GENERAL EQUILIBRIUM IN A SPATIAL ECONOMY

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ABSTRACT

This dissertation examines the theory of general economic equilibrium in a simple spatial context. Two fundamental issues are confronted. First, the problem of reconciling the technical apparatus of general equilibrium theory with a geographical setting is solved by reinterpreting the role of transportation in the economy. Second, the implications of the theory for the behaviour of transportation functions are formally investigated.

The analysis demonstrate that, in the postulated spatial structure, the required behaviour of transportation relationships contradicts empirical evidence. This result is true for both consumption and production sectors, and holds for any spatial configuration of the economy which admits the same interpretaion of transportation.

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1. INTRODUCTION

1.1. Spatial Analysis and The Urban Economy

The systematic study of the urban spatial economy has traditionally captured the interest of scientists from numerous denominations. The recent, exponential growth of scholarly activity in this field is attributable to both the increasing urbanization of society and the dramatic intensification of urban social problems. One of the most vigorous growth areas in this general scientific development is the theoretical approach to the spatial structure of the city, reflecting a widely-felt need for a formal understanding of urban geography. This need itself arises for several reasons. On the one hand the simplification which model-building entails and the ease and economy with which models can be manipulated and tested recommend them as a foundation of applied, policy-oriented investigations. On the other hand, theoretical models, which, by definition, concentrate on a few relationships between a small number of variables, may convey information concerning basic properties of cities. Current research is biased heavily towards the latter trend, and the indications are that this preponderance will continue.

Despite the diversity of subjects and the range of styles, certain features are common to virtually all theoretical urban models and hence provide a source of unification to the field. The most visible feature is the pervasive use of mathematics as a basis for model construction and analysis. Although continually condemned by the scientifically innocent, the logical and practical advantages of a formal approach in gaining insights to spatial processes clearly outweigh any losses in intuitive appeal. Another dominant aspect of these models is the pivotal role of the equilibrium concept. This notion has also been rejected by many who argue that the durability of buildings, etc., prevents equilibrium from being realized. While some components of the city's spatial structure are best understood in historical terms it is equally true that equilibrium analysis is the appropriate tool for examining many urban phenomena. For example, the widespread suburbanization observed in many countries is clearly a response to underlying equilibrating forces. In fact, it can be argued that the most significant contemporary urban problems are all consequences of adaptive processes and can most easily be comprehended through equilibrium analysis.

More specific properties are shared by models in this area. Usually the city is assumed to exist on a flat plain with transportation equally costly in all directions. The city has an exogenously fixed central business district in which all employment is concentrated. The only travel considered is that between residence and workplace. This means that, with few exceptions, locations are characterized by a single dimension, distance from the CBD. The models tend to emphasize either the production or consumption sectors of the urban economy, and unified treatments of both sectors are rare. A small number employ a discrete approach (e.g. Mills (1972c)), which offer advantages in flexibility and ease of use. However, the reliance on numerical methods and the lack of qualitative results which the discrete approach implies has led the overwhelming majority of theoreticians to employ a continuous framework.

The prevailing trend in contemporary research in urban spatial economics is currently towards the construction of normative models, which lay stress upon welfare and optimality aspects, as distinct from the earlier development of positive theories which sought to identify and explain spatial equilibria. The typical normative model seeks an allocation of population, land use, etc., which maximizes some social welfare function.

Cities are distinguished chiefly as places where land use is more intense than elsewhere, hence the market allocation of urban land between competing uses is of fundamental concern to the student of the city's spatial structure. Since prices are the mechanism by which commodities are allocated in the market, then rent, which is the price for land, is of primary importance in any equilibrium model of urban land use. The peculiar characteristic of land is that it is a nonproduced factor of production, and this gave rise to much debate among classical and neoclassical economists in the nineteenth century. While controversy still exists concerning the nature of rent, the emphasis in contemporary urban literature is on the process whereby it is determined, and the manner in which it influences the other elements of the spatial economy.

1.2. General and Partial Equilibrium

Equilibrium models of the type discussed in the above section can be classified either as general or partial, depending on their approach. The vast majority of models are in the partial equilibrium group, which analyses the spatial economy by discussing the behaviour [illegible] [illegible] are: (1) [illegible], who usually receive income through exogenous sources, sale of their services, and profit shares, and spend this income in accordance with their preferences, subject to a budget constraint which depends on prices. (2) *Producers*, who employ inputs from consumers as well as other producers, and with a given technology select a production plan which maximizes profits. (3) *Markets*, in which producers and consumers interact.

Partial analysis examines one market, or the supply or demand conditions for one good, while assuming that incomes, technology, and all other prices remain constant. The justification for the partial approach is that in many circumstances the relative smallness of a firm, household, or market is such that those assumptions are reasonable. By focussing on a few selected relationships and making the necessary simplifying assumptions key elements of the economy may be analyzed in considerable depth in a partial equilibrium model. Among the examples in the spatial context cited by Hoover (1968) are the allocation of land in the CBD, patterns of residential development, selection of retail locations, and the location of industry.

Most of the economic theory of urban land uses and land values exemplify the partial method. The typical model attempts to explain the distribution of rents, residential densities, and related phenomena as consequences of rational behaviour on the part of either producers or consumers.

General equilibrium models, on the other hand, attempt to fit the components of the economy together. They take cognizance simultaneously of all the decision makers and markets and their interdependencies. In the case of a perfectly competitive economy (which is the subject of most investigations) the fundamental problem is that of compatibility. That is, whether the behaviour of each individual is compatible with that of every other individual and with equilibrium in every market. General equilibrium in this sense has not achieved prominence in the spatial literature. The closest approximation to the type of analytical structure discussed here is provided by the interegional trading models of Enke (1951), Samuelson (1952), and Takayama and Judge (1964). These are formulated in linear or quadratic programming terms, and are in general concerned with discovering the equilibrium quantities of imports, exports and local prices for a set of spatially separated markets. The interregional input-output models, such as that of Moses (1960) are in a similar vein. Both approaches are more restrictive than general equilibrium theory (in the sense accepted here) and are less general, more empirically-oriented in design.

In fact the Walrasian school of general equilibrium theory (Walras (1874, 1877)) has developed in almost complete isolation from spatial analysis. Likewise, urban economic theory has traditionally assumed away the problems of compatibility, etc., which constitute the essence of general equilibrium. The fitting together of the components of the spatial economy is clearly a complex matter, due to their interconnections. For example, the demand for housing depends on household income. This income is in turn determined by the price of labour. The wage is itself a function of supply and demand conditions in factor markets. Input demand conditions in turn depend on the prices at which producers can sell their output, which is in turn determined by supply and demand in commodity markets. But this demand depends on household income, and so on.

Interconnections arise between different commodity markets and factor markets. Demand for one good, for example, may depend on prices of other goods. It is consequently impossible to find one equilibrium price, then another, and so on, since each price must be suitable in relation to all other prices. The guiding principle of general equilibrium is, therefore, that all prices and quantities must be determined together. Heretofore the notions of compatibility and simultaneity have been neglected in equilibrium land use models.

1.3. Structure of the Study

This research is motivated by two considerations. On the one hand there is a clear requirement for theoretical constructions of the general equilibrium type in the field of urban spatial economics. The exclusive employment of partial analysis has left many key questions unanswered and a wealth of issues ignored. On the other hand general equilibrium theory, which since Walras has flowered into the most profound and sophisticated branch of economic theory, has effectively ignored any spatial considerations. These two factors are believed to justify the preliminary examination of general equilibrium in a spatial context which the present work represents.

To place this study in perspective and to supply the minimal background for the analysis the following structure has been adopted. First, the literature on partial equilibrium urban land use models is reviewed. There is no suggestion of comprehensiveness, but an attempt is made to indicate the more significant lines of inquiry, and also to provide a coherent conceptual framework which suggests some of the theoretical and historical links between the models. Succeeding the review is a brief critical assessment of this group and a restatement of the significance of general equilibrium in this context.

Secondly, there is an encapsulated presentation of the axiomatic foundations of modern general equilibrium theory, which serves as a platform on which subsequent arguments are developed. In addition to the axioms, the function and relevance of general equilibrium theory are discussed, and major criticisms, both spatial and nonspatial in nature, are enumerated.

Having justified the research and having established its analytical background, the final chapter presents the spatial implications of the theory of general equilibrium. The initial step consists of constructing a simplified geographical environment for the economy. Subsequently, the axioms of the theory are interpreted in this new setting and their spatial repercussions are formally examined. Finally, general equilibrium theory is evaluated in light of the results of the analysis, and possible future research areas are indicated.

1.4. Notation

Vectors and matrices are denoted respectively by single and double underlines, e.g. $\underline{x}$, $\underline{\underline{m}}$. There is no use made of the outer product operation, therefore juxtaposition of vectors indicates an inner product. Whether a given vector is a row or column should be clear from the context. The notation for vector inequalities is as follows: $\underline{x} \gg \underline{y}$ means $x_i > y_i$, all i: $\underline{x} \geq \underline{y}$ means $x_i \geq y_i$, all i; $\underline{x} > \underline{y}$ means $x_i \geq y_i$ all i, and $x_i > y_i$ for at least one i. Functions are indicated by square brackets, e.g. $y = y[\underline{x}]$. Sets are denoted by chain brackets or by capitals; for instance $\{\underline{x}\}$ is the set of all vectors $\underline{x}$, and Y represents the set of all vectors $\underline{y}$. Much of the analysis is concerned with the n-dimensional space of real numbers, with the usual Euclidean metric, which is here denoted as $\mathbb{R}^n$. The nonpositive and nonnegative orthants of this space are given the conventional symbols $\mathbb{R}^n_-$ and $\mathbb{R}^n_+$. The real line is designated by $\mathbb{R}$, with its nonpositive and nonnegative components as $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively. Open, closed, and open-closed intervals on the real line are written in the form (a,b), [a,b], and (a,b] respectively. Ordinary derivatives are denoted both by primes and by $\frac{d}{d}$ notation. Thus $g'[x] = \frac{dg}{dx}$. Set membership is represented by the symbol ε. The end of a proof is marked by the symbol $||$.

All other notation is either of common usage or is defined when it first arises in the text. Theorems, lemmas, corollaries, and definitions are numbered consecutively within each section.

2. PARTIAL EQUILIBRIUM MODELS

2.1. Producer Equilibrium

The basic unit of production is the firm, which is defined as any entity using inputs to produce commodities which are then sold to consumers or to other firms. The problem facing the firm is to decide (1) the quantity of output it wishes to produce, and (2) the combination of inputs used in this production plan. Between the set of inputs used by a firm, denoted $\underline{x}$, and the maximum output y which $\underline{x}$ can yield, there is a fixed technological relationship

$$y = f[\underline{x}]$$

where f is termed the firm's production function, generally assumed to be real and continuously differentiable. In the typical neoclassical problem the prices of the inputs, denoted $\underline{w}$, and p, the output price, are given, and the objective of the firm is to maximize profits π, where

$$\pi = pf[\underline{x}] - \underline{w}\,\underline{x}.$$

The first order conditions for a solution to this problem are given by

$$p \frac{\partial f}{\partial x_i}[x] = w_i. \qquad i = 1, \ldots n$$

Production-based theories of the urban economy usually revolve around a more general model where one of the w_i's, namely the price of land, is a spatially variable unknown instead of a parameter. This apparently minor adjustment releases a torrent of technical and conceptual issues out of which a rich theoretical tradition has developed.

The earliest and most direct application of the theory of the firm to the urban economy is the model of Muth (1961), which introduces many of the characteristics found in ensuing works. Muth commences with a spatially fixed market around which physically homogenous land stretches infinitely. Firms locating on this plain purchase land and non-land (sometimes called "capital") inputs at unit prices r and w and sell their output y at some price p at the point of production. It is assumed that w is fixed and known. The profit function of the firm is given by

$$\pi = pf - wk - rq.$$

For an equilibrium each firm must adjust its input plan to maximize profits, hence the conditions

$$w = p\frac{\partial f}{\partial k},$$

$$r = p\frac{\partial f}{\partial q}$$

must hold. Equilibrium requires that the firm have no incentive to move. Using the first-order conditions, then

$$d\pi = fdp - qdr + pdf - wdk - rdq$$

Muth assumes that price declines with distance from the market as a result of transport costs. Therefore

$$\frac{dr}{ds} = \frac{y}{q}\frac{dp}{ds} < 0.$$

If prices decrease at a decreasing rate with distance, which Muth considers reasonable, then rents will decline at a decreasing rate if output per unit of land also diminishes away from the centre. Since wages are fixed and rent decrease with distance, land will be substituted for capital and output per acre will therefore also decline.

Mills (1972a) presents a similar approach which is more sophisticated than Muth's theory in that specific behaviour of the price function is not required. It is assumed that the output of housing H depends on inputs of land q and capital k, and that output and inputs are related by a linear-homogenous production function of the Cobb-Douglas type i.e.

$$H = a\,q^{\alpha}\,k^{1-\alpha}$$

where $a > 0$ and $0 < \alpha < 1$. It is assumed that the urban area has a predetermined centre, and location is indicated by distance s from this centre. Input and output markets are perfectly competitive. The price of capital, w, is given, but the price of housing, $p_h[s]$, and rent $r[s]$, both depend on location and are determined within the model. The profit function for the producer of housing is therefore given by

$$\pi_h = p_h[s]\, a\, q^{\alpha}\, k^{1-\alpha} - r[s]\, q - wk.$$

maximizing with respect to q and k yields

$$\frac{\alpha\, p_h[s]\, H}{q} = r[s] ,$$

$$\frac{(1-\alpha)\, p_h[s]\, H}{k} = w ,$$

as the first-order conditions. It is assumed that all workers receive the same income y which is determined exogenously, and have the same tastes. The demand for housing per person at s is denoted $h_d[s]$ where

$$h_d[s] = b\, y^{\beta_1}\, p_h[s]^{\beta_2}.$$

and β_1, β_2 are the income and price elasticities of demand for housing respectively. It is assumed that $\beta_1 > 0$, $\beta_2 < 0$. Aggregate demand for housing at s, $H_D[s]$, is given by

$$H_d[s] = h_d[s]\, N[s]$$

where N[s] is the population at s. Hence the equilibrium condition for the housing market is

Furthermore, equilibrium requires that households cannot reduce costs by moving i.e. locational costs must be constant, therefore

$$p_h'[s]\, h_d + t'[s] = 0$$

where t is the transport cost per unit distance. The amount of land used for housing at distance s cannot exceed the land available, and no land may be left unused, hence

$$q[s] = 2\pi s$$

The city will extend only to where the rents bid by household equal agricultural rent r_a, which is assumed to be constant. Consequently

$$r[s^*] = r_a$$

where s^* denotes the city boundary. N is the total population of the city, where

$$N = \int_0^{s^*} N[s]\, ds.$$

Given this set of relationships the first unknown solved for is the distribution of land values, r[s]. Two solutions are possible,

$$r[s] = (r_a^{\gamma} - c(s^* - S))^{\frac{1}{\gamma}}$$

and

$$r[s] = r_a \exp(c(s^* - s))$$

where c and γ are constants. Once the rent distribution has been solved for, the spatial distribution of house prices, housing construction, population, and housing demand follow immediately.

The model can be generalized to allow for more than one income class. For example, suppose that two groups have incomes y_1 and y_2 respectively. The class which lives closer to the C.B.D. has the rent function $r_1[s]$, the more distant group has the function $r_2[s]$. If $\gamma \neq 0$, then $r_2[s]$ is identical to the first solution for r[s] given above. Assuming that the other income group occupies the inner zone, that is from $s = 0$ to some boundary $0 < s^{**} < s^*$, then the boundary condition for this group is that the rents of both groups be equal i.e.

$$r_1[s] = r_2[s] = (r_a^\gamma + c(s^* - s^{**}))^{\frac{1}{\gamma}},$$

therefore

$$r_1[s] = (r_a^\gamma + c(s^* - s^{**}) + c_1(s^{**} - s))^{\frac{1}{\gamma}}.$$

It is easily seen that $r_1[s] > r_2[s]$ for distances $s < s^{**}$ implies $c > c_1$, which in turn implies that $y_2 > y_1$. Hence the lower income group dwells closer to the centre of the city. In general, therefore, the wealthier a class the [illegible]e distant is its residential zone from the C.B.D.

An obvious extension of the single-sector model is to the case where two land-using industries compete. Usually the two are interpreted as housing and transportation, which constitute the two principal occupants of urban land. Such a model has been developed by Mills (1967), and modified subsequently by Mills (1969) and Pines (1970). The assumptions of uniformity of land quality, monocentric city structure, fixed incomes, etc., are all retained. The production functions for housing and transportation are of the linear homogenous Cobb-Douglas class i.e.

$$H = q^{\alpha_1} k^{1-\alpha}$$

$$T = q^{\alpha_2} k^{1-\alpha_2}$$

where H, T are the outputs of housing and transportation respectively, q and k again denote land and capital inputs, and α_1, α_2 are parameters with $1 < \alpha_1, \alpha_2 > 0$. Equating the marginal revenue products of land and capital with marginal cost yields, after some manipulation

$$p_h[s] = c_1 r[s]^{\alpha_1}$$

where r[s] is rent, $p_h[s]$ is the price of housing, and c_1 is a constant. The same procedure in the transportation sector gives

$$p_t[s] = c_2 r[s]^{\alpha_2}$$

where $p_t[s]$ is the price of transportation and c_2 is a constant. It is assumed that per capita demand for housing and transportation are absolutely inelastic with respect to prices. Letting h_d represent the quantity of housing consumed per household, the spatial invariance of locational costs requires

$$p_t[s] + h_d\, p_h'[s] = 0.$$

It is again assumed that the rent at the boundary of the city s* is defined by the (constant) agricultural rent i.e.

$$r[s^*] = r_a.$$

With this information the equilibrium rent function $\bar{r}$ can now be solved for

$$\bar{r}[s] = [r_a^{\alpha_1-\alpha_2} - c_3(s^* - s)]^{\frac{1}{\alpha_1-\alpha_2}}$$

where c_3 is a constant. Rents will decline at a decreasing rate when $\alpha_2 > \alpha_1$. They decrease at an increasing rate when $\alpha_2 < \alpha_1$. The net density of population D[s] is defined as the amount of residential land consumed per person

$$D[s] = c_4[r_a^{\alpha_1-\alpha_2} - c_3(s^* - s)]^{\frac{1}{\alpha_1-\alpha_2}}$$

Both rent and net density functions tend towards exponential forms as α_1 approaches α_2. Both rent and net density functions become exponential when the price of housing is also exponential.

A variant of this analysis is presented by Averous and Lee (1973). In this framework the economy consists of two sectors, one of which produces commodities, while the other provides access to the C.B.D. Given a transport cost function and a land allocation over which he has no control, the representative entrepreneur seeks to maximize his rent-paying ability with the conventional assumptions, and the equilibrium rent distribution is readily obtained.

This model has in turn been generalized to allow for two transport modes. Capozza (1973) has analyzed the interesting situation where a land-using transport mode competes with one which uses no land. An example is a city with a subway system. Linear homogenous Cobb-Douglas production functions for each sector are once again assumed, hence

$$H = q_1^{\alpha_1} k_1^{1-\alpha_1}$$

$$T_1 = q_2^{\alpha_2} k_2^{1-\alpha_2}$$

$$T_2 = k_3^{\alpha_3}$$

where T_1 is the land-using transport mode. Again using the rules for profit-maximization the price function for housing is deduced:

$$p_h[s] = a_1 r[s]^{\alpha_1}$$

where a_1 is a constant. Differentiating with respect to s,

$$p_h'[s] = \alpha_1 a_1 r[s]^{\alpha_1 - 1} r'[s].$$

Allowing individual demand for housing to be unity, then the requirement that locational costs be constant can be written as

$$p_h'[s] - g'[s] = 0$$

where g[s]represents the cost of commuting a distance s from the C.B.D. It is easily shown that

$$p_{t_1} = a_2 r[s]^{\alpha_2},$$

$$p_{t_2} = a_3 .$$

The total cost of commuting from s to the city centre is the integral of transportation costs between s and the centre. If only the first mode is employed, then

$$c[s] = \int_0^s p_{t_1} [s] \, ds, = \int_0^s a_2 \, r[s]^{\alpha_2} \, ds$$

. If the second mode is chosen exclusively, then

$$c[s] = \int_0^s p_{t_2} [s] \, ds = \int_0^s a_3 \, ds = a_3 s.$$

If both modes are selected it is assumed that T_2 is used closer to the centre. The justification for this is that T_2 does not use land and land is more expensive closer to the centre. Given that a commuter switches modes at some distance $\bar{s}$, then total transport costs will be

$$c[s] = \int_0^{\bar{s}} a_3 \, ds + \int_{\bar{s}}^{s} a_2 \, r[s]^{\alpha_2} \, ds.$$

It is assumed that at equilibrium all land is occupied either by housing or the first type of transportation:

$$\bar{q}_1[s] + \bar{q}_2[s] = 2\pi s.$$

As usual, rent at the periphery is equal to the fixed agricultural rent,

$$r[s^*] = r_a,$$

where s^* denotes the urban boundary. At any distance s the demand for transportation is represented by the number of people living further than s from the C.B.D., and at equilibrium the supply of transportation must equal the demand. Hence

$$\bar{T}_1[s] + \bar{T}_2[s] = \int_s^{s^*} H[s] \, ds$$

The model is now fully specified and can be solved for the

[illegible] the first type of

transportation is used the solution is

$$r[s] = (c_1 - c_2 s)^{\frac{1}{\alpha_1 - \alpha_2}}$$

where c_1, c_2 are constant. If only t_2 is involved, then

$$r[s] = (c_3 - c_4 s)^{\frac{1}{\alpha_1}}$$

The corresponding net density functions are respectively

$$D[s] = (c_1 - c_2 s)^{\frac{1-\alpha_1}{\alpha_1 - \alpha_2}} \frac{1}{a_1 \alpha_1}$$

and

$$D[s] = (c_3 - c_4 s)^{\frac{1-\alpha_1}{\alpha_1}} \frac{1}{a_1 \alpha_1} .$$

Where both modes are employed the point at which commuters switch from one mode to another is denoted $\bar{s}$. The resultant rent function is given by

$$r[s] = (\hat{r} + c_5 \bar{s} + c_6 s)^{\frac{1}{\alpha}}$$

where $\hat{r} = r[\bar{s}]$ and $c_5 > 0$, $c_6 < 0$, are constants. Using this result Capozza demonstrates that rents in the inner ring are lowered by the introduction of a land-economizing transportation system. The impact of such an innovation on city size, however, is indeterminate. The increased supply of land will encourage more housing instead of transportation in the inner ring, which implies a smaller city. On the other hand lower rents encourage larger lot sizes, and lower transport costs increase real income and encourage immigration.

The role of economies of scale, so crucial to any understanding of urban economic structure, has been wholly neglected in analyses of the preceding types. Alao (1974) has attempted to inject this essential ingredient into a model which has the additional attractive feature of not requiring a specific form for the production function. In Alao's scheme output depends in the first place on a spatially invariant production function of the form $y = y[q,k]$ where q and k retain their previous interpretation. This has the usually concave properties but does not exhibit constant returns to scale. Secondly, firms experience a positive, distance-dependent externality, denoted $g[s]$. The spatial production function is now defined simply as $g[s]\ y[q,k]$. The demand function d is given to be of the form

$$d = (p + st)^a Z$$

where p is the mill price, t is unit transportation cost for output, and Z is a constant. Hence the problem of the firm is to maximize profits subject to a constraint that supply equal demand, i.e.

$$\underset{q,k,p}{\text{Maximize}} \quad p\ g[s]\ y[q,k] - r[s]\ q - wk$$

$$\text{subject to} \quad g[s]\ y[q,k] = (p + st)^a\ Z.$$

The first-order conditions for this maximum are

$$(p - \lambda)\, g[s]\, \frac{\partial y}{\partial g}\, [q,k] \;=\; r[s]\ ,$$

$$(p - \lambda)\, g[s]\, \frac{\partial y}{\partial k}\, [q,k] \;=\; w\ ,$$

$$g[s]y[q,k] \;=\; -\lambda\, a(p + st)^{a-1}\, Z,$$

$$g[s]\; y[q,k] \;=\; (p + st)^{a}\, Z.$$

where λ is a Lagrange multiplier.

Alao proceeds to analyse this system by conventional comparative statistics. When rents, which are exogenous to this model, are disturbed infinitesimally it is deduced that the price of output alters a corresponding amount in the opposite direction. Furthermore, a change in rent has a negative own effect, and indirectly induces a positve distance-land input substitution effect.

2.2. Consumer Behaviour

The neoclassical problem of the household is that of deciding the quantity of each available good and service it should purchase, given the prices of all commodities and the household's income. Formally, it is assumed that there is a finite number, n, of available commodities, and the quantities of each purchased by a consumer is represented by the vector $\underline{z}$. Each commodity is perfectly divisible so that any nonnegative quantity may be obtained. Thus the set of all possible commodity vectors $\{z\}$, sometimes referred to as the commodity space, is given by

$$Z = \{z \mid \underline{z} \geq 0\}.$$

That is, $\{z\}$ is the nonnegative orthant of $\mathbb{R}^n$. The choice of the consumer is governed by his tastes, which are expressed by the preference relation. For any $\underline{z}^1, \underline{z}^2 \in Z$ the statement

$$\underline{z}^1 \succsim \underline{z}^2$$

is interpreted as saying that $\underline{z}^1$ is preferred or indifferent to $\underline{z}^2$. This relation is assumed to have the properties of being a complete preordering of z and of being continuous. Hence it follows that there exists a continuous real-values function $u[\underline{z}]$defined on Z such that $\underline{z}^1 \succsim \underline{z}^2$ implies $u[\underline{z}^1] \geq u[\underline{z}^2]$. It is traditionally postulated that $u[\underline{z}]$ is continuously differentiable, and in particular, that it is strictly concave.

In the neoclassical framework all money prices, denoted $\underline{p}$, and income y, are known, fixed, and positive. The budget constraint of the household is consequently

$$\underline{p} \cdot \underline{z} \leq y,$$

and the opportunity set is given by

$$C = \{\underline{z} | \underline{p}\ \underline{z} \leq y,\ \underline{z} \geq \underline{0}\}.$$

The problem of the household is that of choosing a most preferred $\underline{z}$ in C. That is, to

$$\underset{z}{\text{maximize}} \quad u[\underline{z}]$$

$$\text{subject to } \underline{p}\ \underline{z} \leq y$$
$$\underline{z} \geq \underline{0}\ .$$

Since $u[\underline{z}]$ is continuous and C is compact, then by the Weierstrass theorem a solution exists. Since u is strictly concave and Z is convex, then this solution is also unique.

Consumer-based theories of urban economic structure are generalizations of this problem to allow for spatial effects. The most general treatment is that of Long (1971) who considers the following problem:

$$\underset{\underline{z}}{\text{Maximize}} \quad u[\underline{z}]$$

$$\text{subject to } \underline{p}[\underline{m},\underline{k},s]\ \underline{z} = y\ .$$

where $\underline{m}$ is a vector of mill prices, $\underline{k}$ is a vector of unit transportation rates, and s denotes distance to the urban core. Thus the known and

is replaced by a known but spatially variable one.

The first order conditions are

$$\frac{\partial u}{\partial z_i} = \lambda p_i[m_i, k_i, s] = 0, \qquad i = 1, \ldots n$$

$$\underline{p}\,\underline{z} - y = 0$$

where λ is a Lagrange multiplier. Differentiation with respect to s and rearrangement yields the effect of locational change on consumption i.e.

$$\frac{\partial z_i}{\partial s} = -\frac{\partial z_i}{\partial y} \sum_j z_j \frac{\partial p_j}{\partial s} + \lambda \sum_j \frac{\Delta_{ij}}{\Delta} \qquad i = 1, \ldots n$$

where Δ is the determinant $\left|\left|\frac{\partial u_i}{\partial z_j}\right|\right|$ and Δ_{ij} is a cofactor. Long interprets the first term on the right hand side as the spatial income effect-the change in income caused by price changes resulting from spatial movement. Similarly Long demonstrates by standard methods, that the second term is the spatial substitution effect-the change in demand for a commodity resulting from price change, real income being held constant.

In this framework the effect on demand of a change in mill price corresponds to the standard, non spatial problem, where mill price and delivered price are identical. The effect is given by

$$\frac{\partial z_i}{\partial m_j} = -\frac{\partial z_i}{\partial y} z_j \frac{\partial p_j}{\partial m_j} + \lambda \frac{\partial p_j}{\partial m_j} \frac{\Delta_{ij}}{\Delta} \qquad (i = 1, \ldots n)$$

$$= -\frac{\partial z_i}{\partial y} z_j + \lambda \frac{\Delta_{ij}}{\Delta},$$

if $\frac{\partial p_j}{\partial m_j} = 1$. Similarly, the effect of a change in a transport rate k_j is summarized by the equation

$$\frac{\partial z_i}{\partial k_j} = -\frac{\partial z_i}{\partial y} z_j \frac{\partial p_j}{\partial k_j} + \lambda \frac{\partial p_j}{\partial k_j} \frac{\Delta_{ij}}{\Delta} \qquad (i = 1, \ldots n)$$

where the first term on the right-hand side indicates the transport income effect, and the second term shows the transport substitution effect. Long indicates that for a linear transport rate function the effect of change in k_i on demand for a specific commodity depends on the distance from the consumer to the mill.

It is clear that an explicit treatment of residential land and its price (rent) would greatly enhance the analysis since these variables are of primary importance in the urban economy. A simple extension of the basic model along these lines leads to the following problem

$$\begin{array}{ll} \underset{z,q}{\text{maximize}} & u[z,q] \\ \text{subject to} & pz + rq + k[s] = y \end{array}$$

where q denotes residential land, r is rent, k[s] is the cost of commuting, and z is now a composite of all commodities except residential land (for convenience z is assumed to be a scalar). This problem has been examined by Muth (1961, 1969) and by Wheaton (1974). The first order conditions for a maximum are

$$u_z - \lambda p = 0 ,$$

$$u_q - \lambda r = 0 ,$$

$$-\lambda \left(q \frac{dr}{ds} + \frac{dk}{ds}\right) = 0 ,$$

$$pz + r[s]q + k[s] - y = 0$$

Muth is especially concerned with the third equation, which when rewritten as

$$q \frac{dr}{ds} = - \frac{dk}{ds}$$

states that, in equilibrium, the commuting costs saved by a shift in location are precisely offset by an increase in residential cost. That is, the household is unable to increase real income by moving. By assumption, commuting costs increase with income, which implies that $\frac{dr}{ds} < 0$. Stability of equilibrium requires that

$$-q \frac{d^2r}{ds^2} - r \frac{\partial q}{\partial s} - \frac{d^2k}{ds^2} \leq 0.$$

At equilibrium, real income is constant under locational change, hence the change in demand for residential land due to such a shift is given by

$$\frac{\partial q}{\partial s} = \left(\frac{\partial q}{\partial r}\right)_{y=\bar{y}} \cdot \frac{\partial r}{\partial s}$$

where $\frac{\partial q}{\partial r}\,_{y=\bar{y}}$ is the slope of the demand curve when real income $\bar{y}$ is held constant. Substituting this into the stability condition yields, after rearrangement

$$\frac{d^2r}{ds^2} \geq \frac{-1}{q}\left(\frac{\partial q}{\partial r}\right)_{y=\bar{y}} \frac{\partial r}{\partial s}^2 + \frac{d^2k}{ds^2} \,.$$

Assuming commuting costs increase at a decreasing rate, and also that housing demand is not perfectly elastic, then $\frac{d^2r}{ds^2} > 0$.

Muth does not employ a specific utility function, and his analytical style leads to relatively weak qualitative results. By contrast, Wheaton, with standard comparatic static methods, produces a wealth of information on this problem. From the first order conditions he derives the equation

$$\frac{u_z}{u_q} = \frac{p}{r}$$

Secondly, he assumes that at equilibrium, households attain a fixed utility level $\bar{\bar{u}}$ so that there is no incentive to move.

$$\bar{u}[z,q] = \bar{\bar{u}} .$$

(Wheaton implicitly assumes that the representative household is at a fixed location s, at which it selects the utility maximizing combination of z and q). The parameters of the problem are $\bar{\bar{u}}$, y, and s, and Wheaton assumes that the unknowns q, z, and r may each be expressed as implicit functions of these parameters. With this equipment he deduces the following qualitative effects of parametric variation on the rent distribution:

$$\frac{\partial r}{\partial y} = \frac{1}{q} > 0 ,$$

$$\frac{\partial r}{\partial s} = \frac{dk[s]}{ds} \frac{1}{q} < 0 ,$$

$$\frac{\partial r}{\partial \bar{\bar{u}}} = \frac{-1}{q} \frac{\partial u}{\partial z} < 0,$$

In like manner, using the condition that rent at the boundary of the city, s*, equal agricultural rent, i.e.

$$r[s^*,\bar{\bar{u}},y] = r_a ,$$

and given the total urban population N where

$$2\pi \int_0^{s^*} \frac{s}{q}\, ds = N \quad ,$$

then by a similar strategy Wheaton demonstrates that increased population diminishes the equilibrium utility level, as does higher agricultural rent. Furthermore, analysis of the impact of commuting costs reveals that an increase in the marginal cost of travel causes shrinkage of the urban boundary and raises rents and population densities at the centre. Finally this model shows that under a given restriction of commuting costs, a rise in income has the effect of increasing the size of the city and lowering central population density and rent.

Solow (1972) employs a structure of the Muth-Wheaton type, but with a specific form for the utility function. He postulates that utility depends in a log-linear fashion on the consumption of land and a composite good. Hence the problem becomes to

$$\text{maximize} \quad u[z,q] = \alpha_1 \ln z + \alpha_2 \ln q$$

$$\text{subject to} \quad y = pz + r\,q + k[s]$$

where α_1, α_2 are constants. Manipulating the first order conditions leads to the differential equation

$$\frac{r'}{r} = - \frac{k'[s]}{\alpha_3 (y-k[s])}$$

where α_3 is a constant. If the transportation cost at the edge of the C.B.D. is zero, then this equation has the solution

$$r[s] = r_1 \left(1 - \frac{k[s]}{y}\right)^{\frac{1}{\alpha}}$$

where r_1 is the rent at the inner boundary. Given $\frac{dk}{ds} > 0$, then rents must decline with distance. This result yields the demand function for residential land,

$$q[s] = \alpha \frac{y}{r_1} \left(1 - \frac{k[s]}{y}\right)^{\frac{1}{1-\alpha}}$$

The primary objective of Solow's analysis is to incorporate the notion of congestion into a land use model. This topic is clearly of great significance in any study of urban spatial structure but is notably absent from most theoretical studies.[3] Solow attempts to introduce congestion by postulating that the cost of travel per person at any point depends on the number of people using transportation at that point and on the land area devoted to transportation at that point. The number of people living beyond a point s, denoted $N[s]$, is given by

$$N[s] = \int_s^{s^*} n[s]\, ds$$

where $n[s]$ is the number of people per unit of radial distance living in the ring at radius s from the centre. The amount of land at distance s devoted to transportation is $2\pi s(1-b[s])$, where $b[s]$ is the amount occupied by housing. Solow assumes that the cost per unit of radial distance of commuting at s is

$$\left(\frac{N[s]}{2\pi s(1 - b[s])}\right)^m ,$$

that is, a power of traffic density. Total commuting costs between s and the C.B.D. boundary are therefore

$$k[s] = \int_{s_0}^{s^*} \left(\frac{N[s]}{2\pi s(1 - b[s])}\right)^m ds .$$

Hence commuting costs depend on the population distribution which in turn depends on commuting costs. This mutual interdependence is the essence of the congestion problem, and analytical solutions have thus far proven elusive. To obtain any results Solow is compelled to resort to numerical methods.

The preceding models make no allowance for households having explicit spatial preferences. When such preferences are admitted the standard model becomes

$$\underset{z,q,s}{\text{maximize}} \quad u[z,q,s]$$

$$\text{subject to} \quad pz + rq + k[s] = y .$$

This problem was first formulated by Alonso (1960, 1964). The objective of the household in this context is to find a location as well as a quantity of the composite good and a quantity of residential land which together maximize utility. Taking the total derivative of the utility function Alonso obtains the following:

$$-\frac{dz}{dq} = \frac{u_q}{u_z} ,$$

$$-\frac{dz}{dk} = \frac{u_s}{u_z} .$$

Similarly, taking the total derivative of the budget constraint he finds that

$$-\frac{dz}{dq} = \frac{r}{p}$$

$$-\frac{dz}{ds} = \frac{1}{p}\left(q\frac{dr}{ds} + \frac{dk}{ds}\right) .$$

Combining the two sets of results leads to

$$\frac{u_q}{u_z} = \frac{r}{p} ,$$

$$\frac{u_s}{u_z} = \frac{1}{p}\left(q\frac{dr}{ds} + \frac{dk}{ds}\right) .$$

The last two equations, together with the budget constraint, implicitly define equilibrium values for z,q, and s. To obtain such a solution however, requires an a priori knowledge of the function $r[s]$, which defeats much of the purpose of this type of model. However, it does illuminate some interesting issues. For example the last equation above states that the marginal rate of substitution between distance and the composite commodity is equal to the ratio of marginal relocation costs ($q \frac{dr}{ds} + \frac{dk}{ds}$) to the price of the composite good. It is assumed that distance and the good have negative and positive marginal utility respectively, and therefore, since $p > 0$, the marginal cost of moving is negative. Assuming $\frac{dk}{ds} < 0$, one is left with the conclusion that $\frac{dr}{ds} < 0$. Furthermore, it is seen that $\frac{dk}{ds} < -q \frac{dr}{ds}$, and hence, at equilibrium, the household only settles where savings from cheaper land exceed increased commuting costs.

The model of Alonso cannot generate continuous equilibrium rent and population distributions owing to the fact that it contains one more unknown (namely rent) than the number of equations. To completely specify the model an additional relationship must be assumed. Casetti (1971) has proposed the condition that, in equilibrium, identical households everywhere obtain an identical level of utility, thus removing any incentive for movement. This condition subsequently appears in the analysis of Wheaton, and in papers by Stull (1973), Menchik (1973) and others. The standard problem therefore becomes, in Casetti's framework, to

$$\underset{\underline{z},q}{\text{maximize}} \quad u[\underline{z},q|s]$$

$$\text{subject to} \quad pz + rq + k[s] = y$$
$$\text{and} \quad u[\bar{z},\bar{q}|s] = \bar{\bar{u}}$$

where $\bar{\bar{u}}$ is a constant and the household is assumed to choose z and q at a given location. The marginal utility conditions yield

$$\bar{z} = \bar{z}(r[s], s,p,y)$$
$$\bar{q} = \bar{q}(r[s], s,p,y)$$

and, after some manipulation

$$\bar{r} = \bar{r}(\bar{u},s,p,y) \quad .$$

Letting $\bar{u} = \bar{\bar{u}}$ which is fixed, then

$$\bar{\bar{r}} = \bar{r}(\bar{\bar{u}},s,p,y)$$

is immediately known. Similarly, substituting $\bar{\bar{r}}$ for $\bar{r}$ yields the spatial equilibrium demand functions q and z. The equilibrium population density distribution is given by $\bar{\bar{q}}^{-1} = \bar{\bar{D}}[s]$ and the equilibrium population is simply $2\pi \int_0^{s^*} \bar{\bar{D}}[s]\, ds$, where s* represents the city boundary.

The Alonso-type structure has been generalized in two aspects. First, the existence of a distribution of income classes is recognized in the work of Beckman (1969). Secondly, the assumption of monocentricity has been relaxed by Papageorgiou (1971a, 1971b).

Beckman assumes a log-linear utility function which is maximized subject to the usual budget constraint and hence derives the demand function for residential land $q[r[s],y,\underline{w}]$, where $\underline{w}$ represents the other parameters of the problem. In addition there is a Pareto distribution of incomes

$$N[y] = a\, y^{-\alpha}$$

where N is the number of households with income greater than or equal to y, and a,α are positive constants. In a band of width ds the population will be $\alpha\, a\, y^{-\alpha-1} \frac{dy}{ds}\, ds$. The supply of land is $2\pi s\, ds$. Equating the supply and demand for residential land yields the differential equation

$$q[r[s],\, y,\, \underline{w}]\, \alpha a\, y^{-\alpha-1} \frac{dy}{ds}\, ds = 2\pi s\, ds$$

which can be solved for r[s]. This model has subsequently been corrected and refined by Delson (1970) and Montesano (1972).

The second generalization of the standard model consists of the relaxation of the assumption of a single urban core. This multicentre approach provides a means whereby urban land use modelling can be connected with Central Place theory. The method of generalization has been developed by Papageorgiou (1971a, 1971b) and is applied to an Alonso-type model in Papageorgiou and Casetti (1971), where a population of identical households resides on a uniform plain on which a hierarchy of urban centres is located. The centres are ranked so that a centre of order i provides all the commodities of centres of order 1 through i-1 as well as other commodities which define its order. It is assumed that households purchase a given commodity from the nearest centre providing it. Thus every residential location can be characterized by a unique vector $\underline{s}$ where $s_i \in \underline{s}$ indicates the shortest distance to a centre providing z_i. When location is given, the consumer equilibrium problem becomes

$$\underset{z,q}{\text{maximize}} \quad u[\underline{z},q|\underline{s}]$$

$$\text{subject to} \quad y = \underline{p}\,\underline{z} + rq + k[\underline{s}] \ ,$$

where $\underline{z}$ is a vector of composite goods with each element z_i corresponding to the commodities which characterize an i^{th} order centre. Using the equilibrium condition $\bar{u}[\underline{z},q|\underline{s}] = \bar{\bar{u}}$ the equilibrium rent profile $\bar{\bar{r}}[\underline{s}]$ is derived, and consequently so is the density distribution of population $\bar{\bar{D}}[s] = \bar{\bar{q}}[s]^{-1}$.

Other aspects of consumer spatial behaviour which are receiving attention include the impact of environmental effects (Nelson (1972), Papageorgiou (1973), Yamada (1972)), the variation in the height of physical structures (Wright (1971)), and the substitution of a "characteristics" space for the commodity space (Walter (1971)). Finally, some major syntheses, such as that by Papageorgiou (1975) which involves both a multicentric environment and a distribution of incomes, are now emerging.

2.3. Multisector Equilibrium

It has already been stated that both classes of model discussed in the two preceding sections share a common characteristic – the assumption that the sector not under consideration is dormant. The initial effort at relaxing this very stringent assumption was made by Muth (1969). However, this model is based on a strict postulate (linearity) about commuting costs. Amson (1972) presents a generalization of Muth's analysis which evicts this requirement. The central thrust of Amson's work is aimed at connecting the population density distribution to the transportation cost function. His primary assumptions are that the profit function of firms producing housing is of the form

$$\pi = p_h H[q,\ell] - rq - w\ell$$

where p_h is the unit price of housing, and r,w denote the unit prices of land inputs q and non-land inputs ℓ respectively. The production function $H[q,\ell]$ is assumed to be Cobb-Douglas. Households possess a utility function

$$u = u[h,z]$$

where h and z correspond to their consumption of housing and a composite commodity. This function is maximized subject to the constraint

$$y = pz + p_h h + k[s]$$

where p is the price of the composite commodity and k[s] indicates commuting costs. By postulating that household size and the value of housing consumed per household are spatially invariant, then, using the conditions for producer equilibrium Amson demonstrates that the density of population D may be represented as

$$D[s] \;=\; \alpha\, p_h[s]^{\beta}$$

where α, β are constants. Assuming that household equilibrium is neutral (i.e. that $\frac{\partial}{\partial s}\,[-\,q\,p_h'[s] - k'[s]] \;=\; 0$, then the price of housing is given by

$$p_h[s] \;=\; [p_o^{\,e+1} - \gamma\,(e+1)\,k[s]]^{\frac{1}{e+1}}$$

where p_o, γ are constants and $e \neq 1$ is the price elasticity of housing demand. If $e = 1$, then

$$p_h[s] \;=\; p_o \exp\,[-\gamma\, k[s]]\;.$$

Thus density of population under these two circumstances follows immediately

$$D[s] \;=\; D_o \exp\,[-\gamma\, k[s]] \text{ if } e = -1\,,$$

$$D[s] \;=\; [a - b\,k[s]]^{\frac{1}{c}} \quad \text{if } e \neq -1\,,$$

where D_0,a,b are positive constants. Amson is therefore able to generate most of the well-known population density functions as special cases of the exponential solution.

A similar approach is adopted by Hochman and Pines (1971) who employ a Mills-like framework as a starting point, but unlike the Mills models the demand for housing and for land used in transportation are both determined endogenously. The household sector in this model is again assumed to be uniform in terms of preferences and income. The representative household maximizes its utility where

$$u = u[H,z]$$

H denoting the quantity of housing consumed. The budget constraint in this instance is given by

$$y = p_h H + pz + k[T[s], p_t[s]]$$

where p_h is the price of housing and commuting costs k depend on the quantity and price of transportation at s, denoted T[s] and $p_t[s]$ respectively. From the first order conditions the differential equation

$$p_h'[s] = \alpha_1 p_t[s]^{\alpha_2} p_h[s]^{\alpha_3}$$

is derived.

On the supply side, the profit functions for producers of housing and transportation are given respectively by

$$\Pi_h = H[q,\ell] - w\ell - r[s]\, q \quad ,$$

$$\Pi_t = T[q,\ell] - w\ell - r[s]\, a \quad ,$$

where ℓ and q are non-land and land inputs and w, r[s] are their corresponding prices. The production functions $H[q,\ell]$, $T[q,\ell]$ are assumed to be Cobb-Douglas. The marginal productivity rules lead to the equations

$$p_h[s] = \beta_1\, r[s]^{\beta_2} \quad ,$$

$$p_t[s] = \gamma_1\, r[s]^{\gamma_2} \quad ,$$

where $\beta_1, \beta_2, \gamma_1, \gamma_2$ are constants. Using the equilibrium conditions for producers and consumers, Hochman and Pines are able to derive the rent, house price, and transportation price function. From producer equilibrium it is seen that

$$p_h'[s] = \beta_1 \beta_2\, r[s]^{\beta_2 - 1} \frac{dr}{ds} \quad .$$

This is substituted into the consumer equilibrium relationship for $p_h'[s]$, and used to solve for r[s]. The general solution is given by

$$r'[s] = A\, r[s]^d$$

where A, d, are constants. If d = 1, then

$$r[s] = r_a \exp(As^*) \exp(-As)$$

where r_a is the rent at s^*, the city limit. Similar solutions are obtained for the price functions of transportation and housing. In all three cases the solutions are decreasing functions of distance. If $d > 1$ the relative rate of decrease of each increases with distance. If $d < 1$, their relative rate of decrease diminishes with distance. If $d = 1$, the relative rates of decrease are all constant—hence the functions are exponential as shown above in the case of rents.

2.4. An Evaluation of the Partial Equilibrium Approach

Partial equilibrium models, both single sector and multisector, have gained considerable success in deriving empirically-supported consequences from simple hypotheses of consumer and producer behaviour. This, however, does not imply that they are wholly satisfactory. In fact, a formidable battery of accusations has been levelled at this field since its inception. Recurrent points of contention include the following: (1) the persistent use of specific functional forms, which greatly restricts the generality of any conclusions. In particular the widespread use of Cobb-Douglas-type functions is especially harmful. The sole justification for this form is its analytical convenience, and in many models the analysis collapses when alternative functions are employed. On the consumption side the use of this function to indicate a preference structure is clearly a major weakness, since it implies unit income elasticities for commodities and a unit elasticity of substitution. (2) The popular assumption of linear homogeneity in both utility and production function compounds the problems associated with selecting specific forms. It is particularly damaging on the production side, where nonlinearities (i.e. nonconstant returns to scale) are of fundamental significance. (3) Most models take the location and volume of employment in the C.B.D. as exogenous. A primary objective of any urban land use model should be to determine these elements endogenously. (4) The analytical structure of this general class of models does not appear amenable to treating congestion, obviously one of the most influential factors in the urban economy. This impotence is amply demonstrated by the failure of Vickerey (1965), Mills (1972b) and Solow (1972) to supply analytical solutions for problems involving congestion. (5) There is a general overemphasis of the suburban aspects of the economy. For completeness it is necessary to consider also the nature of the C.B.D., and to construct theories which involve both the urban core and the suburbs. The literature on optimality is already witnessing an emerging interest in this area (e.g. Livesey (1973)). (6) All the partial equilibrium models reviewed above suffer greatly in terms of realism and potential applicability by their total neglect of the public sector, which is increasingly powerful in the shaping of urban spatial structure.

The purpose of this research however, is not directed towards any of the above-mentioned deficiencies. It is an attempt to examine those issues which motivate the general equilibrium approach, namely the notions of simultaneity and feasibility. The single sector models evade the problem of simultaneity by holding constant either the supply or demand component. The multisector models consider the determination of equilibrium prices and quantities by simultaneous producer-consumer interaction, but only in a limited sense. For example, in the Pines-Hochman model the demand for housing is determined within the model. However, this demand is seen to depend on income, which, as is usual in partial analysis, is fixed. In the general equilibrium scheme, however, the chain of interdependence is complete, and incomes depend on production decisions, which are themselves affected by the demand for housing. The simple interactions which characterize the partial approach are replaced by the more realistic and complex patterns of general equilibrium. The second question, concerning the aggregate attainability of equilibrium solutions, is never considered in partial-equilibrium analysis.

A major concern of this research is to initiate an investigation of these two problems in a spatial context. The vehicle employed for this purpose is the established general equilibrium model, suitably modified to render it compatible with a simple spatial environment.

3. GENERAL EQUILIBRIUM THEORY

3.1. The Objectives of General Equilibrium Theory

The approach embodied in the models of the preceding chapter is essentially that of microeconomics, a field which concentrates on the individual decision-making behaviour of economic agents. It is explicitly assumed that these agents act rationally in accordance with certain choice criteria. The spatial consequences of rational individual behaviour are the concern of partial equilibrium analyses of urban structure. Macroeconomics, on the other hand, examines the entire economy using aggregates such as output level, employment, and the level of prices, and is concerned with such topics as economic growth and inflation.

The theory of general equilibrium uses the formal structure of microeconomics in endeavouring to answer macroeconomic questions. It is related to decentralized economies, and investigates not only individual activity, but also, in contradistinction to partial equilibrium, the interaction of agents of different classes in the economy. This latter gives rise to four critical issues. The first is the question of an equilibrium state. It is of fundamental importance to know the conditions under which such a state can exist. Secondly, there is the question of the uniqueness of an equilibrium position. Thirdly, the stability of an equilibrium is of fundamental importance. Finally, there is the normative problem concerning the optimal properties of an equilibrium. Of these four, only the first issue is dwelt upon in the following discussion and subsequent analysis.

Numerous formulations of the general equilibrium concept have been developed, from its initial expression by Walras (1874, 1877), to the rigorous statement by Debreu (1959). It is not proposed to review at this juncture the historical development of the theory (numerous such reviews are available, for example Arrow (1968)). Rather, it is felt that a more pertinent programme would be to specify the axiomatic foundations of general equilibrium, and to enumerate some of the more prominent criticisms. The justification for this course is found in the ultimate objective of this dissertation, which is to discover the implications of the axiomatic structure for a spatial economy.

For illustrative purposes an analytic framework modified from Arrow and Hahn (1971) is employed. This model epitomizes contemporary general equilibrium theory. All the behavioural assumptions contained therein relate to an economy which exists at a given instant. There is a positive number, ℓ, of commodities. Each commodity is a good or service completely specified physically, temporally, and spatially. Thus, for example, a given good at one location is distinguished from the identical good at any other location. With each commodity i is associated a real number, its price p_i, which may be positive, negative or zero. The economy contains a finite number of agents, classified as firms and households, the former being distinguished from the latter by their ability to produce goods and services. There also exists a finite quantity of resources in the economy, which are distributed among the households. A feasible state of the economy is described as one in which aggregate consumption does not exceed the sum of aggregate production and resources. Given behavioural rules governing producers and consumers, the existence problem reduces to finding a price system which induces reactions in individual agents which in turn generate a feasible state of the economy. The following two sections present the conventional behavioural postulates of the agents.

3.2. The Axioms of Production

Production is organized through a number of separate firms, each of which may use several inputs to produce several outputs. By convention, inputs are represented by negative quantities and outputs by positive quantities. Any specification of possible relations between inputs and outputs for a given firm f is denoted by the vector $\underline{y}_f \in \mathbb{R}^\ell$, which is referred to as a production plan. The set of all such plans open to the firm is called Y_f and is termed the production possibility set for firm f. To reiterate a previous point, a given input or output at different locations in space or time is considered a distinct commodity at each such location.

Given a set of prices for all commodities it is possible to calculate for each production plan its profit, which is the excess of the value of its outputs over the value of its inputs. It is assumed that, at a given price system, each firm chooses an activity which yields at least as much profit as any other open to it. Any sum of individual firms' plans is denoted $\sum_f \underline{y}_f = \underline{y}$ and is referred to as an aggregate production plan. The set of all such plans is the aggregate production possibility set, Y. Both individual and aggregate production sets are assumed to possess certain properties.

First, it is supposed that it is always possible for the firm to engage in no activity.

Assumption 3.1

$\underline{0} \in Y_f$.

This assumption pertains to all the leading models of general equilibrium, with the exception of that by Gale (1955) who employs the weaker assumption that $\underline{0} \in Y$, where Y is the aggregate production set (defined below).

Secondly, if there are technically possible production plans arbitrarily close to a given plan, then the given plan is also in the possibility set.

Assumption 3.2

Y_f *is closed.*

Again, the property of closure is assumed only for aggregate production sets in the models of Debreu (1959), and McKenzie (1959).

Thirdly, it is assumed that production is divisible. That is, if $\underline{y}^1$ is a possible production plan, then so is $\lambda \underline{y}^2$, for $0 < \lambda < 1$. Considerable controversy surrounds the acceptability of this assumption. It is empirically observable that many commodities are indivisible, and consequently production plans containing them are indivisible. Since any convex set containing the origin is divisible, this property is embodied in the following assumption.

Assumption 3.3

Y_f *is convex.*

In addition, it is required that firms be capable of using up commodities without consuming inputs in the process.

Assumption 3.4

If $\underline{y}_f \in Y_f$ *and* $\underline{y}_f^1 < \underline{y}_f$, *then* $\underline{y}_f^1 \in Y_f$.

The aggregate production possibility set is defined as

$$Y = \{\underline{y} = \Sigma\, \underline{y}_f \mid \underline{y}_f \in Y_f \text{ , all } f\}.$$

(The following two assumptions are elegantly combined into a single statement by Arrow and Hahn. They are presented separately here to eliminate the supplementary discussion a direct transcription of the Arrow-Hahn assumption would entail).

The possibility that the economy could produce outputs without using inputs is ruled out explicitly.

Assumption 3.5

$Y \cap \Gamma = \emptyset$.

where Γ is the interior of $\mathbb{R}_+^{\ell}$

Furthermore, no aggregate production plan may be completely undone by another. (To guarantee this all that is necessary is to assume that there exists some non-produced input used in production, e.g. labour).

Assumption *3.6*

$Y \cap (-Y) = \{\underline{0}\}$.

If we note the initial resources of the economy by the vector $\bar{\underline{x}}$, then a final characteristic of aggregate production is that the firms as a whole and the initial endowment can together generate a positive quantity of each commodity.

Assumption 3.*7*

For some $\underline{y} \in Y$, $\underline{y} + \bar{\underline{x}} >> \underline{0}$.

The assumption of irreversibility *(3.6)* is not employed in the works of McKenzie (1959) or Debreu (1962), and a weaker condition is employed by Gale. Free disposal *(Assumption 3.7)* is not explicitly stated in McKenzie (1959) or Debreu (1962), and is assumed only for the aggregate production set in Debreu (1959).

3.3. The Axioms of Consumption

There exists a finite number of households, and $\underline{x}_h$ denotes a consumption plan of household h. Such a plan comprises a listing of the quantities of each commodity used by the household. As stated previously, each household possesses an initial stock of resources, denoted by $\bar{\underline{x}}_h$, which it sells to purchase goods and services. The set of all consumption plans open to a household constitutes its consumption possibility set, X_h. As in the case of producers, certain properties are required of consumption sets.

Assumption 3.8

X_h *is closed and convex;* $\underline{x}_h \geq \underline{0}$ for $\underline{x}_h \in X_h$.

This assumption eliminates the possibility of negative consumption for any commodity. The closure and convexity characteristics are basic to all models of this class. Lower boundedness, however, is not required in the Debreu (1962) model.

Secondly, it is assumed there is a possible consumption plan for each household which is "less than" the households' initial endowment.

Assumption 3.9

There exists an $\bar{\bar{\underline{x}}}_h \in X_h$ *such that:*

$$\bar{\bar{x}}_{hi} \leq \bar{x}_{hi} \quad \text{all } i,$$

$$\bar{\bar{x}}_{hi} < \bar{x}_{hi} \ \textit{if} \ \bar{x}_{hi} > 0.$$

In several works the more stringent requirement that $\bar{\underline{x}} >> \underline{0}$ is employed (e.g. Debreu (1962), Wald (1951)).

The income of the household is derived from the sale of its initial endowment and the share of the household in the profits of firms. It is assumed that household h owns a fraction d_{hf}, of firm f.

Assumption *3.10*

For household h *total income* M_h *at a given set of prices* $\underline{p}$ *and a given set of production decisions* Y_f *is given by*

$$M_h = \underline{p}\,\bar{\underline{x}}_h + \sum_f d_{hf}(\underline{p}\cdot\underline{y}_f),$$

where $d_{hf} \geq 0$, $\sum_h d_{hf} = 1$.

The objective of the household is to choose the most preferred point in its consumption possibility set, given its income and the price system. Preferences are characterized by the following assumption.

Assumption *3.11*

For each household h *there is a relation* $\succsim_h$ *for pairs of elements in the consumption possibility set, with the following properties.*

(a) Transitivity:

$$x_h^1 \succsim_h x_h^2 \textit{ and } x_h^2 \succsim_h x_h^3 \textit{ implies } x_h^1 \succsim_h x_h^3;$$

(b) *Connectedness:*

For all $\underline{x}_h^1, \underline{x}_h^2 \in X_h$, *either* $\underline{x}_h^1 \succsim_h \underline{x}_h^2$ *or* $\underline{x}_h^2 \succsim_h \underline{x}_h^1$;

(c) *Continuity:*

For a given $\underline{x}_h^o$, *the sets* $\{\underline{x}_h | \underline{x}_h \succsim_h \underline{x}_h^o\}$ *and* $\{\underline{x}_h | \underline{x}_h^o \succsim_h \underline{x}_h\}$ *are closed ;*

(d) *Semi-strict Convexity:*

If $\underline{x}_h^1 \succ_h \underline{x}_h^2$, *then* $(1-\lambda)\underline{x}_h^1 + \lambda \underline{x}_h^2 \succ_h \underline{x}_h^2$ *for* $0 < \lambda < 1$;

(e) *Non-satiation:*

There is no $\underline{x}_h^o$ *such that* $\underline{x}_h^o \succsim_h \underline{x}_h$, *for all* $\underline{x}_h \in X_h$.

[The relation $\succ_h$ in (d) means that $\underline{x}_h^1 \succsim_h \underline{x}_h^2$ and not $\underline{x}_h^2 \succsim_h \underline{x}_h^1$].

3.4 A Definition of General Competitive Equilibrium

To permit numerical representation of preferences a real-valued function u_h is defined on X with the property that $\underline{x}_h^1 \succsim_h x_h^2$ if and only if $u[\underline{x}_h^1] \geq u[x_h^2]$. The function is referred to as a utility function. With this final piece of equipment, a competitive equilibrium may be defined as a set of prices $\underline{p}^*$ and production and consumption choices X^*, Y^* such that each firm maximizes profits at the given prices, each household maximizes utility given prices and income, and aggregate consumption does not exceed the sum of resources and aggregate production. Formally, $\underline{p}^*$, X^*, Y^*, constitute a general equilibrium if

(a) $\underline{p}^* > \underline{0}$

(b) $\Sigma \underline{x}^*_h \leq \Sigma \underline{y}^*_f + \bar{\underline{x}}$;

(c) $\underline{y}_f$ maximizes $\underline{p}^* . \underline{y}_f$ subject to $\underline{y}_f \in Y_f$;

(d) $\underline{x}^*_h$ maximizes $u[\underline{x}_h]$ subject to $\underline{p}\, \underline{x}^*_h \leq \underline{p}^* \bar{\underline{x}}_h + \Sigma_f d_{hf} (\underline{p}^* \underline{y}^*_f)$.

From assumptions 3.1 - 3.10, and using standard analytical devices such as the Kakutani fixed-point theorem, the existence of an equilibrium so defined is demonstrated rigorously by Arrow and Hahn (1971).

3.5. Criticisms of General Equilibrium

The analysis of competitive equilibrium as expounded by Arrow, Debreu and others has been severely criticised on numerous grounds including empirical vacuity and implausible axiomatization. It has been censured both on specific issues and in its entirety. A notable example of the latter approach is the evaluation of Kornai (1971) who dismisses general equilibrium as "useless as a real science theory". The major thrust of Kornai's thesis is that the theory provides an inadequate description of the workings of an actual economy. In a rejoinder, Hahn (1973) indicates that the function of the theory is not to describe, but to discover what the economy must look like if a general equilibrium were to exist. In such a manner, claims Hahn, the implications of much traditional and intuitive economic reasoning may be logically deduced. Further, great practical relevance is claimed for the theory. Hahn cites the example of exchange rates. The conventional wisdom is that a floating exchange rate tends to an equilibrium level. General equilibrium theory however, indicates that not only may there not be an equilibrium level, but even should one exist, it need not be unique.

Numerous objections of a specific nature have been directed at general equilibrium. For example, the adoption of perfect competition, deterministic tastes, fixed technology, etc. are all regarded with disapproval. The response of equilibrists is that such objections relate to the state of the art rather than any fundamental flaws in the theory, and they point to work on uncertainty, stochastic equilibria etc., to support their argument. However, it is generally conceded that important weaknesses are visible, exemplified by the inability of the theory which, while rigorously describing equilibrium, fails to discuss how it comes into existence.

One major deficiency which has escaped most critical attacks is the glib assumption that a commodity at different points in space and time may be defined as a distinct commodity at each point. Such a definition trivializes the roles of space and time in an equilibrium setting. It has been pointed out that this treatment gives no insight as to the relationship between a good in one time period and the next. (For example, see Quirk and Saposnik (1968)). From the spatial point of view Starrett (1973) argues that "... the resulting theorems are weak in the sense that they ignore the additional structure implied by spatial arrangements." Starrett proceeds to attack the convexity assumptions of general equilibrium, which, he alleges, eliminate the possibility of any spatialization of the theory. The essence of location theory, according to Starrett, is tied to the notion of increasing returns to scale, which in turn imply non-convexity.

In the following analysis the theory of general competitive equilibrium is re-examined with one vital assumption relaxed: a commodity at different locations will henceforth be considered the same commodity. Assuming that the economy has a particularly simple spatial structure, the following question is posed. Under what circumstances can a general equilibrium exist in this economy when commodities are not spatially specified? It will be demonstrated that the requirements for equilibrium are in general very restrictive and occasionally preposterous.

4. GENERAL EQUILIBRIUM IN A SPATIAL ECONOMY

4.1. The Spatial Structure of The Economy

For expository purposes the geography of the model economy is assumed to be extremely simple. All firms are concentrated in a single urban centre, the "city", which is also the exclusive location for trading and selling commodities. There is a finite population of consuming households distributed in some arbitrary, fixed pattern around this centre. Each household occupies a single residential location, but any given residence may be the home of more than one household.

In this economy commodities differ only in their physical and temporal characteristics. Thus, a given good of a specified vintage is considered to be identical at all locations in the region. This is the critical departure from established thinking.

In general, households purchase goods and services at the centre, and firms purchase inputs from other firms at the centre as well as from households at their places of residence. (The latter class of inputs consists, for example, of labour services.) Any location in this region is characterized by the effort required to supply goods and services at that location. This effort is, of course, manifested in transportation outlays. Rather than considering transportation as a spatially continuous transformation of prices (which leads to analytically intractable difficulties), it is treated in the following manner.

Associated with supplying a unit of some commodity x_i at a location s_i is an expenditure of another commodity t_i, called "transportation". Assuming that p_t is the spatially invariant unit price of transportation, then, if x_i units of the commodity are consumed, the total outlay on transportation is $p_t\ t_i\ x_i$. It is readily seen that to supply a vector $\underline{x}$ of commodities at s_i involves a transportation outlay of $p_t\ \underline{t}\ \underline{x}$. Hence, if we are considering the consumption plan of a household at some residential location, the $\underline{t}\ \underline{x}$ corresponding to that location enters as an element of the plan. Similarly, the input-output vectors of the firms will have an additional input, namely $\underline{t}\ \underline{x}$, which in their case indicates the transportation required to bring labourers, etc. from their homes. The constant p_t occupies a natural place in the price vector corresponding to $\underline{t}\ \underline{x}$ in the production and consumption plans.

With this characterization of location the assumptions of topographic uniformity, geometric regularity of settlements, etc. which are frequently employed in spatial analysis may be dispensed with. The essence of the approach taken here is that every location is identified by the quantity of the commodity "transportation" required to supply a given set of commodities there. Alternatively, one may say that a given commodity differs at various locations only in the input of transportation it demands at each.

Locations in the economy are designated s_i, and the set $\{s_i\} = S$ denotes the spatial extent of the economy.[1]

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[1] In the following sections, elements of convex analysis are occasionally introduced without discussion. Elaboration on these topics is available in the standard texts by Rockafellar (1970), Stoer and Witzgall (1970), Roberts and Varberg (1973), and Valentine (1964). Mangasarian (1969) also covers much of this material.

4.2. Production in The Spatial Economy

All producers are concentrated in a single location. Each obtains inputs from the other producers, and at least some of them receive inputs from the household sector. It is assumed that a producer who employs a transported input pays the entire cost of conveying it from its source to the production centre. Consequently, there is associated with any input-output vector $\underline{v}$ a relationship indicating the minimum outlay on transportation required to obtain the input components of $\underline{v}$.

Definition 4.2.1.

Let $R[\underline{v}]$ *denote the minimum transportation outlay corresponding to a production plan* $\underline{v}$. *Since this relationship associates an input-output vector with a single commodity, then by notational convention:* $R\colon \mathbb{R}^m \to \mathbb{R}$.

Let T denote an actual consumption of transportation. Thus elements of the production set are indicated as $(\underline{v},T)$.

Definition 4.2.2.

The production possibility set Y_f^s *in the spatial economy is*

$$Y_f^s = \{(\underline{v},T) \mid \underline{v} \in Y_f,\ T \leq R[\underline{v}]\}.$$

In the following analysis the implications of Assumptions 3.1 - 3.7 when Y_f^S replaces Y_f are discussed. In addition, the properties of R are examined.

Theorem 4.2.1.

R *is concave.*

Proof: By Assumption 3.3, Y_f^S is convex. Let $\underline{v}^1, \underline{v}^2 \in Y_f$. Then $(\underline{v}^1, R[\underline{v}^1]), (\underline{v}^2, R[\underline{v}^2]) \in Y_f^S$. By the convexity of Y_f^S, $((1-\lambda)\underline{v}^1 + \lambda\underline{v}^2, (1-\lambda)R[\underline{v}^1] + \lambda R[\underline{v}^2]) \in Y_f^S$, for $0 \leq \lambda \leq 1$. Therefore, from the definition of Y_f^S, $R[(1-\lambda)\underline{v}^1 + \lambda\underline{v}^2] \geq R[\underline{v}^1] + \lambda R[\underline{v}^2]$ for $0 \leq \lambda \leq 1$. Therefore R is concave.||

Theorem 4.2.2.

$R[\underline{v}]$ *is continuous on the interior of* Y_f.

Proof: Since $R[\underline{v}]$ is concave, $-R[\underline{v}]$ is convex on Y_f. It can be shown that a convex function is continuous if its domain is open and convex. [see Theorem 4.3.2.] Since Y_f is convex, its interior is convex and, then $-R[\underline{v}]$ is continuous on its interior. Clearly, $R[\underline{v}]$ is also continuous on the interior of Y_f.||

For all firms which do not produce transportation R[v] represents the gross outlay on transportation required to obtain the vector $\underline{v}$. The following theorem applies to all such firms.

Theorem 4.2.3.

The function $R[\underline{v}]$ *has the form*

$$R: \; \mathbb{R}^m \to 0.$$

Proof: By Assumption 3.1, the production plan ($\underline{v} = 0$, $T = 0$), is possible. Choose any $\underline{v} \in Y_f$. By constructing the plan $(\underline{v},0)$, it is clear that $(\underline{v},0) \leq (\underline{0},0)$, and therefore, by Assumption 3.4, $(\underline{v},0) \in Y_f^S$. Since by definition, the range of R is the nonpositive real line, then the minimum transportation outlay associated with any production vector cannot be less than zero. Thus if $(\underline{v},0) \in Y_f^S$, then $R[\underline{v}] = 0$. But $\underline{v}$ is arbitrary, hence for any $\underline{v} \in Y_f$, $R[\underline{v}] = 0$.||

This result is, of course, preposterous, but it does indicate in somewhat spectacular fashion the danger of assuming that general equilibrium theory translates into a spatial theory as a simple consequence of re-interpretation. Once a reasonable spatial structure, however simple, is admitted, then the plausibility of axioms such as that of free disposal can no longer be sustained. The essentially spatial property of a location in the economy is the cost involved in reaching it from any other location. In the instance of a firm located in a city, the distinctive feature of a labour service obtained from a specific household vis-a-vis all others is the transportation involved in bringing members of that household to the plant site. In traditional general equilibrium analyses it is assumed that all such inputs are accessible at no cost. Thus a producer can, in a technical sense, employ an arbitrarily large quantity of any input without producing any goods or services. Having introduced space in a meaningful fashion (i.e. having attached a transportation cost to transported inputs) this possibility no longer exists. To employ any quantity of some input requires, again in a technical sense, a quantity of another input, namely transportation. Thus a firm may still dispose of arbitrarily large amounts of any input, but the constraints imposed by space oblige him to dispose simultaneously of prescribed amounts of transportation.

Thus it is seen that the axiomatic foundations of general equilibrium theory require restructuring if they are to retain credibility in the spatial context. On this particular issue the motivation behind the assumption of free disposal is examined, and a suitable modification is proposed and justified. On examination of the Arrow-Hahn model it is seen that this assumption is used in analysis relating to the continuity of supply correspondences and the differentiability of profit functions. The reason for its employment in this area is that these theorems require that the production possibility set have full dimensionality, a property which is guaranteed once the nonpositive orthant is admitted. It can be shown, however, that a production possibility set which is constrained by a nontrivial transportation function maintains full dimensionality. First we note the following:

Remark:

Since $R[\underline{v}]$ *is concave, then from the properties of concave functions it* *is known that*

$$\{\underline{v} | R[\underline{v}] = -\infty\} = \emptyset .$$

Given that

$$Y_f^s = \{(\underline{v},T) | \underline{v} \in Y_f, \; T \leq R[\underline{v}]\} ,$$

then Y_f^s *is nonempty and there exists at least one* $(\underline{v},T) \in Y_f^s$ *such that* $T < 0$.

Theorem 4.2.4.

The dimensionality of the production possibility set in the spatial economy is not reduced by the introduction of a nontrivial minimum transportation function.

Proof: Let $Y_f^{ss} = \{(\underline{v},T) | \underline{v} \in Y_f, \; T \in \mathbb{R}_-\}$ represent the production possibility set in the spatial economy when minimum transportation outlays are everywhere zero. Y_f^{ss} has dimension $(m + 1)$ by virtue of its containing the nonpositive orthant of $\mathbb{R}^{m+1}$. On the other hand the dimension of Y_f^s is equal to the dimension of its affine hull where

$$\text{aff } Y_f^s = \{w_1(\underline{v}^1,T^1) + w_2(\underline{v}^2,T^2) + \ldots + w_m(\underline{v}^m,T^m) | (\underline{v}^i,T^i) \in Y_f^s, \; \sum_i w_i = 1\}.$$

Since Y_f^S is nonempty, the aff Y_f^S is nonempty and has the dimension of the subspace L which is parallel to it. By definition

$$L = \{(\underline{v},T) - (\underline{v}'.T') | (\underline{v},T), (\underline{v}'T') \in \text{aff } Y_f^S\}$$

where $(\underline{v}',T')$ is some arbitrary, but fixed, element in aff Y_f^S. Since there exists a $(\underline{v},T) \in Y_f^S$ such that $T \neq 0$, then L has dimension $(m + 1)$. Hence Y_f^S also has dimension $(m + 1)$.||

Theorem 4.2.5.

$R[\underline{v}]$ *is linear if and only if the production possibility set* Y_f^S *is a convex cone.*

Proof: This parallels the proof of Corollary 4.3.1.||

No formal discussion is required to show that the assumptions relating to the impossibility of free production has no specific implications for the behaviour of the minimum transport cost function. Similarly, since

$$\{\underline{v} | R[\underline{v}] = -\infty\} = \emptyset$$

as a consequence of concavity, then the introduction of transportation does not eliminate the possibility of the society producing some positive amount of any commodity. (If the possibility $R[\underline{v}] = -\infty$ were admitted, then in this case society, having by assumption only a finite quantity of resources, could not produce any good or service whose production involved transported inputs).

Thus far it has been established that the minimum transportation function $R[\underline{v}]$ facing a given firm is concave and continuous in the space of input-output vectors. Bearing in mind the fact that inputs are designated as negative quantities, then the concavity of the transportation function implies nonincreasing returns to scale in the cost of carrying inputs from residential locations to the centre. This property does not necessarily run counter to empirical evidence on transportation outlays. It is, of course, well known that shipping a specific commodity between two fixed locations leads in almost all documented cases to decreasing unit costs of transportation as the quantity shipped increases. However, $R[\underline{v}]$ refers to the cost of obtaining variable quantities of a set of inputs from a finite set of spatially distinct sources. For a particular firm f the transported elements of an input-output plan $\underline{v}_f$ may therefore be obtained from numerous sources. Corresponding to each source is a function describing the minimum expenditure required to convey variable quantities of $\underline{v}_f$ to the production centre. The function $R[\underline{v}_f]$ is composed in a rather complex fashion from these individual transportation functions. Given a set of specified quantities of all inputs and outputs, denoted $\underline{v}_f$, the firm may distribute its importations from the residential sector in numerous ways. Associated with every such distribution is a resultant aggregate minimum transportation cost. $R[\underline{v}_f]$ refers to the smallest of these minima. This function is therefore by definition the solution to the following problem:

$$\text{minimize} \qquad R_1[\underline{v}_1] + R_2[\underline{v}_2] + \ldots R_n[\underline{v}_n]$$

$$\text{subject to} \qquad \underline{v}_1 + \underline{v}_2 + \ldots \underline{v}_n = \underline{v} .$$

where $R_i[\underline{v}_i]$ denotes the minimum transportation function corresponding to residence s_1.

The function $R[\underline{v}]$ is consequently given by

$$R[\underline{v}] = \sup \{R_1[\underline{v}_1] + R_2[\underline{v}_2] + \ldots R_n[\underline{v}_n] \,|\, \underline{v}_1 + \underline{v}_2 + \ldots \underline{v}_n = \underline{v}\}$$

It has been established that general equilibrium theory requires that $R[\underline{v}]$ be a continuous and concave function. The question now arises as to the properties (if any) of the transportation functions relating to individual household locations which the concavity and continuity of $R[\underline{v}]$ imply. This appears to be a problem of considerable intricacy, and it appears that the constraints on $R[\underline{v}]$ permit a wide range of behaviour in its component functions. It is possible, however, to eliminate at least one configuration of these functions. From empirical evidence one might reasonably hypothesize that for every residential location s_i the corresponding minimum transportation function exhibits increasing returns to scale, i.e. $R_i[\underline{v}_i]$ is convex for all i. It is now shown that if all the components are convex functions, then $R[\underline{v}]$ is also convex.

Theorem 4.2.6.

If $R_i[\underline{v}_i]$ *is convex for all* i, *then* $R[\underline{v}]$ *is convex.*

Proof: By definition $R[\underline{v}] = \sup\{R_1[\underline{v}_1] + R_2[\underline{v}_2] + \ldots + R_n[\underline{v}_n] \mid \sum_i \underline{v}_i = \underline{v}\}$.

Suppose there are only two functions, i.e. $R[\underline{v}] = \sup\{R_1[\underline{v}_1] + R_2[\underline{v} - \underline{v}_1]\}$.

Let $\lambda \geq 0$. Then we can substitute $\lambda\underline{x} + (1 - \lambda)\underline{y} = \underline{v}$, and $\lambda\underline{p} + (1 - \lambda)\ \underline{q} = \underline{v}_1$.

The supremum that ranges over the points $\underline{v}_1$ of an open convex set K may be replaced by a supremum which ranges over the pair $(\underline{p},\underline{q})$ of $K \times K$. Therefore

$$\begin{aligned}
R[\lambda\underline{x} + (1 - \lambda)\ \underline{y}] &= \sup\{R_1[\lambda\underline{p} + (1-\lambda)\underline{q}] + R_2[\lambda\underline{x} + (1 - \lambda)\ \underline{y} - \lambda\underline{p} - \\
&\qquad (1 - \lambda)\ \underline{q}]\} \ . \\
&= \sup\{R_1[\lambda\underline{p} + (1 - \lambda)\ \underline{q}] + R_2[\lambda(\underline{x} - \underline{p}) + \\
&\qquad (1 - \lambda)(\underline{y} - \underline{p})]\} \\
&\leq \sup\{\lambda R_1[\underline{p}] + (1 - \lambda)\ R_1[\underline{q}] + \lambda R_2[\underline{x} - \underline{p}] + \\
&\qquad (1 - \lambda)\ R_2[\underline{y} - \underline{q}]\} \\
&= \lambda \sup\{R_1[\underline{p}] + R_2[\underline{x} - \underline{p}]\} + \\
&\qquad (1 - \lambda) \sup\{R_1[\underline{q}] + R_2[\underline{y} - \underline{q}]\} \ .
\end{aligned}$$

That is

$$R[\lambda \underline{x} + (1 - \lambda)\underline{y}] \leq \lambda R[\underline{x}] + (1 - \lambda) R[\underline{y}]$$

which implies that R is convex. By induction the argument may be extended to the n-dimensional case.||

Thus it is seen that the concavity imposed on $R[\underline{v}]$ by the axioms of general equilibrium theory removes the possibility that the individual transportation functions $R_i[\underline{v}]$ may all be convex, except in the far-fetched situation where all transportation functions are linear and possess the same gradient. However, it is still unresolved as to whether convexity of some of the component functions is compatible with concavity of $R[\underline{v}]$. In conclusion therefore, the contradiction of the required behaviour of $R[\underline{v}]$ by the observed behaviour of transportation functions can not be demonstrated decisively. On the other hand, the reasonable approximation to reality represented by nonstrict convexity of the transportation functions for all residences is not permitted.

4.3. Consumption in the Spatial Economy

The approach of this section is similar to that employed in studying production. However, the analysis is simplified by the fact that the consumer obtains goods and services from a single source, the urban centre. Thus, for any $s_i \in S$, there corresponds to each commodity a unique relationship expressing the minimum effort required to carry that commodity to that residential location from the centre.

Definition 4.3.1.

To any commodity x_i *there corresponds, for* $s_i \in S$*, a function* $g_i = f_i[x_i]x_i$*, known as the minimum transportation function, where* $f_i[x_i]$ *denotes the minimum amount of the commodity transportation required to convey a unit of* x_i *from* s_0 *to* s_i*.*

Assumption 4.3.1.

The minimum transportation function $g_i[x_i]$ *associates a nonnegative outlay of the commodity transportation with any nonnegative quantity of* x_i*. i.e.*

$$g_i : \mathbb{R}_+ \to \mathbb{R}_+$$

Consider now Assumption 3.8. Under this assumption the production possibility set for commodity i would be defined as $\{(x_i,t_i) | x_i,t_i \geq 0\}$, where t_i denotes the level of consumption of transportation. This definition no longer suffices in the spatial context, as may be seen by considering the case where $g_i[x_i] = \alpha x_i$, $\alpha > 0$. Under Assumption 3.8 the consumption plan $(x_i = 1, t_i = 0)$ is feasible. However, for $x_i = 1$, $g_i[x_i] = \alpha$, therefore $t_i < g_i[x_i]$ and $(x_i = 1, t_i = 0)$ is not possible. The consumption possibility set must be redefined to eliminate all such plans.

Definition 4.3.2.

$$X_h^s = \{(x_1,t_1),(x_2,t_2), \ldots (x_n,t_n) | x_i, t_i \in \mathbb{R}_+ ,$$
$$g_i[x_i] \leq t_i, \; i = 1, \ldots n\}$$

is the consumption possibility set for household h in the spatial economy.

By Assumption 3.8, X_h^s is closed and convex. The implications of closure and convexity of X_h^s on the behaviour of transportation functions is now examined. First, the consumption possibility set for household h of a specific commodity i is defined.

Definition 4.3.3.

$$X_{ih}^s = \{(x_i,t_i) | x_i, t_i \in \mathbb{R}_+, t_i \geq g_i[x_i]\} .$$

Lemma 4.3.1.

If X_{ih}^s is convex, then $g_i[x_i]$ is convex.

Proof: By definition, the epigraph of $g_i[x_i]$ defined on $\mathbb{R}_+$ is epi $g_i[x_i] = \{(x_i,t_i) | x_i, t_i \in \mathbb{R}_+, \; g_i[x_i] \leq t_i\}$ obviously, epi $g_i[x_i] \equiv X_{ih}^s$. Let $x_i^1, x_i^2 \in \mathbb{R}_+$. Then $(x_i^1, g_i[x_i^1]), (x_i^2, g_i[x_i^2]) \in$ epi $g_i[x_i]$. By convexity of epi $g_i[x_i]$ it follows that $((1-\lambda)x_i^1 + \lambda x_i^2, (1-\lambda)g_i[x_i^1] + \lambda g_i[x_i^2] \in$ epi $g_i[x_i]$ for $0 \leq \lambda \leq 1$. From the definition of the epigraph,

$(1 - \lambda)\, g_i[x_i^1] + \lambda\, g_i[x_i^2] \geq g_i[(1 - \lambda)x_i^1 + \lambda\, x_i^2]$, therefore $g_i[x_i]$ is convex on $\mathbb{R}_+$. ||

Theorem 4.3.1.

The functions $g_1, g_2, \ldots g_n$, *are convex.*

Proof: For any $((x_1,t_1), (x_2,t_2), \ldots (x_n,t_n)), ((x_1',t_1'), \ldots (x_n',t_n')) \in X_h^s$, it follows from Assumption 3.8 that $\lambda((x_1,t_1), (x_2,t_2), \ldots (x_n,t_n)) + (1 - \lambda)\, ((x_1',t_1'), (x_2',t_2'), \ldots (x_n^1,t_n^1)) \in X_h^s$. Therefore $\lambda(x_1,t_1) + (1 - \lambda)(x_1',t_1'), \lambda(x_2,t_2) + (1 - \lambda)(x_2',t_2'), \ldots \lambda(x_n,t_n) + (1 - \lambda)(x_n',t_n') \in X_h^s$. Consequently $\lambda(x_1,t_1) + (1 - \lambda)(x_1',t_1') \in X_{1h}^s$, $\lambda(x_2,t_2) + (1 - \lambda)\, (x_n',t_n') \in X_{1h}^s$, etc. But $(x_1,t_1), (x_1',t_1') \in X_{1h}^s$, $(x_2,t_2), (x_2',t_2') \in X_{1h}^s$, and so on. Therefore X_{ih}^s is convex, $i=1, \ldots n$. By Lemma 4.3.1., $g_i[x_i]$ is convex, $i = 1, \ldots n$.

The following corollary is immediate.

Corollary 4.3.1.

g_1 *is linear if and only if the consumption possibility set is a convex cone.*

Proof: Let $g_i[x_i]$ be linear, and let $(x_i^1, t_i^1) \varepsilon X_{ih}^s$. Consider $(\beta x_i, \beta t_i)$, where $\beta \geq 0$. Then $t_i \geq g[x_i]$ implies $\beta t_i \geq \beta\ g_i[x_i] = g[\beta x_i]$. Hence $(\beta x_i, \beta t_i) \varepsilon X_{ih}^s$, and X_{ih}^s is therefore a cone. Now consider (x_i^1, t_i^1), $(x_i^2, t_i^2) \varepsilon X_{ih}^s$. For $0 < \lambda < 1$, $\lambda(x_i^1, t_i^1)$, $(1 - \lambda)(x_i^2, t_i^2) \varepsilon X_{ih}^s$. Clearly $\lambda(x_i^1, t_i^1) + (1 - \lambda)(x_i^2, t_i^2) = (\lambda x_i^1 + (1 - \lambda)\ x_i^2,\ \lambda t_i^1 + (1 - \lambda)\ t_i^2) \varepsilon X_{ih}^s$ if and only if $\lambda t_i^1 + (1 - \lambda)\ t_i^2 \geq g_i[\lambda x_i^1 + (1 - \lambda)\ x_i^2] = \lambda g_i[x_i^1] + (1 - \lambda)\ g_i[x_i^2]$. But this condition is satisfied by definition since $t_i^1 \geq g[x_i^1]$ and $t_i^2 \geq g_i[x_i^2]$. Therefore $\lambda(x_i^1, t_i^1) + (1 - \lambda)(x_i^2, t_i^2) \varepsilon X_{ih}^s$, and X_{ih}^s is convex.

Let X_{ih}^s be a convex cone. Then $(x_i^1,\ g_i[x_i^1]) \varepsilon X_{ih}^s$ implies $(\alpha x_i^1,\ \alpha g_i[x_i^1]) \varepsilon X_{ih}^s$. But $(\alpha x_i^1,\ g_i[\alpha x_i^1]) \varepsilon X_{ih}^s$, therefore $\alpha g_i[x_i^1] \geq g_i[\alpha x_i^1]$. Given $(x_i^1,\ \frac{1}{\alpha}\ g_i[\alpha x_i^1]) \varepsilon X_{ih}^s$, then $\frac{1}{\alpha}\ g_i[\alpha x_i^1] \geq g_i[x_i^1]$, which implies $g_i[\alpha x_i^1] \geq \alpha g_i[x_i^1]$. Consequently $g_i[\alpha x_i^1] = \alpha g_i[x_i^1]$. Now let $(x_i^2,\ g_i[x_i^2]) \varepsilon X_{ih}^s$, where $x_i^2 = x_i^1$, and $a \geq 0$. Since X_{ih}^s is a convex cone, then $(x_i^1,\ g_i[x_i^1])$, $(x_i^2,\ g_i[x_i^2]) \varepsilon X_{ih}^s$ imply $(\lambda x_i^1 + (1 - \lambda)\ x_i^2,\ \lambda g_i[x_i^1] + (1 - \lambda)\ g_i[x_i^2]) \varepsilon X_{ih}^s$. Therefore $g[\lambda x_i^1 + (1 - \lambda)\ x_i^2] \leq \lambda g_i[x_i^1] + (1 - \lambda)\ g_i[x_i^2]$. Given $X_i^2 = a\ x_i^1$, then $g_i[\lambda x_i^1 + (1 - \lambda)\ x_i^2] =$

$$g_i[\lambda x_i^1 + (1-\lambda)\, a\, x_i^1] = g_i[x_i^1(\lambda + (1-\lambda)a)] = ((\lambda + (1-\lambda)a)\; g_i[x_i^1]\ .$$

Similarly $\lambda g_i[x_i^1] + (1-\lambda)\, g_i[x_i^2] = (\lambda + (1-\lambda)\, a)\, g_i[x_i^1]$, Therefore

$g_i[\lambda x_i^1 + (1-\lambda)\, x_i^2] = \lambda g_i[x_i^1] + (1-\lambda)\, g_i[x_i^2]$ and hence $g_i[x_i]$ is linear.||

Assumption 3.9 implies other significant properties of the transportation function. Defining Γ as the interior of $\mathbb{R}_+$, then continuity of $g_i[x_i]$ on Γ may be established. First, however, two propositions are stated. Their proofs, being widely available (e.g. Rockafellar (1970), Stoer and Witzgall (1970)), are omitted.

Lemma 4.3.2.

For any convex function $g_i[x_i]$ *and any* $\alpha \in [-\infty, +\infty]$, *the level set* $\{x_i | g_i[x_i] \leq \alpha\}$ *is convex.*

Lemma 4.3.3.

A set $W \subset \mathbb{R}^n$ *is convex if and only if for each integer* $m \geq 1$, *every convex combination of* m *points of* W *is in* W.

Theorem 4.3.2.

The function g_i *is continuous on* Γ.

Proof: This is adapted from Fleming (1965). Let $x_i^1 \in \Gamma$ and let α be the distance from x_i^1 to the closest point in $\mathbb{R}$ not in Γ. Let C be a set such that $C = \{x_i | x_i \in \mathbb{R},\ -\delta \leq x_i - x_i^1 \leq \delta\}$. Letting $\delta < \alpha$

implies $C \subset \Gamma$. Let V denote the set of vertices of C, and let

$$\beta = \max_{x_i \in V} g_i[x_i]$$

By Lemma 4.3.2. the set $\Lambda = \{x_i | x_i \in \Gamma \ , \ g_i[x_i] \leq \beta\}$ is convex. Since C is the convex hull of V and $V \subset \Lambda$, then by Lemma 4.3.3, $C \subset \Lambda$. Let x_i be any point such that $0 < ||x_i - x_i^1|| < \delta$, and let $x_i^2 + u$, $x_i^1 - u$ be on the line through x_i^2 and x_i. Hence we can write $x_i = x_i^1 + \lambda u = \lambda(x_i^1 + u) + (1 - \lambda)\, x_i^1$, $x_i^1 = x_i - \lambda u = x_i + \lambda(x_i^1 - n) - \lambda x_i^1 = \frac{1}{1+\lambda}\, x_i + \frac{\lambda}{1+\lambda}\,(x_i^1 - n)$ where $\lambda = ||x_i - x_i^1||\, \frac{1}{\sigma}$.

Since $g_i[x_i]$ is convex,

$$g_i[x_i] \leq \lambda g_i[x_i^1 + n] + (1 - \lambda)\, g_i[x_i^1] \leq \lambda\beta + (1 - \lambda)\, g_i[x_i^1],$$

$$g_i[x_i^1] \leq \frac{1}{1+\lambda}\, g_i[x_i] + \frac{\lambda}{1+\lambda}\, g_i[x_i^1 - n] \leq \frac{g_i[x_i] + \lambda\beta}{1+\lambda}$$

which together imply that

$$|g_i[x_i] - g_i[x_i^1]| \leq (\beta - g_i[x_i^1])\, \frac{1}{\sigma}||x_i - x_i^2||.$$

Thus for any $\varepsilon > 0$ it follows that $|g_1[x_1] - g_1[x_1^1]| < \varepsilon$ for all x_1 that satisfy $(\beta - g_1[x_1^1])||x_1 - x_1^1|| < \delta$, hence $g_1[x_1]$ is continuous at x_1^1 .

Therefore $g_1[x_1]$ is continuous on the interior of $\mathbb{R}_+$.||

Theorem 4.3.3.

g_1 *possesses a left-hand and a right-hand derivative everywhere on* Γ.

Proof: This follows immediately from the fact that g_1 is a real-valued function of a real variable.||

It should be noted that the left-hand and right-hand derivatives are not necessarily equal everywhere on Γ. Consequently, g_1 is not necessarily differentiable on Γ. The following theorem is also a direct consequence of the convex and real-valued properties of g_1.

Theorem 4.3.4.

The right-hand and left-hand derivatives of g_1 *are monotonically increasing functions of* x_1.

Consider next Assumption 3.9, which states that for a given household there exists a possible consumption plan which is "less than" the household's initial endowment of goods and services. Given that we are considering a particular consumer h at some $s_1 \in S$, then it is reasonable to assume that the area of residential land h is endowed with is at least the minimum necessary for survival at s_1. Similarly, given an initial location at s_1, it is also reasonable to assume that h possesses the minimum transportation necessary to attain the "smallest" possible plan in his consumption set. Letting $\hat{x}_{hs}$, $\hat{t}_h$ denote these two minima respectively, then Assumption 3.9 may be rewritten as follows:

There exists a possible consumption vector $\bar{\bar{x}}_h \in X_h$ *such that*

$\bar{\bar{t}}_h \lesssim \bar{t}_h$; $\bar{\bar{t}}_h < \bar{t}_h$ if $\bar{t}_h > \hat{t}_h$:

$$\bar{\bar{x}}_{hs} \leq \bar{x}_{hs};\ \bar{\bar{x}}_{hs} < \bar{x}_{hs} \text{ if } \bar{x}_{hs} > \hat{x}_{hs};$$

$$\bar{\bar{x}}_{hi} \leq \bar{x}_{hi};\ \bar{\bar{x}}_{hi} < \bar{x}_{hi} \text{ if } \bar{x}_{hi} > 0,\ i \neq s\ .$$

These modifications render Assumption 3.9 plausible in the context of the spatial economy. Furthermore, it is easily seen that these amendments do not in any qualitative way affect the assumption, nor does the assumption require any particular properties of transportation functions, etc.

Assumption 3.10 characterizes the budget constraint. The important feature of this Assumption is that consumption is bounded by $\underline{p}\ \underline{x} \leq M_h$. Given that the vector $\underline{p}$ is constant, then the set of feasible consumption plans (i.e. plans which involve a cost less than or equal to M_h) is a closed and convex set. However, in the spatial economy, expenditures may be separated into those related solely to the purchase of commodities at mill prices, and those related to outlays or transportation. That is for any $s_i \in S$, the budget constraint is given by

$$M_h[\underline{x}] \geq \underline{p}\ \underline{x} + p_t\ \underline{t}[\underline{x}]$$

where $t[\underline{x}]$ denotes the actual expenditure on transportation involved in obtaining $\underline{x}$. The boundary of feasible consumption set is defined by the minimum possible values of the commodities and by the above constraint when the equality holds. The question now arises concerning the properties $t[x]$ must possess to guarantee convexity of $M_h[x]$ and hence of the feasible consumption set.

Definition 4.3.5.

Following Hartman (1959), *a function* f_1 *defined on an open convex set* D *will be called a* d.c. *function if there exists a pair of convex functions* f_2, f_3 *on* D *such that* $f_1 = f_2 - f_3$.

Convexity of $M_h[\underline{x}]$ implies $p_t \; t[\underline{x}] = M_h[\underline{x}] - \underline{p}\,\underline{x}$ is d.c. Hence $t[\underline{x}]$ is d.c. It is important to note that a d.c. function is not necessarily convex. More specifically, it can be shown that on an arbitrary closed interval of its finite domain a d.c. function satisfies the Lipschitz condition

$$\|t[\underline{x}^1] - t[\underline{x}^2]\| \leq Q\|\underline{x}^1 - \underline{x}^2\|$$

where $\underline{x}^1, \underline{x}^2 \; \varepsilon \; [a,b]$, a closed interval on the interior of $\{\underline{x} | t[\underline{x}] < +\infty\}$ and Q is a constant. Natanson (1955) shows that functions satisfying this condition have finite variation. Such a restriction, however, still permits the actual transportation function to exhibit the qualitative structure which empirical evidence suggests, i.e. concavity.

Finally, Assumption 3.11, which concerns the nature of preferences, remains unaltered. The imposition of a spatial structure does not affect the process by which a consumer derives satisfaction and services. The impact of location is merely to constrain the choice of consumption plans open to an individual, and consequently the repercussions of spatializing general equilibrium theory are felt only in those assumptions which relate to physical possibilities facing the consumer.

4.4. Conclusions

To recapitulate, the motivation for this research was two-fold. First, the absence of urban models which were general (in the sense of contemporary economics) generated an interest in examining the interaction of production and consumption in a spatial context. Secondly, the unsatisfactory treatment of locational factors in the general equilibrium literature presented an enticement to investigate the implications of the theory when placed in a spatial setting. Both purposes were accomodated by rejecting the key assumption of general equilibrium theory that a specific commodity may be characterized as different at distinct locations. Locations in the economy were therefore identified by the transportation required to ship commodities to and from a unique industrial centre. For operational purposes transportation was considered to be a spatially-dependent commodity variable with a constant price. This interpretation preserved the differentiating role of transportation while facilitating the use of general equilibrium theory's technical structure. Questions concerning the impact of space were thus resolved into questions surrounding the nature of transportation functions.

It was demonstrated that the axioms of general equilibrium theory imply stringent constraints on the behaviour of transportation functions, and furthermore that the required behaviour contradicts empirical evidence. On the consumption side the axioms imply that as the consumption of a good or service increases the minimum transportation necessary to supply a unit of that good or service increases. This is patently untrue. Empirical research has established that unit transportation costs diminish as the quantity shipped increases. The closest approximation to reality permitted by general equilibrium theory is the assumption of linear transportation relationships for all commodities. On the production side the implications are even less plausible. The literal interpretation of the axiom of free disposal requires that the cost of delivering any input from a residential location to the production centre be zero. When this axiom was replaced by one which performs the same analytical purpose while admitting non-trivial transportation costs it was shown that even then an unacceptable transportation structure was implied. As in consumer theory the axioms require increasing unit transportation costs. Similarly, the adoption of linear transportation functions is the best compromise with reality which the axiomatic system can allow. Some exploratory analysis appears to indicate that even the linearity assumption may in many instances be insufficient to accomodate the technical structure.

Critical weaknesses, from the geographic point of view, have therefore been exposed in the foundations of the general equilibrium model. In a spatial setting it requires assumptions which are confounded by empirical evidence.

The analysis, together with the results, is highly provocative regarding new directions and extensions of the research. First, refinement and clarification of the results indicated above is called for. For example, the implications for individual input-output transportation functions of concavity of their pointwise supremum is still unresolved.

Secondly, the axioms of general equilibrium theory must be examined for the possibility of tailoring them to suit the spatial context. However, as most of the difficulties apparent in the spatial analysis arise from nonconvexities, and as the latter have not yet been successfully accomodated in standard general equilibrium models, this appears to be a monumental task. Nonetheless it may be possible to effect some significant reconstruction, as recorded in the instance of the free disposal axiom.

Thirdly, irrespective of any success in modifying the axioms, the restriction to a single industrial centre and the prohibition of movement on the part of consumers and producers must be lifted. The concept of equilibrium would then be extended to include locational choice as well as consumption and production decisions.

Finally, the proof of existence of an equilibrium in a spatial economy is only the first stage of analysis. Once the existence of an equilibrium state is established there immediately appear questions relating to its uniqueness, the stability of equilibrium, and its optimal properties. In conclusion, the above research represents merely the prelude to a thorough understanding of equilibrium and the spatial economy.

BIBLIOGRAPHY

Alao, N. (1974) "An Approach to Intraurban Location Theory", *Economic Geography*, 50, 59-69.

Alonso, W. (1960) "A Theory of the Urban Land Market", *Papers of the Regional Science Association*, 6, 149-157.

(1964) *Location and Land Use*. Harvard University Press, Cambridge, Mass.

Amson, J.C. (1972) "The Dependence of Population Distribution on Location Costs", *Environment and Planning*, 4, 163-182.

Arrow, K.J. (1968) "Economic Equilibrium", *International Encyclopaedia of the Social Sciences*, Macmillan, London, 376-386.

Arrow, K.J. and F.H. Hahn (1971) *General Competitive Analysis*. Holden Day, San Francisco.

Averous, C.P. and D.B. Lee (1973) "Land Allocation and Transportation Pricing in a Mixed Economy", *Journal of Regional Science*, 13, 173-185.

Beckmann, M.J. (1969) "On the Distribution of Urban Rent and Residential Density", *Journal of Economic Theory*, 1, 60-67.

Capozza, D. (1973) "Subways and Land Use", *Environment and Planning*, 5, 555-576.

Casetti, E. (1971) "Equilibrium Land Values and Population Densities in an Urban Setting", *Economic Geography*, 47, 16-20.

Debreu, G. (1959) *Theory of Value*. Wiley, New York.

(1962) "New Concepts and Techniques for Equilibrium Analysis", *International Economic Review*, 3, 257-273.

Delson, J.K. (1970) "Correction of the Boundary Condition in Beckmann's Model on Urban Rent", *Journal of Economic Theory*, 2, 314-318.

Enke, S. (1951) "Equilibrium among Spatially Separated Markets: Solution by Electric Analogue", *Econometrica*, 19, 40-47.

Fleming, W.H. (1965) *Functions of Several Variables*. Wiley, New York.

Gale, D. (1955) "The Law of Supply and Demand", *Mathematica Scandinavia*, 3, 155-169.

Hahn, F.H. (1973) The Winter of Our Discontent", *Economica*, 40, 322-330.

Hartman, D. (1959) "On Functions Representable as the Difference of Convex Functions", *Pacific Journal of Mathematics*, 9, 707-713.

Hochman, O. and D. Pines (1971) "Competitive Equilibrium of Transportation and Housing in the Residential Ring of an Urban Area", *Environment and Planning*, 3, 51-62.

Hoover, E.M. (1968) "The Partial Equilibrium Approach", in *International Encyclopedia of the Social Sciences*, D.L. Sills, ed., Macmillard, London, 95-100.

Kornai, J. (1971) *Anti-Equilibrium: On Economic Systems Theory and the Tasks of Research.* North-Holland, Amsterdam.

Livesy, D. (1973) "Optimum City Size: A Minimum Congestion Cost Approach", *Journal of Economic Theory*, 6, 144-161.

Long, W. (1971) "Demand in Space: Some Neglected Aspects", Papers of the *Regional Science Association*, 27, 45-60.

Mangasarian, O. (1969) *Nonlinear Programming.* McGraw-Hill, New York.

McKenzie, L.W. (1959) "On the Existence of a General Equilibrium for a Competitive Market", *Econometrica*, 27, 54-71.

Menchik, M.D. (1973) "Income and Residential Location in Alonso's Model of Urban Form", *Northwestern University Studies in Geography*.

Mills, E.S. (1967) "An Aggregate Model of Resource Allocation in a Metropolitan Area", *American Economic Review*, 57, 200-209.

(1969) The Value of Urban Land" in *The Quality of the Urban Environment.* H.S. Perloff, ed., Johns Hopkins University Press, Baltimore, 231-253.

Mills, E.S. (1972a) *Urban Economics*. Scott, Foresman, Glenview, Illinois.

(1972b) *Studies in the Structure of the Urban Economy*. Johns Hopkins, Baltimore.

(1972c) "Markets and Efficient Resource Allocation in Urban Areas", *Swedish Journal of Economics*, 74, 100-117.

Montesano, A. (1972) "A Restatement of Beckmann's Model of Urban Rent and Residential Density", *Journal of Economic Theory*, 4, 329-354.

Moses, L. (1960) "A General Equilibrium Model of Production, Inter-Regional Trade, and Location of Industry", *Review of Economics and Statistics*, 42, 373-397.

Muth, R. (1961a) "The Spatial Structure of the Housing Market", *Papers of the Regional Science Association*, 7, 207-220

(1961b) "Economic Change and Rural-Urban Land Conversions", *Econometrica*, 29, 1-23.

(1969) *Cities and Housing*. University of Chicago Press, Chicago, Illinois.

Natanson, I.P. (1955) *Theory of Functions of a Real Variable*. Ungar, New York.

Nelson, R.H. (1973) "Accessibility and Rent: Applying Becker's Time-Price Concept to the Theory of Residential Location", *Urban Studies*, 10, 83-86.

Papageorgiou, G.J. (1971a) "A Generalization of the Density Gradient Concept", *Geographical Analysis*, 3, 121-127.

(1971b) "The Population Density and Rent Functions within a Multicentre Framework", *Environment and Planning*, 3, 267-282.

(1973) "The Impact of the Environment upon the Spatial Distribution of Population and Land Values", *Economic Geography*, 49, 251-256.

(1975) "Spatial Equilibrium within a Hierarchy of Centres with Distributed Incomes", manuscript, Department of Geography, McMaster University.

Papageorgiou, G.J. and E. Casetti (1971) "Spatial Equilibrium Rent ... atre Setting", *Journal of Regional Science*, Wright, C. (1971) "Residential Location in a Three Dimensional City", *Journal of Political Economy*, 79, 1378-1387.

Yamada, H. (1972) "On the Theory of Residential Location: Accessibility, Space, Leisure, and Environmental Quality", *Papers of the Regional Science Association*, 29, 125-135.