraint (PC) is given as,

$$\int_0^1 A(t(u))g(u; e)\ du - c(e) \geq a_0. \tag{4.2}$$

If the supplier's effort e is verifiable, then no moral hazard issue will exist. However, in reality the effort is usually not verifiable, and thus we are faced with a moral hazard problem. The complete formulation of the moral-hazard problem now involves a second constraint for the buyer, due to information asymmetry (i.e., unverifiable effort). This is indeed an optimization problem for the supplier resulting in the incentive constraint (IC),

$$e \in \arg\max_{\tilde{e}} \left[\int_0^1 A(t(u))g(u; \tilde{e})\ du - c(\tilde{e})\right], \tag{4.3}$$

where the integral is the agent's expected utility and $c(\tilde{e})$ is the disutility if she chooses the effort level $\tilde{e}$.

In the next two sections, we will present the buyer's incentive mechanism and ordering decisions under both the symmetric ("first-best") and asymmetric information ("second-best") cases.

4.2 First-Best Optimal Solution

As a benchmark, we consider the first-best case where the supplier's effort is observable and verifiable. The buyer's problem for inducing a particular level of effort e is to maximize (4.1) subject to (4.2). We denote the buyer's first-best problem as (FB-e), which is given by,

$$\text{(FB-}e\text{):} \qquad \begin{array}{ll} \max_{\{e,Q,t(u)\}} & \Pi\left(Q,t\left(u\right)\right)=\int_0^1 [b(Q,u)-t(u)Q]g(u;e)\ du \\ \text{s. t.} & \int_0^1 A(t(u))g(u;e)\ du - c(e) \geq a_0. \end{array}$$

The above problem will be solved through the following two steps:

(1) Given $Q>0$ and an effort level $e\in[e^-,e^+]$, find the first-best transfer function denoted as $t^*\left(u\mid Q,e\right)$;

(2) Substitute $t^*\left(u\mid Q,e\right)$ into (4.1), which is maximized to obtain the first-best optimal order quantity denoted as Q^* and effort level denoted as e^*, respectively. The first-best optimal transfer function is then computed as $t^*\left(u\mid Q^*,e^*\right)$.

At step 1, we associate a multiplier $\lambda\geq 0$ with the supplier's participation constraint (4.2) and write the Lagrangian function as,

$$\mathcal{L}\left(Q,t(u),\lambda\right)=\int_0^1[b(Q,u)-t(u)Q]g(u;e)\ du-\lambda\left[a_0+c(e)-\int_0^1 A(t(u))g(u;e)\ du\right].$$

Given $Q>0$ and an effort level $e\in[e^-,e^+]$, the first-order condition of $\mathcal{L}\left(Q,t(u),\lambda\right)$ w.r.t. $t\left(u\right)$ is written as,

$$-Qg(u;e)+\lambda A'(t^*\left(u\mid Q,e\right))g(u;e)=0,$$

where $t^*\left(u\mid Q,e\right)$ is the first-best transfer function given Q and e. Isolating λ gives,

$$\lambda=\frac{Q}{A'(t^*\left(u\mid Q,e\right))}, \qquad \text{for } u\in[0,1]. \tag{4.4}$$

It's straightforward to show that the multiplier λ is always positive for any $u\in[0,1]$, given $Q>0$ and $A'>0$. Thus, $t^*\left(u\mid Q,e\right)$ is the same for any yield level $u\in[0,1]$. Note that when $Q=0$, the buyer will not do any business with the supplier and the contractual relationship between the buyer and the supplier ends. The multiplier $\lambda>0$ also suggests the supplier's PC is binding, i.e.,

$$\int_0^1 A(t^*\left(u\mid Q,e\right))g(u;e)\ du-c(e)=a_0.$$

Since $t^*(u \mid Q, e)$ is fixed for any yield, we explicitly find the first-best transfer function as,

$$t^*(u \mid Q, e) = h\left(a_0 + c(e)\right), \tag{4.5}$$

where h is the inverse function of A with $h' > 0$ and $h'' > 0$.

Lemma 16 The first-best transfer function $t^*(u \mid Q, e)$ is increasing and convex in the effort level e, but independent of Q.

Proof. Taking the first- and second-order derivatives of $t^*(u \mid Q, e)$ w.r.t e gives,

$$\begin{aligned} \frac{\partial t^*(u \mid Q, e)}{\partial e} &= h'\left(a_0 + c(e)\right) c'(e) > 0, \\ \frac{\partial^2 t^*(u \mid Q, e)}{\partial e^2} &= h''\left(a_0 + c(e)\right) c'(e) + h'\left(a_0 + c(e)\right) c''(e) > 0, \end{aligned}$$

since $h' > 0$, $h'' > 0$, $c' > 0$ and $c'' > 0$. Also, the explicit form of $t^*(u \mid Q, e)$ in (4.5) indicates that the the first-best transfer function is independent of the order quantity Q. Thus, we proved that the first-best transfer function is increasing and convex in the effort level e but independent of Q. ■

At step 2, we then substitute (4.5) into (4.1), and write the buyer's problem as,

$$\max_{Q,e} \Pi(Q, e) = \int_0^1 [b(Q, u)] g(u; e)\ du - h\left(a_0 + c(e)\right) Q. \tag{4.6}$$

The buyer's expected profit $\Pi(Q, e)$ is jointly concave in Q and e if and only if the Hessian of $\Pi(Q, e)$ is negative semidefinite for $Q > 0$ and $e \in [e^-, e^+]$. If it does, then we can find the first-best optimal order quantity Q^* and effort level e^* through the following first-order conditions,

$$\begin{aligned} \int_0^1 uF\left(Q^* u\right) g\left(u; e^*\right)\ du &= \frac{(r+s) E\left(u; e^*\right) - h\left(a_0 + c(e^*)\right)}{(r+s-v)}, \\ \int_0^1 b(Q^*, u) g_e(u; e) \mid_{e=e^*} du &= \left[h'\left(a_0 + c(e)\right) c'(e)\right] \mid_{e=e^*} Q^*, \end{aligned}$$

where $b(Q^*, u) = \int_0^{Q^* u} [(r-v)x + vQ^* u] f(x)\ dx + \int_{Q^* u}^{\infty} [(r+s)Q^* u - sx] f(x)\ dx$, $g_e(u; e)$ is the partial derivative of $g(u; e)$ w.r.t. e, and $c'(e)$ is the first derivative of $c(e)$.

In order to obtain further managerial insights into the buyer's problem, we make the following assumptions:

(1) The supplier is risk-averse with the utility function $A(t) = \sqrt{t}$.
(2) The supplier's disutility of effort is quadratic, i.e., $c(e) = \gamma e^2$ for $\gamma > 0$.

(3) The demand of the final product follows an exponential distribution with parameter α, i.e., $f(x) = \alpha \exp(-\alpha x)$.
(4) The supplier's yield function takes the form of $g(u; e) = 1 + (u - 1/2)e$, for $e \in (0, 1]$ and $u \in [0, 1]$

Under the above assumptions, the first-best transfer function is computed as,

$$t^*(u \mid Q, e) = \left(a_0 + \gamma e^2\right)^2,$$

which is increasing and convex in e. Substituting it into (4.6), the buyer's expected profit $\Pi(Q, e)$ can be written explicitly as,

$$\begin{aligned}\Pi(Q, e) &= \frac{1}{\alpha} \int_0^1 \left[\alpha Quv - (r + s - v) \exp(-\alpha Qu) + r - v\right] \left[1 + (u - 1/2)e\right] \, du \\ &\quad - \left(a_0 + \gamma e^2\right)^2 Q. \end{aligned} \tag{4.7}$$

The buyer's problem is maximization of his expected profit by choosing the optimal order quantity Q^* and effort level e^*. We now provide an example to illustrate the optimal solution under the first-best case.

Example 9 In this example, we use the following values for the parameters: $\alpha = 0.1$, i.e., $E(X) = 10$ units; $a_0 = 2$; $r = \$50$; $v = \$5$; $s = \$10$ and $\gamma = 0.14$. Given $Q > 0$ and an effort level $e \in [e^-, e^+]$, the first-best transfer function is calculated as

$$t^*(u \mid Q, e) = \left(2 + 0.14e^2\right)^2.$$

Substituting it into (4.7) gives,

$$\begin{aligned}\Pi(Q, e) &= 0.00013Q\left(-147e^4 - 4200e^2 + 3125e - 11250\right) \\ &\quad + \frac{0.00013}{Q^2}\left[2.06250 \times 10^7 eQ + 4.1250Q + 4.1250 \times 10^8 e\right] \exp(-0.1Q) \\ &\quad + \frac{0.00013}{Q^2}\left[3.3750Q^2 + 20.625eQ - 41.250Q - 412.50e\right] \times 10^6.\end{aligned}$$

As shown in Figure 14, the buyer's objective function $\Pi(Q, e)$ is indeed jointly concave in the order quantity Q and effort level e, for $Q > 0$ and $e \in (0, 1]$. Then we can use the first-order conditions to find the first-best optimal solution as follows,

$$Q^* = 54.38 \text{ units}, \qquad e^* = 0.86, \qquad \Pi(Q^*, e^*) = \$291.86.$$

This implies that it's not optimal for the buyer to induce the supplier's highest effort level ($e = 1$) even under the first-best case. A natural question to ask is the following: As the supplier's disutility of effort increases, what will happen to the optimal first-best solution [in terms of Q^*, e^* and $\Pi(Q^*, e^*)$]? To answer this question, we fix $a_0 = 2$ and re-solve the problem for different values of γ. The results are summarized in Table 7, where γ varies from 0.12 to 0.30 in increments of 0.02.

We note that when γ stays at a low level ($\gamma = 0.12$), the optimal effort reaches its maximum possible level of $e^* = 1$. It's optimal for the buyer to induce the supplier's highest effort level for yield improvement. After $\gamma = 0.12$, as γ increases, we find that the optimal effort level e^* decreases but the optimal order quantity Q^* increases. The reasons are as follows: The increasing value of γ brings higher unit transfer cost for the buyer, i.e., $t^*(u \mid Q, e)$ increases; To reduce the costs associated with incentives for yield improvement, the buyer then chooses to induce a lower effort level but to places a larger order. However, the optimal expected profits $\Pi(Q^*, e^*)$ decreases from \$295.80 to \$279.38 as γ increases from 0.12 to 0.30. If the buyer learns that the supplier's disutility of cost increases, then the buyer will choose to induce a lower effort level but places a larger order. ♦

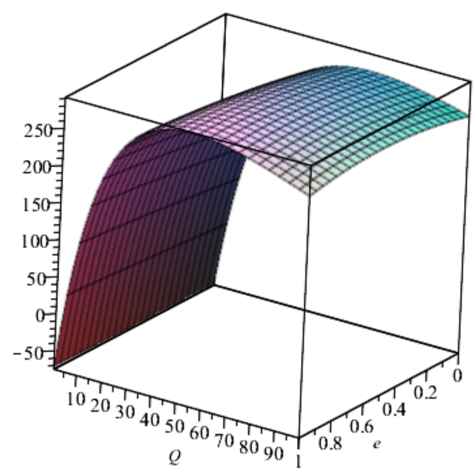

Figure 14. The buyer's objective function given $\gamma = 0.14$ and $a_0 = 2$.

4.3 Second-Best Optimal Solution

In this section, we investigate the buyer's problem under the second-best case where the supplier's effort cannot be observed. We denote the buyer's problem for inducing a particular effort level e as (SB-e), where the buyer maximizes his expected profit in

γ	Q^*	e^*	$\Pi(Q^*, e^*)$
0.12	53.53	1.00	\$295.80
0.14	54.38	0.86	\$291.86
0.16	55.09	0.75	\$288.92
0.18	55.64	0.66	\$286.64
0.20	56.08	0.59	\$284.82
0.22	56.44	0.54	\$283.33
0.24	56.75	0.49	\$282.10
0.26	57.00	0.45	\$281.05
0.28	57.22	0.42	\$280.15
0.30	57.41	0.39	\$279.38

Table 7. The sensitivity analyis of γ on the first-best optimal solution.

(4.1) subject to the two constraints (4.2) and (4.3), i.e.,

$$\text{(SB-}e\text{):}\quad \begin{array}{ll} \max_{\{Q,t(u),e\}} & \Pi(Q, t(u)) = \int_0^1 [b(Q,u) - t(u)Q] g(u;e)\ du, \\ \text{s. t.} & \int_0^1 A(t(u)) g(u;e)\ du - c(e) \geq a_0, \\ & e \in \arg\max_{\tilde{e}} \left[\int_0^1 A(t(u)) g(u;\tilde{e})\ du - c(\tilde{e}) \right]. \end{array}$$

Note that in the above problem the supplier's effort is a continuous variable, which results in an infinite number of incentive constraints (4.3) for $\tilde{e} \geq 0$. One approach to solving this problem, as suggested by Holmström [21] involves replacing (4.3) with simpler "local" incentive constraints in the form

$$\int_0^1 A(t(u)) g_e(u;e)\ du - c'(e) = 0. \tag{4.8}$$

With this simplification, we now obtain another continuous optimization problem denoted as (SB-e)$'$, which is given by,

$$\text{(SB-}e\text{)}'\text{:}\quad \begin{array}{ll} \max_{\{e,Q,t(u)\}} & \Pi(Q, t(u)) = \int_0^1 [b(Q,u) - t(u)Q] g(u;e)\ du, \\ \text{s. t.} & \int_0^1 A(t(u)) g(u;e)\ du - c(e) \geq a_0, \\ & \int_0^1 A(t(u)) g_e(u;e)\ du - c'(e) = 0. \end{array}$$

Letting $\lambda \geq 0$ be the multiplier for (4.2) and μ (free) for (4.8), then we write the Lagrangian function as,

$$\begin{aligned}\mathcal{L}\left(Q,t(u),\lambda,\mu\right) &= \int_0^1 [b(Q,u)-t(u)Q]g(u;e)\ du \\ &\quad -\lambda\left[a_0+c(e)-\int_0^1 A(t(u))g(u;e)\ du\right] \\ &\quad -\mu\left[c'(e)-\int_0^1 A(t(u))g_e(u;e)\ du\right].\end{aligned}$$

Given $Q>0$ and an effort level $e\in[e^-,e^+]$, for $u\in[0,1]$, the first-order condition of the Lagrangian function w.r.t. $t\left(u\right)$ is written as,

$$-Qg(u;e)+\lambda A'(\hat{t}(u\mid e,Q))g(u;e)+\mu A'(\hat{t}(u\mid e,Q))g_e(u;e)=0,$$

where $\hat{t}(u\mid e,Q)$ is denoted as the second-best transfer function for $u\in[0,1]$. Isolating $A'(\hat{t}(u\mid e,Q))$ gives

$$\frac{1}{A'(\hat{t}(u\mid e,Q))}=\frac{1}{Q}\left[\lambda+\mu\frac{g_e(u;e)}{g(u;e)}\right]. \tag{4.9}$$

Note that if $\mu=0$, then (4.9) reduces to (4.4). The buyer's problem will become the first-best case, whose solution is presented at Section 4.2. When $\mu>0$, a real moral hazard problem arises. The second-best transfer function $\hat{t}(u\mid e,Q)$ now depends on $g_e(u;e)/g(u;e)$ given $Q>0$ and $e\in[e^-,e^+]$. Assuming that $g_e(u;e)/g(u;e)$ is increasing in u given an effort level, then the right-hand side of (4.9) also increases in u. To be equal, the left-hand side of (4.9) must increase in u, which results in $\hat{t}(u\mid e,Q)$ increasing in u given that A is an increasing and concave function. Therefore, given $Q>0$ and $e\in[e^-,e^+]$, we have obtained that $\hat{t}(u\mid e,Q)$ is increasing in u.

However, the closed-form solution of $\hat{t}(u\mid e,Q)$ is still unknown, which makes it difficult to find the second-best optimal order quantity $\hat{Q}$ and effort level $\hat{e}$ analytically. One possible approach is to use an approximation whereby the buyer provides a set of the linear contracts to the supplier. For example, let $t\left(u\right)=\eta+\rho u$ where $\rho\geq 0$ (decision variable) and $\eta>0$ (parameter), so that $t\left(u\right)$ is increasing in u and $t\left(u\right)\geq 0$. The closed-form solution of $\hat{t}(u\mid e,Q)$ will be obtained by finding $\hat{\rho}$ given η for implementing a given effort level e. Substituting $\hat{t}(u\mid e,Q)$ into the buyer's expected profit $\Pi\left(Q,t\left(u\right)\right)$, we will find the second-best optimal order quantity $\hat{Q}$ and effort level $\hat{e}$ through the first-order conditions.

In order to obtain further managerial insights into the buyer's problem, we use the same assumptions and parameter values as those under the first-best case in Section 4.2, i.e.,

(1) The supplier is risk-averse with the utility function $A(t) = \sqrt{t}$.
(2) The supplier's disutility of effort is quadratic, i.e., $c(e) = \gamma e^2$ for $\gamma > 0$.
(3) The demand of the final product follows an exponential distribution with parameter α, i.e., $f(x) = \alpha \exp(-\alpha x)$.
(4) The supplier's yield function takes the form of $g(u; e) = 1 + (u - 1/2)e$, for $e \in (0, 1]$ and $u \in [0, 1]$. Given an effort level e, the form $g_e(u; e)/g(u; e)$ is indeed increasing in u since

$$\frac{\partial}{\partial u}\left[\frac{g_e(u; e)}{g(u; e)}\right] = \frac{4}{2\left(eu - e + 2\right)^2} > 0.$$

Under the above assumptions, the supplier's problem given a transfer function $t(u)$ can be written as,

$$\max_e \Pi_S = \int_0^1 \sqrt{t(u)}\left[1 + (u - 1/2)e\right]\ du - \gamma e^2,$$

which is concave in the effort level e since the second-order derivative is negative, i.e.,

$$\frac{d^2\Pi_S}{de^2} = -2\gamma < 0, \qquad \text{for } \gamma > 0.$$

Therefore, we can use the first-order approach to replace the incentive constraint (4.3) with the first-order condition. The buyer's problem is then written as,

$$\begin{array}{ll} \max_{\{e,Q,t(u)\}} & \Pi\left(Q, t\left(u\right)\right) \\ \text{s. t.} & \int_0^1 \sqrt{t(u)}\left[1 + (u - 1/2)e\right]\ du - \gamma e^2 \geq a_0, \\ & \int_0^1 \sqrt{t(u)}(u - 1/2)\ du - 2\gamma e = 0, \end{array}$$

where

$$\begin{aligned} \Pi\left(Q, t\left(u\right)\right) &= \frac{1}{\alpha}\int_0^1 \left[\alpha Q u v - (r + s - v)\exp\left(-\alpha Q u\right) + r - v\right]\left[1 + (u - 1/2)e\right]\ du \\ &\quad -Q\int_0^1 t(u)\left[1 + (u - 1/2)e\right]\ du. \end{aligned}$$

Next, we assume that the buyer provides a set of the linear contracts to the supplier. We define $t(u) = \eta + \rho u$ where $\rho \geq 0$ (decision variable) and $\eta > 0$ (parameter). Then the buyer needs to find the optimal value $\hat{\rho}$ given η for implementing a given effort level e, instead of finding the second-best transfer function $\hat{t}(u \mid e, Q)$ directly. Under such approximation, the buyer's problem becomes another maximization problem, i.e.,

$$\begin{array}{ll} \max_{\{e,Q,\rho\}} & \Pi\left(Q,t\left(u\right)\right) \\ \text{s. t.} & \int_0^1 \sqrt{\eta+\rho u}\left[1+(u-1/2)e\right]\ du-\gamma e^2 \geq a_0, \\ & \int_0^1 \sqrt{\eta+\rho u}(u-1/2)\ du-2\gamma e=0, \end{array}$$

where

$$\begin{aligned} \Pi\left(Q,t\left(u\right)\right) &= \frac{1}{\alpha}\int_0^1 \left[\alpha Quv-(r+s-v)\exp\left(-\alpha Qu\right)+r-v\right]\left[1+(u-1/2)e\right]\ du \\ & -Q\int_0^1 (\eta+\rho u)\left[1+(u-1/2)e\right]\ du. \end{aligned} \tag{4.10}$$

Next, we provide an example to illustrate the results under the second-best case.

Example 10 In this example, the following values for the parameters are used: $\alpha=0.1$, i.e., $E\left(X\right)=10$ units; $a_0=2$; $r=\$50$; $v=\$5$; $s=\$10$; $\gamma=0.14$ and $\eta=0.20$. Then the buyer's transfer function is written as

$$t\left(u\right)=0.20+\rho u.$$

Substituting it into (4.9) gives the buyer's expected profit as,

$$\begin{aligned} \Pi\left(Q,t\left(u\right)\right) &= 0.0167Q\left[-5e\rho+25e-30\rho+138\right] \\ & +\frac{1670}{Q^2}\left[1.65eQ+3.3Q+33e\right]\exp\left(-0.1Q\right) \\ & +\frac{1670}{Q^2}\left[0.27Q^2+1.65eQ-3.3Q-33e\right]. \end{aligned}$$

Maximizing the above profit function subject to the supplier's participation and incentive constraints gives the following second-best optimal solutions as,

$$\hat{\rho}=7.94, \qquad \hat{Q}=51.35 \text{ units}, \qquad \hat{e}=0.64, \qquad \Pi(\hat{Q},\hat{e})=\$270.87.$$

This implies that the buyer cannot implement the first-best effort level, i.e., $e^*=0.86$ in Example 9. The buyer's second-best optimal order quantity is smaller than his first-best optimal order quantity, i.e., $\hat{Q}=51.35<Q^*=54.38$. The buyer's second-best optimal expected profit is also smaller than his first-best optimal expected profit, i.e., $\Pi(\hat{Q},\hat{e})=\$270.87<\Pi\left(Q^*,e^*\right)=\$291.86$.

Next, we perform sensitivity analysis of η and γ on the second-best optimal solution [in terms of $\hat{t}(u\mid e,Q)$, $\hat{Q}$, $\hat{e}$ and $\Pi(\hat{Q},\hat{e})$], respectively. The results of η are summarized in Table 8 where η varies from 0 to 1.00 in increments of 0.10 resulting in $(\hat{t}(u\mid e,Q),\hat{Q},\hat{e},\Pi(\hat{Q},\hat{e}))$ as the second-best optimal solution. We note that when η increases, the coefficient $\hat{\rho}$ decreases, and the second-best transfer function is decreasing in the yield level u. The buyer will induce a lower effort level for yield improvement but places a larger order as η increases. Finally, the buyer's expected profit increases.

The results of γ given $\eta = 0.20$ are summarized in Table 9 where γ varies from 0.12 to 0.30 in increments of 0.02 resulting in $(\hat{t}(u \mid e, Q), \hat{Q}, \hat{e}, \Pi(\hat{Q}, \hat{e}))$ as the second-best optimal solution. We find that when the supplier's disutility of effort increases (γ increases), it's optimal for the buyer to schedule the transfer function with a higher coefficient ($\hat{\rho}$ increases). However, the optimal effort level $\hat{e}$ decreases, which implies that the buyer should induce a lower effort level for yield improvement. Although the buyer places a large order ($\hat{Q}$ increases), the buyer's expected profit still decreases, as γ increases. ♦

η	$\hat{\rho}$	$\hat{t}(u \mid e, Q)$	$\hat{Q}$	$\hat{e}$	$\Pi(\hat{Q}, \hat{e})$
0	8.41	$8.41u$	50.17	0.69	\$268.84
0.10	8.17	$8.17u + 0.10$	50.80	0.66	\$269.94
0.20	7.94	$7.94u + 0.20$	51.35	0.64	\$270.87
0.30	7.72	$7.72u + 0.30$	51.87	0.62	\$271.70
0.40	7.50	$7.50u + 0.40$	52.36	0.60	\$272.44
0.50	7.27	$7.27u + 0.50$	52.84	0.58	\$273.12
0.60	7.05	$7.05u + 0.60$	53.29	0.56	\$273.74
0.70	6.83	$6.83u + 0.70$	53.72	0.54	\$274.30
0.80	6.61	$6.61u + 0.80$	54.14	0.52	\$274.80
0.90	6.40	$6.40u + 0.90$	54.55	0.50	\$275.26
1.00	6.18	$6.18u + 1.00$	54.93	0.48	\$275.67

Table 8. The sensitivity analysis of η on the second-best optimal solution.

4.4 Summary and Concluding Remarks

In this chapter, we study a continuous optimization problem for the buyer when the supplier's effort and yield become continuous variables. Under the first-best case, we find that the optimal transfer function is increasing and convex in the effort level but independent of the order quantity. We run one numerical example and perform the sensitivity analysis to investigate how the supplier's disutility of effort affects the first-best optimal solution. We find that when the supplier's disutility of effort is at a low level, it's optimal for the buyer to induce the supplier's highest effort level for yield improvement. After such a low level, as it's more costly for the supplier to exert effort, we find that the optimal effort level decreases but the optimal order quantity increases. Under the second-best case, we find the buyer's second-

γ	$\hat{\rho}$	$\hat{t}(u \mid e, Q)$	$\hat{Q}$	$\hat{e}$	$\Pi(\hat{Q}, \hat{e})$
0.12	7.86	$7.86u + 0.20$	51.20	0.74	\$275.20
0.14	7.94	$7.94u + 0.20$	51.35	0.64	\$270.87
0.16	8.00	$8.00u + 0.20$	51.48	0.56	\$267.61
0.18	8.05	$8.05u + 0.20$	51.58	0.50	\$265.08
0.20	8.09	$8.09u + 0.20$	51.66	0.45	\$263.05
0.22	8.12	$8.12u + 0.20$	51.74	0.41	\$261.39
0.24	8.14	$8.14u + 0.20$	51.80	0.38	\$260.00
0.26	8.17	$8.17u + 0.20$	51.85	0.35	\$258.83
0.28	8.19	$8.19u + 0.20$	51.90	0.33	\$257.82
0.30	8.20	$8.20u + 0.20$	51.94	0.30	\$256.95

Table 9. The sensitivity analyis of γ on the second-best optimal solution.

best optimal solution through the first-order approach of Holmström [21], where the supplier's maximization problem is concave with respect to the effort. Our numerical example indicates that the buyer cannot implement the first-best effort level when the supplier' effort becomes unobservable. The buyer's second-best optimal order quantity and expected profit are smaller than those under the first-best case. The results of our sensitivity analysis imply that when the supplier's disutility of effort increases, it's optimal for the buyer to schedule the transfer function with a higher coefficient for inducing a lower effort level.

However, the unique solution for the buyer's problem under the first-best approach may not be the same as that for original problem, which results in the incorrect solution by using the first-order approach. The reason is that the first-order condition (4.8) is not always equivalent to the incentive constraint (4.3). In general there are more effort levels satisfying the first-order condition (4.8) than those satisfying the incentive constraint (4.3), since the supplier's problem may not be concave w.r.t. e. When the supplier's problem is not well-behaved, the first-order approach is not valid. One possible approach is to use an approximation whereby the effort is discretized as a finite number of possible effort choices and the yield is also discretized as a finite number of possible yield outcomes. As mentioned in Chapter 3, such discrete model can be solved based on a two-stage process popularized by Grossman and Hart [17].

First, we discretize the effort $e \in [e^-, e^+]$ as e_i for $i = 1, 2, \ldots, m$ and the yield $u \in [0, 1]$ as u_j for $j = 1, \ldots, n$, where $e^- = e_1 < e_2 < \cdots < e_m = e^+$ and $0 \leq u_1 < u_2 \cdots < u_n = 1$. For a fixed value of the effort e_i, we define $p_{ij} = \Pr(U = u_j \mid e = e_i)$ as the probability that the outcome will be u_j for $j = 1, \ldots, n$, thus discretizing the density $g(u; e)$ as well. We also write $t_j \equiv t(u_j)$ as the transfer price to the supplier if u_j is observed and define $\mathbf{t} = (t_1, \ldots, t_n)$. With this approximation, the supplier's participation constraint (4.2) reduces to

$$\sum_{j=1}^{n} A(t_j)p_{ij} - c(e_i) \geq a_0. \tag{4.11}$$

If the supplier chooses an effort level e_i, then the supplier's incentive constraint (4.3) becomes the $m-1$ incentive constraints

$$\sum_{j=1}^{n} A(t_j)p_{ij} - c(e_i) \geq \sum_{j=1}^{n} A(t_j)p_{kj} - c(e_k), \tag{4.12}$$

which must hold for $k = 1, \ldots, m$ and $k \neq i$. Thus, the buyer's continuous optimization problem in Section 4.3 becomes a discrete optimization problem, as we have discussed in Section 3.4 of Chapter 3. We denote the discrete optimization problem as (SB-e_i), which is written as

$$\text{(SB-}e_i\text{):} \qquad \begin{array}{ll} \max_{\{Q,\mathbf{t}\}} & \Pi_i(Q,\mathbf{t}) = \sum_{j=1}^{n}[b_j(Q) - t_jQ]p_{ij} \\ \text{s. t.} & (4.11) \text{ and } (4.12), \end{array}$$

where

$$b_j(Q) = \int_0^{Qu_j} [(r-v)x + vQu_j]f(x)\, dx + \int_{Qu_j}^{\infty} [(r+s)Qu_j - sx]f(x)\, dx,$$

for $j = 1, \ldots, n$.

Next, based on a two-stage process popularized by Grossman and Hart [17], we develop a three-stage process to obtain the optimal solution since the buyer's problem not only involves the incentive mechanism design but also involves the ordering decision. Before presenting the solution procedure, we convert the buyer's maximization problem to a minimization problem given Q and e_i, i.e., $\max_{\{\mathbf{t}\}} \Pi_i(Q,\mathbf{t}) = \sum_{j=1}^{n}[b_j(Q) - t_jQ]p_{ij}$ is equivalent to $\min_{\{\mathbf{t}\}} Q\sum_{j=1}^{n} t_jp_{ij}$, subject to (4.11) and (4.12). Then the three-stage process is summarized as follows:

- Stage 1: Given $Q > 0$, choose a particular effort level $\bar{e}_i$ and find the least total transfer cost of implementing it for the buyer, i.e., choose an incentive contract $\bar{\mathbf{t}}$ from the set of menus $\{\mathbf{t}\}$ to minimize $Q\sum_{j=1}^{n} t_jp_{ij}$ subject to (4.11) and (4.12). We define

  $$C(\bar{e}_i) = \inf\left\{ Q\sum_{j=1}^{n} t_jp_{ij} \mid \mathbf{t} = \bar{\mathbf{t}} \text{ implements } \bar{e}_i \right\},$$

  when the constraint set is nonempty. Otherwise, we write $C(\bar{e}_i) = \infty$. For each effort level $e_i \in [e_1, \ldots, e_m]$, we compute $C(e_i)$.

- Stage 2: Substitute $\bar{\mathbf{t}}$ for implementing $\bar{e}_i$ into the buyer's expected profit function $\Pi_i(Q, t)$, which is maximized to obtain the second-best order quantity $Q(\bar{e}_i)$ for inducing $\bar{e}_i$. For each effort level $e_i \in [e_1, \ldots, e_m]$, we compute $Q(e_i)$.
- Stage 3: Choose the effort level that maximizes the difference between the expected benefit from inducing e_i and the total transfer cost of implementing it, i.e., choose $e_i \in [e_1, \ldots, e_m]$ to maximize $\sum_{j=1}^{n}[b_j(Q(e_i)) - t_j Q(e_i)]p_{ij}$. We denote the second-best optimal effort level as $\hat{e}$, which maximizes $\sum_{j=1}^{n} b_j(Q(e_i))p_{ij} - C(e_i)$ on $e_i \in [e_1, \ldots, e_m]$. The second-best optimal incentive mechanism $\hat{\mathbf{t}}$ is the one that implements the second-best optimal effort $\hat{e}$ at the least total transfer cost, i.e., $Q(\hat{e}) \sum_{j=1}^{n} \hat{t}_j p_{ij} = C(\hat{e})$, and the second-best optimal order quantity $\hat{Q}$ is the one obtained from $Q(\hat{e})$.

Note that the above solution procedure is more general than that provided in Section 3.4.1 of Chapter 3 where the supplier is assumed to be risk-averse with $A(t_j) = \sqrt{t_j}$.

Chapter 5
Conclusion and Future Research

In this thesis, we concentrated our attention on the applications of incentive mechanism design theory to operations and supply chain management (OSCM). In order to understand the significant role of risk aversion inherent in the mechanism design problems, we analyzed a contracting problem when both the principal and the agent are risk-averse under the symmetric information ("first-best") case in Chapter 2. In this chapter, we provided a thorough analysis of the first-best problem when both participants possess non-exponential utility functions and found the optimal wage scheme and the optimal effort level. Then, we applied the theory of incentive mechanism design to the OSCM-related asymmetric information ("second-best") problems. Specifically, in Chapters 3–4, we studied how a buyer designs an incentive mechanism (or a contract) to induce his supplier's effort in improving supply yield and what's the buyer's optimal ordering decision in a setting of random yield for the critical component and uncertain demand for the finished products.

In Chapter 3, we first investigated the buyer's optimal incentive contract and order quantity for inducing the supplier's high level of effort under both first-best and second-best cases. We then constructed a general discrete model with multiple effort choices and yield outcomes under the second-best case. For this general discrete model, we developed an algorithm based on Lagrangian duality to determine the second-best incentive contract given an effort level. We also found the second-best optimal effort level and order quantity by evaluating the buyer's expected profit for each possible level of effort. In Chapter 4, we considered a continuous model where the supplier's effort and yield become continuous variables. The continuous model involves an infinite number of incentive constraints, which makes the problem more difficult to solve. One possible approach is to use Holmström's first-order approach, where the infinite number of incentive constraints is replaced by the first-order condition. Our numerical example indicates that the buyer cannot implement the first-best effort level when the supplier' effort becomes unobservable. The buyer's second-best optimal order quantity and expected profit are smaller than those under the first-best case. We also performed sensitivity analysis to show that when the supplier's disutility of effort increases, it's optimal for the buyer to schedule the transfer function with a higher coefficient for inducing a lower effort level. All results obtained in Chapters 2–4 are based on the assumption that the relationships between buyers and suppliers are one-shot.

In reality, the relationships between buyers and suppliers are often repeated over time. As a realistic and more challenging extension of the single-period model presented in Chapter 3, we will address the dynamic moral hazard problem between the buyer and the supplier in Section 5.1 . In the dynamic model, the buyer is responsible for determining the optimal dynamic ordering policy in a periodic-review inventory setting and the optimal long-term optimal incentive mechanism offered to the supplier. The analysis of such a problem is difficult since the optimal schemes are history dependent and require strategic commitment. We will present a two-period discrete model under moral hazard and will consider the N-period extension of the problem as our future research direction.

5.1 A Two-Period Discrete Model

Recall that in Chapter 3, the buyer not only designs an incentive mechanism (or a contract) to induce his supplier's effort in improving supply yield, but also decides his ordering quantity under random yield and uncertain demand in a single period. Now we consider a case of two periods, where k denotes the time index and $k = 1, 2$. Then the buyer's problem involves designing a long-term incentive mechanism at the beginning of period 1 and choosing his order quantities in each period in a *(dynamic) periodic review framework.*

At the beginning of period k, the supplier (agent) privately chooses an effort level $e_{ki} \in \{e_1, \ldots, e_m\}$. The supplier's yield level takes n possible outcomes, i.e., $u_{kj} \in \{u_1, \ldots, u_n\}$ for $k = 1, 2$. We denote the probability of yield u_{kj} given the effort choice e_{ki} in period k as $p_{ij}(e_{ki}) = \Pr(U = u_{kj} \mid e = e_{ki})$, for $i = 1, \ldots, m$, $j = 1, \ldots, n$ and $k = 1, 2$. As in Chapter 3, we assume that $0 < p_{ij}(e_{ki}) < 1$ for all $e_{ki} \in \{e_1, \ldots, e_m\}$. The buyer's incentive design problem is to determine a long-term contract $\{t_1(u_{1j}), t_2(u_{1j}, u_{2j})\}$, where the transfer function $t_1(u_{1j})$ is associated with the yield u_{1j} in period 1 and transfer function $t_2(u_{1j}, u_{2j})$ is associated with the yields u_{1j} and u_{2j} in periods 1 and 2, respectively.

Besides the incentive mechanism design issue, the buyer also decides his optimal order quantities Q_1 and Q_2 in each period in a periodic review framework. That is, the buyer manages his inventory using a periodic-review system where he places an order every period. Thus, the buyer's inventory is a dynamic process, where y_1 denotes the initial inventory level at the beginning of period 1, y_2 denotes the inventory level at the beginning of period 2 (the end of period 1), and y_3 denotes the inventory level at the end of period 2. Thus, after a yield realization of u_1 and a demand realization of x_1 in period 1, the buyer's inventory level y_2 is computed as $y_2 = y_1 + Q_1 u_1 - x_1$. Similarly, after a yield realization of u_2 and a demand realization of x_2 in period 2, the buyer's inventory level y_3 is computed as $y_3 = y_2 + Q_2 u_2 - x_2$. At the end of each period, the buyer receives his profit and makes a transfer to the supplier according to the terms of the contract, i.e., the transfer contract is executed at the end of each period. Moreover, the buyer's optimal long-term contract offered at the beginning of period 1 is "renegotiation-proof", if at period k "the continuation contract is an optimal solution to the continuation contracting problem for the remaining periods" (Bolton and Dewatripont [5, p.421]). We will find the conditions ensuring that the buyer's optimal contract is renegotiation-proof as our future work. The timing of the contracting and ordering problems is provided in Figure 15.

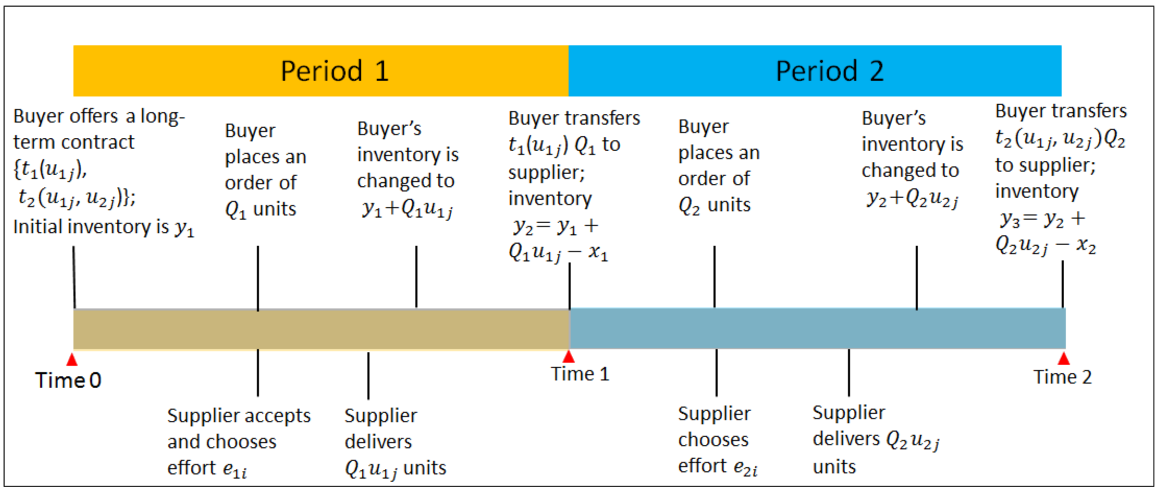

Figure 15. The timing line for the contracting and ordering problems in two periods.

5.1.1 The Supplier's Optimization Problem

As in Chapter 3, We assume that the supplier is risk-averse with an increasing and concave utility function $A(t)$ and $A(0) = 0$, i.e., $A' > 0$ and $A'' < 0$. The buyer's order quantity is assumed to be a positive scalar for the supplier. Thus, given any order $Q_1 > 0$ and $Q_2 > 0$ at periods 1 and 2, respectively, the supplier will be reluctant to accept the buyer's offer, unless her expected utility at the beginning of period 1 is as much as what she can obtain from another, alternate, long-term contract, i.e., her reservation utility $2a_0 > 0$ (no discounting effects). If the supplier accepts the offer and chooses her effort level as e_{ki} in period k, this will result in a disutility level of $c(e_{ki})$, which is an increasing and convex function, i.e., $c' > 0$ and $c'' > 0$.

To maintain the separation between periods, we assume that the supplier's preference is time separable (see Bolton and Dewatripont [5, p. 420]). In addition, we assume that the supplier doesn't have any access to the credit market so that she has to rely on the buyer's total transfer wealth over time. Therefore, if the supplier accepts the buyer's offer and chooses her effort strategy as $[e_{1i}, e_{2i}]$, i.e., the supplier chooses e_{1i} at period 1 and e_{2i} at period 2, then her expected utility at the beginning of period 1 is written as,

$$\begin{aligned}&\Pi_S\left(e_{1i}, e_{2i}\right)\\ =\ &\sum_{j=1}^{n} p_{ij}\left(e_{1i}\right)\left[A\left(t_1\left(u_{1j}\right)\right) - c\left(e_{1i}\right) + \sum_{j=1}^{n} p_{ij}\left(e_{2i}\right)\left(A\left(t_2(u_{1j}, u_{2j})\right) - c\left(e_{2i}\right)\right)\right].\end{aligned}$$

The supplier will accept the buyer's long-term contract only if her participation constraint is satisfied, i.e.,

$$\Pi_S\left(e_{1i}, e_{2i}\right) \geq 2a_0.$$

The supplier's optimization problem involves choosing the effort strategy to maximize her expected utility at the beginning of period 1 given the buyer's long-term contract $\{t_1(u_{1j}), t_2(u_{1j}, u_{2j})\}$. Thus, we write the supplier's problem as,

$$\begin{array}{ll}\max_{\{\tilde{e}_{1i},\tilde{e}_{2i}\}} & \Pi_S\left(\tilde{e}_{1i}, \tilde{e}_{2i} \mid t_1(u_{1j}), t_2(u_{1j}, u_{2j})\right)\\ \text{s. t.} & \Pi_S\left(\tilde{e}_{1i}, \tilde{e}_{2i} \mid t_1(u_{1j}), t_2(u_{1j}, u_{2j})\right) \geq 2a_0,\end{array}$$

where $\tilde{e}_{1i}, \tilde{e}_{2i} \in \{e_1, \ldots, e_m\}$ and $u_{1j}, u_{2j} \in \{u_1, \ldots, u_n\}$.

5.1.2 The Buyer's Optimization Problem

To maintain the separation between periods, we also assume that the buyer's preference is time separable. Then, without any discounting effects, the buyer's expected profit at the beginning of period 1 is written as,

$$\begin{aligned}&\Pi_B\left(Q_1, Q_2, t_1\left(u_{1j}\right), t_2(u_{1j}, u_{2j}) \mid y_1, y_2\right)\\ =\ &\sum_{j=1}^{n} p_{ij}\left(e_{1i}\right)\left[\begin{array}{c} b\left(Q_1, u_{1j} \mid y_1\right) - t_1\left(u_{1j}\right) Q_1 \\ + \sum_{j=1}^{n} p_{ij}\left(e_{2i}\right)\left(b\left(Q_2, u_{2j} \mid y_2\right) - t_2(u_{1j}, u_{2j}) Q_2\right)\end{array}\right],\end{aligned}$$

where

$$\begin{aligned}b(Q_1, u_{1j} \mid y_1) &= \int_0^{y_1+Q_1u_{1j}} \left[(r-v)x + \left(y_1 + Q_1u_{1j}\right)v\right] f(x)\ dx\\ &\quad + \int_{y_1+Q_1u_{1j}}^{\infty} \left[(r+s)\left(y_1 + Q_1u_{1j}\right) - sx\right] f(x)\ dx,\end{aligned}$$

and

$$\begin{aligned} b(Q_2, u_{2j} \mid y_2) &= \int_0^{y_2+Q_2u_{2j}} [(r-v)x + (y_2 + Q_2u_{2j})\, v] f(x)\, dx \\ &\quad + \int_{y_2+Q_2u_{2j}}^{\infty} [(r+s)(y_2 + Q_2u_{2j}) - sx] f(x)\, dx, \end{aligned}$$

for $j = 1, \ldots, n$.

If the supplier's effort cannot be observed by the buyer, and the buyer wants to implement a particular effort strategy $[e_{1i}, e_{2i}]$, then the supplier's optimization problem in Section 5.1.1 turns into the buyer's incentive constraint, i.e.,

$$[e_{1i}, e_{2i}] = \arg \max_{\{\tilde{e}_{1i}, \tilde{e}_{2i}\}} \Pi_S (\tilde{e}_{1i}, \tilde{e}_{2i}).$$

Thus, the buyer's problem is to find the optimal incentive mechanism $(t_1(u_{1j}), t_2(u_{1j}, u_{2j}))$ that will implement the supplier's effort strategy $[e_{1i}, e_{2i}]$ and the best order policy $[Q_1, Q_2]$ that will maximize his expected net profit for inducing the supplier's effort strategy $[e_{1i}, e_{2i}]$, i.e.,

$$\begin{array}{ll} \max_{\{Q_1, Q_2, t_1(u_{1j}), t_2(u_{1j}, u_{2j})\}} & \Pi_B (Q_1, Q_2, t_1(u_{1j}), t_2(u_{1j}, u_{2j}) \mid y_1, y_2) \\ \text{s. t.} & \Pi_S (e_{1i}, e_{2i} \mid t_1(u_{1j}), t_2(u_{1j}, u_{2j})) \geq 2a_0, \\ & [e_{1i}, e_{2i}] = \arg\max_{\{\tilde{e}_{1i}, \tilde{e}_{2i}\}} \Pi_S (\tilde{e}_{1i}, \tilde{e}_{2i} \mid t_1(u_{1j}), t_2(u_{1j}, u_{2j})). \end{array}$$

The above problem is difficult to solve since the transfer scheme is history-dependent of the yield and the order policy is history-dependent of the inventory. Fortunately, Plambeck and Zenios [38] provides a general framework for analyzing a dynamic and stochastic decentralized system within the principal-agent framework. For our two-period dynamic model, we plan to solve the buyer's problem in a two-step backward logic based on Plambeck and Zenios' framework [38]. The solution procedure is summarized as follows: In the first step, given $Q_1 > 0$ and $Q_2 > 0$, consider the supplier's each possible effort strategy and find a minimum transfer cost scheme that implements the supplier's best-response effort strategy. In the second step, the buyer induces the supplier's best-response effort strategy by using the transfer scheme derived in step 1, and then maximizes his expected profit by choosing the optimal induced effort strategy and the optimal order policy. The analysis from the first-step will generate expressions for the per-period transfer cost to the buyer for inducing the supplier's effort strategy. If the per-period transfer cost is the same at each period, then a standard dynamic-programming recursion will be used to solve the buyer's problem.

5.2 N-Period Extension

There are very few papers dealing with the N-period extension of the problem considered in this chapter, but Bolton and Dewatripont [5, Ch. 10] and Laffont and Martimort [28, Ch. 8.2] provide useful starting points for analyzing simplified versions of the problem with few periods and two effort levels; our goal is to extend their results to multiple periods and continuum of efforts. It is also worth mentioning Plambeck and Zenios [38] who provide a valuable modeling framework for analyzing a dynamic and stochastic decentralized system within the principal-agent framework. We believe the N-period extension of the single-period model presented in Chapter 3 is important and challenging, and plan to study it based on Plambeck and Zenios' framework [38] as our future goal.

Note that another possible extension is to consider a continuous-time framework of the buyer's problem presented in Chapter 3. Sannikov [41] provides a good starting point for the use of continuous-time framework for moral-hazard problem. Under certain assumptions, we believe that the continuous-time framework will simplify the analysis of finding the optimal solution and provide insightful implications.

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