THE BINDING OF A Λ PARTICLE

THE BINDING OF A Λ PARTICLE IN NUCLEAR MATTER AND IN $_{\Lambda}He^5$

By

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SCOPE AND CONTENTS:

In this thesis, we are concerned with the outstanding problem of overbinding of a Λ particle in the spin-isospin saturated nuclear medium. The suppression effect in nuclear matter due to the repulsive core has been examined carefully and the single-particle-energy spectrum of the Λ particle is then derived. The two-channel formalism which takes account of the $\Sigma\Lambda$ conversion explicitly is used to study the binding of Λ in nuclear matter and in ${}_{\Lambda}He^5$. With phenomenological ΛN interactions which fit the low energy scattering data, it is shown that in this formalism the experimental binding energy of ${}_{\Lambda}He^5$ can be reproduced.

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CHAPTER I

INTRODUCTION

§1 Motivation

The classification of the fundamental particles within the framework of the SU(3) symmetry or the "eight-fold way"is now well established. Accordingly, the proton (p) and the neutron (n) belong to the octet which consists of p, n, $\Lambda, \Sigma^{+}, \Sigma^{-}, \Sigma^{\circ}$, Ξ^{-} and Ξ°. The proton and the neutron are referred to as the nucleons (N) while the rest of the particles in the octet are referred to as the hyperons (Y). The SU(3) symmetry implies that these eight particles are basically of the same nature. As the interaction among the nucleons is strong enough to form a bound state, namely the atomic nucleus, then it is only natural to expect that the interactions among members of the octet may also be strong enough to bind them together, provided the SU(3) symmetry holds or at least it is not strongly broken. Such systems are called hypernuclei. The earliest experimental evidence for the existence of a hypernucleus was revealed in a photographic emulsion which had been exposed to cosmic ray radiation at high altitude[1]. Since then a number of hypernuclei containing one or two Λ particles have been identified and their binding energies determined. In this thesis, we confine ourselves to those which contain only one Λ particle.

There are at least two motivations for studying hypernuclei. Firstly, we want to understand the basic features of the ΛN interactions, as a natural extension of the studies of the NN interaction. Secondly, we hope to get some new insight into the structure of ordinary nuclei by examining how the nuclei are influenced by the addition of one or more Λ's. Although the Λ particle is similar to the nucleon in many respects, they are still distinctly different from each other. There is no Pauli principle acting between Λ and N. Also Λ is neutral and thus the Coulomb interaction does not come into play in the ΛN system. All these features make Λ a convenient and effective means to probe the structure of the nucleus.

§2 Experimental Basis and Theoretical Calculations in Hypernuclear Physics

A hypernucleus which consists of a Λ particle and an s-shell nuclear core is called an s-shell hypernucleus. The s-shell and p-shell hypernuclei have been studied extensively in photographic emulsion for their π-mesonic decay modes; the binding energies of those hypernuclei are thus determined[2]. In Table I, we list the experimental binding energies for the well established s-shell and p-shell hypernuclei. Here, the symbol ${}_{\Lambda}X^{A}$ is used to denote a hypernucleus formed from the nucleus X; A is the total baryon number of the system including that of the Λ particle. Experimentally no evidence has been found for the existence of a bound state of Λn or Λp. This indicates that the two-body ΛN interaction is not strong enough to form a two-body bound state.

For simplicity, theoretical investigations in the past have been done mostly on the s-shell hypernuclei[3,4]. There, the ΛN interactions involved are mainly in the relative s-state for the s-shell hypernuclei; thus it is possible to link the ΛN interactions directly with the low energy ΛN scattering parameters - the scattering lengths and the effective range - which are now available experimentally[6]. As for the p-shell hypernuclei, one requires a knowledge of the p-state interaction which is only vaguely understood[5].

In nuclear physics, studies of the deuteron and other bound states of nucleons provide valuable information on the nature of the NN interaction. Likewise, we may try to understand the ΛN interaction by analyzing the s-shell hypernuclei. In fact, before the ΛN scattering data were available, the s-shell hypernuclei were the only source of information on the ΛN interaction[3]. A most extensive analysis in this respect was carried out by Herndon and Tang[4]. Using a potential of exponential type, they found that the binding energies of the s-shell hypernuclei, except for ${}_{\Lambda}He^5$, can be fitted by a two-body ΛN potential with a hard core. Their potential parameters do not lead to a two-body ΛN bound state; thus it is consistent with the experimental findings. However, the potential reveals a peculiar feature, i.e. the calculated binding energy of $_{\Lambda}He^5$ was too large as compared with the experimental value.

Recently, some experimental data became available for the ΛN scattering. A natural approach is to set up phenomenological potentials which fit these data, and then calculate the binding energies of various hypernuclei. In setting up phenomenological ΛN potentials, we can be guided by some meson theoretical considerations[88] which have proved to be quite successful in the case of the NN interaction.

For the NN potential, its long-range part is dominated by the one-pion-exchange potential (OPEP). As the interparticle distance becomes shorter, exchanges of two or more pions and also heavier mesons such as η, ω etc. will be important. The most significant difference between the NN and ΛN forces is, perhaps, that the latter has no OPEP tail. This is due to the charge independence of the strong interaction. Hence the long-range part of the ΛN potentials arises from the two-pion exchange. Thus we expect that the range of the ΛN potential is shorter than that of the NN potential.

With the above consideration in mind, one can set up phenomenological ΛN potentials with a "reasonable" range and then adjust the rest of the potential parameters so that the available scattering data can be reproduced. In this approach, one finds that the theoretical calculation of the binding energies of the s-shell hypernuclei are again consistent with the experimental value <u>except for</u> ${}_{\Lambda}He^5$. The calculated binding energy of ${}_{\Lambda}He^5$ in the literature falls into the range from about 6 MeV to 9 MeV, but its experimental value is 3.08 ± 0.02 MeV. This discrepancy between the theoretical and the experimental binding energies of ${}_{\Lambda}He^5$ has provoked many speculations on the peculiar nature of the interaction between or among Λ and nucleons. Before delving into a detailed discussion on this problem, we shall digress for a while and discuss another interesting problem which is intimately related to that of ${}_{\Lambda}He^5$; this is the so-called Λ-particle well depth which is in fact the ground state energy of a Λ particle in nuclear matter.

§3 Overbinding of a Λ Particle in Nuclear Matter and Also in ${}_{\Lambda}He^5$

Nuclear matter is an infinite system consisting of an equal number of protons and neutrons interacting via the nuclear forces. Such a system closely resembles the central part of a heavy nucleus. Because of the translational invariance, the single-particle wave functions in nuclear matter are just plane waves. This makes theoretical calculation very simple. Of course, nuclear matter is an idealized physical system. The "experimental" binding energy of a Λ particle in this system has been estimated by extrapolating the binding energies of the known hypernuclei[7,71]; this is the same process used to estimate the nuclear binding energy of nuclear matter.

There are essentially four methods used so far for theoretical calculations of the binding energy of a Λ particle in nuclear matter - namely: (i) the ordinary perturbation theory which is a series expansion in the strength of the potential, (ii) the Brueckner-Bethe-Goldstone G-matrix approach which replaces an infinite sum of diagrams by the reaction matrix, (iii) the variational approach of Jastrow which introduces a correlated form factor in the wave function for all possible particle-particle interacting pairs, and (iv) the Green's function approach which introduces a propagator at each stage of interaction. Whichever approach is used, it is found that a simple central potential which is in conformity with the low energy scattering data and/or the binding energies of the s-shell hypernuclei always yields a Λ-particle well depth of about 50 MeV or more; whereas the extrapolated experimental well depth is 27 ± 3 MeV[7,71]. Thus, just as is the situation for the binding of $_{\Lambda}He^{5}$, we are confronted with the overbinding of a Λ particle in nuclear matter.

Since the helium nucleus and the nuclear matter resemble each other in that their spin and isospin are both zero, the overbinding of a Λ particle in these systems may result from the same causes. Then any effective means to suppress the overbinding for one of the systems might be equally effective for the other. There are various proposals to account for the overbinding.: (i) The tensor force should be included in addition to the central force. (ii) A repulsive core may be essential in the ΛN interaction. (iii) The effective ΛN force may depend on whether the nucleon is in the free space or it is in a spin-isospin saturated system. (iv) The three-body ΛNN forces or other many-body forces may set in. (v) The rearrangement effect in the nuclear medium may be important because of the presence of the Λ particle.

As noted before, the long-range part of the ΛN force will be dominated by the two-pion-exchange potential, which has been shown to be almost completely central[90]. Hence the tensor force in the ΛN interaction can arise only from exchanges of K-meson and heavier bosons. However, the strong tensor component of the NN force arises from the OPEP[40]. Consequently, the range of the tensor component of the ΛN force will be much shorter than that of the NN force. The effect of such a short-range ΛN tensor force on the s-shell hypernuclei were considered by Buxton and Schrills[9], and Law, Gunye and Bhaduri[10]; and that on the Λ particle in nuclear matter were investigated by Mueller and Clark[11], Bodmer and Rote[12], and Goodfellow and Nogami[13]. All these calculations indicate that the suppression effect due to the tensor forces is very small.

As for the repulsive core, its suppression effect has been analyzed by Herndon and Tang[4], and Ram and James[8]. Their results show that the suppression due to the repulsive core is much more important than that due to the tensor force, but it is still not large enough to resolve the problems of overbinding in ${}_{\Lambda}He^5$ or of a Λ particle in nuclear matter.

The isospin suppression effect was first pointed out by Bodmer and his collaborators[12,14,15]. In an isospin saturated system such as ${}_{\Lambda}He^5$ or a Λ particle in nuclear matter, the isospin of the ground state is $I = 0$. A very large excitation energy is required to attain the next excited energy level with $I = 1$. For example, the first excited state of He^4 is about 20 MeV above the ground state[16]. Hence, the process $\Lambda N \rightarrow \Sigma N$ available in the free scattering is suppressed in the reaction:

$$\Lambda + He^4 \rightarrow \Sigma + He^{4*} .$$

Here, He^{4*} is the excited state of the helium nucleus. According to a procedure suggested by Bodmer[14], the triplet scattering length in an isospin saturated system should be 20% less than that in a free ΛN system. As a result, the binding energy of ${}_{\Lambda}He^5$ is suppressed by about 2 MeV[17]. Thus this is an appreciable effect.

It would be nice if we could understand every physical system just through a simple model based on two-body interactions.

However, two-body forces alone do not seem adequate to produce the required suppression in the binding energies for either ${}_{\Lambda}He^5$ or a Λ particle in nuclear matter. Inevitably, one has to examine the effect of many-body forces. According to the meson theory, all interactions are mediated by the exchange of mesons. For a system with three or more fundamental particles, the mesons emitted by each pair of particles may lose their memory on their parent particles and the mesons may wander around to form a meson cloud. Certainly, this is a special feature of a many-particle system. In addition, the mesons themselves can interact strongly to emit another chain of mesons. The interaction picture will become more and more complicated in this fashion, and one has to resort to some kind of feasible approximation. Thus far, only simple models of three-body forces have been developed[18,19]. It is difficult to draw any clear-cut conclusions about the effect of the three-body ΛNN forces, because it depends on the cutoff of the potential, the NN correlations and also the specific choice of particular diagrams of interaction. The best estimates are that its suppression in the binding of ${}_{\Lambda}He^5$ is likely to be 1.5 to 3.0 MeV[17,19,20].

To separate a Λ particle from a system of nucleons, the energy required may not be equal to the Λ single-particle energy which we have derived from the energy spectrum of Λ in the energy denominator of the Bethe-Goldstone equation. In the absence of the Λ particle inside the nuclear medium, there is a possibility for the nucleons to rearrange themselves to an energetically more favorable state and this will result in a release of the so-called rearrangement energy. The re-arrangement effect for pure nuclear matter was estimated to be about 10 MeV[89]. An effect of almost the same order of magnitude for a Λ particle in nuclear matter has also been indicated by the works of Dabrowski and his collaborators[78,79].

§4 Scope of the Present Work

The purpose of this thesis is to present a systematic study of the binding mechanism of a Λ particle in a nucleus, by using simple model potentials for the hyperon-nucleon interaction. The focal point of our analysis is the outstanding problem of the overbinding of Λ in ${}_{\Lambda}He^5$ and in nuclear matter. For the hyperon-nucleon interaction, we assume nonlocal separable potentials of the Yamaguchi type[21] whose simple forms greatly facilitate the treatment of a many-body system.

We shall carry out a detailed analysis on the well depth of a Λ particle and also on the binding energy of a Λ particle in ${}_{\Lambda}He^5$ by the use of the so-called two-channel formalism (TCF). Within the framework of this formalism, it has been illustrated by Nogami[19] that the isospin suppression effect and the ΛNN forces may be attributed to the same mechanism. Thus the TCF is very efficient in dealing with the problem of overbinding. The underlying idea of the TCF will be discussed in connection with the analysis on the ΛN scattering in Chapter

II.

Since there are several methods to handle the many-body problem, we shall outline the essence of each method in Chapter III. Bishop's claim of the superiority of the Green function method over the Bethe-Goldstone method will be critically examined. In addition, the self-consistency problem on the energy denominator will be clarified.

In Chapter IV, we shall study the suppression effect due to a repulsive core in the ΛN interaction. At the same time, the single-particle-energy spectrum of a Λ particle in nuclear matter is extracted from a self-consistent calculation. The validity of various approximations such as the energy-gap approximation and the effective-mass approximation will be examined. As for the suppression due to the short-range repulsive core, we find that it can be as large as 10 MeV for the binding energy of a Λ particle in nuclear matter.

The two-channel formalism is employed in Chapters V and VI to study the binding energy of a Λ particle in nuclear matter and in helium. In this formalism, the suppression of the well depth in nuclear matter due to the $\Sigma\Lambda$ conversion is found to be about 15 MeV. As for the binding energy of ${}_{\Lambda}He^5$, we are able to produce its experimental value with some of our model potentials. This is indeed a prime success of the two-channel formalism.

Table I

Experimental binding energies of Λ particle

Hypernuclei	Binding energy (MeV)
${}_{\Lambda}H^{3}$	0.06 ± 0.06
${}_{\Lambda}H^{4}$	2.02 ± 0.05
${}_{\Lambda}He^{4}$	2.31 ± 0.03
${}_{\Lambda}He^{5}$	3.08 ± 0.02
${}_{\Lambda}He^{6}$	4.28 ± 0.15
${}_{\Lambda}Li^{7}$	5.56 ± 0.06
${}_{\Lambda}Be^{7}$	5.09 ± 0.12
${}_{\Lambda}Li^{8}$	6.80 ± 0.05
${}_{\Lambda}Be^{8}$	6.81 ± 0.07
${}_{\Lambda}Li^{9}$	8.25 ± 0.13
${}_{\Lambda}Be^{9}$	6.63 ± 0.04
${}_{\Lambda}B^{10}$	8.62 ± 0.20
${}_{\Lambda}B^{11}$	10.19 ± 0.10
${}_{\Lambda}B^{12}$	11.06 ± 0.10
${}_{\Lambda}C^{13}$	11.32 ± 0.15

CHAPTER II

ON THE ΛN SCATTERING

§1 Introduction

In order to understand fully the mechanism of interaction between particles, we have to study the interaction of the particles when they are at a distance, i.e. the scattering in free space, as well as when they are in a bound state if it exists. Normally, the interaction potential of a two-particle system is derived from scattering phase shifts. Its validity is subject to a further test on its ability to reproduce physical quantities such as binding energy, excited state energies, magnetic moment etc. of a system of particles. Unfortunately, the phase shifts at very high energies are generally not known. Thus a potential derived in this way is not based on the entire knowledge of phase shifts and hence is not devoid of ambiguity. However, even if the phase shift is known for all the energies, extensive works on the potential scattering theory reveal that the potential is still not uniquely determined[84]. Therefore, a check on its consistency with other information is always essential.

Thus far, the phase shift analysis has been applied most extensively in the studies of nucleon-nucleon forces.

The phase shifts for the NN scattering are known in the laboratory energy up to about 350 MeV. There are now several sets of "realistic" potentials, such as the Hamada-Johnston potential[81], the Yale-potential[82] and the Reid soft-core potential[83], that can reproduce these scattering data. As for the ΛN scattering, only scanty scattering data in the centre of mass momentum range 110 ∿ 330 MeV/c and a few more at higher momenta are available. These data are scarcely enough to determine the low energy scattering parameters: the scattering length and the effective range.

Currently, there are two sets of experimental data available for the ΛN scattering[6,55]. Analyses were done using the s-wave scattering cross-section

$$\sigma(\Lambda p) = \frac{\pi}{(-\frac{1}{a_s} + \frac{1}{2} r_s k^2)^2 + k^2} + \frac{3\pi}{(-\frac{1}{a_t} + \frac{1}{2} r_t k^2)^2 + k^2}$$

where a_s, a_t, r_s and r_t are the singlet and triplet scattering lengths and effective ranges, respectively. The best parameters obtained by Alexander et al.[6] are

$$\begin{aligned} a_s &= -1.8 \text{ fm} \quad ; \quad a_t = -1.6 \text{ fm} \\ r_s &= \ \ 2.8 \text{ fm} \quad ; \quad r_t = 3.3 \text{ fm} \end{aligned} \qquad \text{(II-1)}$$

and that by Sechi-Zorn et al.[55] are

$$\begin{aligned} a_s &= -2.0 \text{ fm} \quad ; \quad a_t = -2.2 \text{ fm} \\ r_s &= \ \ 2.8 \text{ fm} \quad ; \quad r_t = \ \ 3.3 \text{ fm} \end{aligned} \qquad \text{(II-2)}$$

where $1 \text{ fm} = 10^{-13}$ cm.

As was pointed out by the experimental groups themselves and others[25,26], there is a large ambiguity concerning the experimental values of the scattering parameters. This ambiguity can be removed if the ΛN scattering experiment is done with polarized Λ particles so that the singlet and triplet interactions can be analyzed separately[25,27].

In our analysis on the ΛN interaction, we shall take the set of parameters given by Alexander et al.[6]. Since these parameters are nearly spin independent, we shall use their spin average $\bar{a}$ and $\bar{r}_o$ such that the total scattering cross-section σ and its derivative with respect to the energy E, $\frac{d\sigma}{dE}$, at E = 0 remain the same. Thus, we have the relations

$$\bar{a}^2 = \frac{1}{4}\left(a_s^2 + 3a_t^2\right)$$
$$\bar{a}^3(\bar{a}-\bar{r}_o) = \frac{1}{4}\{a_s^3(a_s-r_s) + 3a_t^3(a_s-r_s)\}$$

which yield

$$\begin{aligned} \bar{a} &= -1.6523 \text{ fm} \\ \bar{r}_o &= \ \ 3.1717 \text{ fm} \quad . \end{aligned} \qquad \text{(II-3)}$$

Now, we can try to relate these parameters to the potential parameters. A phenomenological potential is then established. The various potentials that we use to analyze the Λ-particle binding energy in nuclear matter and also in ${}_{\Lambda}He^5$ will be derived in Chapters IV and V. Meanwhile, we shall discuss some general aspects of the ΛN scattering in the one-channel formalism and the two-channel formalism.

§2 One-channel Formalism

When the only open channel in a scattering process is the entrance channel, we can work within the framework of the so-called one-channel formalism where the wave function consists only of a single component. The studies of the ΛN scattering in this formalism do not include the inelastic processes explicitly.

For the potentials used in this thesis, we assume that they are represented by nonlocal separable potentials so that the scattering problem is greatly simplified. Now the Schrödinger equation in momentum space takes the form

$$\left(\frac{k^2}{2\mu} - \frac{p^2}{2\mu}\right)\psi^k(\underset{\sim}{p}) = \int \langle p|V|p'\rangle \psi^k(\underset{\sim}{p}')d^3p' \qquad \text{(II-4)}$$

where μ is the reduced mass; and the separable potential is taken as:

$$\langle \underset{\sim}{p}|V|\underset{\sim}{p}'\rangle = -\lambda g(\underset{\sim}{p})g(\underset{\sim}{p}') \qquad \text{(II-5)}$$

Here, λ is the strength of the potential and $g(\underset{\sim}{p})$ is called a form factor. Unless otherwise specified, the units $\hbar = c = 1$ will be used. If $g(\underset{\sim}{p})$ is a function of the magnitude of the relative momentum $\underset{\sim}{p}$ only, we will show later that the scattering amplitude is spherical symmetric. Thus we are in effect considering an s-wave interaction.

By imposing appropriate boundary conditions, the scattering solution of (II-4) can be written as

$$\psi^k(\underset{\sim}{p}) = \phi^k(\underset{\sim}{p}) + \frac{2\mu}{k^2-p^2+i\varepsilon}\int <\underset{\sim}{p}|V|\underset{\sim}{p}'>\psi^k(\underset{\sim}{p}')d^3p' \qquad \text{(II-6)}$$

where $\phi^k(\underset{\sim}{p}) = \delta(\underset{\sim}{p}-\underset{\sim}{k})$ corresponds to the incident plane wave with momentum $\underset{\sim}{k}$ and ε is a positive infinitesimal just to ensure an out-going scattered wave in our final solution.

According to Lippman and Schwinger[28], the T-matrix is defined as

$$<p|T|k> = <\phi^p|V|\psi^k> . \qquad \text{(II-7)}$$

Now, we can use either the Lippman-Schwinger equation

$$T = V + V\frac{1}{E-H_o+i\varepsilon}T \qquad \text{(II-8)}$$

or (II-6) directly to obtain

$$<p|T|k> = <p|V|k> + 2\mu\int\frac{<p|V|p'><p'|T|k>d^3p'}{(k^2-p'^2+i\varepsilon)} \qquad \text{(II-9)}$$

where E is the energy of the system and H_o is the kinetic energy part of the Hamiltonian operator. This integral equation is readily solved by introducing an auxiliary function

$$X = \int\frac{g(\underset{\sim}{p}')<p'|T|k>d^3p'}{k^2-p'^2+i\varepsilon} . \qquad \text{(II-10)}$$

We obtain finally from (II-9) and (II-10)

$$<p|T|k> = -\frac{\lambda g(\underset{\sim}{p})g(\underset{\sim}{k})}{1+\lambda J(k)} \qquad \text{(II-11)}$$

where

$$J(k) = 2\mu\int\frac{g^2(\underset{\sim}{p})d^3p}{k^2-p^2+i\varepsilon} .$$

The scattering amplitude f(k) and the phase shift δ are related to the T-matrix by the following expression:

$$f(k) = e^{i\delta}\sin\delta/k$$
$$= -4\pi^2\langle k|T|k\rangle$$
$$= 4\pi^2\lambda g(\underset{\sim}{k})^2/[1 + \lambda J(k)] \qquad \text{(II-12)}$$

Thus it is independent of the scattering angle if $g(\underset{\sim}{k}) = g(k)$. In this case, we have picked up a purely s-wave scattering amplitude.

Now we can relate the scattering length a and effective range r_o by the relation

$$k\cot\delta = -\frac{1}{a} + \frac{1}{2} r_o k^2 + \ldots. \qquad \text{(II-13)}$$

For simplicity, let us take the Yamaguchi-type [21] form factor:

$$g(p) = \frac{1}{p^2+\beta^2} \qquad \text{(II-14)}$$

then we obtain

$$J(k) = 4\pi^2\mu g(k)^2[\frac{k^2-\beta^2}{2\beta} - ik]$$

and finally (II-12) and (II-13) yield

$$\frac{1}{a} = \frac{\beta}{2}\left(1 - \frac{\beta^3}{2\pi^2\mu\lambda}\right) \qquad \text{(II-15)}$$

$$r_o = \frac{1}{\beta}\left(1 + \frac{\beta^3}{\pi^2\mu\lambda}\right) \qquad \text{(II-16)}$$

Since the Fourier transform of g(p) gives

$$\int d^3p\, g(p)e^{i\underset{\sim}{p}\cdot\underset{\sim}{r}} \sim \frac{e^{-\beta r}}{r}$$

we shall interpret $1/\beta$ as the range of the potential. The strength λ and the range $1/\beta$ for the ΛN scattering may now be determined by combining the results of (II-2), (II-15) and (II-16).

Here the fitting of the potential parameters is done phenomenologically and only the low energy scattering phase shifts have been taken into account. However the high-energy behaviour of the phase shifts is important if we want to get a more realistic potential from the scattering data. In fact the off-the-energy-shell matrix element of the scattering amplitude can be quite different for potentials with different phase shift behaviour at high energies. This poses a serious problem in the studies of interaction potential by a phenomenological approach because the off-the-energy-shell matrix element is essential to determine the bound state properties of a many-body system.

Of course, there are many other fundamental problems in the analysis of potential scattering, even if the experimental phase shifts are known to all energies. It is known that phase shift alone does not guarantee a unique potential. However it is possible to obtain a unique potential[84] provided we restrict ourselves to a one-term nonlocal separable potential. The procedure to deduce a potential in this fashion is called the inverse problem. A general discussion of the inverse problem for a nonlocal separable potential may be found in the works of Tabakin[84].

§3 Two-channel Formalism

The discussion of the ΛN scattering is incomplete if we ignore the inelastic processes $\Lambda N \longleftrightarrow \Sigma N$ which involve a mass difference of about 78 MeV. Even if the ΛN system is at an energy well below the Σ-production threshold, the virtual processes could play an important role because the form of the ΛN scattering matrix will be modified by these processes.

In order to take into account the ΣN channel, we have to introduce a two-component Schrödinger equation[29,30]:

$$(\hat{T} + V)\Psi = E\,\Psi \tag{II-17}$$

where Ψ is a two-component wave function which will represent the mixture of ΛN and ΣN components in the system. $\hat{T}$ consists of the kinetic energy operators of ΛN and ΣN and the potential V contains diagonal as well as nondiagonal terms. The diagonal terms $V_{\Lambda\Lambda}$ and $V_{\Sigma\Sigma}$ are responsible for the processes $\Lambda N \rightarrow \Lambda N$ and $\Sigma N \rightarrow \Sigma N$ respectively, and the nondiagonal term $V_{\Lambda\Sigma}$ converts a Λ into a Σ and vice versa. To be more specific, we write

$$\Psi = \begin{pmatrix} \psi_\Lambda \\ \psi_\Sigma \end{pmatrix}$$

$$\hat{T} = \begin{pmatrix} T_\Lambda & 0 \\ 0 & T_\Sigma \end{pmatrix} = \begin{pmatrix} -\dfrac{\nabla_\Lambda^{\,2}}{2\mu_\Lambda} & 0 \\ 0 & -\dfrac{\nabla_\Sigma^{\,2}}{2\mu_\Sigma} \end{pmatrix}$$

and

$$V = \begin{pmatrix} V_{\Lambda\Lambda} & V_{\Lambda\Sigma} \\ V_{\Sigma\Lambda} & V_{\Sigma\Sigma} \end{pmatrix}$$

where $\mu_\Lambda = M_\Lambda M_N/(M_\Lambda + M_N)$; $\mu_\Sigma = M_\Sigma M_N/(M_\Sigma + M_N)$ and M_Σ is the mass of the Σ particle. It is now possible to write (II-17) as two coupled equations:

$$T_\Sigma \psi_\Sigma + (V_{\Sigma\Sigma} - E_\Sigma)\psi_\Sigma = -V_{\Sigma\Lambda}\psi_\Lambda \qquad \text{(II-18)}$$

and

$$T_\Lambda \psi_\Lambda + (V_{\Lambda\Lambda} - E_\Lambda)\psi_\Lambda = -V_{\Lambda\Sigma}\psi_\Sigma \qquad \text{(II-19)}$$

where E_Σ and E_Λ are the energies of the ΣN and ΛN systems. Conservation of energy requires that

$$E_\Lambda - E_\Sigma = M_\Sigma - M_\Lambda \equiv \Delta.$$

Below the Σ-production threshold, Fast et al.[31] have shown that one can obtain an effective one-channel potential for the ΛN system. To see the effect of a two-channel potential, we shall derive such an effective one-channel potential.

From (II-18), we get

$$\psi_\Sigma = -(T_\Sigma + V_{\Sigma\Sigma} - E_\Sigma)^{-1} V_{\Sigma\Lambda}\psi_\Lambda .$$

Substituting this into (II-19), we obtain the effective one-channel Schrödinger equation

$$(T_\Lambda + V_{\Lambda N})\psi_\Lambda = E_\Lambda \psi_\Lambda \qquad \text{(II-20)}$$

where

$$V_{\Lambda N} = V_{\Lambda\Lambda} - V_{\Lambda\Sigma}\ (T_\Sigma + V_{\Sigma\Sigma} - E_\Sigma)^{-1} V_{\Sigma\Lambda}\ . \tag{II-21}$$

Let us introduce the free-particle Green's function

$$G_\Sigma = (T_\Sigma - E_\Sigma)^{-1}$$

Then we have the following expansion:

$$V_{\Lambda N} = V_{\Lambda\Lambda} - V_{\Lambda\Sigma} G_\Sigma V_{\Sigma\Lambda} + V_{\Lambda\Sigma} G_\Sigma V_{\Sigma\Sigma} G_\Sigma V_{\Sigma\Lambda} + \ldots.. \tag{II-22}$$

Consequently we can interpret the effective interaction $V_{\Lambda N}$ in terms of a family of Feynman diagrams. The graphical representation of this effective interaction is shown in Fig. 1. Here, the over all ΛN interaction is represented by a blob and

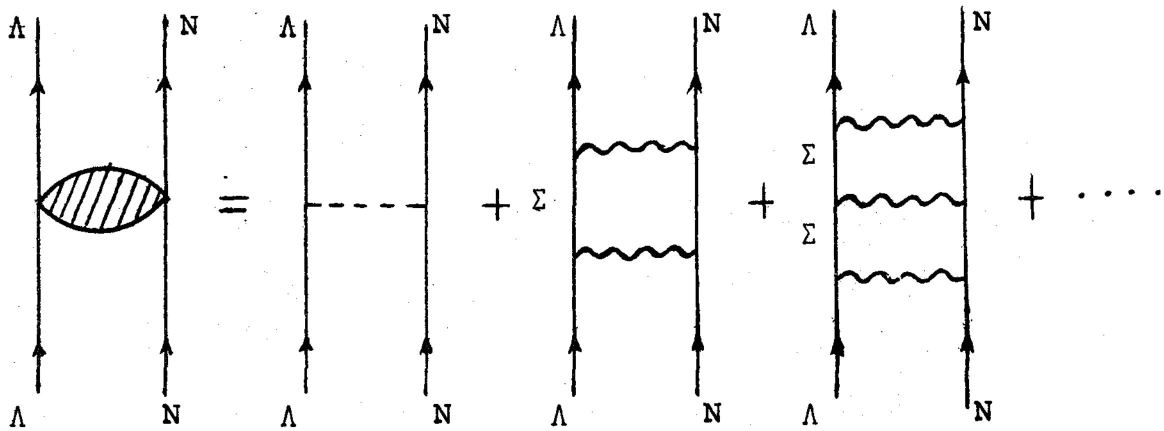

Fig. 1

a dashed line is used to include interaction of ΛN other than the process of the ΛΣ conversion. In addition we have used a wavy line to represent the interaction which causes the ΛΣ conversion as well as that which gives rise to the elastic ΣN scattering.

For a Λ particle in a many-body system, such as nuclear matter, (II-22) is slightly modified. The effective OCF (one-channel formalism) potential $V_{\Lambda N}$ becomes

$$V_{\Lambda N} = \sum_i V_{\Lambda\Lambda}^{(i)} - \sum_i V_{\Lambda\Sigma}^{(i)} G_\Sigma V_{\Sigma\Lambda}^{(i)} - \sum_{i\neq j} V_{\Lambda\Sigma}^{(i)} G_\Sigma V_{\Sigma\Lambda}^{(j)} + \ldots \quad \text{(II-23)}$$

where the superscript i is to identify the nucleons with which the Λ or Σ particle is interacting. The first and the second term are just interactions involving two-body forces but a three-body ΛNN force appears in the third term. The various

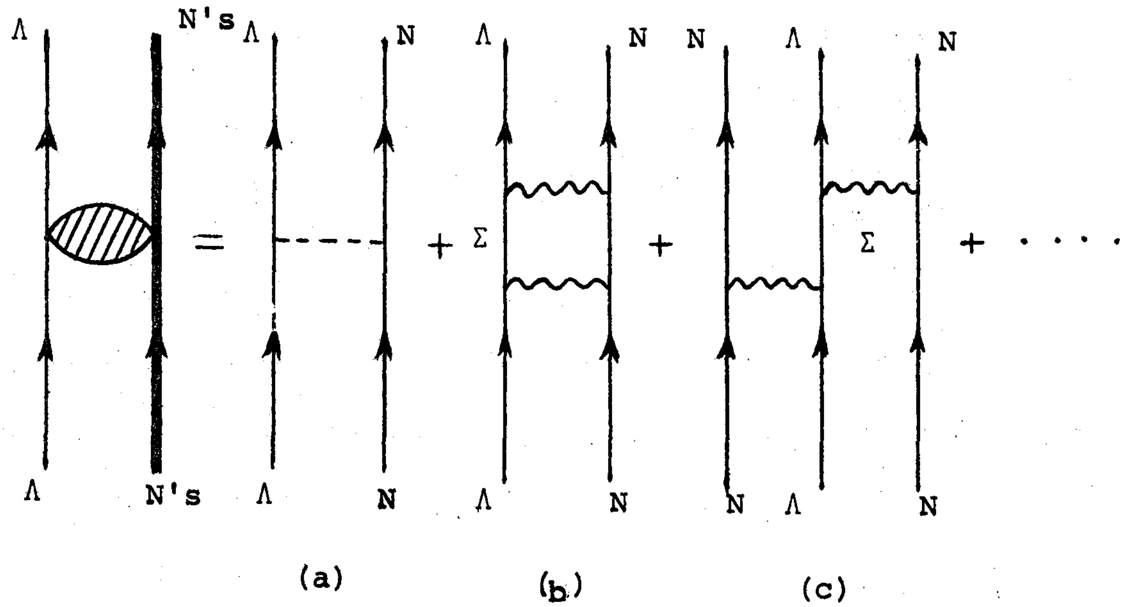

Fig. 2

interactions are shown graphically in Fig. 2. In this diagram we use a heavy line to represent the many-nucleon system. We see, therefore, from Fig. 2 how a three-body ΛNN force comes into play in an effective OCF potential. Figs. 2(b) and 2(c) contain new features of an OCF potential derived from a two-channel formalism. They serve as correction terms to the normal OCF potential where only the Λ particle appears in the intermediate states.

Let us examine the second-order term of (II-23) more closely. From the field theoretical point of view, we have to include the isospin of the nucleon in the representation of $V_{\Lambda\Sigma}$ and $V_{\Sigma\Lambda}$. The modifications are as follows:

$$V_{\Lambda\Sigma}^{(i)} \rightarrow \tilde{\Lambda}^{+}\tilde{\underset{\sim}{\Sigma}}\cdot\underset{\sim}{\tau}^{(i)}V_{\Lambda\Sigma}^{(i)}/\sqrt{3}$$

and

$$V_{\Sigma\Lambda}^{(i)} \rightarrow \underset{\sim}{\tau}^{(i)}\cdot\tilde{\underset{\sim}{\Sigma}}^{+}\tilde{\Lambda}V_{\Sigma\Lambda}^{(i)}/\sqrt{3}$$

where $\tilde{\Lambda}$ $\tilde{\Lambda}^{+}$ and $\tilde{\underset{\sim}{\Sigma}}$, $\tilde{\underset{\sim}{\Sigma}}^{+}$ are the annihilation and creation operators for Λ and Σ particles respectively ; and $\underset{\sim}{\tau}^{(i)}$ is the isospin of the i^{th} nucleon. With this in effect, the second term of (II-23) remains the same but the third term becomes

$$-\frac{1}{3}\sum_{i\neq j}\underset{\sim}{\tau}^{(i)}\cdot\underset{\sim}{\tau}^{(j)}V_{\Lambda\Sigma}^{(i)}G_{\Sigma}V_{\Sigma\Lambda}^{(j)}$$

where we have omitted the annihilation and creation operators. For an isospin saturated system such as He^4 or nuclear matter, the ΛNN force contributes only from the exchange term because expectation value of $\underset{\sim}{\tau}^{(i)} \cdot \underset{\sim}{\tau}^{(j)}$ vanishes for the direct term. Therefore we have a repulsive ΛNN force which is essential for getting the suppressive effect that we are looking for.

We have seen, therefore, that the two-channel formalism includes in essence the three-body forces. We shall use this formalism in our studies on the binding of a Λ particle in nuclear matter and also in $_{\Lambda}He^{5}$.

Of course there are other three-body forces which result from intermediate states involving Y_1^* (1385) etc. One particular diagram is shown in Fig. 3. The contributions from

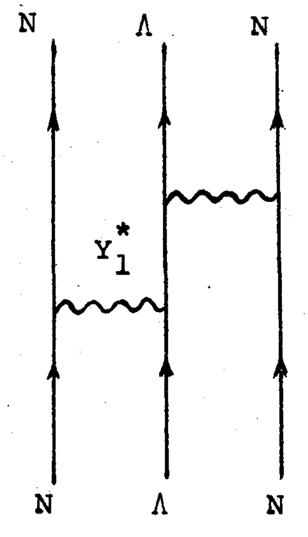

Fig. 3

the various types of three-body forces may be estimated by calculating the S-matrices of the corresponding Feynman diagrams; the Feynman rules for evaluating the S-matrix can be found in the works of Miyazawa, Fujita and Miyazawa[32]. In particular, one can calculate the contribution from a two-pion or a three-pion exchange potential of the three-body ΛNN force. This has already been done by Bhaduri et al. and Loiseau[20] in a static approximation. For the binding of $_{\Lambda}He^{5}$, their findings show that as much as 3 MeV suppression can result from the two-pion-exchange three body ΛNN forces.

CHAPTER III

MANY-BODY PROBLEMS

§1 Introduction

The treatment of many-body problems in a finite system has encountered many technical difficulties. In particular, it is by no means simple to obtain the exact wave functions of the system and hence a knowledge of the exact correlation of the particles is normally not available. Consequently, various models and approximations have been introduced. For example, we may like to study the central portion of heavy nuclei. It will consist of almost equal number of neutrons and protons which are distributed uniformly inside the nucleus. A generalization of this configuration of the nucleons leads to the concept of nuclear matter.

Nuclear matter is defined as a system containing equal numbers of neutrons and protons which fill the infinite space with uniform density. The normal density of nuclear matter is taken to be 0.17 nucleons/fm^3 which yields a Fermi momentum $k_F = 1.36$ fm^{-1}. Of course, we have to switch off the Coulomb force. Otherwise a configuration of nucleons with mass number much larger than 200 will be unstable.

The many-body problem of an infinite system such as nuclear matter has some very simple characteristic features.

There will be no surface effect, and translational invariance of the system implies that the single-particle wave functions are just plane waves. We can then use the methods described in Sec. 3 of this chapter to calculate the physical properties of the system.

Although our works will be guided by those on pure nuclear-matter calculations, we will not study nuclear matter as a problem by itself. In fact, we assume that it has been solved by pure nuclear-matter calculations. Our main concern will be to investigate the nuclear-matter system with an imbedded Λ particle. We hope that the studies of this hypothetical system will provide further information on the basic ΛN interaction as well as the many-body effect in finite hypernuclei.

The experimental basis of such a system will be discussed in Sec. 2. We show that the binding energy of a Λ particle in nuclear matter is around 30 MeV. In Sec. 3, we present various approaches to the many-body problems. It is interesting to examine the relation between the Bethe-Goldstone approach and the Galistskii method. Recently Bishop[50] has claimed that these two methods are not equivalent. However, we shall show that they are essentially the same.

§2 Well Depth of a Λ Particle in Nuclear Matter

The Weizsacker semi-empirical mass formula can be used to deduce the binding energy of a nucleon inside nuclear matter, but for a Λ particle in nuclear matter an equivalent formula is not available to extrapolate its binding energy. To circumvent the difficulty, we make use of the following simple physical property of this Λ particle system. Since we are considering only one Λ particle inside the nuclear medium and besides it is distinguishable from the nucleons, the lowest energy level available for the Λ particle will be at $k_\Lambda = 0$, where k_Λ is the momentum of the Λ particle. Thus for the ground state, the Λ particle will stay at the bottom of the potential generated by the interaction of the Λ particle with nuclear matter. Therefore, its binding energy is just equal to the well depth of this potential. Based on this observation, one can invoke the following procedures to extract the binding energy of a Λ particle in nuclear matter.

One may try to reproduce the binding energies of various known hypernuclei by using a model potential for the Λ-nucleus interaction. The well depth of this model potential for a very large nucleus will then be identified as the Λ particle binding energy in nuclear matter. The procedures, which have been used in the literature, are based mostly on the assumption that the range of the ΛN potential is much smaller than the radius of the nuclear core and the shape of the potential will therefore vary roughly according to the nuclear density.

Let us consider first the model of Walecka[34]. Here, we assume that the depth of the potential well is independent of the mass number and the adding of extra nucleons into the system only causes the increase of the range of the potential, R, which is given by

$$R = r_o(A-1)^{1/3}$$

where A is the total mass number including that of the Λ particle; and $r_o = 1.12$ fm is a size parameter determined from the electron scattering of atomic nuclei such that $3/(4\pi r_o^3)$ gives the average nucleon density[35]. By so doing, we have generated a square-well potential and the binding energies of the hypernuclei will be given by[34]

$$E(A) = D_\Lambda + \frac{\pi^2 A^{-2/3}}{2M_\Lambda r_o^2} \left(1 - \frac{2}{S} + \frac{3}{S^2}\right) \qquad \text{(III-1)}$$

where $S = [2r_o^2 A^{2/3} M_\Lambda E(A)]^{1/2}$. This equation was employed by Lagnaux et al.[7] to analyze the π-mesonic decay of the spallation hypernuclei resulting from the K^- capture of silver and bromine in nuclear emulsion. The events which they observed correspond to the hypernuclei with mass number $40 < A < 100$. The energy released by the process of π-mesonic decay could then be used to calculate the binding energies of these hypernuclei. The best fit to their experimental data yields $D_\Lambda = 27 \pm 3$ MeV. A refined experimental analysis by Lemonne et al.[71] gives $D_\Lambda = 27 \pm 1.5$ MeV.

It is also possible to extract the value of D_Λ by studying the family of the p-shell hypernuclei[4,14,25,36,38]. In this connection, the Λ particle is assumed to move in a Wood-Saxon potential well:

$$V(r_\Lambda) = - V_o/[1 + \exp(r_\Lambda - R_o/S_o)]$$

It is found that the binding energies of the family of the p-shell hypernuclei can be fitted by the following parameters[25]:

$$V_o = 30.0 \text{ MeV}$$

$$R_o = 1.08(A-1)^{1/3} \text{ fm}$$

$$S_o = 0.50 \text{ fm} .$$

Thus the extrapolated well depth is $D_\Lambda = 30.0$ MeV which is slightly higher than the value obtained from the studies of the spallation hypernuclei with $40 < A < 100$.

In the methods presented above, we have assumed that the shape of the potential follows roughly the nuclear density. Hence, the uncertainties in the shape of nucleon distribution will in turn affect the uncertainties in the shape of the potential of Λ particle interacting with the nuclear core. Recently Downs and Kunz[39] estimated these effects by considering the hypernuclei with $A = 45\sim90$. They found that the well depth should lie within the range of the values given below:

$$27 \lesssim D \lesssim 35 \text{ MeV} .$$

This is a rather large uncertainty. In addition, if three-body forces are important, which may well be the case for ΛNN system, the Λ-nucleus potential may not be well represented by a Wood-Saxon shape[19,41]. Furthermore, it has also been pointed out by Bhaduri et al.[41] that the present extrapolation procedure depends on the smoothness of the curve E(A) as a function of $A^{-2/3}$. In fact, because of the spin and isospin dependence of the ΛN force and the presence of ΛNN forces, the curve may not be very smooth for the p-shell hypernuclei. Explicit calculations including ΛNN forces on heavier hypernuclei ${}_{\Lambda}Si^{29}$ and ${}_{\Lambda}Ca^{41}$ show that the binding energies lie far below the extrapolation curve of the uniform model.

Currently, most of the estimated well depths are around 27 ± 3 MeV. As a basis for our theoretical investigation, we will accept this value as the "experimental" binding energy of a Λ particle in nuclear matter.

§3 Methods of Calculation in Many-Body Problem

Our analysis on the problem of a Λ particle in nuclear medium has benefited by the extensive works so far done on pure nuclear matter. The techniques of calculation for nuclear matter can be employed to study the well depth for a Λ particle. Of course, here the Λ particle is distinct from the nucleons and the eigen-state of each ΛN pair need not be antisymmetrized.

Before we proceed to calculate the Λ-particle well depth, we shall discuss briefly the following methods that are available in nuclear matter calculation: (a) Rayleigh-Schrödinger perturbation theory, (b) Brueckner theory, (c) Jastrow correlation approach, and (d) Green's function method of Galitskii and Migdal[42].

(a) <u>Rayleigh-Schrödinger perturbation theory</u>

The Rayleigh-Schrödinger perturbation theory is one of the simplest of all perturbation theories. The method is valid on the assumption that the interaction potential is weak. To see this, we will work out the perturbation expansion of this theory up to the second order.

Suppose the Hamiltonian H of our Schrödinger equation can be separated into two parts

$$H = H_o + H_1 \tag{III-2}$$

and

$$H|\Psi_o\rangle = E_o|\Psi_o\rangle \tag{III-3}$$

where $|\Psi_o\rangle$ is the exact ground state wave function and E_o the ground state energy of the total Hamiltonian. The unperturbed Hamiltonian H_o is chosen to have simple form so that its Schrödinger equation can be solved exactly to yield

$$H_o|\phi_n\rangle = W_n|\phi_n\rangle \tag{III-4}$$

where $|\phi_n\rangle$ is the eigen-wave function of the unperturbed Hamiltonian with the corresponding eigen-value W_n. As usual, the state with n=0 will be taken as the ground state. To keep track of the order of perturbation, we introduce the parameter λ_o into the Hamiltonian, so that

$$H = H_o + \lambda_o H_1$$

and λ_o will eventually be set equal to 1.

The perturbed wave function and energy level will now be written as

$$|\Psi_o\rangle = |\phi_o\rangle + \lambda_o|\Psi_1\rangle + \lambda_o^2|\Psi_2\rangle + \dots$$

$$E_o = W_o + \lambda_o\Delta_1 + \lambda_o^2\Delta_2 + \dots \tag{III-5}$$

where $|\Psi_i\rangle$ and Δ_i are the i^{th} order correction to the wave function and the energy respectively. Substituting (III-5) into (III-3) and comparing terms of every order of the expansion coefficient λ_o we get

$$E_o = W_o + \langle\phi_o|H_1|\phi_o\rangle + \sum_n{}' \frac{\langle\phi_o|H_1|\phi_n\rangle\langle\phi_n|H_1|\phi_o\rangle}{W_o - W_n} + \dots . \tag{III-6}$$

where the prime over the summation symbol is to exclude the ground state from appearing in the intermediate states.

From (III-6), we can obtain the Λ-particle well depth by considering just the contribution from the ΛN pairs. Assuming that H_o contains the kinetic energy part only, the ground state of a Λ particle in nuclear matter will then be given by

$$D_\Lambda = -\sum_{k_N \leq k_F} \langle k_N k_\Lambda|H_1|k_N k_\Lambda\rangle - \sum_{\substack{k_N \leq k_F,\, k_N' > k_F \\ k_\Lambda'}} \frac{\langle k_N k_\Lambda|H_1|k_N' k_\Lambda'\rangle\langle k_N' k_\Lambda'|H_1|k_N k_\Lambda\rangle}{W_N' + W_\Lambda' - W_N - W_\Lambda}$$

$$+ \dots\dots \tag{III-7}$$

where k_F is the Fermi momentum of the nucleons; W_N and W_Λ are the single-particle energies of the nucleon and Λ respectively; and $|k_N k_\Lambda\rangle$ is the product of plane waves for the nucleon and Λ particle with linear momentum k_N and k_Λ. In arriving at (III-7), we have used the fact that $k_\Lambda = 0$ for the ground state of the Λ particle, therefore the term, W_o, corresponding to the kinetic energy does not appear there.

In the Rayleigh-Schrödinger perturbation theory, the expansion is one in powers of the matrix elements of the potential, and hence the series will converge rapidly, provided the interaction potential is very weak. In the case of a potential with strong repulsive core, the theory breaks down because the series of the expansion diverges and each of the matrix elements tends to infinity when a hard core is used. This imposes a severe restriction on its application. Nevertheless, Bodmer and Sampanthar, and Bodmer and Rote[12,18] did apply the theory to the problem of the well depth of a Λ particle as an illustrative calculation by using Yukawa potentials.

(b) Brueckner theory and Brillouin-Wigner perturbation theory

The best known method in nuclear-matter calculation is based on the Brueckner theory[43]. Its mathematical foundation is laid by the Goldstone expansion[44] which is a linked-cluster perturbation series for the ground state energy of a many-body system. The physical contents of the theory and the method of calculation have been investigated extensively by Bethe and his collaborators. Reviews on nuclear matter calculations are available in the literature[45], and the method will be discussed briefly in this subsection.

To begin with, we assume that only the two-body interaction is effective and the Hamiltonian for our many-body system takes the form

$$H = \sum_i T_i + \sum_{i<j} V_{ij}$$

$$= H_o + H_1$$

where

$$H_o = \sum_i (T_i + U_i)$$

$$H_1 = \sum_{i<j} V_{ij} - \sum_i U_i \ .$$

Here, T_i and V_{ij} are the kinetic energy operator and the two-body interaction potential respectively. U_i is the single-particle potential which is chosen in such a way that it may cancel some of the higher-order terms in the perturbation series[72,73] and thus makes the lower-order expansion a better approximation.

In a simplified version of the Brueckner theory, there is a so-called independent-pair approximation in which each pair of particles is treated independently[70] and the rest of nuclear matter is supposed not to affect the (virtual) scattering of the pair. The particles inside the Fermi sea will serve only as a background for the interaction of the pair and their sole effect is to exclude the independent pair from scattering into the intermediate states with $k_N \leq k_F$ because physically all these states are occupied.

Basically, the method is similar to that of Rayleigh-Schrödinger. But instead of treating each term of the infinite series of ladder diagrams shown in Fig. 4 separately, we sum the whole series by introducing the Brueckner G-matrix. Here in Fig. 4, a wavy line stands for G and a dotted line for each interaction that is induced by a two-body potential V. Formally,

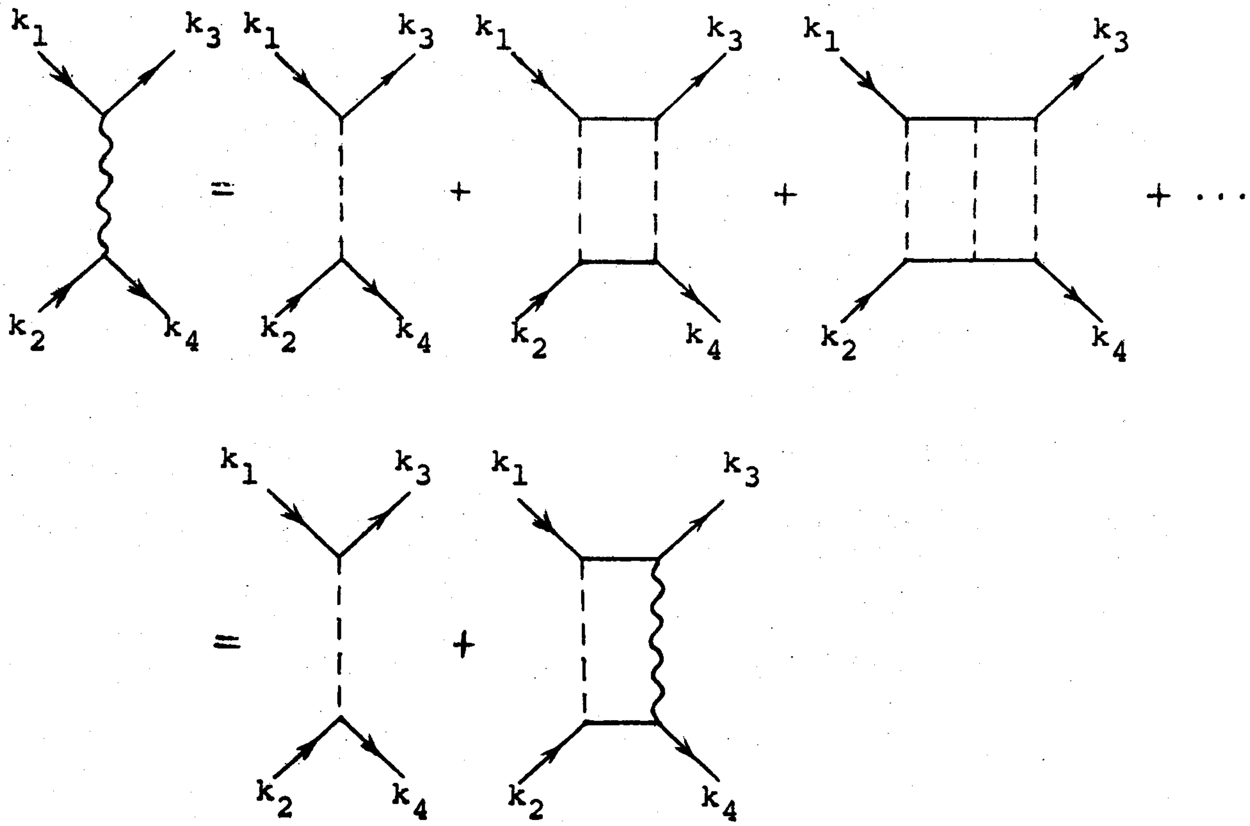

Fig. 4

the G-matrix is defined by

$$G = V - V \frac{Q}{e} G \tag{III-9}$$

where Q is the Pauli operator which will ensure that the momenta of the nucleons in the intermediate states are larger than k_F, and e is the energy denominator which is given by

$$e = H_o - W \tag{III-10}$$

W is the starting energy which will be defined as the sum of the single-particle energies of the initial state of the pair of interacting particles. In this respect, we say that the G-matrix is evaluated on the energy shell. This definition of the G-matrix is related to the choice of the single-particle potential U_i. We hope that the choice would result in the cancellation of some higher-order diagrams. In connection with the Hartree-Fock variational calculation, we may choose U_i self-consistently so that

$$U(k_N) = \sum_{k_N' \le k_F} \langle k_N k_N' | G | k_N k_N' - k_N' k_N \rangle \tag{III-11}$$

where $|k_N k_N'\rangle$ is the product of the single-particle wave functions of the Hamiltonian H_o.

Brueckner and Goldman[72] and also Bethe, Brandow and Petschek[73] found that the third-order diagrams given in Fig. 5 cancel each other if $U(k_N)$ is calculated on the energy shell.

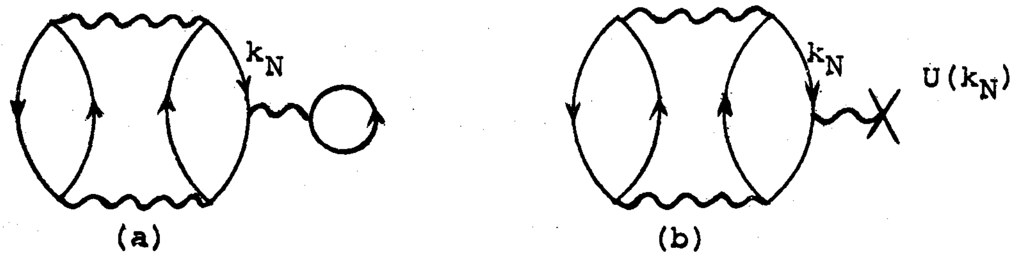

Fig. 5

Of course, there are other third-order diagrams such as those shown in Fig. 6. However, nuclear-matter calculation done by

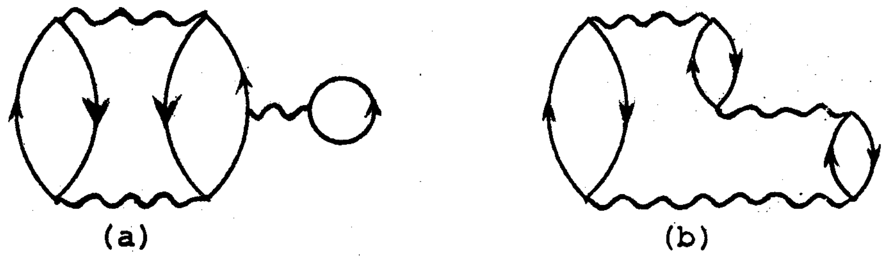

Fig. 6

Dahlblom[74] shows that they are unimportant. Since the contribution from the self-energy insertion of Fig. 6(a) is small, the single-particle potential in the intermediate state will be taken as zero in the so-called standard nuclear-matter calculation. Following this standard procedure, Banerjee and Sprung[66] have recently carried out a calculation on pure nuclear matter by using Reid's soft-core potential. Their analysis shows that the single-particle potential for the nucleon with momentum less than k_F can be represented by the following expression:

$$U(k_N) = (85.17-12.08\ k_N^2)\ \text{MeV} \quad (k_N<k_F) \qquad \text{(III-12)}$$

where the nucleon momentum k_N is in fm^{-1}. This single-particle potential will be adopted throughout our works on the Λ-particle well depth.

There are many advantages of introducing the G-matrix in nuclear-matter calculation. The matrix elements of G are finite even if we are dealing with a hard-core potential, and each G-insertion will account for an infinite sequence of ladder diagrams of all orders for the two-body interaction, as it is clearly shown in Fig. 4. Besides, the second-order diagrams of Fig. 7 are in effect already included in the first-order diagrams of Fig. 8 by the very definition of the G matrices.

Fig. 7

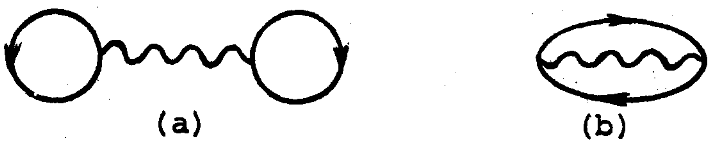

Fig. 8

Thus there is no second-order diagram in the Brueckner theory.

Furthermore, the single-particle potential U_i is at our disposal so that we can make the third-order contribution vanishingly small. Therefore a mere consideration of the first-order diagrams may have its validity well extended up to the third order. In fact, most works on nuclear-matter calculations are done on the line of the first-order diagrams corresponding to Fig. 8. The binding energy up to this order is given by

$$E = \sum_m T_m + \frac{1}{2} \sum_{m,n} \langle k_m k_n | G | k_m k_n - k_n k_m \rangle . \qquad \text{(III-13)}$$

Since nuclear-matter density in the unit of nucleons/fm^3 is much less than 1, it is justifiable to consider only the first-order Brueckner-Goldstone expansion[45]. Indeed, Day[45] has shown that each additional hole line to a diagram reduces its contribution by a factor proportional to the nuclear density. Consequently, higher-order diagrams which inevitably contain more hole lines are unimportant.

Enough has been said on the underlying principle of the Brueckner-Goldstone expansion theory. We shall turn to the studies of a specific example by using the nonlocal separable potential in which the G-matrix can be solved exactly. First, let us derive some of the fundamental formulae. Here, the Hamiltonian is again split into two parts so that $H = H_o + H_1$. H_o is now purely the kinetic energy operator and $H_1 = V$.

Then

$$H_o|\phi_n\rangle = W_n|\phi_n\rangle \tag{III-14}$$

and

$$H|\Psi_o\rangle = E_o|\Psi_o\rangle \tag{III-15}$$

From (III-14) and (III-15), we have for the ground state

$$(E_o-W_o)\langle\phi_o|\Psi_o\rangle = \langle\phi_o|H_1|\Psi_o\rangle\ . \tag{III-16}$$

If we choose the normalization, $\langle\phi_o|\Psi_o\rangle = 1$, then (III-16) becomes

$$E_o = W_o + \langle\phi_o|H_1|\Psi_o\rangle\ . \tag{III-17}$$

In order to calculate the ground state energy, we have to know $|\Psi_o\rangle$. From (III-15), we may split the total Hamiltonian to get

$$H_1|\Psi_o\rangle = (E_o-H_o)|\Psi_o\rangle\ . \tag{III-18}$$

As $(H_o-W_o)|\phi_o\rangle = 0$, we can rewrite (III-18) to get the Bethe-Goldstone equation

$$|\Psi_o\rangle = \frac{W_o-H_o}{H_o-E_o}|\phi_o\rangle - \frac{1}{H_o-E_o}H_1|\Psi_o\rangle$$

$$= |\phi_o\rangle - \frac{1}{H_o-E_o}\{H_1|\Psi_o\rangle - \langle\phi_o|H_1|\Psi_o\rangle|\phi_o\rangle\}$$

$$= |\phi_o\rangle - \frac{Q}{H_o-E_o}H_1|\Psi_o\rangle \tag{III-19}$$

where Q is the Pauli operator which may be defined explicitly as follows

$$Q = \mathbb{1} - |\phi_o\rangle\langle\phi_o|\ .$$

By an iteration of (III-19), the so-called Brillouin-Wigner perturbation expansion of the wave function can be generated. Multiplying this expansion or (III-19) by the operator H_1, one gets

$$H_1|\Psi_o\rangle = H_1|\phi_o\rangle - H_1 \frac{Q}{H_o-E_o} H_1|\Psi_o\rangle$$

$$= H_1|\phi_o\rangle - H_1 \frac{Q}{H_o-E_o} H_1|\phi_o\rangle + H_1 \frac{Q}{H_o-E_o} H_1 \frac{Q}{H_o-E_o} H_1|\phi_o\rangle + \ldots \text{(III-20)}$$

$$= G|\phi_o\rangle \qquad \text{(III-20)}$$

where in the last line we have used the definition of the G-matrix

$$G = H_1 - H_1 \frac{Q}{H_o-E_o} G.$$

Consequently, the ground state energy given by (III-17) becomes

$$E_o = W_o + \langle\phi_o|G|\phi_o\rangle \qquad \text{(III-21)}$$

where $\langle\phi_o|G|\phi_o\rangle$ is the diagonal matrix element of G.

To obtain an explicit expression for the ground state energy E_o, we will now evaluate the nondiagonal as well as the diagonal G-matrix elements. For the sake of clarity, a superscript will be attached to each eigen-state to specify its energy eigenvalue. Multiplying (III-18) by $\langle\phi_o^p|H_1$, we get

$$\langle\phi_o^p|H_1|\Psi_o^k\rangle = \langle\phi_o^p|H_1|\phi_o^k\rangle - \langle\phi_o^p|H_1 \frac{Q}{H_o-E_o} H_1|\Psi_o^k\rangle$$

We may now insert a complete set to the above equation to yield

$$<\phi_o^p|H_1|\Psi_o^k> = <\phi_o^p|H_1|\phi_o^k> - \int \frac{Q(p')<\phi_o^p|H_1|\phi_o^{p'}>d^3p'<\phi_o^{p'}|H_1|\Psi_o^k>}{W(p')-E_o} \quad \text{(III-22)}$$

where W(p') is the energy of the intermediate state and we have explicitly written down the p' dependence of the operator Q.

We are now ready to solve the G-matrix with the non-local separable potential of the following form:

$$<\phi_o^p|H_1|\phi_o^{p'}> = <p|V|p'>$$

$$= -\lambda g(p)g(p') \quad \text{(III-23)}$$

In order to solve (III-22), it is convenient to define

$$X = \int \frac{Q(p)g(p)<\phi_o^p|H_1|\Psi_o^k>d^3p}{W(p)-E_o}\;. \quad \text{(III-24)}$$

Then with the aid of (III-24), the exact solution of (III-22) can be obtained if we use the nonlocal separable potential of (III-23). The result of this manipulation gives

$$<\phi_o^p|G|\phi_o^k> = <\phi_o^p|H_1|\Psi_o^k>$$

$$= -\frac{\lambda g(p)g(k)}{1-\lambda J'(E_o)} \quad \text{(III-25)}$$

where

$$J'(E_o) = \int \frac{Q(p)g^2(p)d^3p}{W(p)-E_o}\;. \quad \text{(III-26)}$$

Therefore, given the form factor g(p) of a nonlocal separable potential, one will be able to calculate the binding energy by the use of (III-21) and (III-25).

Of course, here we have the G-matrix element for only one of the independent pairs of particles. When the total ground state energy of a many-body system is required, we have to take a sum over all the possible pairs. For instance, the binding energy of the Λ particle in nuclear matter will be

$$D_\Lambda = -4 \int_{k_N \leq k_F} \langle \phi_o^k | G | \phi_o^k \rangle \; d^3k_N$$

$$= 4\lambda \int_{k_N \leq k_F} \frac{g^2(k)}{1-\lambda J'(E_o)} \; d^3k_N \qquad \text{(III-27)}$$

where we have used the fact that the contribution from the kinetic energy is zero in this particular case. Since the summation over the spin and isospin of nucleons has to be included, a factor of 4 appears in (III-27).

(c) Jastrow Approach

The self-consistent method of Hartree-Fock cannot be applied to a system of particles interacting with strong forces, because the matrix element involved in such a calculation will be very large and it tends to infinity when a potential with a hard core is used. The flaw of the method lies in its assumption that the total wave function is just a Slater determinant of the single-particle wave functions. In the presence of strong forces, the product of single-particle wave functions could be a very poor approximation for the total wave function. On physical grounds, we expect that the wave function will be drastically modified at short distances and hence a strong correlation should occur.

To overcome all these difficulties, a procedure has been suggested by Jastrow[47]. According to his suggestion, the wave function of a system of N particles is written as

$$\psi(\underset{\sim}{r}_1 \ldots \underset{\sim}{r}_N) = S(\underset{\sim}{r}_1 \ldots \underset{\sim}{r}_N) \prod_{i<j}^{N} f(\underset{\sim}{r}_{ij}) \qquad \text{(III-28)}$$

where $S(\underset{\sim}{r}_1 \ldots \underset{\sim}{r}_N)$ is the product of single-particle wave functions with the appropriate symmetrization for the system of particles; $f(\underset{\sim}{r}_{ij})$ is the correlation function which depends on the relative co-ordinates $\underset{\sim}{r}_{ij}$ and some arbitrary parameters. Physically, the wave functions should vanish inside the hard core and recover as plane waves at infinity. Therefore the correlation will be chosen in such a way that it is zero inside the hard core and tends to 1 at large distances.

To obtain the ground state energy of the system, we do a variational calculation for

$$\langle E \rangle = \frac{\int \psi(\underset{\sim}{r}_1 \ldots \underset{\sim}{r}_N) H(\underset{\sim}{r}_1 \ldots \underset{\sim}{r}_N) \psi^*(\underset{\sim}{r}_1 \ldots \underset{\sim}{r}_N) d^3r_1 \ldots d^3r_N}{\int \psi(\underset{\sim}{r}_1 \ldots \underset{\sim}{r}_N) \psi^*(\underset{\sim}{r}_1 \ldots \underset{\sim}{r}_N) d^3r_1 \ldots d^3r_N} \qquad \text{(III-29)}$$

until a minimum is attained by varying the arbitrary parameters in the correlation functions. Here, $H(\underset{\sim}{r}_1 \ldots \underset{\sim}{r}_N)$ is the Hamiltonian for the N-particle system. In order to obtain a meaningful result and to make our variational calculation to converge faster, it was found by Emery[91] that it is necessary to impose restriction on the correlation functions, such as[11]

$$\int d^3r[f^2(r)-1] = 0 \ . \qquad \text{(III-30)}$$

Otherwise, the correlation functions are completely arbitrary. One unsatisfactory feature of the method is that the calculation depends on the choice of correlation functions.

Instead of suppressing the binding of a Λ particle in nuclear matter the Jastrow approach has consistently yielded a well depth of 60 MeV or more[8,11,48], whereas the independent-pair approximation (IPA) version of Brueckner theory gives values at about 50 MeV[12]. In particular, using the same central potential, Ram and James[8] obtained $D_\Lambda = 62.3$ MeV for the Jastrow approach and $D_\Lambda = 46.4$ MeV for the IPA. Thus there is a discrepancy of about 16 MeV between the two approaches. Attempts have been made to resolve the discrepancy without much success[76,77]. Since the two approaches look completely different, one has to be very careful to place any judgement by comparing just their final results. It remains to be seen whether the Jastrow approach has too strong a correlation among the particles or the IPA does not have enough of this sort of correlation.

(d) Galitskii method

The Green's function method of Galitskii and Migdal[42] has recently been used to calculate the Λ-particle well depth[49,50]. We shall outline the method in this subsection.

The Green's function for a Λ particle is defined as

$$G_\Lambda(x-x') = -i\langle\Psi_o|T(\tilde{\Lambda}(x)\tilde{\Lambda}^+(x'))|\Psi_o\rangle \qquad \text{(III-31)}$$

where $\tilde{\Lambda}(x)$ is the annihilation field operator for the Λ particle in the Heisenberg representation; $|\Psi_o\rangle$ is the true ground state; and T is Wick's chronological ordering operator.*

*In this subsection, the co-ordinates x and the momenta P, p, p', p_1 and p_2 are vectors in 4-space, i.e. $x=(\underset{\sim}{x},t)$, $p=(\underset{\sim}{p},p_o)$ etc.

We have shown in Appendix E that in the energy-momentum space, it takes the form

$$G_\Lambda(p) = [p_o - \frac{\underset{\sim}{p}^2}{2M_\Lambda} - \Sigma_\Lambda(p) + i\varepsilon]^{-1} \qquad \text{(III-32)}$$

where p_o is the energy part of the 4-vector p; $\Sigma_\Lambda(p)$ is the so-called compact or proper self-energy of the Λ particle; and ε is a positive infinitesmal. Of course, there exists a similar expression for the nucleon.

Let us consider the Feynman graphs of Fig. 9 which are actually the ladder diagrams for the self-energy Σ_Λ. At the moment we shall neglect the correlation of the particle in the intermediate states. Thus we shall use the free-particle Green's function propagators for the Λ particle and the nucleon N in the intermediate states, namely

$$G^o_\Lambda(p) = [p_o - \frac{\underset{\sim}{p}^2}{2M_\Lambda} + i\varepsilon]^{-1} \qquad \text{(III-33)}$$

$$G^o_N(p) = [p_o - \frac{\underset{\sim}{p}^2}{2M_N} + i\varepsilon \; \mathrm{sgn}(|\underset{\sim}{p}|-k_F)]^{-1} \qquad \text{(III-34)}$$

where

$$\mathrm{sgn}\;(|\underset{\sim}{p}|-k_F) = \begin{cases} +1 & \underset{\sim}{p}^2 > k_F^2 \\ -1 & \text{otherwise} \end{cases}$$

To evaluate $\Sigma_\Lambda(p)$, it is convenient to study first the Feynman diagrams of the scattering process shown in Fig. 10. Here, we have used Γ_o to represent the totality of scat-

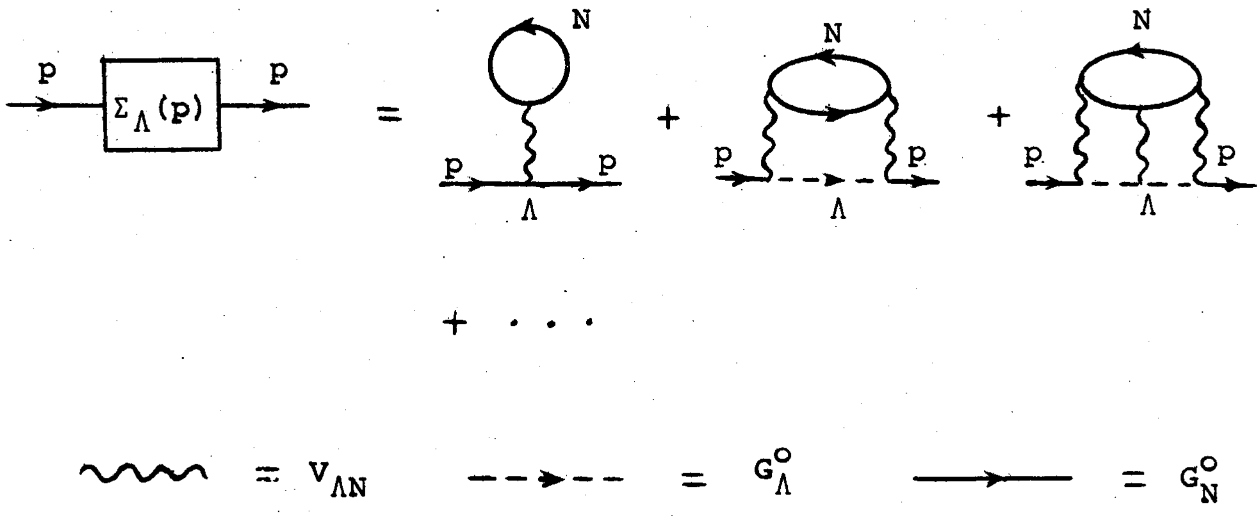
p
$\Sigma_\Lambda(p)$
p
=
N
p
p
Λ
+
N
p
p
Λ
+
N
p
p
Λ
+ . . .
= $V_{\Lambda N}$
= G^{o}_{Λ}
= G^{o}_{N}

Fig. 9

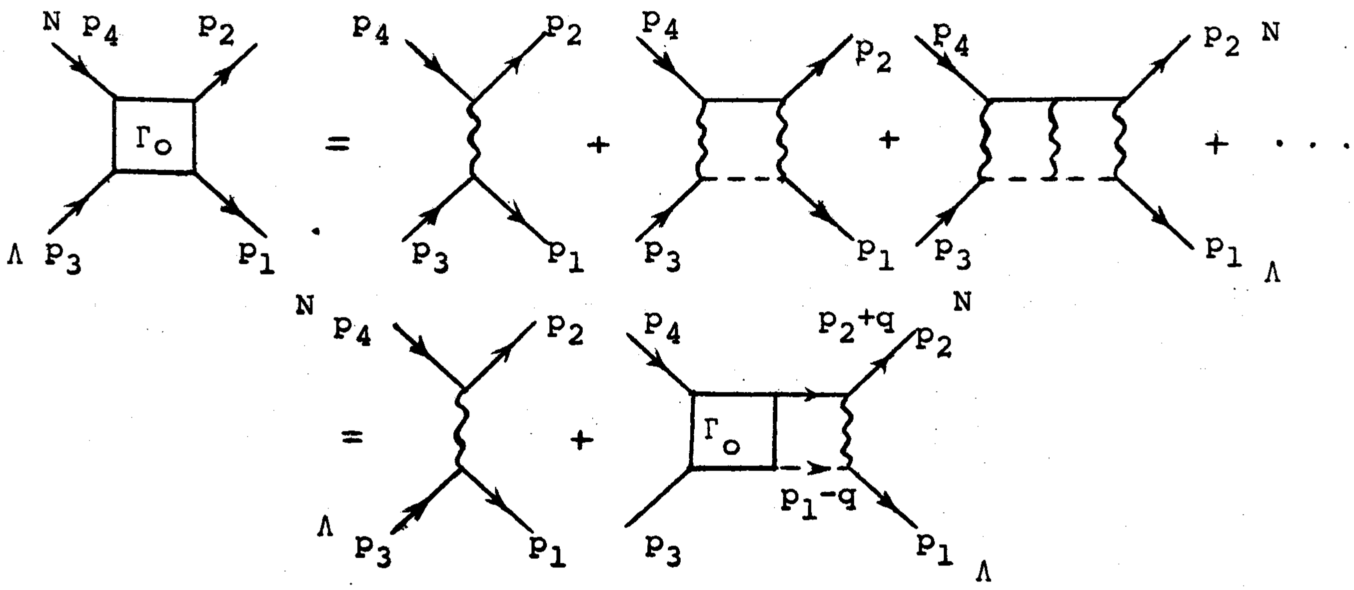
N p_4
p_2
Γ_o
Λ p_3
p_1
=
+
+
+ . . .
N
Λ
p_2+q
p_1-q
Λ

Fig. 10

tering processes in our graph. Comparing Figs. 9 and 10, we note that we can close the nucleon line of the latter to get the former.

Schematically, we introduce an integral equation for Fig. 10,

$$i\Gamma_o(p_1,p_2;p_3,p_4) = iV(p_1,p_2;p_3,p_4) - \int \frac{d^4q}{(2\pi)^4}\,\Gamma_o(p_1-q,p_2+q;p_3,p_4)$$

$$\times\, G_N^o(p_2+q)\;G_\Lambda^o(p_1-q)\,V(p_1,p_2;p_1-q,p_2+q) \qquad \text{(III-35)}$$

It has been shown by Migdal that the $\Gamma_o(p_1,p_2;p_3,p_4)$ here is just the scattering amplitude[42]. Since the total momentum is conserved for each pair of particles, Γ_o can be expressed in terms of the total momentum P and the relative momenta p and p' in the centre of mass system, where

$$P = p_1+p_2 = p_3+p_4$$

$$p = \frac{1}{M_N+M_\Lambda}\,(M_N p_1 - M_\Lambda p_2)$$

$$p' = \frac{1}{M_N+M_\Lambda}\,(M_N p_3 - M_\Lambda p_4)\,. \qquad \text{(III-36)}$$

Furthermore, to be consistent with the requirement of conservation of total momentum, we have

$$V(p_1,p_2;p_3,p_4) = \langle \underset{\sim}{p}|V|\underset{\sim}{p}'\rangle \qquad \text{(III-37)}$$

which is a function of the relative momenta only. Then (III-35) and (III-37) imply that Γ_o is independent of p_o and p_o'. Hence we can introduce a new notation

$$\Gamma_o(\underset{\sim}{p},\underset{\sim}{p}';P) = \Gamma_o(p_1,p_2;p_3,p_4) \ . \tag{III-38}$$

Before we solve (III-35), we have to evaluate the integral

$$I = \oint_c \frac{dq_o}{(2\pi)} G_N^o(p_2+q)\,G_\Lambda^o(p_1-q)$$

where the contour c consists of the whole real axis and the infinite semi-circle on the lower half-plane; and the integration is performed in the clockwise direction. Explicitly, we have

$$I = \oint_c \frac{dq_o}{(2\pi)} \frac{1}{\{(p_2+q)_o - \frac{(\underset{\sim}{p}_2+\underset{\sim}{q})^2}{2M_N} + i\varepsilon\ \mathrm{sgn}(|\underset{\sim}{p}_2+\underset{\sim}{q}|-k_F)\}} \times$$

$$\times \frac{1}{[(p_1-q)_o - \frac{(\underset{\sim}{p}_1-\underset{\sim}{q})^2}{2M_\Lambda} + i\varepsilon]}$$

and the only contribution comes from the pole

$$q_o = -p_{20} + \frac{(\underset{\sim}{p}_2+\underset{\sim}{q})^2}{2M_N} - i\varepsilon\ \mathrm{sgn}(|\underset{\sim}{p}_2+\underset{\sim}{q}|-k_F)$$

provided $|\underset{\sim}{p}_2+\underset{\sim}{q}|>k_F$. Thus, (III-29) becomes

$$\Gamma_o(\underset{\sim}{p},\underset{\sim}{p}';P) = \langle\underset{\sim}{p}|V|\underset{\sim}{p}'\rangle + \int \frac{d^3q}{(2\pi)^3} \Gamma_o(\underset{\sim}{p}-\underset{\sim}{q},p';P)\langle\underset{\sim}{p}|V|\underset{\sim}{p}-\underset{\sim}{q}\rangle$$

$$\times\ \theta\left(\left|\frac{M_N\underset{\sim}{P}}{M_\Lambda+M_N} - \underset{\sim}{p}+\underset{\sim}{q}\right|-k_F\right) \times \frac{1}{D_o} \tag{III-39}$$

where we have used $\underset{\sim}{p}-\underset{\sim}{q}$ for the relative momentum of $\underset{\sim}{p}_1-\underset{\sim}{q}$ and $\underset{\sim}{p}_2+\underset{\sim}{q}$, and the fact that $p_2+q = \frac{M_N}{M_N+M_\Lambda} P-p+q$. The step function θ is defined by

$$\theta(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{otherwise} \end{cases}$$

and we have set

$$D_o = (p_1+p_2)_o - \frac{(\underset{\sim}{p}_2+\underset{\sim}{q})^2}{2M_N} - \frac{(\underset{\sim}{p}_1-\underset{\sim}{q})^2}{2M_\Lambda} + i\varepsilon$$

$$= P_o - \frac{\underset{\sim}{P}^2}{2(M_N+M_\Lambda)} - \frac{(\underset{\sim}{p}-\underset{\sim}{q})^2}{2\mu} + i\varepsilon$$

where μ is the reduced mass of the ΛN system. By changing the integration variable and introducing

$$\varepsilon_o = P_o - \frac{\underset{\sim}{P}^2}{2(M_N+M_\Lambda)} \qquad \text{(III-40)}$$

we obtain from (III-39)

$$\Gamma_o(\underset{\sim}{p},\underset{\sim}{p}';P) = \langle\underset{\sim}{p}|V|\underset{\sim}{p}'\rangle + \int \frac{d^3k}{(2\pi)^3} \frac{\langle\underset{\sim}{p}|V|\underset{\sim}{k}\rangle\theta(|X_N\underset{\sim}{P}-\underset{\sim}{k}|-k_F)\Gamma_o(\underset{\sim}{k},\underset{\sim}{p}';P)}{(\varepsilon_o - \frac{k^2}{2\mu} + i\varepsilon)}$$

(III-41)

where $X_N = M_N/(M_N+M_\Lambda)$. If the nonlocal separable potential of (II-5) is used, we can solve (III-41) exactly to get

$$\Gamma_o(\underset{\sim}{p},\underset{\sim}{p}';P) = - \frac{\lambda g(\underset{\sim}{p})g(\underset{\sim}{p}')}{1-\lambda J''(P_o)} \qquad \text{(III-42)}$$

where

$$J''(P_o) = \int \frac{\theta(|X_N\underset{\sim}{P}-\underset{\sim}{p}|-k_F)g^2(\underset{\sim}{p})}{(\frac{\underset{\sim}{p}^2}{2\mu} - \varepsilon_o - i\varepsilon)} \frac{d^3p}{(2\pi)^3} . \qquad \text{(III-43)}$$

The self-energy $\sigma(\underset{\sim}{p}_1, \underset{\sim}{p}_2)$ for the Λ particle of momentum $\underset{\sim}{p}_1$ which interacts with the nucleon of momentum $\underset{\sim}{p}_2$ is now given by

$$\sigma(\underset{\sim}{p}_1, \underset{\sim}{p}_2) = -i \oint_c \frac{dp_{20}}{(2\pi)} G_N(p_2)\Gamma_o(p_1, p_2; p_1, p_2)$$

$$= - \frac{\lambda g^2(\underset{\sim}{p})\theta(k_F - |\underset{\sim}{p}_2|)}{1 - \lambda J''(P_o)} \qquad \text{(III-44)}$$

where we have to close the contour c in the upper half-plane because here a nucleon hole line is involved. In arriving at (III-44), the exact Green's function for the nucleon has been used. Since we have

$$G_N(p_2) = [p_{20} - \frac{\underset{\sim}{p}_2^2}{2M_N} - \Sigma_N(p_2) + i\varepsilon \,\mathrm{sgn}(|\underset{\sim}{p}_2| - k_F)]^{-1} \qquad \text{(III-45)}$$

p_{20} of (III-44) takes on the value

$$p_{20} = \frac{\underset{\sim}{p}_2^2}{2M_N} + \Sigma_N(p_2). \qquad \text{(III-46)}$$

In practice, the self-energy $\Sigma_N(p)$ of a nucleon inside the Fermi sea will be taken from pure nuclear-matter calculation.

To obtain the proper self-energy of the Λ particle in nuclear matter, we have to sum $\sigma(\underset{\sim}{p}_1, \underset{\sim}{p}_2)$ over all the possible ΛN pairs. Therefore

$$\Sigma_\Lambda(\underset{\sim}{p}_1) = 4 \int \sigma(\underset{\sim}{p}_1, \underset{\sim}{p}_2) d^3p_2$$

$$= - 4\lambda \int \frac{\theta(k_F - |\underset{\sim}{p}_2|) g^2(\underset{\sim}{p})}{1 - \lambda J''(P_o)} \frac{d^3p_2}{(2\pi)^3} \qquad \text{(III-47)}$$

We can now close the external Λ particle line to get the binding energy of the Λ particle. It follows that

$$-D_\Lambda(\underset{\sim}{p}_1) = -i \oint_c \frac{dp_{10}}{(2\pi)} G_\Lambda(p_1)\Sigma_\Lambda(\underset{\sim}{p}_1) \qquad \text{(III-48)}$$

where the Green's function for the hole is given by

$$G_\Lambda(p_1) = [p_{10} - \frac{\underset{\sim}{p}_1^2}{2M_\Lambda} - \Sigma_\Lambda(p_1) - i\varepsilon]^{-1} . \qquad \text{(III-49)}$$

Consequently, we have

$$D_\Lambda(\underset{\sim}{p}_1) = 4\lambda \int \frac{\theta(k_F-|\underset{\sim}{p}_2|)g^2(\underset{\sim}{p})}{1-\lambda\ J''(P_o)} \frac{d^3p_2}{(2\pi)^3} \qquad \text{(III-50)}$$

with

$$p_{10} = \frac{\underset{\sim}{p}_1^2}{2M_\Lambda} + \Sigma_\Lambda(p_1). \qquad \text{(III-51)}$$

The well depth can then be identified as

$$D_\Lambda = D_\Lambda(\underset{\sim}{o}). \qquad \text{(III-52)}$$

As for the energy denominator that appears in (III-50), we recall that

$$J''(P_o) = \int \frac{\theta(|X_N\underset{\sim}{P}-\underset{\sim}{q}|-k_F)g^2(\underset{\sim}{q})}{(\frac{\underset{\sim}{q}^2}{2\mu} - \varepsilon_o - i\varepsilon)} \frac{d^3q}{(2\pi)^3} \qquad \text{(III-53)}$$

and

$$\varepsilon_o = P_o - \frac{\underset{\sim}{P}^2}{2(M_N+M_\Lambda)} .$$

But from (III-46) and (III-51), we have

$$P_o = \frac{\underset{\sim}{p}_1^2}{2M_\Lambda} + \frac{\underset{\sim}{p}_2^2}{2M_N} + \Sigma_\Lambda(p_1) + \Sigma_N(p_2)$$

$$= \frac{\underset{\sim}{p}^2}{2\mu} + \frac{\underset{\sim}{P}^2}{2(M_N+M_\Lambda)} + \Sigma_\Lambda(p_1) + \Sigma_N(p_2)$$

which implies that

$$\varepsilon_o = \frac{\underset{\sim}{p}^2}{2\mu} + \Sigma_\Lambda(p_1) + \Sigma_N(p_2) \qquad \text{(III-54)}$$

Hence $J''(P_o)$ of (III-53) and $J'(E_o)$ of (III-26) are exactly the same except for the normalization factor of $(2\pi)^3$. This in turn implies that (III-27) and (III-52) are equivalent. We conclude therefore that the well depths obtained from the method of Brueckner-Bethe-Goldstone and that of Galitskii-Migdal are identical.

It is possible to improve the Galitskii method by modifying the self-energy diagrams of Fig. 9. We may include the correlation of the Λ particle in the intermediate states with the sea of nucleons. This can be done by considering diagrams of Fig. 11. In this respect, the $G^o_\Lambda(p)$ that appears

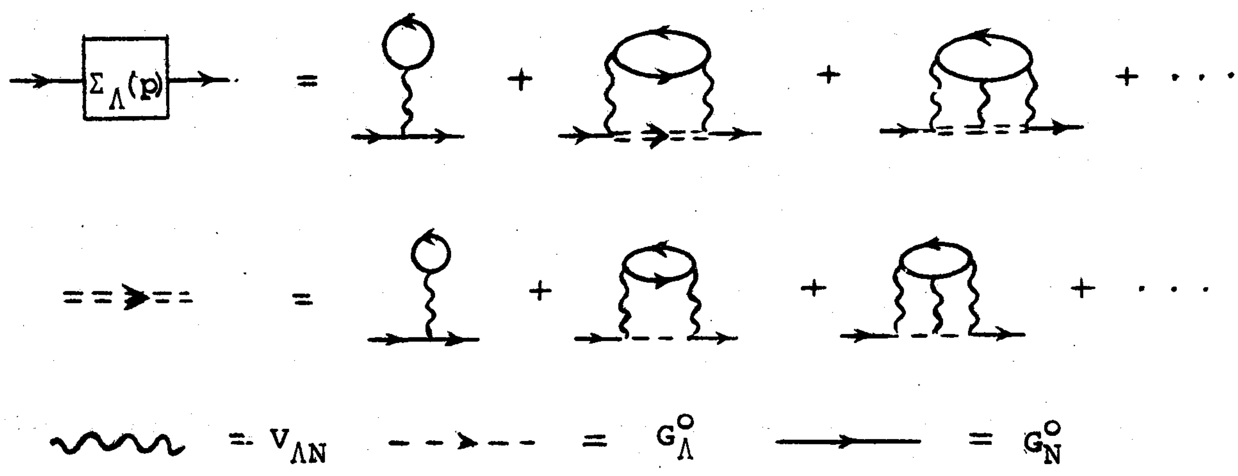

Fig. 11

in (III-35) should be replaced by the exact Green's function G_Λ of (III-32). We find that the detailed calculation is exactly similar to the case with the use of free Λ-particle Green's function and all the formulae remain the same except for G' which now becomes

$$G' = \int \frac{\theta(|x_N \underset{\sim}{p} - \underset{\sim}{q}| - k_F) g^2(\underset{\sim}{q})}{(\frac{q^2}{2\mu} + \Sigma_\Lambda(p_3) - \varepsilon_o - i\varepsilon)} \frac{d^3q}{(2\pi)^3} \qquad \text{(III-55)}$$

Thus the energy denominator is further modified by the appearance of the self-energy $\Sigma_\Lambda(p)$ in the intermediate states.

To calculate the Λ-particle well depth, we have arrived at a self-consistency problem. If we do not consider correlation of the Λ particle with the nucleons in its intermediate state, we have to calculate $J''(P_o)$ self-consistently only for the self-energy of the Λ particle in its initial state. In this case, so far as the Λ particle is concerned, only its ground state energy will appear in the energy denominator. The ground state energy in this approximation is usually referred to as the energy gap parameter.

Certainly, a correct treatment of the problem should be such that correlation in the intermediate states be included. Then a self-consistent calculation of the self-energies for both the initial and the final states is required in the energy denominator.

CHAPTER IV

SINGLE-PARTICLE-ENERGY SPECTRUM OF A Λ PARTICLE IN NUCLEAR MATTER

§1 Introduction

The extrapolated experimental well-depth of a Λ particle in nuclear matter is found to be 27±3 MeV[7,71], but model calculations using simple central potentials without short-range repulsion that fit the low energy ΛN scattering data or the binding energies of the s-shell hypernuclei have invariably yielded values of well depth in between 50 and 60 MeV[12]. This discrepancy has provoked many theoretical speculations. As we have mentioned in Chapter I, the disparity may be caused by overlooking the following effects in our simple model: (i) tensor forces; (ii) repulsive core; (iii) isospin suppression; (iv) three-body forces and (iv) rearrangement energy. The prospects of arriving at a reconciliation between the theory and the experiment by imposing these respective effects in our calculations have already been discussed in Chapter I. We shall in particular study the effect of a repulsive core in this chapter.

From the analysis of the NN scattering phase shifts, we learn that there exists a repulsive core in the basic NN interaction. If baryon-baryon interactions possess a universal feature in their interactions as is suggested by the SU(3) invariance, it is natural to include a repulsive core in the ΛN interaction. We shall use a two-term nonlocal separable potential to examine the effect of a repulsive core on the Λ-particle well depth.

In the preceding chapter, we have shown that the calculation of the well depth of a Λ particle in nuclear matter is a self-consistency problem. In a crude approximation we may ignore the correlation of the Λ particle with the nucleons in the intermediate states, the energy denominator should then be calculated self-consistently with the presence of a gap parameter. This gap parameter is just the well depth because it corresponds to the single-particle-energy of the Λ particle in the ground state. On the other hand, the intermediate states of the Λ particle may be modified through a correlation of the particles inside the nuclear medium. In this respect, the single-particle-energy spectrum of the Λ particle will also have to be calculated self-consistently in the intermediate states.

Before discussing the problem of the Λ-particle well depth, we shall express the scattering length and effective range in terms of the potential parameters of a nonlocal separable potential of rank two. Then we can use the low energy scattering parameters to determine a potential for the ΛN interaction.

§2 ΛN Scattering in a Nonlocal Separable Potential of Rank Two

Let us assume that the ΛN interaction is represented by a nonlocal separable potential of the following form:

$$\langle k|V|k'\rangle = -\{\lambda_1 g_1(k) g_1(k') + \lambda_2 g_2(k) g_2(k')\} \qquad \text{(IV-1)}$$

where the form factor $g_i(k)$ is taken to be

$$g_i(k) = 1/(k^2+\beta_i^2) \quad , \quad i = 1,2$$

and λ_i and β_i are constants to be determined by the ΛN scattering data.

According to Appendix A, the wave function is given by

$$\psi^k(p) = \delta(\underset{\sim}{p}-\underset{\sim}{k}) - 2\mu \sum_{i=1}^{2} \frac{\lambda_i Z_i g_i(p)}{k^2-p^2+i\varepsilon} \qquad \text{(IV-2)}$$

where

$$Z_1 = \{g_1(k)[B_{22}(k)\lambda_2+1] - g_2(k)B_{12}(k)\lambda_2\}/\Delta' \qquad \text{(IV-3a)}$$

$$Z_2 = \{g_2(k)[B_{11}(k)\lambda_1+1] - g_1(k)B_{21}(k)\lambda_2\}/\Delta' \qquad \text{(IV-3b)}$$

$$\Delta' = 1+B_{11}(k)\lambda_1+B_{22}(k)\lambda_2 + [B_{11}(k)B_{22}(k)-B_{12}(k)^2]\lambda_1\lambda_2 \qquad \text{(IV-3c)}$$

and

$$B_{ij}(k) = 2\mu \int d^3p \, \frac{g_i(p)g_j(p)}{k^2-p^2+i\varepsilon} \, . \qquad \text{(IV-3d)}$$

Consequently, we obtain for the scattering amplitude

$$\begin{aligned} f(k) &= e^{i\delta}\sin\delta/k \\ &= 4\pi^2\mu \sum_{i=1}^{2} \lambda_i Z_i g_i(k) \qquad \text{(IV-4)} \\ &= 2\pi^2 \, N/D \end{aligned}$$

where

$$N = -\lambda_1\lambda_2[g_1(k)^2R_{22}+g_2(k)^2R_{11}-2g_1(k)g_2(k)R_{12}]+\lambda_1 g_1(k)^2+\lambda_2 g_2(k)^2$$

$$D = \lambda_1\lambda_2(R_{11}R_{22}-R_{12}{}^2) - \lambda_1R_{11}-\lambda_2R_{22} + 1 - 2\pi^2 ikN$$

and

$$R_{ij} = R_{ij}(k) = P\int 2\mu\{g_i(p)g_j(p)/(p^2-k^2)\}d^3p \quad . \qquad \text{(IV-5)}$$

The reduced mass of the ΛN system is denoted by μ, and the P in front of the integral stands for the principal part of the integral. Using the relation $k\cot\delta = -1/a + r_o k^2/2$ for the scattering length a and effective range r_o, we get

$$a = 2\,\frac{\beta_1+\beta_2}{\beta_1\beta_2}\cdot\frac{y}{z} \qquad \text{(IV-6)}$$

$$r_o = \frac{-2}{a(\beta_1\beta_2)^2}\left(\frac{y'}{y} - \frac{z'}{z}\right) \qquad \text{(IV-7)}$$

where

$$y = \gamma_1\gamma_2(\beta_1-\beta_2)^2 - (\beta_1+\beta_2)(\gamma_1\beta_2^4 + \gamma_2\beta_1^4)$$

$$z = \gamma_1\gamma_2(\beta_1-\beta_2)^2 - (\beta_1+\beta_2)^2\{\gamma_1\beta_2^3 + \gamma_2\beta_1^3 - (\beta_1\beta_2)^3\}$$

$$y' = \gamma_1\gamma_2(\beta_1-\beta_2)^2\{2(\beta_1+\beta_2)^2-3\beta_1\beta_2\}-2(\beta_1+\beta_2)(\gamma_1\beta_2^6 +\gamma_2\beta_1^6)$$

$$z' = \gamma_1\gamma_2(\beta_1-\beta_2)^2\{3(\beta_1+\beta_2)^2-2\beta_1\beta_2\}-3(\beta_1+\beta_2)^2(\gamma_1\beta_2^5 + \gamma_2\beta_1^5)$$

and

$$\gamma_i = 2\pi^2\mu\lambda_i \quad , \quad i = 1,2 \ .$$

The potential parameters λ and β can now be fitted to the average scattering length and effective range of (II-3). We have four parameters at our disposal. We first assume some reasonable values for the range-parameters β_1 and β_2. λ_1 and λ_2 are then determined. We take $\lambda_1 > 0$ and $\lambda_2 < 0$ so that they correspond to attractive and repulsive part of the potential, respectively. For β's, we assume $\beta_1 < \beta_2$ so that the range of the attractive part is longer than that of the repulsive part.

In our calculation, we take $M_N = 938.9$ MeV and $M_\Lambda = 1115.6$ MeV. The values of the potential parameters are displayed in Table II. In the last column of the table, we indicate the energy E_s at which the ΛN scattering phase shift changes sign in the centre of mass system. Thus far, experimental value of E_s is still not available because the scattering data are too meagre to yield a definite value for E_s. Once E_s is known experimentally, we have less ambiguous choice among the phenomenological potentials.

§3 Λ Particle in Nuclear Matter

In nuclear matter, all the states of the nucleons below the Fermi momentum are not available for scattering, therefore we have to impose a restriction due to the Pauli exclusion principle in the intermediate states. In this connection, the Bethe-Goldstone wave function may be obtained by a slight modification of (IV-2). We should have therefore

$$\psi'^{k}_{(p)} = \delta(\underset{\sim}{p}-\underset{\sim}{k}) - \sum_{i=1}^{2} \frac{\lambda_i Z_i' g_i(p)}{e(p,k)} \tag{IV-8}$$

where Z_i' is obtained from (IV-3) by replacing $B_{ij}(k)$ everywhere by

$$B_{ij}'(k) = \int d^3p \, \frac{Q(p) g_i(p) g_j(p)}{e(p,k)} \tag{IV-9}$$

Here Q(p) is the Pauli operator and e(p,k) is the energy denominator to be defined later.

We can now solve the G-matrix to get

$$\begin{aligned} \langle\phi^k|G|\phi^k\rangle &= \langle\phi^k|V|\psi'^k\rangle \\ &= N'/D' \end{aligned} \tag{IV-10}$$

where

$$N' = -\lambda_1\lambda_2[g_1(k)^2 B_{22}'(k) + g_2(k)^2 B_{11}'(k) - 2g_1(k)g_2(k)B_{12}'(k)] + \lambda_1 g_1(k)^2 + \lambda_2 g_2(k)^2$$

$$D' = \lambda_1\lambda_2[B_{11}'(k)B_{22}'(k) - B_{12}'(k)^2] - \lambda_1 B_{11}'(k) - \lambda_2 B_{22}'(k) + 1$$

and ϕ^k is the relative plane wave state of the ΛN system.

Thus the single-particle energy of a Λ particle in nuclear matter is given by

$$E_\Lambda(k_\Lambda) = \frac{k_\Lambda^2}{2M_\Lambda} - D_\Lambda(k_\Lambda) \tag{IV-11}$$

with

$$D_\Lambda(k_\Lambda) = -4 \int_{k_N<k_F} d^3k_N \langle\phi^k|G|\phi^k\rangle \tag{IV-12}$$

Here k_N and K_Λ are the momenta of the nucleon and Λ particle respectively. Hence the relative momentum $\underset{\sim}{k}$ can be written as

$$\underset{\sim}{k} = X_\Lambda \underset{\sim}{k}_N - X_N \underset{\sim}{k}_\Lambda$$

where $X_\Lambda = M_\Lambda/(M_N+M_\Lambda)$; $X_N = M_N/(M_N+M_\Lambda)$.

The ground state of the Λ particle can now be identified as

$$D_\Lambda = -E_\Lambda(o) = D_\Lambda(o) \qquad \text{(IV-14)}$$

To calculate the G-matrix, we have to evaluate $B'_{ij}(k)$ of (IV-9). Its integration can be done with the help of the result worked out in Appendix B, ie.

$$\int d^3pQ(p) = 4\pi\left[\int_{k_F+X_NK}^{\infty} p^2dp + \frac{1}{4X_NK}\int_{|k_F-X_NK|}^{k_F+X_NK}\{(p+X_NK)^2-k_F^2\}pdp\right.$$

$$\left.+\ \theta(-k_F + X_NK)\int_{o}^{-k_F+X_NK} p^2dp\right]$$

where $\underset{\sim}{K} = \underset{\sim}{k}_N + \underset{\sim}{k}_\Lambda$ is the total momentum.

Now, the energy denominator that appears in (IV-9) has to be specified. We define

$$e(p,k) = P^2/2\mu - k^2/2\mu + \Delta(U_N + U_\Lambda) \qquad \text{(IV-15)}$$

where ΔU_N and ΔU_Λ are the differences between the intermediate and initial single-particle potential energies of the nucleon and Λ particle respectively. As for ΔU_N, we adopt the standard prescription in nuclear-matter calculation by setting $U_N=0$ in the intermediate states. In accordance with (III-12), we have

$$\Delta U_N = (85.17-12.08\ k_N^2)\ \text{MeV}$$

$$= (0.4316-0.06122\ k_N^2)\,\text{fm}^{-1}\quad (k_N<k_F) \qquad \text{(IV-16)}$$

Now the Λ particle may be treated in the same way. By so doing, we have

$$\Delta U_\Lambda = D_\Lambda \qquad \text{(IV-17)}$$

which is referred to as the gap parameter. Certainly, this has to be calculated self-consistently.

In addition we may like to include correlation in the intermediate states for the Λ particle. Then the result of (III-55) indicates that the single-particle-energy spectrum $E_\Lambda(k_\Lambda)$ will also appear in the intermediate states. Consequently, $E_\Lambda(k_\Lambda)$ or $D_\Lambda(k_\Lambda)$ must be evaluated self-consistently in the energy denominator for the initial states as well as the intermediate states. It is rather difficult to achieve an exact self-consistency. However, we find that $D_\Lambda(k_\Lambda)$ can be well represented by

$$D_\Lambda(k_\Lambda) = D_\Lambda \exp[-\eta k_\Lambda^2] \qquad \text{(IV-18)}$$

where η is a constant. The error involved is about 1% or less for $k_\Lambda^2 < 1.5\ \text{fm}^{-2}$. For larger values of k_Λ, the form of (IV-18) is less accurate, but D_Λ, the binding energy of Λ particle, is not sensitive to $D_\Lambda(k_\Lambda)$ at large k_Λ. Hence we shall use (IV-18) in our calculation. In this case, we get

$$\Delta U = -D_\Lambda e^{-\eta p_\Lambda^2} + D_\Lambda e^{-\eta k_\Lambda^2} \qquad \text{(IV-19)}$$

where p_Λ is the momentum of a Λ particle in the intermediate states.

§4 Results of Calculations

We have considered two sets of potentials. Those of the first set fit a and r_o of (II-3), while those of the second set were given by Schick[80], who chose to fit a = -1.95 fm and r_o = 3.50 fm. The values of the potential parameters, β's and r's, are listed in Tables II and III. As is seen from the energy where the phase shift changes sign, the strength of the short-range repulsion becomes weaker in the order of I, II, III and IV in the first set, and A, B, ... and N in the second set. Potentials IV and N are of rank one and have no repulsive part at all.

These potentials can now be applied to (IV-12) to calculate $D_\Lambda(k_\Lambda)$. We first consider the case where ΔU_Λ is given by (IV-19). In the representation $D_\Lambda(k_\Lambda) = D_\Lambda \exp[-\eta k_\Lambda^2]$, the first approximation of D_Λ is obtained by setting $\Delta U_\Lambda = 0$ in (IV-15). In the next and subsequent iteration $\Delta U_\Lambda = D_\Lambda\{\exp(-\eta k_\Lambda^2) - \exp(-\eta p_\Lambda^2)\}$ is used. D_Λ and η are adjusted until a self-consistency is achieved within 1% for the range of $k_\Lambda^2 < 1.5\ \text{fm}^{-2}$. Normally, about three iterations have been enough for our purpose.

We have listed the values of D_Λ and η for various potentials in Table IV. As a reference for comparison, we include also the value B_o which is the well depth of a Λ particle in the case where $\Delta U_\Lambda = 0$. B_o is of course the crudest approximation for the well depth because the Λ particle is taken to be free both in the initial and the intermediate states. Because η is quite small, we may expand $U_\Lambda(k_\Lambda)$ of (IV-18) with respect to η for small values of k_Λ^2. The single-particle energy of the Λ particle then becomes

$$E_\Lambda(k_\Lambda) = -D_\Lambda + \{(1/2M_\Lambda) + D_\Lambda\eta\}k_\Lambda^2 + \ldots$$

$$= - D_\Lambda + k_\Lambda^2/(2M_\Lambda^*) + \ldots \qquad \text{(IV-20)}$$

where

$$M_\Lambda^* = M_\Lambda/(1 + 2M_\Lambda D_\Lambda\eta) \qquad \text{(IV-21)}$$

In this context, we have obtained an effective-mass approximation for the single-particle-energy spectrum. Certainly, a Gaussian approximation in the form of (IV-18) is better than an effective-mass approximation. As we expect physically, $E_\Lambda(k_\Lambda)$ at high energies should be dominated by the kinetic energy term. Notwithstanding, the value of the effective mass M_Λ^* for various potentials are listed in Table IV.

In Table V, we show the values of the well depth B_g obtained from an energy-gap approximation. Here,we assume that $\Delta U_\Lambda = D_\Lambda$.

There is another version of the methods of approximation. One takes the effective masses that appear in the single-particle-energy spectra literally[50] and replaces all the physical masses in the Hamiltonian as well as in any relevant formulae by these effective masses. Of course, the effective mass is only meant to be a convenient approximation for the shape of the single-particle-energy spectrum. Assigning a definite physical content to this value may overstress its importance. Nevertheless, calculations in this fashion are often found in the literature. To see the outcome of such an approximation, we have calculated the well depth B_m by setting

$$e(p,k) = \left(\frac{p_N^2}{2M_N} + \frac{p_\Lambda^2}{2M_\Lambda^*}\right) - \left(\frac{k_N^2}{2M_N} + \frac{k_\Lambda^2}{2M_\Lambda^*} - 0.4316+0.06122\ k_N^2\right) \qquad \text{(IV-22)}$$

Here M_Λ^* will be calculated self-consistently. According to (IV-22), $M_N^* = 0.6319\ M_N$ when the nucleons are below the Fermi sea. In the intermediate state, we have $M_N^* = M_N$ because we assume that the nucleons are then free. The results of our calculation for B_m and M_Λ^* are shown in Table V.

§5 Conclusion

Our results of Table IV indicate that B_o-D_Λ is larger for stronger repulsive core. For a weak repulsive core, $B_o - D_\Lambda$ is negligibly small. Therefore whenever we are dealing with a strong repulsive potential, the single-particle-energy spectrum has to be treated properly. With reference to a simple one-term nonlocal separable potential such as potential IV or N in Table IV, the suppression effect of the repulsive core can be as large as 10 MeV which suggests that this effect may play a very important role in the status of a Λ particle in nuclear matter. A further detailed analysis on the ΛN scattering phase shift is necessary to confirm such an indication because its effectiveness depends on where the phase shift changes its sign.

The well depth calculated with the energy-gap approximation is on the average about 0.5 MeV smaller than that calculated with the use of single-particle-energy spectrum. This can be seen by comparing D_Λ and B_g of Tables IV and V. Thus a self-energy insertion in the intermediate state of a Λ particle does not make a major contribution. A similar conclusion has also been reached in pure nuclear-matter calculation[74].

In an effective-mass approximation, a drastic suppression effect is possible to attain, provided we take the effective mass literally. For instance, comparing the well depths B_m's of potentials A and N in Table V, we obtain a suppression of about 30 MeV. However, we mentioned before that this approximation procedure is not justifiable.

Table II

Potential parameters of the ΛN interaction

Potentials	$\beta_1(fm^{-1})$	$\beta_2(fm^{-1})$	$\lambda_1(fm^{-2})$	$-\lambda_2(fm^{-2})$	E_s(MeV)
I	3.27	4.0	51.133	2842.9	82.8
II	3.20	3.5	36.126	55.848	84.9
III	2.80	3.0	10.948	13.915	93.4
IV	1.4663	-	0.033867	0.0	∞

Table III

Potential parameters of the ΛN interaction adopted from Schick[80]

Potentials	$\beta_1(fm^{-1})$	$\beta_2(fm^{-1})$	$\lambda_1(fm^{-2})$	$-\lambda_2(fm^{-2})$	E_s(MeV)
A	3.1250	10/3	363.77	2146.1	65.8
B	2.9851	10/3	59.745	151.48	66.3
C	2.8571	10/3	22.364	57.588	67.0
D	2.1053	10/3	0.66242	3.2609	77.4
E	1.6260	10/3	0.098660	0.63519	116.6
N	1.3060	-	0.024462	0.0	∞

Table IV

Λ-particle well depths and parameters for the single-particle-energy spectrum

Potentials	B_o(MeV)	D_Λ(MeV)	η(fm^2)	M^*_Λ/M_Λ
I	45.80	41.53	0.190	0.689
II	49.78	47.55	0.171	0.682
III	52.22	51.68	0.155	0.685
IV	50.85	50.48	0.116	0.749
A	44.69	38.38	0.264	0.633
B	47.50	42.21	0.255	0.618
C	49.54	45.21	0.239	0.618
D	55.25	54.60	0.193	0.623
E	55.18	54.98	0.163	0.661
N	53.99	53.60	0.138	0.702

Table V

Λ-particle well depths in the energy-gap and the effective-mass approximations

Potentials	B_g (MeV)	B_m (MeV)	M^*_Λ/M_Λ
I	41.13	27.84	0.770
II	47.26	39.62	0.741
III	50.91	48.56	0.744
IV	49.95	47.42	0.838
A	37.95	17.07	0.708
B	41.93	23.82	0.679
C	45.00	29.90	0.656
D	54.53	52.75	0.644
E	54.72	53.04	0.717
N	53.14	50.43	0.795

CHAPTER V

Λ-PARTICLE WELL DEPTH IN THE TWO-CHANNEL FORMALISM

§1 Introduction

In this chapter, we will examine another aspect of the suppression effect in the well depth of a Λ particle in nuclear matter. Here, we will study the effect of $\Sigma\Lambda$ conversion in the so-called two-channel formalism. In this formalism, the inelastic process $\Lambda N \longleftrightarrow \Sigma N$ is treated explicitly. In addition, it takes care of not only the isospin suppression effect in an isospin saturated system but also the effect of three-body ΛNN forces[19]. Some detailed discussion on this point has already been given in Chapter II and it will not be elaborated here.

As we know, in the one-channel formalism, only the scattering data of $\Lambda N \to \Lambda N$ have been used to fit the potential parameters and nothing can be said on the process $\Lambda N \longleftrightarrow \Sigma N$. When we are working in the two-channel formalism, besides the scattering data of $\Lambda N \to \Lambda N$, the potential parameters must now be consistent with the scattering data of $\Lambda N \leftrightarrow \Sigma N$ and $\Sigma N \to \Sigma N$. Therefore, by so doing, we can derive a less ambiguous phenomenological potential. Since the scattering data for the hyperons are scarce, especially at high energies, we are still left with some uncertainty on the potential parameters. However, this difficulty has been relieved to a certain extent by the recent discovery of a ^3S Λp resonance below the ΣN threshold[61,62], though the mass M^* and the width Γ of the resonance have not yet been determined very accurately.

Currently, there are two sets of Λp resonance data[61,62] - namely:

(A) $M^* = 2126$ MeV , $\Gamma \lesssim 10$ MeV

(B) $M^* = 2110$ MeV , $\Gamma \simeq 20$ MeV .

The ambiguity in the potential parameters is greatly reduced by the requirement that the resonance be reproduced by the potential. In passing, we would like to mention that the Λp resonance is the first baryon-baryon resonance ever observed experimentally. It may be interpreted as a quasi-bound state of ΣN due to a strong ΣN attractive interaction. Indeed, meson-theoretical potential models which reproduce the Λp resonance as a quasi-bound state of ΣN in the ^3S state have already been proposed[33,63].

Since it will be interesting to investigate the outcome of various models with different physical boundary conditions, we shall present the results on the calculated Λ-particle well depth with and without the Λp resonance being taken into account. In the following, we shall follow essentially the works of Nogami and Satoh[22].

§2 Model Potentials in the Two-channel Formalism

We will use a simple one-term nonlocal separable potential for each stage of the interactions: $\Lambda N \to \Lambda N$, $\Lambda N \longleftrightarrow \Sigma N$ and $\Sigma N \to \Sigma N$. This simplified feature of the potential will enable us to have a clear picture on the effect of a potential in the two-channel formalism. Thus the potential in momentum space is taken as

$$\begin{aligned} \langle k|V|k'\rangle &= \begin{pmatrix} \langle k|V_{\Lambda\Lambda}|k'\rangle & \langle k|V_{\Lambda\Sigma}|k'\rangle \\ \langle k|V_{\Sigma\Lambda}|k'\rangle & \langle k|V_{\Sigma\Sigma}|k'\rangle \end{pmatrix} \\ &= \begin{pmatrix} -\lambda_\Lambda g_\Lambda(k) g_\Lambda(k') & -\lambda_\chi g_\Lambda(k) g_\Sigma(k'_\Sigma) \\ -\lambda_\chi g_\Sigma(k_\Sigma) g_\Lambda(k') & -\lambda_\Sigma g_\Sigma(k_\Sigma) g_\Sigma(k'_\Sigma) \end{pmatrix} \end{aligned} \qquad \text{(V-1)}$$

Here, λ_Λ, λ_χ and λ_Σ are the strengths of the potential; $g_i(k) = 1/(k^2+\beta_i^2)$ with $i = \Lambda$ or Σ. The β_i appearing in the form factor $g_i(k)$ is the inverse of the range of interaction for the respective channel. One should also note that k stands for the relative momentum of the ΛN system and k_Σ for that of the ΣN system. In our calculation, the potential parameters will be chosen in such a way that the calculated scattering cross-sections for the various channels are compatible with the experiment data. Furthermore, if we consider the Λp resonance as well, the potential has to reproduce the resonant data.

In order to derive a relationship between the scattering cross-sections and the potential parameters, we have to solve the Schrödinger equation

$$(\hat{T} + V)\Psi = E\Psi \tag{V-2}$$

with

$$\Psi = \begin{pmatrix} \psi_\Lambda \\ \psi_\Sigma \end{pmatrix} \quad ; \quad \hat{T} = \begin{pmatrix} T_\Lambda & 0 \\ 0 & T_\Sigma \end{pmatrix}$$

and

$$T_\Lambda = -\frac{\nabla_\Lambda^{\,2}}{2\mu_\Lambda} \quad ; \quad T_\Sigma = -\frac{\nabla_\Sigma^{\,2}}{2\mu_\Sigma} \quad .$$

Next, the 2×2 matrix equation of (V-2) can be written as two coupled equations

$$(T_\Lambda + V_{\Lambda\Lambda})\psi_\Lambda + V_{\Lambda\Sigma}\psi_\Sigma = E_\Lambda\psi_\Lambda \tag{V-3}$$

$$(T_\Sigma + V_{\Sigma\Sigma})\psi_\Sigma + V_{\Sigma\Lambda}\psi_\Lambda = (E_\Lambda - \Delta)\psi_\Sigma \tag{V-4}$$

where $\Delta \equiv M_\Sigma - M_\Lambda$. When a nonlocal separable potential of the form given by (V-1) is used, (V-3) and (V-4) become

$$\begin{aligned}(\frac{p^2}{2\mu_\Lambda} - E_\Lambda)\psi_\Lambda^k(p) &- \lambda_\Lambda g_\Lambda(p) \int g_\Lambda(p')\psi_\Lambda^k(p')d^3p' \\ &-\lambda_x g_\Lambda(p) \int g_\Sigma(p_\Sigma')\psi_\Sigma^{k_\Sigma}(p_\Sigma')d^3p_\Sigma' = 0\end{aligned} \tag{V-5}$$

$$\begin{aligned}(\frac{p_\Sigma^2}{2\mu_\Sigma} - E_\Lambda + \Delta)\psi_\Sigma^{k_\Sigma}(p_\Sigma) &- \lambda_\Sigma g_\Sigma(p_\Sigma) \int g_\Sigma(p_\Sigma')\psi_\Sigma^{k_\Sigma}(p_\Sigma')d^3p_\Sigma' \\ &-\lambda_x g_\Sigma(p_\Sigma) \int g_\Lambda(p')\psi_\Lambda^k(p')d^3p' = 0\end{aligned} \tag{V-6}$$

where $E_\Lambda = \frac{k^2}{2\mu_\Lambda}$ and $E_\Lambda - \Delta = \frac{k_\Sigma^2}{2\mu_\Sigma}$. According to Appendix C, the scattering solution of these equations with ΛN as the entrance channel will be

$$\psi_\Lambda^k(p) = \delta(\underset{\sim}{p}-\underset{\sim}{k}) + T_{\Lambda\Lambda}(p,k)/[4\pi^2\mu_\Lambda e_\Lambda(p,k)] \tag{V-7}$$

$$\psi_\Sigma^{k_\Sigma}(p_\Sigma) = T_{\Sigma\Lambda}(k_\Sigma,k)/[4\pi^2\mu_\Sigma e_\Sigma(p_\Sigma,k_\Sigma)] \tag{V-8}$$

where

$$T_{\Lambda\Lambda}(p,k) = 4\pi^2\mu_\Lambda\{\lambda_\Lambda - d(\lambda)J_\Sigma\}g_\Lambda(k)g_\Lambda(p)/D(k,k_\Sigma)$$

$$T_{\Sigma\Lambda}(p_\Sigma,k) = 4\pi^2\mu_\Sigma\lambda_x g_\Lambda(k)g_\Sigma(p_\Sigma)/D(k,k_\Sigma)$$

$$d(\lambda) = \lambda_\Lambda\lambda_\Sigma - \lambda_x^2$$

$$D(k,k_\Sigma) = \begin{vmatrix} 1-\lambda_\Lambda J_\Lambda & -\lambda_x J_\Lambda \\ -\lambda_x J_\Sigma & 1-\lambda_\Sigma J_\Sigma \end{vmatrix}$$

with

$$J_\Lambda = \int d^3p\ g_\Lambda^2(p)/e_\Lambda(p,k)$$

and

$$J_\Sigma = \int d^3p\ g_\Sigma^2(p)/e_\Sigma(p_\Sigma,k_\Sigma)$$

Here, the energy denominators for scattering are defined by

$$e_\Lambda(p,k) = (p^2-k^2-i\varepsilon)/2\mu_\Lambda$$

$$e_\Sigma(p_\Sigma,k_\Sigma) = (p_\Sigma^2 - k_\Sigma^2 - i\varepsilon)/2\mu_\Sigma \ .$$

The physical meaning of $T_{\Lambda\Lambda}(p,k)$ and $T_{\Sigma\Lambda}(p_\Sigma,k)$ can be better understood if (V-7) and (V-8) are Fourier transformed into the co-ordinate space. Then the asymptotic wave functions become

$$\psi_\Lambda \sim e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}_\Lambda} + T_{\Lambda\Lambda}(k,k)e^{ikr_\Lambda}/r_\Lambda \qquad \text{(V-9)}$$

$$\psi_\Sigma \sim T_{\Sigma\Lambda}(k_\Sigma,k)e^{ik_\Sigma r_\Sigma}/r_\Sigma \quad . \qquad \text{(V-10)}$$

In getting (V-10), we have assumed that $k_\Sigma^2 > 0$. We see that $T_{\Lambda\Lambda}(k,k)$ and $T_{\Sigma\Lambda}(k_\Sigma,k)$ are just the scattering amplitudes of $\Lambda N \to \Lambda N$ and $\Lambda N \to \Sigma N$ respectively. They are related to the total scattering cross-sections by[64]

$$\sigma(\Lambda N\to\Lambda N) = 4\pi|T_{\Lambda\Lambda}(k,k)|^2 \qquad \text{(V-11)}$$

$$\sigma(\Lambda N\to\Sigma N) = 4\pi|T_{\Sigma\Lambda}(k_\Sigma,k)|^2(v_\Sigma/v_\Lambda)\theta(E_\Lambda-\Delta) \qquad \text{(V-12)}$$

where v_Σ and v_Λ are the relative velocities of ΣN and ΛN respectively; and the step function $\theta(E_\Lambda-\Delta)$ will ensure that the scattering process of $\Lambda N \to \Sigma N$ can occur only when the total energy of the system is above the ΣN threshold.

Suppose ΣN is the entrance channel, we then arrive at another set of total scattering cross-sections:

$$\sigma(\Sigma N\to\Sigma N) = 4\pi|T_{\Sigma\Sigma}(k_\Sigma,k_\Sigma)|^2 \qquad \text{(V-13)}$$

$$\sigma(\Sigma N\to\Lambda N) = 4\pi|T_{\Lambda\Sigma}(k,k_\Sigma)|^2(v_\Lambda/v_\Sigma) \qquad \text{(V-14)}$$

Here, $T_{\Sigma\Sigma}(k_\Sigma,k_\Sigma)$ and $T_{\Lambda\Sigma}(k,k_\Sigma)$ are obtained from $T_{\Lambda\Lambda}(k,k)$ and $T_{\Sigma\Lambda}(k_\Sigma,k)$ by changing Λ into Σ and vice versa. In the above manipulation, one should note specifically that k actually stands for k_Λ.

At low energies, the phase shift δ for the entrance channel ΛN is real and is related to the ΛN scattering amplitude by

$$T_{\Lambda\Lambda}(k,k) = e^{i\delta}\sin\delta/k \ . \qquad \text{(V-15)}$$

This will allow us to obtain the scattering length a and the effective range r_o by the relation

$$k\cot\delta = -1/a + \frac{1}{2} r_o k^2 + \ldots$$

If we assume that the ranges of the interactions are equal, i.e. $\beta_\Lambda = \beta_\Sigma = \beta$, we have from Appendix C

$$\frac{1}{a} = \frac{1}{2}\beta\{1 - \frac{\beta^3[\beta(\beta+\kappa')^2 - \gamma_\Sigma]}{\gamma_\Lambda\beta(\beta+\kappa')^2 - d(\gamma)}\} \qquad \text{(V-16)}$$

$$r_o = \frac{3}{\beta} - \frac{4}{a\beta^2} - \frac{\mu_\Sigma\gamma_x^2\beta^5(\beta+\kappa')}{\mu_\Lambda\kappa'[\gamma_\Lambda\beta(\beta+\kappa')^2 - d(\gamma)]^2} \qquad \text{(V-17)}$$

where $\gamma_\Lambda = 2\pi^2\mu_\Lambda\lambda_\Lambda$, $\gamma_\Sigma = 2\pi^2\mu_\Sigma\lambda_\Sigma$, $\gamma_x = 2\pi^2(\mu_\Lambda\mu_\Sigma)^{\frac{1}{2}}\lambda_x$, $d(\gamma) = \gamma_\Lambda\gamma_\Sigma - \gamma_x^2$ and $\kappa' = (2\mu_\Sigma\Delta)^{\frac{1}{2}}$. Again, we will fit the low energy scattering parameters a and r_o by their spin average values given by (II-3). The masses are taken to be $M_N = 938.9$ MeV, $M_\Lambda = 1115.6$ MeV and $M_\Sigma = 1193.1$ MeV. Besides the scattering cross-section of the entrance channel, the potential parameters must also be chosen in such a way that they are consistent with the experimental scattering cross-sections $\sigma(\Sigma N \to \Sigma N)$ and $\sigma(\Sigma N \to \Lambda N)$.

Thus far only meagre data on the scattering cross-sections for Σ^-p and Σ^+p are available. Most of the data analysed are within the range of the momenta $p_\Sigma = 110 \sim 170$ MeV/c in the centre of mass system. Let us quote the values for $p_\Sigma \approx 150$ MeV/c.

$$\sigma(\Sigma^-p \to \Lambda n) = 147 \pm 19 \text{ mb}$$

$$\sigma(\Sigma^-p \to \Sigma^-p) = 198 \pm 48 \text{ mb}$$

$$\sigma(\Sigma^-p \to \Sigma^0 n) = 111 \pm 19 \text{ mb}$$

$$\sigma(\Sigma^+p \to \Sigma^+p) = 203 \pm 117 \text{ mb} \ . \qquad \text{(V-18)}$$

However, some of these are not pure isospin scattering cross-sections. Under the hypothesis of charge independence, there are only two independent amplitudes which describe the scattering in all the charge states of ΣN scattering. These are the isospin $I = \frac{1}{2}$ and $\frac{3}{2}$ amplitudes, because the Σ-hyperons belong to the $I = 1$ multiplet and the nucleons to the $I = \frac{1}{2}$ multiplet. Since the ΛN system consists only of the isospin $I = \frac{1}{2}$ state, the contribution to the total scattering cross-sections of (V-18) from the $I = \frac{3}{2}$ state should be abstracted. Therefore, we shall analyze the scattering in terms of the pure isospin scattering amplitude.

First let us consider the scattering process

$$\Sigma^-p \to \Sigma^-p.$$

Its total scattering amplitude $<\Sigma^- p|T_{\Sigma\Sigma}|\Sigma^- p>$ may be expanded in terms of the isospin state $|I,I_z>$, thus

$$<\Sigma^- p|T_{\Sigma\Sigma}|\Sigma^- p> = \sum_{\substack{I\ ,I' \\ I_z, I_z'}} <\Sigma^- p|I,I_z><I,I_z|T_{\Sigma\Sigma}|I',I_z'><I',I_z'|\Sigma^- p>$$

Since the isospin is conserved in the scattering and the scattering amplitude does not depend on I_z, we write

$$<\Sigma^- p|T_{\Sigma\Sigma}|\Sigma^- p> = \sum_I <\Sigma^- p|I,-\tfrac{1}{2}><I|T_{\Sigma\Sigma}|I><I,-\tfrac{1}{2}|\Sigma^- p> \quad \text{(V-19)}$$

In this equation, we have put down explicitly I_z of $\Sigma^- p$, which is $-\frac{1}{2}$. The values of the Clebsch-Gordon coefficients $<\Sigma^- p|I,-\frac{1}{2}>$ can be found in Condon and Shortley[87]. Then (V-19) becomes

$$<\Sigma^- p|T_{\Sigma\Sigma}|\Sigma^- p> = \tfrac{1}{3}(2<\tfrac{1}{2}|T_{\Sigma\Sigma}|\tfrac{1}{2}> + <\tfrac{3}{2}|T_{\Sigma\Sigma}|\tfrac{3}{2}>) \quad \text{(V-20)}$$

A similar procedure leads to the scattering amplitudes for $\Sigma^- p \to \Sigma^o n$ and $\Sigma^+ p \to \Sigma^+ p$. The results are:

$$<\Sigma^- p|T_{\Sigma\Sigma}|\Sigma^o n> = \frac{\sqrt{2}}{3}\ (-<\tfrac{1}{2}|T_{\Sigma\Sigma}|\tfrac{1}{2}> + <\tfrac{3}{2}|T_{\Sigma\Sigma}|\tfrac{3}{2}>) \quad \text{(V-21)}$$

$$<\Sigma^+ p|T_{\Sigma\Sigma}|\Sigma^+ p> = <\tfrac{3}{2}|T_{\Sigma\Sigma}|\tfrac{3}{2}> \quad \text{(V-22)}$$

Since the scattering cross-section is proportional to the square of the absolute value of the scattering amplitude, we get

$$\sigma(\Sigma^- p \to \Sigma^- p) + \sigma(\Sigma^- p \to \Sigma^o n) = \tfrac{1}{3}(\sigma_3 + 2\sigma_1) \quad \text{(V-23)}$$

where $\sigma_{2I} = |<I|T_{\Sigma\Sigma}|I>|^2$ is the pure isospin scattering cross-section. From (V-22) and (V-23), we obtain the $I = \frac{1}{2}$ scattering cross-section,

$$\sigma_1 = \frac{1}{2}\{3[\sigma(\Sigma^- p \to \Sigma^- p) + \sigma(\Sigma^- n \to \Sigma^o n)] - \sigma(\Sigma^+ n \to \Sigma^+ n)\}. \quad \text{(V-24)}$$

If we ignore the errors for the moment in (V-18), we get

$$\sigma_1 = 362 \text{ mb} .$$

Because of the errors in (V-18), σ_1 may be as small as 200 mb, or as large as 500 mb. Such a large uncertainty on σ_1 will induce a considerable uncertainty in the potential parameters. One can only hope that more scattering data on the hyperon-nucleon scattering will be compiled in the near future so that the error is reduced.

§3 Nuclear Matter

After the potential parameters have been fixed by using the free scattering data, we can proceed to solve the Bethe-Goldstone equation in the two-channel formalism. We write down again some of the basic relations for the G-matrix. They are:

$$G = V - V \frac{Q}{e} G$$

and

$$G\Phi = V\Psi' .$$

Of course, we are now dealing with a two-component wave function. Hence

$$\Phi = \begin{pmatrix} \phi_\Lambda \\ 0 \end{pmatrix}$$

and

$$\Psi' = \begin{pmatrix} \psi'_\Lambda \\ \psi'_\Sigma \end{pmatrix}$$

where ϕ_Λ is just the plane wave state of the ΛN system and ψ'_Λ and ψ'_Σ are the ΛN and ΣN components of the Bethe-Goldstone wave functions. Except for the presence of the Pauli operator Q, the Bethe-Goldstone wave functions take the form of (V-7) and (V-8). To be explicit, we have

$$\psi'^{k}_{\Lambda}(p) = \delta(\underset{\sim}{p}-\underset{\sim}{k}) + T'_{\Lambda\Lambda}(p,k)Q/[4\pi^2\mu_\Lambda e'_\Lambda(p,k)] \qquad \text{(V-25)}$$

$$\psi'^{\Sigma}_{\Sigma}(p_\Sigma) = T'_{\Sigma\Lambda}(p_\Sigma,k_\Lambda)Q/[4\pi^2\mu_\Sigma e'_\Sigma(p_\Sigma,k_\Sigma)]. \qquad \text{(V-26)}$$

The scattering amplitudes $T'_{\Lambda\Lambda}(p,k)$ and $T'_{\Sigma\Lambda}(p_\Sigma,k)$ are obtained from $T_{\Lambda\Lambda}(p,k)$ and $T_{\Sigma\Lambda}(p_\Sigma,k)$ by changing J_Λ and J_Σ into J'_Λ and J'_Σ. And we define

$$J'_\Lambda = \int d^3p\ Q\ g^2_\Lambda(p)/e'_\Lambda(p,k)$$

$$J'_\Sigma = \int d^3p\ Q\ g^2_\Sigma(p)/e'_\Sigma(p,k_\Sigma) \qquad \text{(V-27)}$$

The angular integration of these integrals can be evaluated by using the results of Appendix B. As for the energy denominators, we take the same $U_N(k)$ as in the preceeding chapters. For simplicity, the Λ and the Σ hyperons are taken to be free in the initial and final states. In view of our calculation demonstrated in Chapter IV, it is justifiable to make this simplification, because the potential considered here contains no repulsive part. Thus we have

$$e'_\Lambda(p,k) = e_\Lambda(p,k) + U_N$$
$$e'_\Sigma(p_\Sigma,k) = e_\Sigma(p_\Sigma,k) + U_N$$

We can now evaluate the G-matrix. It may be written as

$$<k|G|k'> = <\Phi^k|G|\Phi^{k'}>$$
$$= <\Phi^k|V|\Psi'^{k'}>$$
$$= <\phi_\Lambda^k|V_{\Lambda\Lambda}|\psi'^{k'}_\Lambda> + <\phi_\Lambda^k|V_{\Lambda\Sigma}|\psi'^{k'}_\Sigma> . \tag{V-29}$$

Together with (V-1), (V-25) and (V-26), it yields

$$<k|G|k'> = g_\Lambda(k)g_\Lambda(k')\{\lambda_\Lambda - d(\lambda)J'_\Sigma\}/D' \tag{V-30}$$

with

$$D' = 1 - \lambda_\Lambda J'_\Lambda - \lambda_\Sigma J'_\Sigma + d(\lambda)J'_\Lambda J'_\Sigma .$$

When we sum the G-matrix over all possible nucleon states, we get the Λ-particle well depth

$$D_\Lambda = -\frac{4}{x_\Lambda^3}\int_{k<x_\Lambda k_F} <k|G|k>d^3k \tag{V-31}$$

where $x_\Lambda = M_\Lambda/(M_N+M_\Lambda)$. To obtain (V-31), we have used the relation $\underset{\sim}{k} = M_N/(M_N+M_\Lambda)\underset{\sim}{k}_\Lambda + M_\Lambda/M_N+M_\Lambda)\underset{\sim}{k}_N$ and the fact that $\underset{\sim}{k}_\Lambda = 0$ for the ground state of a Λ particle in nuclear matter.

§4 Results of Calculations

We display the potential parameters in Table VI. Here, potential III and IX are adopted from Schick and Toepfer[23]. We do not take into account the ^3S resonance of Λp in potential I to IV. If in addition we have to consider the Λp resonance just below the ΣN threshold, the triplet scattering phase shift will have to satisfy $k\cot\delta = 0$ at the resonance energy E_R. And the width of the resonance is obtained by expanding $k\cot\delta$ around the resonance energy.

To achieve this, we start with the expansion

$$k\cot\delta = C_o(E_\Lambda - E_R) + \ldots\ldots \qquad \text{(V-32)}$$

where

$$C_o = 2\mu_\Lambda \frac{d}{dk^2}(k\cot\delta)\Big|_{k=k_R}$$

and

$$E_R \equiv k_R^2/2\mu_\Lambda \, .$$

Next, we have for the total scattering cross-section

$$\sigma(\Lambda N \to \Lambda N) = \frac{4\pi}{(k\cot\delta)^2 + k^2} \qquad \text{(V-33)}$$

According to the Breit-Wigner formula, the scattering cross-section near the resonance can be expressed in terms of the width Γ and the resonance energy E_R as follows

$$\sigma(\Lambda N \to \Lambda N) = \pi(\Gamma/k)^2\{(E_\Lambda - E_R)^2 + (\Gamma/2)^2\}^{-1} \qquad \text{(V-34)}$$

Comparing (V-33) and (V-34), and making use of (V-32), we get

$$\Gamma^2 = 4k_R^2/C_o^2 \quad . \tag{V-35}$$

With the use of (V-35) and the condition $k\cot\delta|_{k=k_R} = 0$, we deduced potentials V and VI as shown in Table VI. Likewise, potentials VII and VIII have been obtained recently by Satoh[65] with the additional assumption that $\beta_\Lambda \neq \beta_\Sigma$. The calculated well depths for the various potentials listed in Table VI are shown in Table VII. The suppression energy ΔD_Λ with respect to the OCF potential is also indicated in the last column of Table VII. It is obtained by the following definition:

$$\Delta D_\Lambda \equiv D_\Lambda(\mathrm{TCF}) - D_\Lambda(\mathrm{OCF}) . \tag{V-36}$$

It is clear from Table VII that the suppression effect is appreciable when the TCF potentials are used except for potential II which is spin independent. We find that potential II yields a suppression of only 3.38 MeV. On the other extreme, we get a suppression of 27.74 MeV for potential IX. In this particular potential, it has been deliberately assumed that $\lambda_\Lambda = \lambda_\Sigma = 0$ in its triplet interaction so that the $\Sigma\Lambda$-conversion process is predominant in the triplet state. For the rest of the TCF potentials, the suppression falls into the range

$$14 \text{ MeV} < \Delta D_\Lambda < 20 \text{ MeV}$$

which indicates that we can attain a large suppression by working in the two-channel formalism.

We conclude, therefore, that the $\Sigma\Lambda$-conversion can play a decisive role in bringing into accord the experimental and the calculated Λ-particle well depths. Hence it is clear why all the previous calculations of the Λ-particle well depth by using OCP potentials did not yield satisfactory results. Some further studies on this particular aspect will be discussed in the next chapter for the binding of ${}_{\Lambda}He^{5}$.

Table VI

Potential parameters in the two-channel formalism

Potentials	Singlet	Triplet	Remarks
I $\beta(fm^{-1})$	1.4663		OCF
$\lambda(10^{-2}fm^{-2})$	3.3867		
II $\beta(fm^{-1})$	1.4402		TCF
λ_Λ	2.8596		
$\lambda_x(10^{-2}fm^{-2})$	2.5		
λ_Σ	3.5		
$\sigma(\Sigma N\to\Lambda N)$ (mb)	183		
$\sigma(\Sigma N\to\Sigma N)$ (mb)	105		
III $\beta(fm^{-1})$	1.1428	1.0685	OCF
$\lambda(10^{-2}fm^{-2})$	1.7102	1.2563	
IV $\beta(fm^{-1})$	1.4623	1.3233	TCF
λ_Λ	3.3030	0.7222	
$\lambda_x(10^{-2}fm^{-2})$	1.0	5.5	
λ_Σ	4.0	1.0	
$\sigma(\Sigma N\to\Lambda N)$ (mb)	143		
$\sigma(\Sigma N\to\Sigma N)$ (mb)	121		
V $\beta(fm^{-1})$	1.4295	1.3179	TCF
λ_Λ	2.5949	1.2491	M^*=2126 MeV
$\lambda_x(10^{-2}fm^{-2})$	3.3	3.5671	Γ =9.7 MeV
λ_Σ	1.0	7.5	
$\sigma(\Sigma N\to\Lambda N)$ (mb)	124		
$\sigma(\Sigma N\to\Sigma N)$ (mb)	477		

(continued next page)

Table VI (continued)

Potentials	Singlet	Triplet	Remarks
VI $\beta(fm^{-1})$	1.2854	1.3252	TCF
λ_Λ	0.5290	1.6413	M*=2110 MeV
$\lambda_x(10^{-2}fm^{-2})$	4.8	2.4926	Γ =8.7 MeV
λ_Σ	4.0	11.02	
$\sigma(\Sigma N\to\Lambda N)$ (mb)	90		
$\sigma(\Sigma N\to\Sigma N)$ (mb)	484		
VII $\beta_\Lambda(fm^{-1})$	1.4456	1.3239	TCF
$\beta_\Sigma(fm^{-1})$	0.7228	0.6619	M*=2126 MeV
λ_Λ	2.9979	1.3982	Γ =9.9 MeV
$\lambda_x(10^{-2}fm^{-2})$	1.2	2.0161	
λ_Σ	0.1	1.45	
$\sigma(\Sigma N\to\Lambda N)$ (mb)	125		
$\sigma(\Sigma N\to\Sigma N)$ (mb)	613		
VIII $\beta_\Lambda(fm^{-1})$	1.3807	1.2774	TCF
$\beta_\Sigma(fm^{-1})$	0.6904	0.6387	M*=2110 MeV
λ_Λ	1.9159	1.2078	Γ =11.9 MeV
$\lambda_x(10^{-2}fm^{-2})$	2.2	1.6039	
λ_Σ	0.2	2.4	
$\sigma(\Sigma N\to\Lambda N)$ (mb)	84.7		
$\sigma(\Sigma N\to\Sigma N)$ (mb)	724		

(continued next page)

Table VI (continued)

Potentials		Singlet	Triplet	Remarks
IX	$\beta(\mathrm{fm}^{-1})$	1.1364	0.9803	TCF
	λ_Λ	1.5963	0.0	
	$\lambda_x(10^{-2}\mathrm{fm}^{-2})$	1.0199	3.1978	
	λ_Σ	1.5963	0.0	
	$\sigma(\Sigma N\rightarrow\Lambda N)$ (mb)	139		
	$\sigma(\Sigma N\rightarrow\Sigma N)$ (mb)	120		

TABLE VII

Binding energies of a Λ particle in the two-channel formalism

Potentials	D_Λ (MeV)	ΔD_Λ (MeV)
I	50.85	-
II	47.47	3.38
III	53.82	-
IV	34.17	16.68
V	36.32	14.53
VI	34.38	16.47
VII	36.01	14.84
VIII	32.17	18.68
IX	26.08	27.74

CHAPTER VI

BINDING OF $_{\Lambda}He^{5}$

§1 Introduction

In the last two chapters, we have discussed the problem of the overbinding of a Λ particle in nuclear matter. We found that both the two-channel formalism - which includes in effect the Σ-suppression and the three-body ΛNN forces - and the repulsive core give substantial suppression in the well depth of the Λ particle. Hence it suggests a very promising procedure to handle the problem of overbinding in $_{\Lambda}He^{5}$ where suppression effects are also being sought to account for the disparity between the theoretical and experimental binding energies. Especially, we know that the $_{\Lambda}He^{5}$ system is very similar to the system of a Λ particle in nuclear matter. Both of them are spin-isospin saturated systems which may have rather important consequences in the effective forces that are operating among the particles. For instance, the inelastic channel $\Lambda N \rightarrow \Sigma N$ is open in the free scattering if the energy of the system is above the ΣN threshold, but isospin conservation hinders the occurrence of this process in a spin-isospin saturated system.

The problem of overbinding in $_{\Lambda}He^{5}$ is of long standing. When a two-body central potential derived from the free ΛN

scattering data is used, it is found consistently that the calculated binding energy of $_{\Lambda}He^5$ is larger than the experimental value by about 2 MeV or more. However, for the other s-shell hypernuclei, the central potential deduced from the ΛN scattering phase shifts is capable to reproduce their respective experimental binding energies. This indicates that a spin-isospin saturated system such as the $_{\Lambda}He^5$ may require a special treatment.

As the central potential fails to yield a correct binding for $_{\Lambda}He^5$, the first remedy one may suggest is to include the tensor force. This aspect has been examined quite thoroughly by Law et al.[10] using a Hartree-Fock calculation. Since the tensor force component of the ΛN force has not been extracted from the meagre data of the ΛN scattering and it has only been indicated from the theoretical studies of the K-meson and η-meson exchange forces that it has a range of the order of 0.4 fm or less, Law et al. have used in their calculation the Yukawa potentials and also the cut-off Yukawa potentials corresponding to the exchange of the K-meson and η-meson which have masses of 493.8 and 548.7 MeV respectively. In their calculation, deformation of the nucleus He^4 in the presence of the Λ particle is allowed so that the effect of the tensor force shows up even in the first order. With reasonable ranges and strengths for the tensor forces, they find that the tensor forces do not play any important role in

${}_{\Lambda}He^{5}$, and the overbinding in ${}_{\Lambda}He^{5}$ persists. A similar conclusion has also been reached for the studies of a Λ particle in nuclear matter[11,12,13].

Thus it becomes necessary to seek an alternative approach to the problem. We shall proceed to consider the effects of the $\Sigma\Lambda$ conversion and the three-body ΛNN forces. As it has been shown by Nogami[19], we can account for most of the effects by working in the two-channel formalism. We have already seen in Chapter V that the calculated well depth in the two-channel formalism is very close to the experimental Λ-particle well depth. Hence we expect that the Σ-suppression effect and the ΛNN force may be equally important in the binding of ${}_{\Lambda}He^{5}$. Indeed, our calculations indicate that the problem of the overbinding of ${}_{\Lambda}He^{5}$ may be resolved if these factors are taken into account.

Recently, Law and Nguyen[24] have studied the effect of Σ-suppression in their free G-matrix expansion model. We will discuss their model and results in Secs. 2 and 3. The model will be modified in Sec. 4 so that we can study the binding of ${}_{\Lambda}He^{5}$ with use of the two-channel nonlocal separable potentials.

§2 The Free G-matrix Expansion Model

The motivation for the proposed free G-matrix expansion model was originally to calculate the well depth directly from the ΛN scattering cross-section[51]. This approach has the advantage that it does not rely on any phenomenological potential. Calculation can be done by using the given scattering phase shifts alone; the normal procedure of extracting a potential from the scattering data is avoided. In the original work of Bhaduri and Law, it is essential to assume that the forces are nearly spin independent, so that they can work with the total scattering cross-section in their calculation. If, however, the singlet and triplet scattering cross-sections are known separately, the restriction on the model owing to the assumption of spin independence can be removed[24].

The relevant equation for the free G-matrix expansion series is obtained by solving the G-matrix in the nuclear medium and the G-matrix for free scattering simultaneously. The G-matrix for free scattering is given by

$$K = V - V \frac{P}{e_o} K \tag{VI-1}$$

where P stands for the principal part and e_o is the energy denominator for scattering, which is just the difference of the kinetic energies of the intermediate and the initial states. As for the G-matrix in nuclear medium, we have

$$G = V - V \frac{Q'}{e} G \tag{VI-2}$$

where Q' is the Pauli operator for finite nucleus and e is the energy denominator defined to be the difference of the unperturbed energy of the intermediate state and the perturbed energy of the initial state.

From (VII-1) and (VI-2), we can deduce the relation

$$G = K + K \left(\frac{P}{e_o} - \frac{Q'}{e}\right) G$$

$$= K + K \left(\frac{P}{e_o} - \frac{Q'}{e}\right) K + \ldots\ldots \quad . \qquad \text{(VI-3)}$$

When K is small, the expansion series will converge rapidly and G can be determined quite accurately by doing a calculation up to the second order.

In order to see the magnitude of K and the relation of K to the scattering cross-section, we follow Otter[53] to obtain the diagonal elements of the s-wave free reaction matrix, thus

$$\langle k|K|k\rangle = -\frac{4\pi\hbar^2}{2\mu} \frac{\tan\delta_o}{k}$$

where k is the relative momentum; δ_o is the s-wave phase shift; and μ is the reduced mass. Assuming δ_o is small, we can write

$$\langle k|K|k\rangle = -\frac{4\pi\hbar^2}{2\mu} \cdot \frac{\delta_o}{k} \qquad \text{(VI-4)}$$

Next, we note that the total cross-section is given by

$$\sigma_T(k) = \frac{4\pi}{k^2} \sin^2\delta_o \ .$$

Therefore,

$$\langle k|K|k\rangle = -\frac{4\pi\hbar^2}{2\mu k} \sin^{-1} \left(k\sqrt{\frac{\sigma_T}{4\pi}}\right)$$

$$\approx -\sqrt{4\pi}\, \frac{\hbar^2}{2\mu} \sqrt{\sigma_T} \qquad \text{(VI-5)}$$

and the last step follows by the assumption that σ_T is small.

It is now clear that the free G-matrix expansion may be good for the ΛN system but not for the NN system, because the scattering cross-section of the latter is very much larger than that of the former.

§3 Application of the Free G-matrix Expansion in ${}_{\Lambda}He^{5}$

The free G-matrix expansion in the one-channel formalism was first applied by Bhaduri and Law[51] to calculate the Λ-particle well depth and, more recently, similar calculation was extended to the two-channel formalism to include the Σ-channel[52]. All of the above calculations are done with the assumption that the forces are nearly spin independent. This spin independent model is not an unreasonable approach for the ΛN system in the sense that the singlet and the triplet ΛN scattering phase shifts have not been well differentiated. However, as a theoretical model, it is also interesting to incorporate the spin dependence in our calculation. Therefore in this section a model, in which the spin dependence and the Σ-channel are included, will be discussed and applied to the ${}_{\Lambda}He^{5}$ system. We will not present here the full details of the results and the calculation, as they can be found in the works of Law and Nguyen[24].

Let us introduce a superscript $\gamma(\gamma = s;t)$ in (VI-3) to distinguish the singlet and triplet reaction matrices, then we have

$$G^\gamma = K^\gamma + K^\gamma \left(\frac{P}{e_o} - \frac{Q'}{e}\right) G^\gamma \qquad \text{(VI-6)}$$

where G^γ and K^γ are 2×2 matrices. For example, we can write explicitly

$$G^\gamma = \begin{pmatrix} G^\gamma_{\Lambda\Lambda} & G^\gamma_{\Lambda\Sigma} \\ G^\gamma_{\Sigma\Lambda} & G^\gamma_{\Sigma\Sigma} \end{pmatrix}$$

with its respective matrix elements corresponding to the processes: $\Lambda N \to \Lambda N$; $\Lambda N \leftrightarrow \Sigma N$; and $\Sigma N \to \Sigma N$. In this representation, we have

$$G^\gamma_{\Lambda\Lambda} = K^\gamma_{\Lambda\Lambda} + K^\gamma_{\Lambda\Lambda} \left(\frac{P}{e_o^\Lambda} - \frac{Q'}{e^\Lambda}\right) G^\gamma_{\Lambda\Lambda} + K^\gamma_{\Lambda\Sigma} \left(\frac{P}{e_o^\Sigma} - \frac{Q'}{e^\Sigma}\right) G^\gamma_{\Sigma\Lambda} \ . \qquad \text{(VI-7)}$$

To obtain the singlet and triplet free reaction matrix elements, we proceed with the following argument. The total scattering cross-section is given by

$$\sigma_T(k) = \frac{1}{4}\sigma_s(k) + \frac{3}{4}\sigma_t \ .$$

Since no detailed experimental information on the singlet and triplet scattering cross-sections separately is available, we shall assume that they are related by a scaling factor ζ, such that

$$\zeta = \frac{\sigma_s}{\sigma_t} \ .$$

We have then

$$\sigma_s(k) = \frac{4\zeta}{3+\zeta}\,\sigma_T(k) \tag{VI-8}$$

$$\sigma_t(k) = \frac{4}{3+\zeta}\,\sigma_T(k)\ . \tag{VI-9}$$

Next, it is found that the ΛN scattering data of Alexander et al.[54] and Sechi-Zorn et al.[55] can be reasonably fitted by the following expression[51]:

$$\sigma_T(k) = 30.0\ \exp(-3.7638\ k^2)\ \mathrm{fm}^2 \tag{VI-10}$$

If we assume that the total free reaction matrix $K^T_{\Lambda\Lambda}$ is local and energy independent, then[51]

$$\begin{aligned}\langle k|K^T_{\Lambda\Lambda}|k\rangle &= -\frac{\hbar^2}{2\mu}\ \sqrt{4\pi\sigma_T} \\ &= \int_0^\infty dr\ j_0^2(kr)K^T_{\Lambda\Lambda}(r)\end{aligned} \tag{VI-11}$$

where $j_0(kr)$ is the spherical Bessel function. Now $K^T_{\Lambda\Lambda}(r)$ can be determined by the inversion[56] of (VI-11) to give

$$K^T_{\Lambda\Lambda}(r) = 25.8(1-1.061\ r^2)\exp(-0.5305\ r^2)\ \mathrm{MeV} \tag{VI-12}$$

Because of the relations given in (VI-8) and (VI-9), we have finally

$$\begin{aligned}K^s_{\Lambda\Lambda}(r) &= \sqrt{\frac{4\zeta}{3+\zeta}}\ K^T_{\Lambda\Lambda}(r) \\ K^t_{\Lambda\Lambda}(r) &= \sqrt{\frac{4}{3+\zeta}}\ K^T_{\Lambda\Lambda}(r)\ .\end{aligned} \tag{VI-13}$$

Similarly, $K^T_{\Lambda\Sigma}(r)$ can be derived by fitting the $\Sigma^- p \to \Lambda N$ scattering data of Engelmann et al.[57]. According to Law[52], it is given by

$$K^T_{\Lambda\Sigma}(r) = \sqrt{\frac{3}{2}} \times \frac{18.6 \exp(-0.7\ r)}{0.7\ r}\ \text{MeV} \qquad \text{(VI-14)}$$

It is found that the binding energy of ${}_\Lambda He^5$ is not sensitive to the spin dependent factor ζ in the Σ-channel hence we will work directly on the total free reaction matrix $K^T_{\Lambda\Sigma}$.

To calculate the binding energy of ${}_\Lambda He^5$, we will use harmonic oscillator wave functions as our basic states. Let us first estimate the contribution to the binding energy $B(A = 5)$ of ${}_\Lambda He^5$ from the kinetic energy alone. The kinetic energy operator with the correction for the motion of the centre of mass will be

$$T = \sum_{i=1}^{4} \frac{p_i^2}{2M_N} + \frac{p_\Lambda^2}{2M_\Lambda} - \left(\sum_{i=1}^{4} \underset{\sim}{p}_i + \underset{\sim}{p}_\Lambda\right)^2/(2M) \qquad \text{(VI-15)}$$

where $\underset{\sim}{p}_i$ and $\underset{\sim}{p}_\Lambda$ are the momenta of the nucleon and Λ particle respectively; and $M = 4M_N + M_\Lambda$. Rearranging terms in (VI-15), we get

$$T = \sum_{i=1}^{4} \frac{M-M_N}{M} \cdot \frac{p_i^2}{2M_N} + \frac{M-M_\Lambda}{2M} \cdot \frac{p_\Lambda^2}{2M_\Lambda} - \sum_{i\neq j} (\underset{\sim}{p}_i \cdot \underset{\sim}{p}_j + \underset{\sim}{p}_i \cdot \underset{\sim}{p}_\Lambda)/(2M). \qquad \text{(VI-16)}$$

Its matrix element with respect to the ground state oscillator wave function yields

$$\langle 0|T|0\rangle = 3\ \frac{M-M_N}{M_\Lambda} \hbar\omega_N + \frac{3}{4}\ \frac{M-M_\Lambda}{M} \hbar\omega_\Lambda \qquad \text{(VI-17)}$$

where ω_N and ω_Λ are the oscillator frequencies of the nucleon and Λ particle respectively. To obtain (VI-17), we have used the fact that the expectation value of a kinetic energy operator in harmonic oscillator representation is $\frac{3}{4}\hbar\omega$, and $\langle O|\underset{\sim}{p}_i \cdot \underset{\sim}{p}_j|O\rangle = 0$ when $i \neq j$[58].

The binding energy can now be written as

$$-B(A=5) = 3\,\frac{M-M_N}{M}\,\hbar\omega_N + 3\langle\phi_N\phi_N|G^s_{NN} + G^t_{NN}|\phi_N\phi_N\rangle$$

$$+ \frac{3}{4}\,\frac{M-M_\Lambda}{M}\,\hbar\omega_\Lambda + \langle\phi_N\phi_\Lambda|3G^t_{\Lambda\Lambda} + G^s_{\Lambda\Lambda}|\phi_N\phi_\Lambda\rangle \qquad \text{(VI-18)}$$

where ϕ_N and ϕ_Λ are the ground state oscillator wave functions for the nucleon and Λ particle respectively; and G^γ_{NN} is the G-matrix for the nucleons. For the ΛN pair, the proper weight 3 to 1 appears for the triplet and singlet states as indicated in the formula. However, for the nucleons inside He^4 nucleus, each pair of nucleons can only be in either a spin-singlet-isospin-triplet state or a spin-triplet-isospin-singlet state. Hence, a factor of 3 appears for both the spin singlet and the spin triplet G-matrices.

In this calculation, the nucleon-nucleon reaction matrices are approximated by using Volkov's soft-core effective potential[59],

$$G^s_{NN} = G^t_{NN} \simeq -83.34\,\exp\left(-\frac{r}{1.6}\right)^2 + 144.86\,\exp\left(-\frac{r}{0.82}\right)^2\ \text{MeV} \qquad \text{(VI-19)}$$

which will lead to a binding energy of 27.89 MeV for He^4 with $\hbar\omega_N$ = 22.0 MeV. As for the ΛN reaction matrices, retaining up to the second-order terms of (VI-7), we get

$$3G^t_{\Lambda\Lambda}+G^s_{\Lambda\Lambda}=(3\sqrt{\frac{4}{3+\zeta}}+\sqrt{\frac{4\zeta}{3+\zeta}})K^T_{\Lambda\Lambda}(r)+4K^T_{\Lambda\Lambda}(r)(\frac{P}{e^\Lambda_o}-\frac{Q'}{e^\Lambda})K^T_{\Lambda\Lambda}(r)$$

$$+\ 4K^T_{\Lambda\Sigma}(r)(\frac{P}{e^\Sigma_o}-\frac{Q'}{e^\Sigma})K^T_{\Sigma\Lambda}(r) \qquad \text{(VI-20)}$$

We notice that the second-order terms are independent of the scaling factor ζ.

We can now solve (VI-18) with the aid of (VI-19) and (VI-20) by a variational calculation. The oscillator frequencies ω_N and ω_Λ are varied until a minimum is attained for B(A=5). The Λ-particle binding energy (or more appropriately the separation energy) in ${}_\Lambda He^5$ is then identified as

$$B_\Lambda = B(A=5) - B(A=4) \qquad \text{(VI-21)}$$

where

$$-B(A=4) = \frac{9}{4}\hbar\omega_N + 3\langle\phi_N\phi_N|G^s_{NN} + G^t_{NN}|\phi_N\phi_N\rangle \qquad \text{(VI-22)}$$

which is the ground state energy of He^4.

The final results of the above calculation show that if the Σ-channel is not included, a large spin dependent force with $\zeta \simeq 11$ is required to bring into accord the theoretical and the experimental values of the binding energy of the Λ particle in ${}_\Lambda He^5$. However, if the Σ-channel is introduced, the spin dependence is reduced to $\zeta \simeq 4.5$.

The calculation using free G-matrix expansion is not devoid of any criticism. For instance, the off energy shell matrix elements of K have to be obtained by the Born approximation with the assumption that the matrix is local and energy independent. It is also found that the results are sensitive to the choice of the range of the reaction matrix $K^T_{\Lambda\Sigma}$ [52].

As a remedy, we will modify the free G-matrix expansion model in the following section so that it can be adapted to the nonlocal separable potentials discussed in Chapter V. The binding energy of ${}_\Lambda He^5$ will then be calculated by using this new model. By so doing, we are able to compare the suppression effects in the two-channel formalism for the binding energy of ${}_\Lambda He^5$ and that of the Λ particle in nuclear matter.

§4 Binding Energy of ${}_\Lambda He^5$ with Two-channel Nonlocal Separable Potentials

We will continue to use the model with the harmonic oscillator wave functions as the basic eigenvectors, and hence we still have the following expression for the binding energy of ${}_\Lambda He^5$:

$$-B(A=5) = 3\,\frac{M-M_N}{M}\,\hbar\omega_N + 3\langle\phi_N\phi_N|G^s_{NN}+G^t_{NN}|\phi_N\phi_N\rangle$$
$$+\frac{3}{4}\,\frac{M-M_\Lambda}{M}\,\hbar\omega_\Lambda + \langle\phi_N\phi_\Lambda|3G^t_{\Lambda\Lambda}+G^s_{\Lambda\Lambda}|\phi_N\phi_\Lambda\rangle \; . \qquad \text{(VI-23)}$$

Again, G^s_{NN} and G^t_{NN} are approximated by the same expression as given by (VI-19), but $G^\gamma_{\Lambda\Lambda}$ will be obtained by solving the Bethe-Goldstone equation in a two-channel nonlocal separable potential.

The relevant matrix elements of the 2×2 Bethe-Goldstone equation

$$G = V - V\frac{Q'}{e}G$$

are given by

$$G_{\Lambda\Lambda} = V_{\Lambda\Lambda} - V_{\Lambda\Lambda}\frac{Q'}{e^{\Lambda}}G_{\Lambda\Lambda} - V_{\Lambda\Sigma}\frac{Q'}{e^{\Sigma}}G_{\Sigma\Lambda}$$

$$G_{\Sigma\Lambda} = V_{\Sigma\Lambda} - V_{\Sigma\Lambda}\frac{Q'}{e^{\Lambda}}G_{\Lambda\Lambda} - V_{\Sigma\Sigma}\frac{Q'}{e^{\Sigma}}G_{\Sigma\Lambda}\ . \qquad \text{(VI-24)}$$

Here we have dropped the superscripts s and t for simplicity. We shall restore them whenever it is necessary to distinguish the singlet and triplet G-matrices. Now if we assume that the nonlocal separable potential takes the form

$$\langle k|V|k'\rangle \quad \begin{pmatrix} -\lambda_{\Lambda}g_{\Lambda}(k)g_{\Lambda}(k') & -\lambda_{x}g_{\Lambda}(k)g_{\Sigma}(k') \\ & \\ -\lambda_{x}g_{\Sigma}(k)g_{\Lambda}(k') & -\lambda_{\Sigma}g_{\Sigma}(k)g_{\Sigma}(k') \end{pmatrix} \qquad \text{(VI-25)}$$

the G-matrix can readily be solved in the momentum representation. As it is shown in Appendix D, we can insert intermediate states in (VI-24) to get

$$\langle k|G_{\Lambda\Lambda}(k)|k'\rangle = -g_{\Lambda}(k)g_{\Lambda}(k')[\lambda_{\Lambda} - d(\lambda)J_{\Sigma}]/D(\underset{\sim}{K}) \qquad \text{(VI-26)}$$

where

$$d(\lambda) = \lambda_{\Lambda}\lambda_{\Sigma} - \lambda_{x}^{2}$$

$$D(\underset{\sim}{K}) = 1 - \lambda_{\Lambda}J_{\Lambda} - \lambda_{\Sigma}J_{\Sigma} + d(\lambda)J_{\Lambda}J_{\Sigma}$$

$$J_{\Lambda} = \int d^{3}p\ g_{\Lambda}^{2}(p)\ \frac{Q_{\Lambda}'(\underset{\sim}{p},\underset{\sim}{K})}{e^{\Lambda}(\underset{\sim}{p},\underset{\sim}{K})}$$

and

$$J_{\Sigma} = \int d^3p_{\Sigma} g_{\Sigma}^2(p) \frac{Q_{\Sigma}'(\underset{\sim}{p},\underset{\sim}{k})}{e^{\Sigma}(\underset{\sim}{p},\underset{\sim}{k})} .$$

The Pauli operators which appear in the above integrands have the following properties:

$$Q_{\Lambda}'(\underset{\sim}{p},\underset{\sim}{K}) = 1 \quad \text{if} \quad |\underset{\sim}{p} + \frac{M_N}{M_N+M_{\Lambda}} \underset{\sim}{K}| > k_F$$

$$= 0 \quad \text{otherwise}$$

and

$$Q_{\Sigma}'(\underset{\sim}{p},K) = 1 \quad \text{if} \quad |\underset{\sim}{p} + \frac{M_N}{M_N+M_{\Lambda}} \underset{\sim}{K}| > k_F$$

$$= 0 \quad \text{otherwise}$$

where $\underset{\sim}{K}$ is the total momentum.

As for the Fermi momentum k_F, it may be obtained in two ways[86]. In the first method, we can define an equivalent density of the He^4 nucleus by $\rho = 4/[(4\pi/3)R^3]$, where R is the root-mean-square radius such that $R = \sqrt{5\hbar/(2M_N\omega_N)}$. Now, using the relation $k_F = [(3\pi^2/2)\rho]^{1/3}$ from the uniform model of the nucleus, one gets $k_F = (9\pi/2)^{1/3} \cdot \sqrt{2M_N\omega_N/(5\hbar)}$. In the other method, we note that the kinetic energy of an s-shell nucleon is $\frac{3}{4}\hbar\omega_N$, which can be equated, in the Fermi gas model, to the average kinetic energy of a particle, $(3/10)(\hbar^2k_F^2/M_N)$. This gives $k_F = \sqrt{5M_N\omega_N/(2\hbar)}$ which is numerically almost identical to the value obtained by the first method.

For the energy denominator, we assume that the single-particle potential is zero in the intermediate state. As long as we are not considering an interaction with a repulsive core, we have demonstrated in Chapter IV that our assumption is reasonable. Thus we take (in the units of $\hbar = c = 1$),

$$e^{\Lambda}(\underset{\sim}{p},\underset{\sim}{K}) = \frac{\underset{\sim}{p}^2}{2\mu_\Lambda} + \frac{\underset{\sim}{K}^2}{2(M_N+M_\Lambda)} - W$$

and

$$e^{\Sigma}(\underset{\sim}{p},\underset{\sim}{K}) = \frac{\underset{\sim}{p}^2}{2\mu_\Sigma} + \frac{\underset{\sim}{K}^2}{2(M_N+M_\Lambda)} + (M_\Sigma - M_\Lambda) - W$$

where μ_Λ and μ_Σ are the ΛN and ΣN reduced masses, respectively; and W is the starting energy which corresponds to the self-consistent sum of the Λ and N kinetic and potential energies in the ground state. In our calculation, the starting energy can be written as

$$W = \frac{3}{4}(\hbar\omega_N + \hbar\omega_\Lambda) + \frac{3}{4}\langle\phi_N\phi_N|G^t_{NN} + G^s_{NN}|\phi_N\phi_N\rangle$$
$$+ \langle\phi_N\phi_\Lambda|G^3_{\Lambda\Lambda} + 3G^t_{\Lambda\Lambda}|\phi_N\phi_\Lambda\rangle \qquad \text{(VI-27)}$$

Now we can proceed to evaluate the matrix element $\langle\phi_N\phi_\Lambda|G_{\Lambda\Lambda}|\phi_N\phi_\Lambda\rangle$. In the momentum space representation, we have

$$\langle\phi_N\phi_\Lambda|G_{\Lambda\Lambda}|\phi_N\phi_\Lambda\rangle = \int d^3k_N \int d^3k_\Lambda \int d^3k'_N \int d^3k'_\Lambda$$
$$\phi^*_N(k_N)\phi^*_\Lambda(k\ \ )\langle k_N k_\Lambda|G_{\Lambda\Lambda}|k'_N k'_\Lambda\rangle\phi_N(k'_N)\phi_\Lambda(k'_\Lambda) \qquad \text{(VI-28)}$$

For the s state, the harmonic oscillator wave function is given by

$$\phi(k) = N \exp(-k^2/2\alpha^2)$$

where

$$N = (\alpha\sqrt{\pi})^{-3/2}$$

$$\alpha^2 = m\omega/\hbar$$

and

$$m = M_N \text{ or } M_\Lambda \ .$$

To simplify (VI-28), we introduce the transformation:

$$\underset{\sim}{K} = \underset{\sim}{k}_N + \underset{\sim}{k}_\Lambda$$

$$\underset{\sim}{k} = X_\Lambda \underset{\sim}{k}_N + X_N \underset{\sim}{k}_\Lambda$$

with $X_N = M_N/(M_N+M_\Lambda)$ and $X_\Lambda = M_\Lambda/(M_N+M_\Lambda)$. Since the total momentum is conserved, we have

$$\langle k_N k_\Lambda | G_{\Lambda\Lambda} | k_N' k_\Lambda' \rangle = \delta(\underset{\sim}{K}-\underset{\sim}{K}')\langle k | G_{\Lambda\Lambda}(\underset{\sim}{K}) | k' \rangle \quad .$$

Also we note that $\underset{\sim}{k}_N = \underset{\sim}{k} + X_N \underset{\sim}{K}$ and $\underset{\sim}{k}_\Lambda = -\underset{\sim}{k} + X_\Lambda \underset{\sim}{K}$; therefore the product of the nucleon and the Λ particle wave functions may now be written as

$$\phi_N(\underset{\sim}{k}+X_N\underset{\sim}{K})\phi_\Lambda(-\underset{\sim}{k}+X_\Lambda\underset{\sim}{K}) = N_N N_\Lambda \exp[-\tfrac{1}{2}(A\underset{\sim}{k}^2+B\underset{\sim}{K}^2+2C(\underset{\sim}{k}\cdot\underset{\sim}{K})]$$

where

$$A = \frac{1}{2}\left(\frac{1}{\alpha_N^2} + \frac{1}{\alpha_\Lambda^2}\right)$$

$$B = \frac{1}{2}\left(\frac{X_N^2}{\alpha_N^2} + \frac{X_\Lambda^2}{\alpha_\Lambda^2}\right)$$

and

$$C = \frac{1}{2}\left(\frac{X_N}{\alpha_N^2} - \frac{X_\Lambda}{\alpha_\Lambda^2}\right) .$$

Thus (VI-28) can be reduced to

$$\langle\phi_N\phi_\Lambda|G_{\Lambda\Lambda}|\phi_N\phi_\Lambda\rangle$$
$$= N_N^2N_\Lambda^2 \int d^3k \int d^3k' \int d^3K \exp[-A(k^2+k'^2)-2B\underset{\sim}{K}^2-2C\underset{\sim}{K}\cdot(\underset{\sim}{k}+\underset{\sim}{k}')]\times$$
$$\langle k|G_{\Lambda\Lambda}(K)|k'\rangle . \qquad \text{(VI-29)}$$

With further rearrangements as shown in Appendix D, the matrix element $\langle\phi_N\phi_\Lambda|G_{\Lambda\Lambda}|\phi_N\phi_\Lambda\rangle$ can be simplified so that it takes the form

$$\langle\phi_N\phi_\Lambda|G_{\Lambda\Lambda}|\phi_N\phi_\Lambda\rangle = -\frac{16}{C\alpha_N^3\alpha_\Lambda^3}\int_0^\infty f(q)H(q)\exp(-\tfrac{1}{2}Aq^2)\,dq \qquad \text{(VI-30)}$$

where

$$H(q) = \int_0^\infty \frac{p\exp(-2Ap^2)}{(p^2+(\frac{q}{2})^2+\beta_\Lambda^2)}\,\ell n\left|\frac{(p+q/2)^2+\beta_\Lambda^2}{(p-q/2)^2+\beta_\Lambda^2}\right|\,dp$$

and

$$f(q) = \int_0^\infty K\sinh(2CKq)\,\frac{\lambda_\Lambda-d(\lambda)J_\Sigma}{D(\underset{\sim}{K})}\exp(-2B\underset{\sim}{K}^2)\,dK$$

Now we have everything ready for carrying out a detailed calculation. In the last chapter, the two-channel nonlocal separable potentials deduced from the ΛN and ΣN scattering have been applied to the problem of a Λ particle in nuclear matter. They can now be cast into our formulae for the ${}_{\Lambda}He^5$ system.

The process of getting the binding energy of ${}_{\Lambda}He^5$ from (VI-23) is to minimize B(A=5) by varying $\hbar\omega_N$ and $\hbar\omega_\Lambda$ and insert the starting energy of (VI-27) self-consistently into the energy denominators. In most cases, we find that $\hbar\omega_N$'s is not very much different from $\hbar\omega_N$ = 22.0 MeV which corresponds to the oscillator energy of He^4 nucleus alone. In fact, minima are attained with $\hbar\omega_N$ = 22.0 MeV for all the TCF potentials except potential II of Schick and Toepfer[23]. This indicates that the He^4 nucleus is rigid in the sense that the core of the nucleus does not deform in the presence of a Λ particle.

After we have found the binding energy of ${}_{\Lambda}He^5$, the separation energy of the Λ particle in the ${}_{\Lambda}He^5$ system is obtained by using the definition given in (VI-21). The results of our calculations are displayed in Table VIII. We see that there is a tremendous overbinding of the Λ particle in ${}_{\Lambda}He^5$ when the OCF potentials I and III are used in the calculation. This result is consistent with all the previous calculations given in the literature[4,17,36,60]. Here, potential III of Schick and Toepfer yields slightly better results than potential I because the potential parameters of the former is moderately spin dependent.

Working with the TCF potentials, we obtain about 2 MeV suppression for potential II but a tremendous amount for the rest of the two-channel potentials. One could ascribe the discrepancy to the way in which the potentials are being fitted. In potential II, the singlet and the triplet potentials are assumed to be the same, whereas the other TCF potentials are derived with the assumption that the singlet and the triplet states are interacting differently, even though only the spin average scattering length and effective range of (II-3) have been used. In addition, one may be tempted to ascribe the difference as due to the "weak" and the "strong" coupling of the ΛN channel to the ΣN channel. However, this is somewhat misleading. On the contrary, at P_{Σ} = 150 MeV/c, the scattering cross-sections $\sigma(\Sigma N \rightarrow \Lambda N)$ are 183 mb for potential II and 143 mb for potential IV - the experimental value is 147±19 mb; therefore one would describe potential II to be stronger than potential IV and not vice versa. Nevertheless our analysis indicates that a spin dependent force is necessary in order to get the right amount of suppression.

We have also examined the special case of TCF potential of Schick and Toepfer[23] with $\lambda_{\Lambda}=\lambda_{\Sigma}=0$. However this potential produces so large a suppression as not to bind the Λ particle to the core of the He^4 nucleus. In fact, we have seen that this particular potential has a drastic suppression effect in the case of a Λ particle in nuclear matter. The suppression is 27.74 MeV as compared with about 15 MeV for the other TCP spin dependent potentials listed in Table VII.

If the ^3S ΛP resonance is included to fit the potential parameters as has been done for potentials V and VI , we can obtain reasonable value for the binding energy of $_{\Lambda}He^5$. Moreover, when the ranges β_{Λ} and β_{Σ} of the potentials are allowed to be different, instead of overbinding, we may underestimate the binding energy of $_{\Lambda}He^5$ as we can see from the results of Table VIII for potentials VII and VIII. However, these qualitative results are obtained with the sacrifice of the accuracy of fitting $\sigma(\Sigma N \rightarrow \Lambda N)$. For instance, potential VIII is fitted with $\sigma(\Sigma N \rightarrow \Lambda N)$ = 84.7 mb, whereas its experimental value is 147±19 mb.

In summary, we note that there is a close correlation for the suppression effects in the binding of $_{\Lambda}He^5$ and that of Λ-hypernuclear matter. Within the accuracy of our method, a potential which gives almost the correct experimental binding energy of $_{\Lambda}He^5$ will produce a suppression of 15 to 16 MeV for the Λ-particle well depth in nuclear matter.

Thus within the framework of the two-channel formalism, when the Σ-suppression and the ΛNN forces are incorporated implicitly, we are able to reproduce the experimental binding energy of $_{\Lambda}He^5$ from the scattering data. Hence we conclude that those effects are indeed important for this particular s-shell hypernucleus.

Table VIII

Binding energies of ${}_{\Lambda}He^5$

Potentials	$\hbar\omega_N$ (MeV)	$\hbar\omega_\Lambda$ (MeV)	k_F^{-1} (fm^{-1})	B(A=5) (MeV)	B_Λ (MeV)
I	24.5	23.2	1.2153	37.41	9.52
II	24.0	21.7	1.2029	35.74	7.85
III	23.0	18.5	1.1776	35.12	7.23
IV	22.0	16.5	1.1517	30.63	2.74
V	22.0	16.5	1.1517	30.88	2.99
VI	22.0	15.5	1.1517	29.98	2.09
VII	22.0	16.0	1.1517	30.36	2.47
VIII	22.0	14.0	1.1517	28.64	0.75

CHAPTER VII

CONCLUSION AND DISCUSSION

In this thesis, we have endeavoured to understand the basic ΛN and the Λ-nucleus interactions through analysis of the binding of a Λ particle in nuclear matter and also in the He^4 nucleus. The overbinding of the Λ particle in both of these systems is an outstanding problem which serves as a valuable source of detailed information on not only the nature of the two-body ΛN forces but also the many-body effects in a many-body system.

Of course, prior to our studies of the problem on the overbinding of a Λ particle, we have to seek a reliable method of calculation on many-body problems. Therefore, we have devoted part of our effort to reviewing various calculational methods. At the same time, we have attempted to clarify some of the confusions concerning the approximation procedures that have been adopted by various authors. For instance, Bishop[50] claims that the Brueckner-Bethe-Goldstone method is an approximation which can be derived as a limiting case of the Galitskii-Migdal method. Thus he means that the latter is more reliable than the former. However, Bishop's assertion is based on a drastic approximation in his treatment of the Brueckner-Bethe-Goldstone method. In a systematic analysis, we have demonstrated that, indeed, the two methods are equivalent.

The Green's function method of Galitskii is further developed in this disseration to include the correlation of the particles in the intermediate states. As a result, we arrive at a self-consistency problem on the single-particle-energy spectra for both the intermediate and initial states. Thus the energy denominator in the Bethe-Goldstone equation becomes somewhat involved. However in a simpler approximation if we do not take account of correlation in the intermediate states, only the single-particle-energy spectrum of the initial states is calculated self-consistently in the energy denominator. We show that this crude approximation corresponds to introducing an energy gap parameter in the energy denominator for our calculation of the Λ-particle well depth. Also we point out that the so-called effective-mass approximation which has been used quite commonly in the literature can lead to a gross error. It is an approximation only for the single-particle-energy spectrum, and thus it is incorrect to replace the physical mass in the Hamiltonian by the effective mass deduced from this approximation.

The single-particle-energy spectrum of a Λ particle in nuclear matter has been extracted from a self-consistency calculation. We show that the well depth is not very sensitive to the detailed shape of the spectrum at high energies and also that a self-consistent insertion of the single-particle-energy spectra in the energy denominator is essential if we are dealing with a strong repulsive interaction at short distances.

Guided by our analysis on the calculation methods of many-body problems, we investigate the effects of a repulsive core on the binding of a Λ particle in nuclear matter. We have found that the repulsive core in the ΛN potential can suppress the Λ binding energy by as much as 10 MeV. For instance, potential A of Table IV yields a well depth of 38.38 MeV. This is a substantial suppression, though the value is still too large as compared with the "experimental value" of about 30 MeV.

The two-channel formalism has also been employed to study the Λ-particle well depth. This formalism includes in effect a part of the suppression due to the three-body forces and the isospin mechanism[19]. Here the suppression in the well depth is found to be around 15 MeV which indicates that the three-body forces and the isospin mechanism are important factors to account for the overbinding.

In the light of these findings with nuclear matter, the overbinding of a Λ particle in ${}_{\Lambda}He^{5}$ is investigated by using the two-channel formalism. The analysis is based on a modified model of Law and Nguyen[24]. The basic states are represented by the harmonic oscillator wave functions. The reaction matrix element is calculated self-consistently until a minimum in the binding energy of $_{\Lambda}He^5$ is attained by varying the oscillator frequencies. We find that the oscillator frequency for the nucleon in the He^4 nucleus remains almost the same before and after the Λ particle is introduced into the system; namely the He^4 core remains rigid even with the presence of a Λ-hyperon.

The best value which we obtain for the binding energy of a Λ particle in $_{\Lambda}He^5$ is 2.99 MeV as shown in potential V of Table VIII. This is very close to the experimental value which is 3.08±0.02 MeV. The success of our model calculation is indeed a great step forward in understanding the overbinding of the $_{\Lambda}He^5$ system. Now it is clear that the reason for the failure in producing the correct binding energy in the previous literature is due to the omission of the three-body forces and of the isospin-suppression mechanism.

APPENDIX A

NONLOCAL SEPARABLE POTENTIAL OF RANK TWO

The scattering from a nonlocal separable potential of rank two is discussed in this appendix. As usual, we shall work in the momentum space representation. The potential is assumed to have the form

$$\langle k|V|k'\rangle = -\{\lambda_1 g_1(k)g_1(k') + \lambda_2 g_2(k)g_2(k')\}. \quad \text{(A-1)}$$

Here we are confining ourselves to the studies of s-wave interaction because a nonlocal separable potential which is a function of the magnitude of the momenta gives rise to an s-wave scattering amplitude.

Let us substitute (A-1) into our Schrödinger equation

$$(k^2-p^2)\psi(p) = 2\mu \int d^3p' \langle p|V|p'\rangle\psi(p') \quad \text{(A-2)}$$

where μ is the reduced mass. The units are such that $\hbar$=c=1. We get

$$(k^2-p^2)\psi(p) = -2\mu \sum_{i=1}^{2} \lambda_i Z_i g_i(p) \quad \text{(A-3)}$$

where

$$Z_i = \int d^3p\, g_i(p)\psi(p).$$

For the scattering of an incident plane wave, (A-3) takes the form

$$\psi(p) = \delta(\underset{\sim}{p}-\underset{\sim}{k}) - 2\mu \sum_{i=1}^{2} \frac{\lambda_i Z_i g_i(p)}{k^2-p^2+i\varepsilon}. \quad \text{(A-4)}$$

This implies

$$Z_i = g_i - \sum_{j=1}^{2} \lambda_j Z_j B_{ij}(k) \qquad \text{(A-5)}$$

where

$$B_{ij}(k) = 2\mu \int d^3p \, \frac{g_i(p) g_j(p)}{k^2 - p^2 + i\varepsilon} \, .$$

We can now solve (A-5) for Z_i. There are two simultaneous equations to be considered here, namely:

$$(B_{11}(k)\lambda_1 + 1) Z_1 + B_{12}(k)\lambda_2 Z_2 = g_1(k) \qquad \text{(A-6)}$$

$$B_{21}(k)\lambda_1 Z_1 + (B_{22}(k)\lambda_2 + 1) Z_2 = g_2(k) \, . \qquad \text{(A-7)}$$

Therefore, we have

$$Z_1 = \{g_1(k)[B_{22}(k)\lambda_2 + 1] - g_2(k) B_{12}(k)\lambda_2\}/\Delta' \qquad \text{(A-8)}$$

$$Z_2 = \{g_2(k)[B_{11}(k)\lambda_1 + 1] - g_1(k) B_{12}(k)\lambda_1\}/\Delta' \qquad \text{(A-9)}$$

with

$$\Delta' = 1 + B_{11}(k)\lambda_1 + B_{22}(k)\lambda_2 + [B_{11}(k) B_{22}(k) - B_{12}(k)^2]\lambda_1\lambda_2 \, .$$

If $g_i(k) = 1/(k^2 + \beta_i^2)$, we get

$$B_{ij}(k) = 4\pi^2 \mu g_i(k) g_j(k) \left[\frac{k^2 - \beta_i\beta_j}{\beta_i + \beta_j} - ik\right] . \qquad \text{(A-10)}$$

Now the asymptotic form of (A-4) in the co-ordinate space is given by

$$\psi(r) \sim e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} + 4\pi^2 \mu \sum_{i=1}^{2} \lambda_i Z_i g_i(k) \, \frac{e^{ikr}}{r} \qquad \text{(A-11)}$$

Hence the s-wave scattering amplitude is given by

$$f(k) = e^{i\delta}\sin\delta/k$$
$$= 4\pi^2\mu \sum_{i=1}^{2} \lambda_i Z_i g_i(k) \tag{A-12}$$

where δ is the s-wave phase shift. To simplify (A-12), we introduce

$$R_{ij} = P\int 2\mu\{g_i(p)g_j(p)/(p^2-k^2)\}d^3p \tag{A-13}$$

which, according to (A-10), gives

$$R_{ij} = 4\pi^2\mu g_i(k)g_j(k)(k^2-\beta_i\beta_j)/(\beta_i+\beta_j) \quad . \tag{A-14}$$

Thus the s-wave scattering amplitude becomes

$$f(k) = 2\pi^2 N/D \tag{A-15}$$

where

$$N = -\lambda_1\lambda_2[g_1(k)^2R_{22}+g_2(k)^2R_{11}-2g_1(k)g_2(k)R_{12}]+\lambda_1 g_1(k)^2+\lambda_2 g_2(k)^2$$

$$D = \lambda_1\lambda_2(R_{11}R_{22}-R_{12}^2) - \lambda_1R_{11}-\lambda_2R_{22} + 1 - 2\pi^2 ikN$$

Now the scattering length and the effective range are defined by

$$k\cot\delta = -1/a + r_o k^2/2 + \ldots \, . \tag{A-16}$$

We can compare this equation with

$$f(k) = 1/[k\cot\delta-ik] \tag{A-17}$$

to get

$$a = 2\,\frac{\beta_1+\beta_2}{\beta_1\beta_2}\;\frac{y}{z} \qquad \text{(A-18)}$$

$$r_o = \frac{-2}{a(\beta_1\beta_2)^2}\;\left(\frac{y'}{y} - \frac{z'}{z}\right) \qquad \text{(A-19)}$$

where

$$y = \gamma_1\gamma_2(\beta_1-\beta_2)^2 - (\beta_1+\beta_2)(\gamma_1\beta_2^4 + \gamma_2\,\beta_1^4)$$

$$z = \gamma_1\gamma_2(\beta_1-\beta_2)^2 - (\beta_1+\beta_2)^2\{\gamma_1\beta_2^3 + \gamma_2\beta_1^3 - (\beta_1\beta_2)^3\}$$

$$y' = \gamma_1\gamma_2(\beta_1-\beta_2)^2\{2(\beta_1+\beta_2)^2-3\beta_1\beta_2\}-2(\beta_1+\beta_2)(\gamma_1\beta_2^6+\gamma_2\,\beta_1^6)$$

$$z' = \gamma_1\gamma_2(\beta_1-\beta_2)^2\{3(\beta_1+\beta_2)^2-2\beta_1\beta_2\}-3(\beta_1+\beta_2)^2(\gamma_1\beta_2^5+\gamma_2\beta_1^5)$$

and

$$\gamma_i = 2\pi^2\mu\lambda_i \quad , \quad i = 1,2.$$

APPENDIX B

ANGULAR INTEGRATION IN THE PRESENCE OF THE PAULI OPERATOR

When we have the Pauli operator in the integrand, the volume integral in the momentum space can be carried out in the following manner.

Let us denote the momenta of Λ and nucleon in the initial and intermediate states by $\underset{\sim}{k}_\Lambda$ and $\underset{\sim}{k}_N$, and $\underset{\sim}{k}'_\Lambda$ and $\underset{\sim}{k}'_N$, respectively. The relative momenta in the initial and intermediate states will then be given by

$$\underset{\sim}{k} = X_\Lambda \underset{\sim}{k}_N - X_N \underset{\sim}{k}_\Lambda \qquad \text{(B-1)}$$

$$\underset{\sim}{k}' = X_\Lambda \underset{\sim}{k}'_N - X_N \underset{\sim}{k}'_\Lambda \ . \qquad \text{(B-2)}$$

Here, $X_\Lambda = M_\Lambda/(M_\Lambda+M_N)$ and $X_N = M_N/(M_N+M_\Lambda)$. The total momentum is conserved. Thus

$$\underset{\sim}{K} = \underset{\sim}{k}_\Lambda + \underset{\sim}{k}_N = \underset{\sim}{k}'_\Lambda + \underset{\sim}{k}'_N \qquad \text{(B-3)}$$

From (B-1) and (B-2), we obtain

$$\underset{\sim}{k}'_N = \underset{\sim}{k}' + X_N \underset{\sim}{K}. \qquad \text{(B-4)}$$

Now we have to deal with an integration of the following type:

$$\int d^3k' \ Q(k',K)$$

where $Q(k',K)$ is the Pauli operator which requires that $|\underset{\sim}{k}'_N| > k_F$. If θ' is the angle between $\underset{\sim}{k}'$ and $\underset{\sim}{K}$, we should have the condition that

$$\cos\theta' > Z \equiv [k_F^2 - k'^2 - (X_N K)^2]/(2X_N k' K). \qquad (B-5)$$

The integration over k' can be split into three parts, corresponding to Z>1, 1>Z>-1 and -1>Z, respectively.

(i) If Z>1, i.e. $0<k'<k_F-\alpha_N K$, there is no contribution.

(ii) If 1>Z>-1, i.e. $|k_F-X_N K|<k'<k_F+X_N K$, the integration becomes

$$2\pi \int_{k_F-X_N K}^{k_F+X_N K} \int_{Z}^{1} k'^2 dk' d\cos\theta'$$

(iii) If -1>Z, i.e. $k'>k_F+X_N K$ or $0<k'<-k_F+X_N K$, we have

$$2\pi [\int_{k_F+X_N K}^{\infty} \int_{-1}^{1} k'^2 dk' d\cos\theta' + \theta(-k_F+X_N K).$$

$$\int_{0}^{-k_F+X_N K} \int_{-1}^{1} k'^2 dk' d\cos\theta']$$

where $\theta(-k_F+X_N K)$ is a step function such that

$$\theta(x) = 1 \quad \text{if} \quad x > 0$$
$$= 0 \quad \text{otherwise.}$$

Summarizing the results, we get

$$\int d^3k'Q(k',K) = 2\pi\left[\int_{k_F+X_NK}^{\infty}\int_{-1}^{1} k'^2dk' + \int_{|k_F-X_NK|}^{k_F+X_NK}\int_{Z}^{1} k'^2dk'\right.$$

$$\left.+\theta(-k_F+X_NK)\int_{o}^{-k_F+X_NK}\int_{-1}^{1} k'^2dk'\right]d\cos\theta'. \qquad \text{(B-6)}$$

If the integrand is independent of θ', we have for (B-6),

$$\int d^3k'Q(k',K) = 4\pi\left[\int_{k_F+X_NK}^{\infty} k'^2dk' +\right.$$

$$+\frac{1}{4X_NK}\int_{|k_F-X_NK|}^{k_F+X_NK} \{(k'+X_NK)^2-k_F{}^2\}k'dk'$$

$$\left.+\theta(-k_F+X_NK)\int_{o}^{-k_F+X_NK} k'^2dk'\right] \qquad \text{(B-7)}$$

APPENDIX C

SCATTERING MATRIX IN THE TWO-CHANNEL FORMALISM

We shall present in this appendix a detailed calculation of the scattering matrix in the two-channel formalism when a nonlocal separable potential is used. Subsequently, the expressions for the scattering length and the effective range relating to the potential parameters are deduced from the scattering phase shifts.

The two-channel coupled Schrödinger equations are given by

$$\left(\frac{p^2}{2\mu_\Lambda} - E_\Lambda\right)\psi_\Lambda^k(p) - \lambda_\Lambda g_\Lambda(p)\int g_\Lambda(p')\psi_\Lambda^k(p')d^3p'$$

$$-\lambda_x g_\Lambda(p)\int g_\Sigma(p_\Sigma')\psi_\Sigma^{k_\Sigma}(p_\Sigma')d^3p_\Sigma' = 0 \qquad \text{(C-1)}$$

$$\left(\frac{p_\Sigma^2}{2\mu_\Sigma} - E_\Lambda+\Delta\right)\psi_\Sigma^{k_\Sigma}(p_\Sigma) - \lambda_\Sigma g_\Sigma(p_\Sigma)\int g_\Sigma(p_\Sigma')\psi_\Sigma^{k_\Sigma}(p_\Sigma')d^3p_\Sigma'$$

$$-\lambda_x g_\Sigma(p_\Sigma)\int g_\Lambda(p')\psi_\Lambda^k(p')d^3p' = 0 \qquad \text{(C-2)}$$

where μ_Λ and μ_Σ are the ΛN and ΣN reduced masses; $E_\Lambda = \frac{k^2}{2\mu_\Lambda}$, $E_\Lambda - \Delta = \frac{k_\Sigma^2}{2\mu_\Sigma}$ and $\Delta = M_\Sigma - M_\Lambda$. For the ΛN energy below the ΣN threshold, $E_\Lambda - \Delta < 0$ and $k_\Sigma^2 < 0$.

Let us introduce Z_Λ and Z_Σ. Thus

$$Z_\Lambda = \int g_\Lambda(p)\psi_\Lambda^k(p)d^3p$$

$$Z_\Sigma = \int g_\Sigma(p)\psi_\Sigma^{k_\Sigma}(p)d^3p .$$

Then our Schrödinger equation becomes

$$\left(\frac{p^2}{2\mu_\Lambda} - \frac{k^2}{2\mu_\Lambda}\right)\psi_\Lambda^k(p) - (\lambda_\Lambda Z_\Lambda + \lambda_x Z_\Sigma)g_\Lambda(p) = 0 \qquad \text{(C-3)}$$

$$\left(\frac{p_\Sigma^2}{2\mu_\Sigma} - \frac{k_\Sigma^2}{2\mu_\Sigma}\right)\psi_\Sigma^{k_\Sigma}(p_\Sigma) - (\lambda_\Sigma Z_\Sigma + \lambda_x Z_\Lambda)g_\Sigma(p_\Sigma) = 0. \qquad \text{(C-4)}$$

With ΛN as the incident channel, the physical solutions of (C-3) and (C-4) will be

$$\psi_\Lambda^k(p) = \delta(\underset{\sim}{p}-\underset{\sim}{k}) + \frac{\lambda_\Lambda Z_\Lambda + \lambda_x Z_\Sigma}{e_\Lambda(p,k)} g_\Lambda(p) \qquad \text{(C-5)}$$

$$\psi_\Sigma^{k_\Sigma}(p\ ) = \frac{\lambda_\Sigma Z_\Sigma + \lambda_x Z_\Lambda}{e_\Sigma(p_\Sigma,k_\Sigma)} g_\Sigma(p_\Sigma) \qquad \text{(C-6)}$$

where

$$e_\Lambda(p,k) = (p^2-k^2-i\varepsilon)/2\mu_\Lambda$$

$$e_\Sigma(p_\Sigma,k_\Sigma) = (p_\Sigma^2-k_\Sigma^2)/2\mu_\Sigma .$$

If $k_\Sigma^2<0$, there is no pole in the energy denominator of $e_\Sigma(p_\Sigma,k_\Sigma)$. Otherwise, we have to introduce $i\varepsilon$ into $e_\Sigma(p_\Sigma,k_\Sigma)$. From (C-5) and (C-6), we can solve for Z_Λ and Z_Σ to yield

$$Z_\Lambda = [1-\lambda_\Sigma J_\Sigma]g_\Lambda(k)/D(k,k_\Sigma) \qquad \text{(C-7)}$$

$$Z_\Sigma = \lambda_x J_\Sigma g_\Lambda(k)/D(k,k_\Sigma) \qquad \text{(C-8)}$$

where

$$D(k,k_\Sigma) = \begin{vmatrix} 1-\lambda_\Lambda J_\Lambda & -\lambda_x J_\Lambda \\ \lambda_x J_\Sigma & 1-\lambda_\Sigma J_\Sigma \end{vmatrix} \qquad \text{(C-9)}$$

with

$$J_\Lambda = \int \frac{g_\Lambda^2(p)d^3p}{e_\Lambda(p,k)} \qquad \text{(C-10)}$$

$$J_\Sigma = \int \frac{g_\Sigma^2(p)}{e_\Sigma(p,k_\Sigma)} d^3p \ . \qquad \text{(C-11)}$$

Thus we have

$$\psi_\Lambda^k(p) = \delta(\underset{\sim}{p}-\underset{\sim}{k}) + \frac{(\lambda_\Lambda - d(\lambda)J_\Sigma)g_\Lambda(k)g_\Lambda(p)}{D(k,k_\Sigma)} \cdot \frac{1}{e_\Lambda(p,k)}$$

$$\equiv \delta(\underset{\sim}{p}-\underset{\sim}{k}) + T_{\Lambda\Lambda}(p,k)/[4\pi^2\mu_\Lambda e_\Lambda(p,k)] \qquad \text{(C-12)}$$

and

$$\psi_\Sigma^{k_\Sigma}(p_\Sigma) = \frac{\lambda_x g_\Lambda(k)g_\Sigma(p_\Sigma)}{D(k,k_\Sigma)} \cdot \frac{1}{e_\Sigma(p_\Sigma,k_\Sigma)}$$

$$\equiv T_{\Sigma\Lambda}(p_\Sigma,k)/[4\pi^2\mu_\Sigma e_\Sigma(p_\Sigma,k_\Sigma)] \qquad \text{(C-13)}$$

where $d(\lambda) = \lambda_\Lambda\lambda_\Sigma - \lambda_x^2$; and $T_{\Lambda\Lambda}(p,k)$ and $T_{\Sigma\Lambda}(p_\Sigma,k)$ are the scattering amplitudes for $\Lambda N\to\Lambda N$ and $\Lambda N\to\Sigma N$, respectively. The scattering cross-sections are then given by

$$\sigma(\Lambda N\to\Lambda N) = 4\pi|T_{\Lambda\Lambda}(k,k)|^2$$

$$\sigma(\Lambda N\to\Sigma N) = 4\pi|T_{\Sigma\Lambda}(k_\Sigma,k)|^2(v_\Sigma/v_\Lambda)\theta(E_\Lambda-\Delta) \qquad \text{(C-14)}$$

with $v_\Sigma=k_\Sigma/\mu_\Sigma$ and $v_\Lambda=k/\mu_\Lambda$. Here, the step function $\theta(x)$ will ensure that the inelastic scattering $\Lambda N\to\Sigma N$ is open if the energy of the system is above the ΣN threshold.

In the case when ΣN is the entrance channel, we have

$$\sigma(\Sigma N \to \Sigma N) = 4\pi |T_{\Sigma\Sigma}(k_\Sigma, k_\Sigma)|^2$$
$$\sigma(\Sigma N \to \Lambda N) = 4\pi |T_{\Lambda\Sigma}(k, k_\Sigma)|^2 (v_\Lambda / v_\Sigma) \qquad \text{(C-15)}$$

where $T_{\Sigma\Sigma}(k_\Sigma, k_\Sigma)$ and $T_{\Lambda\Sigma}(k, k_\Sigma)$ are obtained from $T_{\Lambda\Lambda}(k,k)$ and $T_{\Sigma\Lambda}(k_\Sigma, k)$ by changing Λ into Σ and vice versa.

Now, let us obtain the ΛN scattering length and effective range in terms of the potential parameters. The scattering amplitude and the phase shift are related by

$$T_{\Lambda\Lambda} = e^{i\delta} \sin\delta / k$$

or alternatively

$$T_{\Lambda\Lambda}^{-1} = -ik + k\cot\delta. \qquad \text{(C-16)}$$

But from (C-12), we have

$$T_{\Lambda\Lambda}^{-1} = \frac{1}{4\pi^2 \mu_\Lambda g_\Lambda^2(k)} \cdot \frac{D}{A_o} \qquad \text{(C-17)}$$

where $A_o \equiv \lambda_\Lambda - d(\lambda) J_\Sigma$, and for simplicity we have replaced $D(k, k_\Sigma)$ by D in the formula. Furthermore we have

$$D/A_o = -J_\Lambda + (1 - \lambda_\Sigma J_\Sigma)/A_o \qquad \text{(C-18)}$$

and we can rewrite (C-17) as follows

$$T_{\Lambda\Lambda}^{-1} = \frac{1}{4\pi^2 \mu_\Lambda g_\Lambda^2(k)} \{J_\Lambda + \frac{1 - \lambda_\Sigma J_\Sigma}{A_o}\} \qquad \text{(C-19)}$$

Now

$$J_\Lambda = \int d^3p \, \frac{g_\Lambda^2(p)}{e_\Lambda(p,k)}$$

$$= 4\pi P \int_0^\infty dp \, \frac{p^2 g_\Lambda^2(p)}{e_\Lambda(p,k)} + i\{4\pi^2 \mu_\Lambda k g_\Lambda^2(k)\} \; . \qquad \text{(C-20)}$$

Since we are considering the case where $k_\Sigma^2 < 0$, J_Σ and hence A_o are real. Thus we get

$$\text{Im}\,(T_{\Lambda\Lambda}^{-1}) = -k \qquad \text{(C-21)}$$

and

$$\text{Re}\,(T_{\Lambda\Lambda}^{-1}) = \frac{1}{4\pi^2 \mu_\Lambda g_\Lambda^2(k)} \{-4\pi P \int_0^\infty dp \, \frac{p^2 g_\Lambda^2(p)}{e_\Lambda(p,k)} + \frac{1-\lambda_\Sigma J_\Sigma}{A_o}\} \qquad \text{(C-22)}$$

We see that (C-16) and (C-21) are consistent. Comparing (C-16) and (C-22), we have

$$k\cot\delta = \frac{1}{4\pi^2 \mu_\Lambda g_\Lambda^2(k)} \{-4\pi P \int_0^\infty dp \, \frac{p^2 g_\Lambda^2(p)}{e_\Lambda(p,k)} + \frac{1-\lambda_\Sigma J_\Sigma}{A_o}\} \; . \qquad \text{(C-23)}$$

We shall take

$$g_{\Lambda,\Sigma}(k) = \frac{1}{k^2+\beta^2} \; .$$

Here, we assume that $\beta_\Lambda = \beta_\Sigma = \beta$ for simplicity. Thus we get

$$P \int_0^\infty dp \, \frac{p^2 g_\Lambda^2(p)}{e_\Lambda(p,k)} = - \frac{\pi \mu_\Lambda (k^2-\beta^2)}{2\beta (k^2+\beta^2)^2} \qquad \text{(C-24)}$$

$$J_\Sigma = \int d^3p \, \frac{g_\Sigma^2(p)}{e_\Sigma(p,k_\Sigma)} = \frac{2\pi^2 \mu_\Sigma}{\beta (K_\Sigma+\beta)^2} \qquad \text{(C-25)}$$

where

$$K_\Sigma^2 = 2\mu_\Sigma(\Delta - k^2/2\mu_\Lambda) \quad . \tag{C-26}$$

Substituting these into (C-23), we have

$$k\cot\delta = -\beta + \frac{\beta^2+k^2}{2\beta} + \frac{(\beta^2+k^2)^2}{4\pi^2\mu_\Lambda\lambda_\Lambda} \; \Theta \tag{C-27}$$

where

$$\Theta = \frac{1-\lambda_\Sigma J_\Sigma}{1-\lambda' J_\Sigma} = \frac{\beta(\beta+K_\Sigma)^2 - 2\pi^2\mu_\Sigma\lambda_\Sigma}{\beta(\beta+K_\Sigma)^2-2\pi^2\mu_\Sigma\lambda'} \tag{C-28}$$

with $\lambda' = d(\lambda)/\lambda_\Lambda$. In order to expand the RHS of (C-27) in a power series of k_Λ, we have to convert K_Σ into a function of k_Λ . To this end, we introduce κ and κ_Σ as follows

$$\Delta \equiv \frac{\kappa^2}{2\mu_\Lambda} = \frac{\kappa_\Sigma^2}{2\mu_\Sigma}$$

from which we get

$$\begin{aligned} K_\Sigma &= \kappa_\Sigma(1-x^2)^{1/2} \\ &= \kappa_\Sigma(1 - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 + \dots) \end{aligned} \tag{C-29}$$

with

$$x \equiv k/\kappa .$$

Consequently, H becomes

$$\begin{aligned}\Theta &= \frac{\beta\{\beta^2+2\beta\kappa_\Sigma(1-\frac{1}{2}x^2-\frac{1}{8}x^4+\dots)+\kappa_\Sigma^2(1-x^2)\}-2\pi^2\mu_\Sigma\lambda_\Sigma}{\beta\{\beta^2+2\beta\kappa_\Sigma(1-\frac{1}{2}x^2-\frac{1}{8}x^4+\dots)+\kappa_\Sigma^2(1-x^2)\}-2\pi^2\mu_\Sigma\lambda'} \\ &\simeq \frac{A_1}{A_2}\{1-\frac{\beta\kappa_\Sigma(\beta+\kappa_\Sigma)}{A_1}x^2 - \frac{\beta^2\kappa_\Sigma}{4A_1}x^4\}\{1-\frac{\beta\kappa_\Sigma(\beta+\kappa_\Sigma)}{A_2}x^2\end{aligned}$$

$$- \frac{\beta^2 \kappa_\Sigma}{4A_2} x^4 \}^{-1}$$

$$= \frac{A_1}{A_2} (1-a_1 x^2 - a_2 x^2)(1+a_3 x^2 + a_4 x^4) \qquad \text{(C-30)}$$

where

$$A_1 = \beta(\beta+\kappa_\Sigma)^2 - 2\pi^2 \mu_\Sigma \lambda_\Sigma$$

$$A_2 = \beta(\beta+\kappa_\Sigma)^2 - 2\pi^2 \mu_\Sigma \lambda'$$

$$a_1 = \beta\kappa_\Sigma(\beta+\kappa_\Sigma)/A_1; \quad a_2 = \beta^2\kappa_\Sigma/4A_1$$

$$a_3 = \beta\kappa_\Sigma(\beta+\kappa_\Sigma)/A_2; \quad a_4 = \beta^2\kappa_\Sigma^2(\beta+\kappa_\Sigma)^2/A_2 + \beta^2\kappa_\Sigma/4A_2.$$

Thus, the third term on the RHS of (C-27) gives

$$\frac{(\beta^2+k^2)^2}{4\pi^2\mu_\Lambda\lambda_\Lambda} \Theta = \frac{1}{4\pi^2\mu_\Lambda\lambda_\Lambda} \cdot \frac{A_1}{A_2} \{\beta^4 + [2\beta^2\kappa^2 + (a_3-a_1)\beta^4]x^2$$

$$+ [(a_4 - a_1a_3 - a_2)\beta^4 + 2(a_3-a_1)\beta^2\kappa^2 + \kappa^4]x^4$$

$$+ [-(a_1a_4 + a_2a_3)\beta^4 + 2\beta^2\kappa^2(a_4 - a_1a_2 - a_2)$$

$$+ (a_3-a_1)\kappa^4]x^6 + \ldots\ldots \} . \qquad \text{(C-31)}$$

Now the low energy scattering parameters are defined by the following expansion formula

$$k\cot\delta = -\frac{1}{a} + \frac{1}{2} r_o k^2 + P_o r_o^3 k^4 + Q_o r_o^5 k^6 + \ldots . \qquad \text{(C-32)}$$

Comparing the coefficients of (C-27) and (C-32) we obtain

$$\frac{1}{a} = \frac{\beta}{2} - \frac{\beta^4}{4\pi^2\mu_\Lambda\lambda_\Lambda} \quad \frac{A_1}{A_2}$$

$$r_o = \frac{1}{\beta} + \frac{[2\beta^2\kappa^2+(a_3-a_1)\beta^4]}{\pi^2\mu_\Lambda\lambda_\Lambda} \cdot \frac{A_1}{A_2}$$

which may be written more explicitly in terms of the potential parameters as

$$\frac{1}{a} = \frac{\beta}{2}\{1 - \frac{\beta^3[\beta(\beta+\kappa')^2-\gamma_\Sigma]}{\gamma_\Lambda\beta(\beta+\kappa')^2-d(\gamma)}\} \tag{C-33}$$

$$r_o = \frac{3}{\beta} - \frac{4}{a\beta^2} - \frac{\mu_\Sigma\gamma_x\beta^5(\beta+\kappa')}{\mu_\Lambda\kappa'[\gamma_\Lambda\beta(\beta+\kappa')^2-d(\gamma)]^2} \tag{C-34}$$

where $\gamma_\Lambda = 2\pi^2\mu_\Lambda\lambda_\Lambda$; $\gamma_\Sigma = 2\pi^2\mu_\Sigma\lambda_\Sigma$; $\gamma_x = 2\pi^2(\mu_\Lambda\mu_\Sigma)^{1/2}\lambda_x$; $d(\gamma) = \gamma_\Lambda\gamma_\Sigma-\gamma_x^2$; and $\kappa' \equiv \kappa_\Sigma = (2\mu_\Sigma\Delta)^{1/2}$.

APPENDIX D

REACTION MATRIX WITH A NONLOCAL SEPARABLE POTENTIAL

We shall set up a model to calculate the reaction matrix elements of the ${}_{\Lambda}He^{5}$ system. Here, again a nonlocal seperable potential will be used. In the two-channel formalism, we assume that V takes the form

$$\langle k|V|k'\rangle = \begin{pmatrix} -\lambda_{\Lambda} g_{\Lambda}(k) g_{\Lambda}(k') & -\lambda_{x} g_{\Lambda}(k) g_{\Sigma}(k') \\ \\ -\lambda_{x} g_{\Sigma}(k) g_{\Lambda}(k') & -\lambda_{\Sigma} g_{\Lambda}(k) g_{\Sigma}(k') \end{pmatrix} \qquad \text{(D=1)}$$

where λ_{Λ}, λ_{x} and λ_{Σ} are the strengths of the potential and $g_i(k) = 1/(k^2+\beta_i^2)$, $(i = \Lambda \text{ or } \Sigma)$; β_i's is the inverse of the range of the potential. Now we have for the G-matrix

$$G = V - V \frac{Q'}{e} G. \qquad \text{(D-2)}$$

From (D-1) and (D-2), we get

$$\langle k|G_{\Lambda\Lambda}|k'\rangle = \langle k|V_{\Lambda\Lambda}|k'\rangle - \langle k|V_{\Lambda\Lambda}\frac{Q'}{e^{\Lambda}} G_{\Lambda\Lambda}|k'\rangle - \langle k|V_{\Lambda\Sigma}\frac{Q'}{e^{\Sigma}} G_{\Sigma\Lambda}|k'\rangle \qquad \text{(D-3)}$$

and

$$\langle k|G_{\Sigma\Lambda}|k'\rangle = \langle k|V_{\Sigma\Lambda}|k'\rangle - \langle k|V_{\Sigma\Lambda}\frac{Q'}{e^{\Lambda}} G_{\Lambda\Lambda}|k'\rangle - \langle k|V_{\Sigma\Sigma}\frac{Q'}{e^{\Sigma}} G_{\Sigma\Lambda}|k'\rangle \, . \qquad \text{(D-4)}$$

Since V is separable, we observe that the G-matrix is also separable. By examining (D-3), (D-4) and the other matrix elements of G, we can write

$$<k|G|k'> = \begin{pmatrix} -\lambda_\Lambda g_\Lambda(k)X_\Lambda(k') & -\lambda_x g_\Lambda(k)\tilde{X}_\Sigma(k') \\ & \\ -\lambda_x g_\Sigma(k)\tilde{X}_\Lambda(k') & -\lambda_\Sigma g_\Sigma(k)X_\Sigma(k') \end{pmatrix} \quad \text{(D-5)}$$

Where the X's are to be determined. We can now make use of (D-2) and (D-5) to solve (D-3) and (D-4) by inserting a complete set of intermediate states. For (D-3), we get

$$-\lambda_\Lambda g_\Lambda(k)X_\Lambda(k') = -\lambda_\Lambda g_\Lambda(k)g_\Lambda(k') - \lambda_\Lambda^2 g_\Lambda(k)X_\Lambda(k') \int d^3p \, \frac{Q_\Lambda'}{e^\Lambda} g_\Lambda^2(p)$$

$$-\lambda_x^2 g_\Lambda(k)\tilde{X}_\Lambda(k') \int d^3p \, \frac{Q_\Sigma'}{e^\Sigma} g_\Sigma^2(p)$$

or

$$\lambda_\Lambda X_\Lambda(k') = \lambda_\Lambda g_\Lambda(k') + \lambda_\Lambda^2 J_\Lambda X_\Lambda(k') + \lambda_x^2 J_\Sigma \tilde{X}_\Lambda(k') \quad \text{(D-6)}$$

where

$$J_\Lambda = \int d^3p \, g_\Lambda^2(p) \, \frac{Q_\Lambda'(\underset{\sim}{p},\underset{\sim}{K})}{e^\Lambda(\underset{\sim}{p},\underset{\sim}{K})} \quad \text{(D-6a)}$$

$$J_\Sigma = \int d^3p \, g_\Sigma^2(p) \, \frac{Q_\Sigma'(\underset{\sim}{p},\underset{\sim}{K})}{e^\Sigma(\underset{\sim}{p},\underset{\sim}{K})} \quad \text{(D-6b)}$$

$$e^\Lambda(\underset{\sim}{p},\underset{\sim}{K}) = \frac{\underset{\sim}{p}^2}{2\mu_\Lambda} + \frac{\underset{\sim}{K}^2}{2M^\Lambda} - W$$

and

$$e^\Sigma(\underset{\sim}{p},\underset{\sim}{K}) = \frac{\underset{\sim}{p}^2}{2\mu_\Sigma} + \frac{\underset{\sim}{K}^2}{2M^\Sigma} + M_\Sigma - M_\Lambda - W$$

with $\mu_\Lambda = \frac{M_\Lambda M_N}{M^\Lambda}$, $\mu_\Sigma = \frac{M_\Sigma M_N}{M^\Sigma}$, $M^\Lambda = M_\Lambda + M_N$ and $M^\Sigma = M_\Sigma + M_N$.

Here, $\underset{\sim}{K}$ is the total momentum and W the starting energy. Following a similar procedure, we get from (D-4)

$$(\lambda_x-\lambda_x\lambda_\Sigma J_\Sigma)\tilde{X}_\Lambda(k) = \lambda_x g_\Lambda(k)+\lambda_x\lambda_\Lambda J_\Lambda X_\Lambda(k). \tag{D-7}$$

The simultaneous first-order equations of (D-6) and (D-7) for $X_\Lambda(k)$ and $\tilde{X}_\Lambda(k)$ yield

$$\lambda_\Lambda X_\Lambda(k) = \frac{[\lambda_\Lambda+(\lambda_x^2-\lambda_\Lambda\lambda_\Sigma)J_\Sigma]g_\Lambda(k)}{[1-\lambda_\Lambda J_\Lambda-\lambda_\Sigma J_\Sigma+(\lambda_\Lambda\lambda_\Sigma-\lambda_x^2)J_\Sigma J_\Lambda]} \tag{D-8}$$

and

$$\tilde{X}_\Lambda(k) = \frac{g_\Lambda(k)}{[1-\lambda_\Lambda J_\Lambda-\lambda_\Sigma J_\Sigma+(\lambda_\Lambda\lambda_\Sigma-\lambda_x^2)J_\Sigma J_\Lambda]}. \tag{D-9}$$

Therefore, the G-matrix elements are given by

$$\langle k|G_{\Lambda\Lambda}|k'\rangle = -\frac{[\lambda_\Lambda-d(\lambda)J_\Sigma]g_\Lambda(k)g_\Lambda(k')}{D(\underset{\sim}{K})} \tag{D-10}$$

and

$$\langle k|G_{\Sigma\Lambda}|k'\rangle = -\frac{\lambda_x g_\Sigma(k)g_\Lambda(k)}{D(\underset{\sim}{K})} \tag{D-11}$$

where

$$d(\lambda) = \lambda_\Lambda\lambda_\Sigma-\lambda_x^2$$

$$D(\underset{\sim}{K}) = 1-\lambda_\Lambda J_\Lambda-\lambda_\Sigma J_\Sigma + d(\lambda)J_\Lambda J_\Sigma .$$

We can now begin to evaluate (VI-29). We have

$$\langle\phi_N\phi_\Lambda|G_{\Lambda\Lambda}|\phi_N\phi_\Lambda\rangle$$

$$= N_N^2N_\Lambda^2\int d^3k\int d^3k'\int d^3K\ \exp[-A(k^2+k'^2)-2BK^2-2C(\underset{\sim}{\ }\cdot(\underset{\sim}{k}+\underset{\sim}{k}'))]\times$$

$$\cdot\langle k|G_{\Lambda\Lambda}|k'\rangle$$

$$= -N_N^2N_\Lambda^2\int d^3k\ e^{-Ak^2}g_\Lambda(k)\int d^3k'\ e^{-Ak'^2}g_\Lambda(k')\int d^3K\ \frac{[\lambda_\Lambda-d(\lambda)J_\Sigma]}{D(\underset{\sim}{K})}\times$$

$$e^{-2BK^2-2CK\cdot(\underset{\sim}{k}+\underset{\sim}{k}')} \tag{D-12}$$

Since $[\lambda_\lambda - d(\lambda)J_\Sigma]/D(\underset{\sim}{K})$ is a function of $\underset{\sim}{K}$ only, we have for the $\underset{\sim}{K}$ space integration

$$\int d^3K\ e^{-2C\underset{\sim}{K}\cdot(\underset{\sim}{k}+\underset{\sim}{k}')} \frac{[\lambda_\Lambda - d(\lambda)J_\Sigma]}{D(\underset{\sim}{K})}\ e^{-2BK^2}$$

$$= \frac{2\pi}{C}\int_0^\infty dK\ K\ \frac{\sinh(2CK|\underset{\sim}{k}+\underset{\sim}{k}'|)\,[\lambda_\Lambda - d(\lambda)J_\Sigma]}{|\underset{\sim}{k}+\underset{\sim}{k}'|\,D(\underset{\sim}{K})} e^{-2BK^2} .$$

Let us introduce the variable $\underset{\sim}{q} = \underset{\sim}{k}+\underset{\sim}{k}'$ into (D-12). Consequently, we get

$$\langle\phi_N\phi_\Lambda|G_{\Lambda\Lambda}|\phi_N\phi_\Lambda\rangle$$

$$= -\frac{N_N^2N_\Lambda^2}{C}\cdot 2\pi\int d^3k\ e^{-Ak^2} g_\Lambda(k)\int d^3q\ e^{-A(\underset{\sim}{q}-\underset{\sim}{k})^2}\ g_\Lambda(\underset{\sim}{q}-\underset{\sim}{k})\ \times$$

$$\int_0^\infty dK\ K\ e^{-2BK^2}\ \sinh(2CKq)\,[\lambda_\Lambda - d(\lambda)J_\Sigma]/D(\underset{\sim}{K})$$

$$= -\frac{2\pi N_N^2N_\Lambda^2}{C}\int d^3q\ \frac{f(q)\,Z(q)}{q} \qquad \text{(D-13)}$$

where

$$f(q) = \int_0^\infty dK\ Ke^{-2BK^2}\ \sinh(2CKq)\,[\lambda_\Lambda - d(\lambda)J_\Sigma]/D(\underset{\sim}{K})$$

$$Z(q) = \int d^3k\ g_\Lambda(k)\,g_\Lambda(\underset{\sim}{q}-\underset{\sim}{k})\,e^{-A[k^2+(\underset{\sim}{q}-\underset{\sim}{k})^2]}$$

Since $g_\Lambda(k) = 1/(k^2+\beta_\Lambda^2)$, we have

$$Z(q) = \int d^3p \; g_\Lambda(\underset{\sim}{p}+\frac{\underset{\sim}{q}}{2})\, g_\Lambda(\frac{\underset{\sim}{q}}{2} - \underset{\sim}{p})\, e^{-A(\underset{\sim}{p}+\underset{\sim}{q}/2)^2} \times e^{-A(\underset{\sim}{q}/2-\underset{\sim}{p})^2}$$

$$= 2\pi \int_0^\infty p^2 dp \int_{-1}^{1} dx \; \frac{e^{-2A(p^2+q^2/4)}}{[p^2+(\frac{q}{2})^2+\beta_\Lambda^2+pqx]\,[p^2+(\frac{q}{2})^2+\beta_\Lambda^2-pqx]}$$

$$= \frac{2\pi e^{-\frac{1}{2}Aq^2}}{q} \int_0^\infty dp \; \frac{pe^{-2Ap^2}}{(p^2+\frac{q^2}{4}+\beta_\Lambda^2)} \; \ell n \left| \frac{(p+q/2)^2+\beta_\Lambda^2}{(p-q/2)^2+\beta_\Lambda^2} \right| . \qquad \text{(D-14)}$$

Finally, (D-13) can be written as

$$\langle \phi_N \phi_\Lambda | G_{\Lambda\Lambda} | \phi_N \phi_\Lambda \rangle$$

$$= - \frac{2(2\pi)^3 N_N^2 N_\Lambda^2}{C} \int_0^\infty dq \; f(q) \; H(q) e^{-\frac{1}{2} Aq^2} \qquad \text{(D-15)}$$

where

$$H(q) = \int_0^\infty dp \; \frac{pe^{-2Ap^2}}{(p^2+\frac{q^2}{4}+\beta_\Lambda^2)} \; \ell n \left| \frac{(p+q/2)^2+\beta_\Lambda^2}{(p-q/2)^2+\beta_\Lambda^2} \right|$$

APPENDIX E

GREEN'S FUNCTION

§1. Introduction

The Green's function technique has been a very powerful and elegant method for the studies of many-body problems. The prescription is one that derived from quantum field theory. Therefore, we will use the language of second quantization and associate the process of scattering or propagation of a particle (hole) with the creation of a particle (hole) at a given space-time point and the annihilation of the same particle (hole) at another space-time point.

Suppose now we add a particle to an N-body system. The new equation with N+1 particles can be described by the state vector

$$|\Psi(\underset{\sim}{r},t)\rangle = \psi^{+}(\underset{\sim}{r},t)|\Psi_{o}\rangle \qquad \text{(E-1)}$$

where we have assumed that the initial N-body system is in its exact ground state $|\Psi_o\rangle$, and $\psi^+(\underset{\sim}{r},t)$ is a creation operator, in the Heisenberg picture, for the particle under consideration. If the particle does not interact with the initial many-body system, it will propagate as a free particle. However, when it is allowed to interact with other particles, it acquires self-energy through its interaction with the cloud of particles. A renormalized or effective mass may have to be introduced to express the physical state of this particle. Usually, we refer to such a particle as a dressed particle, physical particle or quasi particle. Of course, we can also have a state with N-1 particles, such as

$$|\Psi(\underset{\sim}{r},t)\rangle = \psi(\underset{\sim}{r},t)|\Psi_o\rangle \tag{E-2}$$

where $\psi(\underset{\sim}{r},t)$ is an annihilation operator. This state may be interpreted as having a hole in the initial many-body system. It can be shown that the physical properties of the quasi particle (hole) are related to the single-particle Green's function

$$G(\underset{\sim}{r}_1,t_1;\underset{\sim}{r}_2,t_2) = -i\langle\Psi_o|T(\psi(\underset{\sim}{r}_1,t_1)\psi^+(\underset{\sim}{r}_2,t_2))|\Psi_o\rangle \tag{E-3}$$

which, apart from some constant factor, is closely related to the probability amplitude because it is proportional to the scalar product of (E-1) or of (E-2). In the above expression, T is Wick's chronological ordering operator just to bring the physical contents of Green's function into conformity with the causality principle such that for the fermions

$$T(\psi(\underset{\sim}{r}_1,t_1)\psi^+(\underset{\sim}{r}_2,t_2)) = \begin{cases} \psi(\underset{\sim}{r}_1,t_1)\psi^+(\underset{\sim}{r}_2,t_2) & \text{if } t_1>t_2 \\ -\psi^+(\underset{\sim}{r}_2,t_2)\psi(\underset{\sim}{r}_1,t_1) & \text{if } t_1<t_2 \end{cases} \tag{E-4}$$

Of course, for the bosons we would have the positive sign instead of the negative sign when $t_1<t_2$. In this appendix, we will consider only the case of fermions.

If we want to study collective modes of the many-body system, we have to introduce the many-particle Green's functions.

In particular, we may consider the two-particle Green's function which will be related to the density fluctuation in the many-body system and also to the pair correlation of particles.

§2 Free-particle Green's Function and its Generalization

It is important to get familiar with the free-particle Green's function because it illustrates some of the physical and mathematical properties of the exact Green's function which can be regarded as a straightforward generalization of the free-particle Green's function.

Let us assume that the total Hamiltonian is given by

$$H = H_0 + H_1$$

where H_0 consists of the kinetic energy and H_1 the interaction. For the case with $H_1 = 0$, we introduce the free-particle Green's function

$$\begin{aligned} G_o(\underset{\sim}{r}_1,t_1;\underset{\sim}{r}_2,t_2) &= -i\langle\Phi_o|T(\psi_o(\underset{\sim}{r}_1,t_1)\psi_o^+(\underset{\sim}{r}_2,t_2))|\Phi_o\rangle \\ &= -i\theta(t_1-t_2)\langle\Phi_o|\psi_o(x_1)\psi_o^+(x_2)|\Phi_o\rangle \\ &\quad + i\theta(t_2-t_1)\langle\Phi_o|\psi_o^+(x_2)\psi_o(x_1)|\Phi_o\rangle \end{aligned} \tag{E-5}$$

where θ is the step function; $|\Phi_o\rangle$ the non-interacting many-body ground state and $x \equiv (\underset{\sim}{r},t)$ is a 4-vector. Now, $\psi_o(x)$ satisfies

$$\frac{\hbar}{i}\frac{\partial\psi_o(x)}{\partial t} - \frac{\hbar^2}{2M}\nabla^2\psi_o(x) = 0. \tag{E-6}$$

Using the indentity $\frac{\partial\theta(t)}{\partial t} = \delta(t)$, one finds

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial t_1} - \frac{\hbar^2}{2M}\nabla_1^2\right)G_o(x_1;x_2) = -\hbar\delta(t_1-t_2)\langle\Phi_o|\{\psi_o(x_1),\psi_o^+(x_2)\}|\Phi_o\rangle \ .$$

Since the anticommutator at equal time gives $\delta(\underset{\sim}{r}_1-\underset{\sim}{r}_2)$, we get

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial t_1} - \frac{\hbar^2}{2M}\nabla_1^2\right)G_o(x_1;x_2) = -\hbar\delta(x_1-x_2) \qquad \text{(E-7)}$$

Thus, $G_o(x_1;x_2)$ is indeed Green's function in the mathematical sense. We can now generalize our definition of Green's function from (E-5). When $H_1 \neq 0$, we define the one-particle Green's function as

$$G(x_1;x_2) = -i\langle\Psi_o|T(\psi(x_1)\psi^+(x_2))|\Psi_o\rangle \ . \qquad \text{(E-8)}$$

Here, $\psi(x)$ is a field operator satisfying

$$\frac{\hbar}{i}\frac{\partial\psi(x)}{\partial t} - \frac{\hbar^2}{2M}\nabla^2\psi(x) + \int \psi^+(\underset{\sim}{r}',t)V(|\underset{\sim}{r}-\underset{\sim}{r}'|)\psi(\underset{\sim}{r}',t)d^3r'\psi(x) \qquad \text{(E-9)}$$

where $V(|\underset{\sim}{r}-\underset{\sim}{r}'|)$ is a two-body interaction potential. It is possible to obtain an integro-differential equation similar to that of (E-7) but we will do this after we develop a more compact mathematical operator in the next section.

Let us now turn to the momentum representation of Green's function. We know that the field operator in the Heisenberg picture has the following expansion[67]

$$\psi(\underset{\sim}{r},t) = \frac{(2\pi)^{3/2}}{\Omega}\sum_{\underset{\sim}{k}}\xi_{\underset{\sim}{k}}(t)e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \qquad \text{(E-10)}$$

where $\xi_{\underset{\sim}{k}}(t)$ is the annihilation operator which can be obtained from the annihilation operator $\xi_{\underset{\sim}{k}}$ in the Schrödinger picture by virtue of the transformation

$$\xi_{\underset{\sim}{k}}(t) = e^{iHt/\hbar}\,\xi_{\underset{\sim}{k}}\,e^{-iHt/\hbar} \tag{E-11}$$

where $\xi_{\underset{\sim}{k}}$ satisfies anticommutation relations:

$$\{\xi_{\underset{\sim}{k}},\xi^{+}_{\underset{\sim}{k}'}\} = \xi_{\underset{\sim}{k}}\xi^{+}_{\underset{\sim}{k}'} + \xi^{+}_{\underset{\sim}{k}'}\xi_{\underset{\sim}{k}}$$

$$= \frac{\Omega}{(2\pi)^3}\,\delta_{\underset{\sim}{k},\underset{\sim}{k}'} \tag{E-12}$$

$$\{\xi_{\underset{\sim}{k}},\xi_{\underset{\sim}{k}'}\} = \{\xi^{+}_{\underset{\sim}{k}},\xi^{+}_{\underset{\sim}{k}'}\}$$

$$= 0 \tag{E-13}$$

Here Ω is the volume of the space under consideration. Thus, (E-8) can be rewritten as

$$G(x_1;x_2) = -\frac{i(2\pi)^3}{\Omega^2}\sum_{\underset{\sim}{k}',\underset{\sim}{k}''}\langle\Psi_o|T(\xi_{\underset{\sim}{k}'}(t_1)\xi^{+}_{\underset{\sim}{k}''}(t_2))|\Psi_o\rangle e^{-(\underset{\sim}{k}'\cdot\underset{\sim}{r}_1-\underset{\sim}{k}''\cdot\underset{\sim}{r}_2)}$$

$$= -\frac{i(2\pi)^3}{\Omega^2}\sum_{\underset{\sim}{k}'}\langle\Psi_o|T(\xi_{\underset{\sim}{k}'}(t_1)\xi^{+}_{\underset{\sim}{k}'}(t_2))|\Psi_o\rangle e^{i\underset{\sim}{k}'\cdot(\underset{\sim}{r}_1-\underset{\sim}{r}_2)}. \tag{E-14}$$

In the last step we have used the equation

$$\langle\Psi_o|T(\xi_{\underset{\sim}{k}'}(t_1)\xi^{+}_{\underset{\sim}{k}''}(t_2))|\Psi_o\rangle=\delta_{\underset{\sim}{k}',\underset{\sim}{k}''}\langle\Psi_o|T(\xi_{\underset{\sim}{k}'}(t_1)\xi^{+}_{\underset{\sim}{k}''}(t_2))|\Psi_o\rangle\ .$$

If the interaction is to conserve the energy and momentum, Green's function can depend only on the difference of the co-ordinates. Then, we can write

$$G(x_1;x_2) \equiv G(\underset{\sim}{r}_1-\underset{\sim}{r}_2;t_1-t_2).$$

Now, the Fourier transform of $G(\underset{\sim}{r},t)$ with respect to the co-ordinates yields the momentum representation of Green's function

$$G(\underset{\sim}{k},t) = \int G(\underset{\sim}{r},t)e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} d^3r .$$

Therefore, we get from (E-14)

$$G(\underset{\sim}{k},t_1-t_2) = -\frac{i(2\pi)^3}{\Omega} \langle\Psi_o|T(\xi_{\underset{\sim}{k}}(t_1)\xi^+_{\underset{\sim}{k}}(t_2))|\Psi_o\rangle . \qquad \text{(E-15)}$$

For the ground state of a many-body system, all the single-particle states below the Fermi momentum k_F are occupied. If $k = q > k_F$, we observe from (E-15) that a particle is created at t_2 and annihilated at t_1, provided $t_2 < t_1$. Otherwise, Green's function vanishes because we have $\xi_{\underset{\sim}{k}}(t_1)|\Psi_o\rangle = 0$. Alternatively, if $k = \ell < k_F$, then at t_1 a hole is created and subsequently annihilated at t_2, provided $t_1 < t_2$. Otherwise $G(\underset{\sim}{\ell},t_1-t_2) = 0$.

We will now evaluate the free-particle Green's function $G_o(\underset{\sim}{k},t)$. If $k = q > k_F$ and $t \equiv t_1-t_2 > 0$, we have

$$G_o(\underset{\sim}{q},t) = -\frac{i(2\pi)^3}{\Omega} \langle\Psi_o|e^{iH_ot_1/\hbar}\xi_{\underset{\sim}{q}}e^{-iH_o(t_1-t_2)/\hbar}\xi^+_{\underset{\sim}{q}}e^{-iH_ot_2/\hbar}|\Phi_o\rangle$$

$$= -i\frac{(2\pi)^3}{\Omega} e^{it\varepsilon_o/\hbar}\langle\Phi_o|\xi_{\underset{\sim}{q}} e^{-iH_ot/\hbar} \xi^+_q|\Phi_o\rangle \qquad \text{(E-16)}$$

where we have made use of the equation

$$H_o|\Phi_n\rangle = \varepsilon_n|\Phi_n\rangle$$

and ε_o is the ground state energy. Inserting a complete set of states $\sum_n |\Phi_n\rangle\langle\Phi_n|$ into (E-16) we obtain

$$G_o(\underset{\sim}{q},t) = -ie^{-i(\varepsilon_q-\varepsilon_o)t/\hbar} \qquad \text{(E-17)}$$

where $\varepsilon_q-\varepsilon_o \equiv \hbar^2 q^2/2M$ is the free single-particle energy with respect to the ground state energy. If $k = \ell < k_F$ and $t < 0$, we will get

$$G_o(\underset{\sim}{\ell},t) = -ie^{-i(\varepsilon_\ell-\varepsilon_o)t/\hbar} \qquad \text{(E-18)}$$

with $\varepsilon_\ell-\varepsilon_o \equiv -\hbar^2\ell^2/2M$. In summary, we have

$$G_o(\underset{\sim}{k},t) = i[-\theta(k-k_F)\theta(t)+\theta(k_F-k)\theta(-t)] \times$$
$$\exp[-i(\varepsilon_k-\varepsilon_o)t/\hbar]. \qquad \text{(E-19)}$$

The analytical properties of Green's function becomes more transparent if we perform another Fourier transform of (E-19) with respect to time. The full space-time Fourier transform of Green's function is defined by

$$G(\underset{\sim}{k},\hbar\omega) = \int_{-\infty}^{\infty} G(\underset{\sim}{k},t)e^{i\omega t}\, dt. \qquad \text{(E-20)}$$

Now, the Fourier transform of $\theta(t)e^{-i\zeta t}$ should be

$$\int_{-\infty}^{\infty}\theta(t)e^{-i(\zeta-\omega)t}dt = \lim_{\varepsilon\to+0} \frac{i}{\omega-\zeta+i\varepsilon} . \qquad \text{(E-21)}$$

This identity can be justified by taking the inverse of the Fourier transform which will indeed regenerate the function $\theta(t)e^{-i\zeta t}$. A rigorous justification of the identity will require the use of generalized functions or distributions[69]. With the use of (E-21), the Fourier transform of (E-19) gives

$$G_o(\underset{\sim}{k},\hbar\omega) = \theta(k-k_F)\,\frac{\hbar}{\hbar\omega-(\varepsilon_k-\varepsilon_o)+i\varepsilon}$$

$$+\ \theta(k_F-k)\,\frac{\hbar}{\hbar\omega-(\varepsilon_k-\varepsilon_o)-i\varepsilon} \qquad \text{(E-22)}$$

which is equivalent to

$$G_o(\underset{\sim}{k},\hbar\omega) = \frac{\hbar}{\hbar\omega-(\varepsilon_k-\varepsilon_o)+i\varepsilon\ \mathrm{sgn}(k-k_F)} \qquad \text{(E-23)}$$

where

$$\mathrm{sgn}(k-k_F) = \begin{cases} +1 & \text{if} \quad k > k_F \\ -1 & \phantom{\text{if}} \quad k < k_F \end{cases}$$

From (E-23), we see that the contribution from the quasi particle to Green's function comes from the lower half of the ω-plane and the quasi hole contributes only in the upper half of the ω-plane. Thus, when we are considering a Feynman diagram in the energy-momentum representation, an appropriate contour in the ω-plane has to be chosen in accordance with the physical state. We have to close the contour in the lower (upper) half of the ω-plane, if we are dealing with a particle (hole) line of a Feynman diagram.

When interaction is allowed, the one-particle Green's function is not as compact as the one given by (E-23) though the general structure remains the same. The Green's function will be modified by the presence of spectral functions. However, in practice we normally employ the perturbation theory to evaluate Green's function by summing a relevant set of diagrams. Thus, instead of studying its spectral representation, we will discuss the diagram technique. We will show how this can be done.

§3 Feynman Diagrams

In this section, we would like to discuss the perturbation theory by using Feynman diagrams. It is often convenient to work with the interaction picture wherefore the interaction part of the Hamiltonian can be singled out. Following Roman[67], the Green's function in the Heisenberg picture can be transformed into the interaction picture to give

$$G(x_1-x_2) = -i\,\frac{\langle\Psi_o|T(\psi'(x_1)\psi'^{+}(x_2)S)|\psi_o\rangle}{\langle\Psi_o|S|\Psi_o\rangle} \qquad \text{(E-24)}$$

where

$$S = T\exp[-(i/\hbar)\int_{-\infty}^{\infty} H_1'(t)\,dt]$$

is the S-matrix; $\psi'(x)$ is the field operator in the interaction picture and $H_1'(t)$ is defined by

$$H_1'(t) = \frac{1}{2}\int \psi^{+}(\underset{\sim}{r}',t)\psi^{+}(\underset{\sim}{r}'',t)V(|\underset{\sim}{r}'-\underset{\sim}{r}''|)\psi(\underset{\sim}{r}'',t)\psi(\underset{\sim}{r}',t)d^3r'd^3r''.$$

Expanding the S-matrix, we can rewrite (E-24) as the following expansion series

$$G(x_1-x_2) = -\frac{i}{<\Psi_o|S|\Psi_o>} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar}\right)^n \times$$

$$\int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} <\Psi_o|T(\psi'(x_1)\psi'^+(x_2)H_1'(t') \ \ldots\ldots \ H_1'(t^n))|\Psi_o>$$

$$dt^n \ \ldots\ldots \ dt' \qquad \text{(E-25)}$$

Hence, the zeroth order of $G(x_1-x_2)$ will be

$$G^{(o)}(x_1-x_2) = -\ i<\Psi_o|T(\psi'(x_1)\psi'^+(x_2))|\Psi_o> \qquad \text{(E-26)}$$

where it is understood that $<\Psi_o|S|\Psi_o> = 1$ in the zeroth order. But the matrix element of the time ordered Heisenberg operators and that of interaction operators have the relation

$$<\Phi_o|T(\psi(x_1)\ldots.\psi^+(x_n))|\Psi_o>=<\Psi_o|S^+T(\psi'(x_1)\ldots.\psi'^+(x_n)S)|\Psi_o>$$

and also

$$|\Psi_o> = S|\Phi_o>$$

it follows that

$$<\Psi_o|T(\psi'(x_1)\psi'^+(x_2))|\Psi_o> = <\Phi_o|S^+T(\psi'(x_1)\psi'^+(x_2)S)|\Phi_o>$$

$$= <\Phi_o|T(\psi(x_1)\psi^+(x_2)|\Phi_o>$$

$$= i\ G_o(x_1-x_2)\ . \qquad \text{(E-27)}$$

We have from (E-26) and (E-27)

$$G^{(o)}(x_1-x_2) = G_o(x_1-x_2)\ . \qquad \text{(E-28)}$$

As for the first order, we have

$$G^{(1)}(x_1-x_2) = -\frac{1}{\hbar\langle\Psi_o|S|\Psi_o\rangle}\int_{-\infty}^{\infty}\langle\Psi_o|T(\psi'(x_1)\psi'^+(x_2)H_1'(t'))|\Psi_o\rangle dt'$$

$$= -\frac{1}{2\hbar\langle\Psi_o|S|\Psi_o\rangle}\iiint_{-\infty}^{\infty}\langle\Psi_o|T(\psi'(x_1)\psi'^+(x_2)\cdot$$

$$\psi'^+(\underset{\sim}{r}',t')\psi'^+(\underset{\sim}{r}'',t)V(|\underset{\sim}{r}'-\underset{\sim}{r}''|)\psi'(\underset{\sim}{r}'',t')\psi'(\underset{\sim}{r}',t'))|\Psi_o\rangle\cdot$$

$$d^3r'd^3r''dt'. \qquad \text{(E-29)}$$

To put this into a full space-time integral, we introduce

$$W(x'-x'') = V(|\underset{\sim}{r}'-\underset{\sim}{r}''|)\delta(t'-t'')$$

then

$$G^{(1)}(x_1-x_2) = -\frac{1}{2\hbar\langle\Psi_o|S|\Psi_o\rangle}\iint\langle\Psi_o|T(\psi'(x_1)\psi'^+(x_2)\psi'^+(x')$$

$$\psi'^+(x'')\psi'(x'')\psi'(x'))|\Psi_o\rangle W(x'-x'')d^4x'd^4x'' . \qquad \text{(E-30)}$$

As a consequence of Wick's theorem, we have

$$\langle\Psi_o|T(\psi'(x_1)\psi'^+(x_2)\psi'^+(x')\psi'^+(x'')\psi'(x'')\psi'(x'))|\Psi_o\rangle$$

$$= \langle\Psi_o|T(\psi'(x_1)\psi'^+(x'))|\Psi_o\rangle\langle\Psi_o|T(\psi'(x'')\psi'^+(x_2))|\Psi_o\rangle\langle\Psi_o|T(\psi'(x')$$

$$\psi'^+(x''))|\Psi_o\rangle$$

$$- \langle\Psi_o|T(\psi'(x_1)\psi'^+(x'))|\Psi_o\rangle\langle\Psi_o|T(\psi'(x'')\psi'^+(x''))|\Psi_o\rangle\langle\Psi_o|T(\psi'(x')$$

$$\psi'^+(x_2))|\Psi_o\rangle$$

$$- <\Psi_o|T\ (\psi'(x_1)\psi'^+(x''))|\Psi_o><\Psi_o|T(\psi'(x'')\psi'^+(x_2\ ))|\Psi_o><\Psi_o|T(\psi'(x')\psi'^+(x'))|\Psi_o>$$

$$+ <\Psi_o|T(\psi'(x_1)\psi'^+(x''))|\Psi_o><\Psi_o|T(\psi'(x'')\psi'^+(x'))|\Psi_o><\Psi_o|T(\psi'(x')\psi'^+(x_2))|\Psi_o>$$

$$- <\Psi_o|T(\psi'(x_1)\psi'^+(x_2))|\Psi_o><\Psi_o|T(\psi'(x'')\psi'^+(x'))|\Psi_o><\Psi_o|T(\psi'(x')\psi'^+(x''))|\Psi_o>$$

$$+ <\Psi_o|T(\psi'(x_1)\psi'^+(x_2))|\Psi_o><\Psi_o|T(\psi'(x'')\psi'^+(x''))|\Psi_o><\Psi_o|T(\psi'(x')\psi'^+(x'))|\Psi_o>$$

$$= - i\ G_o(x_1-x')G_o(x''-x_2)\ G_o(x'-x'') + i\ G_o(x_1-x')G_o(x''-x'')\ G_o(x'-x_2)$$
$$+ i\ G_o(x_1-x'')\ G_o(x''-x_2)\ G_o(x'-x') - i\ G_o(x_1-x'')\ G_o(x''-x')\ G_o(x'-x_2)$$
$$+ i\ G_o(x_1-x_2)\ G_o(x''-x')\ G_o(x'-x'') - i\ G_o(x_1-x_2)\ G_o(x''-x'')\ G_o(x'-x').$$

(E-31)

Therefore, the first-order term of $G(x_1-x_2)$ can also be expressed in terms of free-particle Green's function. One may proceed to show that this is true for any order and one can associate Green's function at every order with a set of Feynman diagrams. Once we have set up the Fenyman rules, we can calculate Green's function to any order by applying these rules to the Feynman diagrams. If we use a solid line to represent the free-particle Green's function and a dashed line for the interaction $W(x'-x'')$, a diagramatic representation of the first-order Green's function is given in Fig.12. The diagrams are arranged in the order of the terms appearing in (E-31).

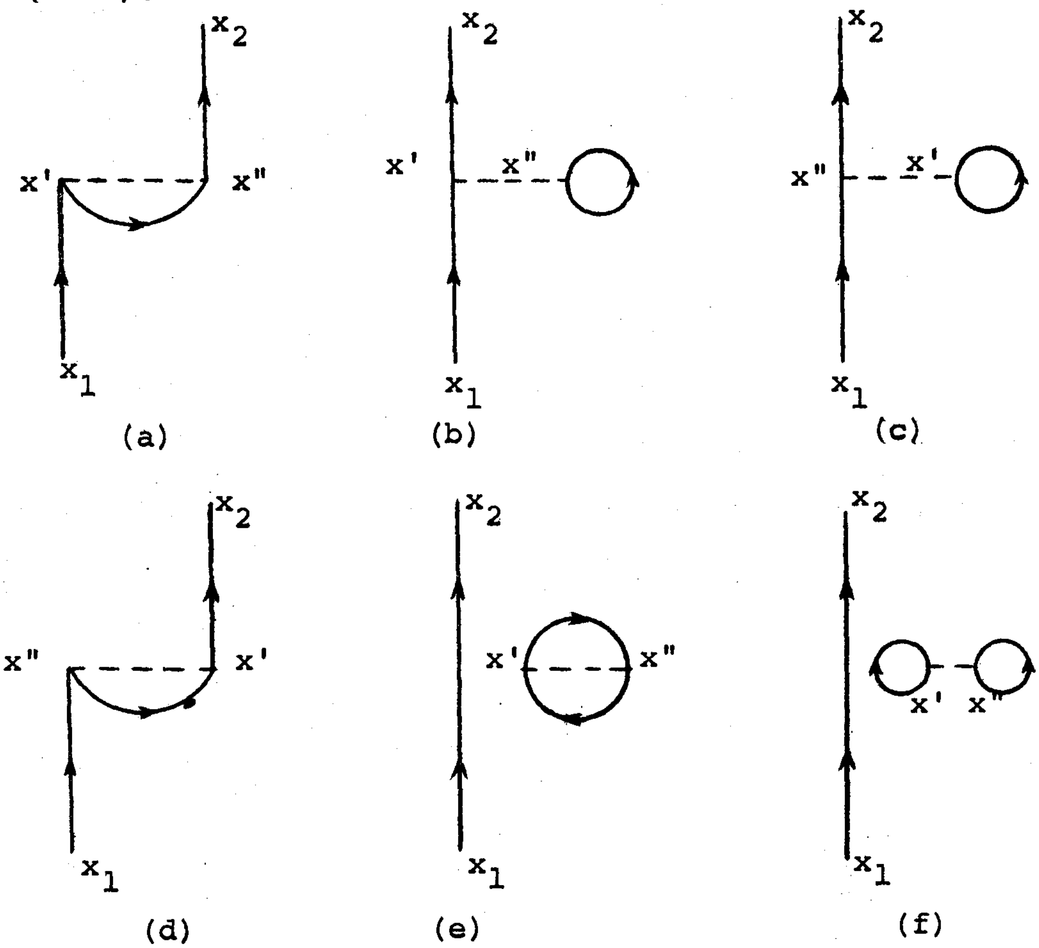

Fig. 12

Since x' and x" are integration variables and W(x'-x") is symmetric, diagrams (a) and (d) as well as diagrams (b) and (c) give the same contribution to $G^{(1)}(x_1-x_2)$. For diagrams of order n, there will be n! topologically equivalent graphs. Furthermore, diagrams (e) and (f) are disconnected graphs and the rest are connected ones. If we keep the connected graphs of a particular order fixed and sum over all the disconnected graphs, then the disconnected graphs will yield a factor $<\Psi_o|S|\Psi_o>$ which will cancel the same factor in the denominator of Green's function. If only one of the topologically equivalent connected graphs is taken, then we have to include a factor $n!<\Psi_o|S|\Psi_o>$ in the Green's function. Therefore $G(x_1-x_2)$ can be written as

$$G(x_1-x_2) = -i \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar}\right)^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} <\Psi_o\, T(\psi'(x_1)\psi'^{+}(x_2)H_1'(t')\ldots H_1'(t^n))\,|\Psi_o>_c \; dt'\ldots dt^n \qquad \text{(E-32)}$$

where the subscript c indicates that only the connected and topologically non-equivalent graphs will be considered. When the graphical technique is used, it is often convenient to work in the energy-momentum space. This can be achieved by taking a full space-time Fourier transform of (E-32). For example, the contribution from diagram (a) of Fig.12 in the energy-momentum space would be

$$\frac{i}{\hbar}\,\frac{G_o(p)}{(2\pi)^4}\int G_o(p-q)W(q)\,d^4q\; G_o(p)$$

with

$$W(q) = \int V(|\underset{\sim}{r}'-\underset{\sim}{r}''|)\,e^{-i\underset{\sim}{q}\cdot(\underset{\sim}{r}'-\underset{\sim}{r}'')}\,d^3(\underset{\sim}{r}'-\underset{\sim}{r}'').$$

Here, we have introduced a 4-vector p such that $p = (\underset{\sim}{p}, p_o) \equiv (\underset{\sim}{p}, \hbar\omega)$. As a result, diagram (a) of Fig. 12 is replaced by Fig. 13.

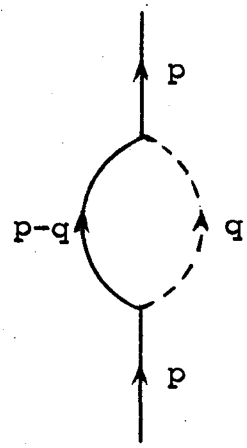

Fig.13

Of course, there are infinite numbers of diagrams that have to be summed. For the self-energy type diagram, however, it is enough to consider only the compact graphs. A compact graph is defined as the graph which cannot be separated into two parts by just breaking a single propagating line. Formally, all the possible graphs could be included in a way shown by Fig. 14. The block $\Sigma(p)$ consists of the totality of all the

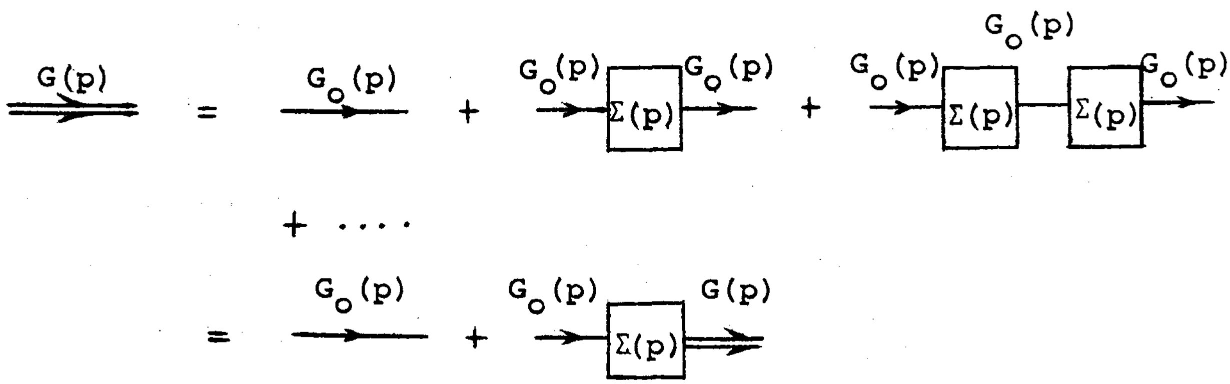

Fig. 14

nonequivalent compact self-energy graphs. The mathematical equivalent of Fig. 14 is

$$G(p) = G_o(p) + G_o(p)\Sigma(p)G(p). \tag{E-33}$$

This is the so-called Dyson's equation which may be rewritten as

$$G(p) = \frac{G_o(p)}{1-G_o(p)\Sigma(p)} \tag{E-34}$$

or alternatively

$$\begin{aligned} G(p)^{-1} &= G_o(p)^{-1} - \Sigma(p) \\ &= \frac{1}{\hbar}\left[p_o - \frac{\hbar^2 \underset{\sim}{p}^2}{2M} + i\varepsilon\, \mathrm{sgn}(|\underset{\sim}{p}| - k_F)\right] - \Sigma(p) \end{aligned} \tag{E-35}$$

where we have used (E-23) for the $G_o(p)$. Thus we arrive at a very simple relation for Green's function and the compact self-energy parts of the propagator, and the studies of Green's function reduce to that of $\Sigma(p)$. An explicit example of summing compact ladder diagrams will be demonstrated in Chapter III.

Before concluding this section,we would like to show that the generalized Green's function is indeed Green's function of an integro-differential equation. In space-time representation, (E-33) takes the form

$$G(x_1-x_2) = G_o(x_1-x_2) + \int d^4x' d^4x''\, G_o(x_1-x')\Sigma(x'-x'')G(x''-x_2) \tag{E-36}$$

Operating this equation by $(\frac{\hbar}{i}\frac{\partial}{\partial t_1} - \frac{\hbar^2}{2M}\nabla_1^2)$ and using the result of (E-7), we obtain finally the desired integro-differential equation

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial t_1} - \frac{\hbar^2}{2M}\nabla_1^2\right)G(x_1-x_2) = -\hbar\delta(x_1-x_2) - \hbar\int d^4x''\Sigma(x_1-x'')G(x''-x_2) \quad \text{(E-37)}$$

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