CONTINUOUS TIME APPROACH OF STOCK-FLOW CONSISTENT MACROECONOMIC MODELS

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Abstract

The aim of this thesis is to provide tools in order to simplify and extend the analysis of stock-flow consistent macroeconomic models. For models in continuous time, we will focus on stability and steady state solutions, considering constant exogenous parameters.

For models in discrete time, we will derive systems with considerable reduction in complexity and size from rather big linear systems of equations set up by previous authors into much simpler systems of difference equations. These can then also be analyzed from a continuous perspective, if their limits are taken.

We will also see how an alternative continuous layout of the models proposed by us, further reduces the number of dimensions with respect to their counterparts in discrete time, in some cases.

Aknowledgements

My journey through graduate school at McMaster University has truly been a rewarding and life changing experience. It is definitely one of the best decisions that I have ever taken. As I always do, I want to first thank God for giving me this opportunity and for also giving me the courage and the intelligence to face this challenge with the best of myself. I have grown in many different areas as a person, not only academically or professionally. I can only hope that I have been a positive influence as well to the people that I am about to mention.

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To my parents, Gaspar and Lilia

and to my siblings,

César Daniel, Liliana and Paulina.

Contents

List of Tables

List of Figures

Introduction

Following the financial crisis of 2008, the current paradigm in economics began to be heavily questioned, while the defenders of alternative theories started to gain prominence. Prior to that, lack of active intervention by the government was seen as the healthiest way to let an economy grow. After the crisis, the ideas of Keynes came back to mind, which is no surprise, given the fact that the government ended up intervening to save the system from a collapse of immense proportions, after experienced its worst shock in decades.

The crisis was initially of financial nature with banks owning assets related to subprime mortgages, but where output and productivity were not the main concern. Nevertheless, we saw the spread of this crisis to the other sectors of the economy, and it ended up affecting output and employment in such a way that still many of the world's major economies are dealing with the consequences.

Within the post-Keynesian economic school, stock-flow consistent models are of particular interest, since they offer a comprehensive framework for how the different sectors of the economy are connected and how relevant decisions and policies can be analyzed. As we will see, these models support some of the post-Keynesian main ideas, the most important of them being the intervention of the government through spending. A deficit in the government accounts is not seen as undesirable, but as an opportunity to grow the economy, and stock-flow consistent models show how an increase of government spending can lead to an increase in the GDP of the economy in a sustainable way.

Stock-flow consistent models are useful to track the monetary evolution of the balance sheets through time, whose summation of both columns and rows containing assets, liabilities and capital, by the laws of accounting add up the total capital (or net worth) of firms, sectors or entire economies. The evolution of the value of the assets, liabilities and capital in the economy depends on the setup of the transactions and flow of funds matrices. Following accounting rules, in the balance sheet, assets are entered in the matrix with a plus (+) sign and liabilities with a minus (-) sign. In consequence, the sum over the columns equals the capital of each sector.

The transactions matrices model all the flow of transactions between the sectors that arise from personal or corporate decisions, as well as contracts agreed between different sectors. Any transaction implying an income of funds is entered with a plus (+) sign, while any transaction implying a use of funds is entered with a minus (-) sign. The flow of funds matrices model how the stock of each asset is eroded or increased from such transactions, and in a similar way, funds used are entered with a minus (-) sign and funds received are entered with a plus (+) sign. The flow of funds matrices track changes made directly to the balance sheet through time. This implies that, for each sector, changes in any column (sectors) of the transactions must equal the changes the respective column in the flow of funds matrix. Stock-flow consistent models thus follow the principle that "all the money comes from somewhere and goes somewhere", which implies that each column of the transaction matrix plus its respective column in the flow of funds matrix must add up to zero. In a similar fashion, every row must add up to zero, since revenues for one sector imply the same level spending from the rest of them.

Our main focus will be on continuous time models. The main difference on the construction style is that the interest payments in discrete time take into consideration the time elapsed between the periods, while continuous time models consider instantaneous flows of payment. For the first four chapters, both continuous and discrete variants are provided.

For all the models, we analyze two cases: perfect foresight and expectations of the expected disposable income of households, which directly impacts their consumption, since households make decisions on consumption based on their expected income. For perfect foresight, the expected disposable income equals the realized disposable income, which is equal to say that households make decisions on their consumption based on their actual realized income. These expectations only play a role in the consumption function for our models.

We will also analyze steady states of systems with constant parameters. Even though most of the exogenous parameters could vary over time if we consider more realistic assumptions, the analysis of the system subject to constant exogenous parameters will give us a glimpse of how the system behaves. Moreover, according to Turnovsky, the steady state of the system subject to constant parameters should be treated as a reference point (7) [4].

In the first chapter, we introduce a very simple model, where the only asset in the economy is cash and the economy is divided in three sectors: households, production and government. We analyze two cases, a model with expectations and with perfect foresight. Along the continuous time model, an alternative discrete time analogous model is provided for the reader. For both the continuous and discrete case we derive a one dimensional system for the case with perfect foresight and a two dimensional model for the case with expectations.

The second chapter is an extension of the first model, adding the central bank as a sector and bills as a new form of asset. For the case with perfect foresight, a two dimensional system is derived in discrete time and a one dimensional system is derived in continuous time. For the case with expectations, a two dimensional linear system of differential equations is derived in continuous time and a three dimensional linear system of difference equations in the discrete time version.

The third chapter is basically another extension of the first two models, adding bonds (long term securities) to the basket of assets. In continuous time, there is still only one state in the case of perfect foresight and a two dimensional linear system of ordinary differential equations for the case with expectations. While we have a similar behaviour of the state variables for the two dimensional linear systems in case with expectations in the first two chapters, the phase diagram in the case with expectations shows a different behaviour. In the analogous models in discrete time, we end up with a three dimensional system of difference equations for the case of perfect foresight and with a four dimensional system of difference equations for the case with expectations.

The fourth chapter introduces the concepts of private banks, where money deposits carry interest rate. This is not an extension of the previous model in the sense that the previous assets (high-powered money, bills and bonds) are not taken into consideration. In this chapter we also model transactions within one sector, which affect the whole system, and in order to analyze this, we consider these transactions between sectors as transactions between different accounts (current and capital) inside such sector. For the discrete time variant we derive a two dimensional system of difference equations, and a one dimensional linear system for the model in continuous time proposed.

The fifth chapter consists of a model that comprises the models of all the previous chapters and where only the continuous time variant is analyzed. A three dimensional linear system or ordinary differential equations is derived for the case with expectations and a two dimensional system is derived for the case of perfect foresight.

The ideas of this thesis rely heavily on the work of Wynne Godley and Marc Lavoie from their book "Monetary Economics", which will be referenced in numerous occasions throughout this thesis.

It is important to mention that our goal is to focus on the analysis of continuous time models. The discrete time models proposed by Godley and Lavoie and that are presented along with our continuous time models, are only provided as a reference for the reader who might be more familiar with their work and with discrete time models, as well as to show that there is a much more efficient way to present their own models. Nevertheless, they shouldn't be treated as the exact discrete time version of our continuous time models, since in some cases, we will make different assumptions for our models in continuous time which will result in a substantial simplification of the systems derived. If we considered the exact same assumptions, then the systems derived would have the same number of dimensions in both discrete and continuous time in all cases.

1 Government Money

1.1 Balance Sheet Matrix, Transactions Matrix and Linear System of Equations

We will start this chapter with a very basic model in which the only asset is *cash*, or *high-powered* money, as economists often call, and for which reason, we will denote it by the letter H.

In this model we divide the whole economic universe into just three different sectors: households, production firms and the government. With respect to the balance sheet matrix of this model, it is assumed that only households own high-powered money, so the whole stock of money is entered with a plus sign (+) in the households' column. It is also assumed this to be a "pure service economy", so there is no need for firms to have capital for the production of goods and services, which is instantaneous and demand-led (thus eliminating the need of inventories). Moreover, all sales revenue is transferred to the households through wages. In summary, the production sector doesn't hold any assets. High-powered money is a financial asset and all financial assets must have a liability counterpart. In this case, since high-powered money is issued by the government (often in the form of banknotes), it represents a liability for them, and thus it is entered with a minus sign (-) in the government column. It is also assumed that high-powered money carries no interest. Since money is a financial asset and is the only asset in the economy, the net value of the economy is nil.

The first three columns of the transactions matrix as it has been set up, describe the variables which correspond, in principle, to the components of the National In-

Table 1: Government Money: Matrices

Balance Sheet	Households	Production	Government	Σ
Money stock	$+H$		$-H$	0
Transactions				
Consumption	$-C$	$+C$		0
Govt. expenditures		$+G$	$-G$	
Wages	$+W$	$-W$		0
Taxes	$-T$		$+T$	0
Σ	S_h	0	S_g	0
Flow of Funds				
Money stock	$+\frac{dH}{dt}$ $(+\Delta H)$		$-\frac{dH}{dt}$ $(+\Delta H)$	0
Change in net worth (Σ)	S_h		S_g	0

come and Product Accounts (NIPA) arranged as transactions between sectors and which take place in some defined unit of time, such as a quarter or a year (60) [2]. The transactions entered with a minus sign (-) imply a payment and those entered with a plus sign (+) imply a receipt. The last row of this matrix, which is the summation of the transactions, gives us the accumulation of wealth by each sector, where S_h constitutes the savings of the households and S_g constitutes the surplus of the government. Since these two must add up to zero, it is easy to see that, in this model, the level of saving of the households equals the deficit of the government. Note that we have defined the continuous rate of change for H as $\frac{dH}{dt}$ and the discrete change as ΔH. We will focus on continuous time models, but we will make reference to their discrete time counterpart when necessary.

In the flow of funds matrix, since the only asset in the economy is high-powered money, it follows that both the saving of households and the deficit of the government equal the change in the stock of high-powered money.

Because our models are stock-flow consistent, we can always dismiss one redundant equation. From columns of the transactions and the flow of funds, we can deduct the following equations:

$$\frac{dH}{dt} = W - C - T, \quad t > 0 \tag{1.1}$$

$$W = C + G \tag{1.2}$$

Equation (1.2) has the same equivalent in discrete time, since all the variables in the equation are considered to be from the same time t. The discrete-time variation of (1.1) is

$$\Delta H = H_t - H_{t-1} = W_t - C_t - T_t \tag{1.3}$$

We may dismiss the equation arising from the last column, since the system would become redundant if it were considered, given the fact that it is a linear combination of the first two equations (1.1 and 1.2), and it provides no new information:

$$\frac{dH}{dt} = G - T, \quad t > 0$$

which can be written in discrete time as follows:

$$\Delta H = H_t - H_{t-1} = G - T \tag{1.4}$$

Before being able to make the system solvable, we will define how the variables W, C, T and G behave, but first, we will define the disposable income

$$Y_D = W - T \tag{1.5}$$

We will now introduce the concept of expected disposable income, and we will define it with the following differential equation:

$$\dot{Y_D^e} = \eta(Y_D - Y_D^e)$$

where η is an exogenous parameter. This differential equation can be expressed in integral form as

$$Y_D^e(t) = e^{\eta(t_0 - t)} Y_D^e(t_0) + \eta \int_{t_0}^{t} e^{\eta(s-t)} Y_D(s) ds, \quad t > 0$$

From the integral form, we can observe how the definition of the expected disposable income consists of the exponential averaged disposable income controlled by parameter η, in the sense that for larger values of η, the expected disposable income Y_D^e will be closer to the realized disposable income Y_D than for smaller values of η. Considering $\eta = 1$, the rate of the expected disposable income becomes

$$\dot{Y_D^e} = Y_D - Y_D^e, \quad t > 0 \tag{1.6}$$

with the analogous discrete equation being

$$\Delta Y_D^e = Y_{Dt-1} - Y_{Dt-1}^e \tag{1.7}$$

which will make equation (1.7) mean that the expected disposable income equals the disposable income from the previous period, as Godley and Lavoie considers (81)[2].

Assume that

$$C = \alpha_1 Y_D^e + \alpha_2 H \tag{1.8}$$

where C stands for consumption, and its behaviour will be defined based on what Godley and Lavoie suggest (79)[2], as a proportion α_1 of the expected disposable income Y_D^e plus a proportion α_2 of their wealth, which in this case is fully made of high-powered money. Parameters α_1 and α_2 are exogenous.

Equation (1.8) can be written in discrete time as follows:

$$C_t = \alpha_1 Y_{Dt}^e + \alpha_2 H_{t-1} \tag{1.9}$$

where the assumption is that households consume a proportion of their disposable income on the same period plus a proportion of their level of wealth held at the end of the previous period.

Alternatively, under the assumption of perfect foresight, that is $Y_D^e = Y_D$ at all time, equation (1.8) will take the form

$$C = \alpha_1 Y_D + \alpha_2 H \tag{1.10}$$

while its analogous in discrete time, equation (1.9), will take the following form:

$$C_t = \alpha_1 Y_{Dt} + \alpha_2 H_{t-1} \tag{1.11}$$

Next, assume that

$$T = \theta W \tag{1.12}$$

where T stands for taxes, and it is defined as a fixed exogenous proportion θ of the income (wages received by households in this case). This proportion is defined by the government and is thus exogenous. Finally set

$$W = \mathrm{wNE} \tag{1.13}$$

where W is defined as the total wage bill paid out by firms to households, w is the wage rate (wage per person), N is the total economically active population and E is the employment rate.

In this model, we will also make the assumption that N and the wage rate w are exogenously determined, thus making the employment rate E endogenous.

Finally, we will consider the government expenditures represented by G as exogenous. This will close the model and will enable us to find a solution.

1.2 System of Differential Equations and Steady State Solutions under Constant Parameters

The setting up of this system resembles closely that of Godley and Lavoie "Simplest Model with Government Money with Expectations", and whose system of difference equations can be found on Appendix 3.2 from their book (92)[2] - they deal with a discrete time model which is takes into account the alternative set of difference equations defined earlier. The approach suggested by them is to analyze the behaviour of this system over time based on the characteristics of the linear system of equations derived from the model.

Nevertheless, it is possible to derive, from the linear system of equations just presented, a two dimensional system of ordinary differential equations in terms of exogenous parameters, whose behaviour will define the behaviour of all the other endogenous variables of the system, and thus for the system as a whole. In other words, it is possible to derive explicit expressions of all the other endogenous variables in terms of the state variables from the system of ordinary differential equations. In the case of the model with perfect foresight, that is, where $Y_D^e = Y_D$, instead of a two dimensional system, it is possible to derive a single differential equation.

Substituting equation (1.8) into equation (1.2), we get

$$W = \alpha_1 Y_D^e + \alpha_2 H + G \tag{1.14}$$

which is an endogenous variable expressed in terms of H and Y_D^e, later found to be state variables, and exogenous variables. It is easy to prove that we can define the rest the endogenous variables of the system in terms of H, Y_D^e and exogenous variables.

Moreover, substituting equations (1.2), (1.10) and (1.12) in equation (1.1), we get the following differential equation:

$$\frac{dH}{dt} = G - \theta(\alpha_1 Y_D^e + \alpha_2 H + G) \tag{1.15}$$

which together with equation (1.6), determine the solution of the state variables H and Y_D^e.

Equations (1.6) and (1.15) form a linear system of ordinary differential equations represented by

$$\dot{X} = AX + b$$

where

$$A = \begin{pmatrix} C_1 & C_2 \\ C_4 & C_5 \end{pmatrix}$$

$$b = \begin{pmatrix} C_3 \\ C_6 \end{pmatrix}$$

$$X = \begin{pmatrix} Y_D^e \\ H \end{pmatrix}$$

where

$$C_1 = (1-\theta)\alpha_1 - 1$$

$$C_2 = (1-\theta)\alpha_2$$

$$C_3 = (1-\theta)G$$

$$C_4 = -\theta\alpha_1$$

$$C_5 = -\theta\alpha_2$$

$$C_6 = (1-\theta)G$$

Following the same process but using the alternative discrete time equations, we derive the following difference equations:

$$\Delta Y_D = ((1-\theta)\alpha_1 - 1)Y_{Dt-1} + ((1-\theta))\alpha_2 H_{t-1} + (1-\theta)G$$

$$\Delta H = -\theta\alpha_1 Y_{Dt-1} - \theta\alpha_2 H_{t-1} + (1-\theta)G$$

We can see that taking the limit when $\Delta t \to 0$, the alternative discrete time model converges to a system that resembles the model in continuous time, although not exactly equal, due to the subtle differences in the continuous functions (1.8) and (1.6) with respect with their analogous equations in discrete time (1.9) and (1.9) and (1.7).

The steady state solutions for these two state variables when G is constant are given by the equilibrium point of the system. We should note in first place that the equilibrium point is not centered at (0,0) for most cases. We will denote the equilibrium point coordinates by ($\bar{Y_D^e}$,$\bar{H}$), with

$$\bar{Y_D^e} = -\frac{C_3C_5 - C_2C_6}{C_1C_5 - C_2C_4}$$

$$\bar{H} = \frac{C_1C_6 - C_3C_4}{C_1C_5 - C_2C_4}$$

Note that for this model, the total GDP is equal to the total wage bill W. Considering the steady state solutions, from equation (1.14), it follows that

$$\bar{W} = \alpha_1\bar{Y_D^e} + \alpha_2\bar{H} + G \tag{1.16}$$

Considering now the case of perfect foresight, where the expected disposable income equals the actual disposable income, that is $Y_D^e = Y_D$, we should consider equation (1.10) instead equation (1.8). Combining equations (1.2), (1.10), (1.12) and (1.5), and isolating for W, we get

$$W = \frac{\alpha_2H + G}{1 - \alpha_1(1-\theta)} \tag{1.17}$$

Combining equations (1.1), (1.17) and (1.10), we obtain

$$\frac{dH}{dt} + \frac{\alpha_2\theta}{1-\alpha_1(1-\theta)}H = \frac{(1-\alpha)(1-\theta)G}{1-\alpha_1(1-\theta)} \tag{1.18}$$

whose steady state is defined by

$$\bar{H} = \frac{(1-\alpha_1)(1-\theta)G}{\alpha_2\theta}$$

From equations (1.17):

$$\bar{W} = \frac{G+\alpha_2\bar{H}}{1-\alpha_1(1-\theta)}$$

Moreover, the steady state for the employment rate becomes

$$\bar{E} = \frac{G+\alpha_2\bar{H}}{(1-\alpha_1(1-\theta))\mathrm{wN}}$$

1.3 Solution and Analysis of the System of Differential Equations

In this section, we will analyze the stability behaviour. Let's start with the simplest case, where there is perfect foresight for the expected disposable income, and therefore the system reduces to a single differential equation. Equation (1.18) is a differential equation of the form

$$\dot{H} + AH = b \tag{1.19}$$

where

$$A = \frac{\alpha_2\theta}{1-\alpha_1(1-\theta)}$$

$$b = \frac{(1-\alpha_1)(1-\theta)G}{1-\alpha_1(1-\theta)}$$

Equation (1.18) is a first order non-homogenous differential equation, and its solution is found considering the integrating factor

$$e^{\int_{t_0}^{t} Ads}$$

and after some manipulation of equation 1.16 together with this integrating factor, we arrive to the solution

$$H = \frac{(1-\alpha_1)(1-\theta)G}{\alpha_2\theta} + \left(H_0 + \frac{(1-\alpha)(1-\theta)G}{\alpha_2\theta}\right) e^{\frac{-\alpha_2\theta}{1-\alpha_1(1-\theta)}t} \tag{1.20}$$

The stability of equation (1.20) is assured as long as the following constraint holds:

$$0 < \frac{\alpha_2\theta}{1-\alpha_1(1-\theta)} \tag{1.21}$$

It is worth noting that the system is always stable if we consider the following constraints proposed by Godley and Lavoie regarding the tax rate and consumption: $0 < \theta < 1$ and $0 < \alpha_2 < \alpha_1 < 0$ (91)[2].

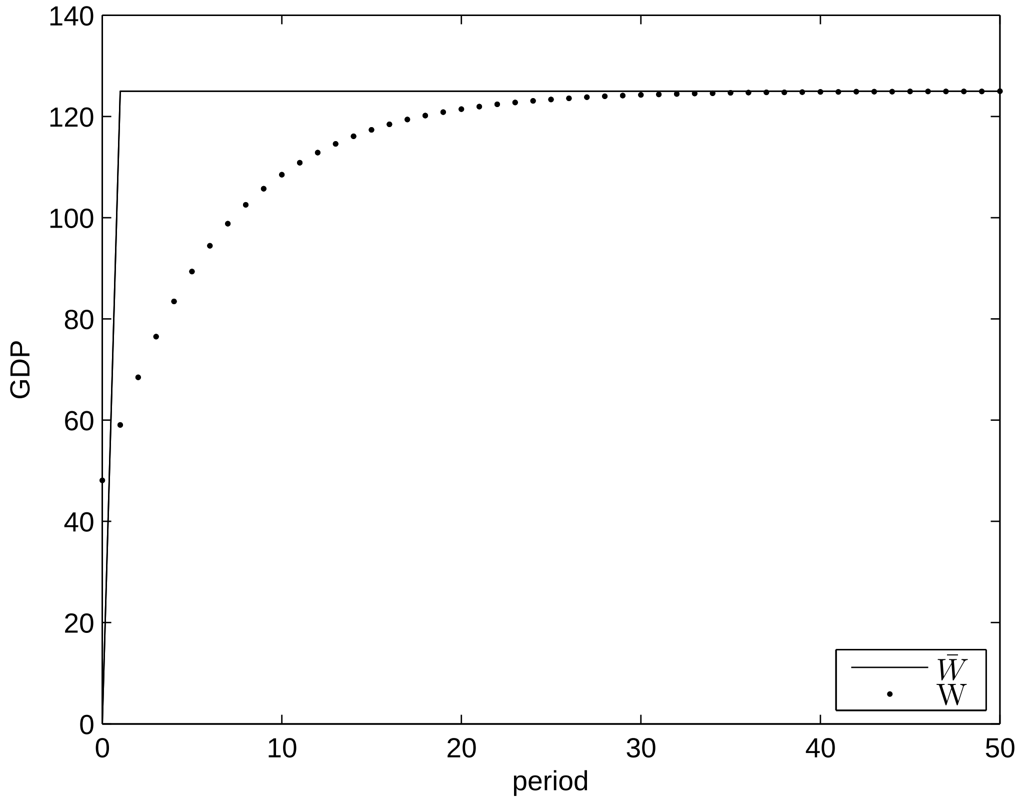

Figure 1: Government Money: Evolution of GDP with perfect foresight

Figure 1 shows the evolution of the GDP W for $0 < t < 100$ as well as the steady state $\bar{W}$, for the case with perfect foresight (no expectations). It uses the following values of exogenous variables: $G = 20$, $\alpha_1 = 0.6$, $\alpha_2 = 0.4$, $\theta = 0.2$, wN $= 1$, $H_0 = 0$.

For the case where there is no perfect foresight and the behaviour of the expected disposable income is given by equation (1.6), the system of ordinary differential equations formed by equations (1.6) and (1.18) consists of a non-homogeneous linear two dimensional system. In general, for systems with not necessarily constant exogenous parameters, the solution of such a non-homogeneous linear system is given by the variation of constants formula. Thus, any solution would be of the following form:

$$X(t) = \Omega(t)\Omega^{-1}(t_0)X_0 + \int_{t_0}^{t} \Omega(t)\Omega^{-1}(s)b(s)ds \tag{1.22}$$

where $\Omega(t)$ is a fundamental matrix of $A(t)$.

The stability of such system is limited by the existence of constants k and $\beta > 0$, such that

$$||\Omega(t)\Omega(s)^{-1}||_\infty \leq Ke^{-\beta(t-s)} \quad t \geq s \geq 0$$

In the case where $A(t)$ is constant (in which case we will denote it as A) or diagonal, the fundamental matrix $\Omega(t)$ is given by the exponential matrix $e^{tA} = e^{tTJT^{-1}} = Te^{tJ}T^{-1}$, where T is the matrix containing the eigenvectors associated with the matrix A and its eigenvalues. Matrix J is the matrix containing the eigenvalues in diagonal canonical form. The eigenvalues for this constant matrix are given by

$$\Lambda_{1,2} = \frac{(C_1 + C_5) \pm \sqrt{(C_1 + C_5)^2 - 4(C_1C_5 - C_2C_4)}}{2} \tag{1.23}$$

For the stability analysis of a system with constant parameters that is not centered at zero, the following change of variables will create a new system with an equilibrium point located at (0,0) and the stability of that system will guarantee the stability of the original one. Define

$$Y_D^e = y + \bar{Y}_D^e$$

$$H = z + \bar{H}$$

This change of variable gives rise the new system

$$\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ C_4 & C_5 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = A \begin{pmatrix} y \\ z \end{pmatrix}$$

For the point (0,0) to be stable in this new system, and thus the point $(\bar{Y_D^e},\bar{H})$ in the original system to be stable as well, the real part of the eigenvalues of A must be negative. (By stable, we mean asymptotically stable).

From equation (1.23), we know that we need both $(C_1 + C_5) < 0$ and $4(C_1C_5 - C_2C_4) > 0$ as conditions for stability of the system.

Let us express the first condition $(C_1 + C_5) < 0$ in terms of exogenous variables doing the proper substitutions in C_1 and C_5.

$$\begin{aligned} C_1 + C_5 &= (1-\theta)\alpha_1 - \theta\alpha_2 - 1 \\ &< 0 \end{aligned} \tag{1.24}$$

It is not difficult to see that this inequality always holds when $0 < \theta$, $0 < \alpha_1 < 1$ and $0 < \alpha_2$.

The second condition that needs to be hold is $C_1C_5 - C_2C_4 > 0$ to hold, so expressing this fully in exogenous variables, we get

$$C_1C_5 - C_2C_4 = ((1-\theta)\alpha_1 - 1)(-\theta\alpha_2) - (1-\theta)\alpha_2(-\theta\alpha_1) > 0 \tag{1.25}$$

which reduces to the condition

$$0 < \alpha_2$$

As before, we see that the system is always stable if we consider the following constraints proposed by Godley and Lavoie regarding the tax rate and consumption function: $0 < \theta < 1$ and $0 < \alpha_2 < \alpha_1 < 1$. (91)[2].

Finally, it is important to mention that economically meaningful models require Y_D^e and H to be positive at all times.

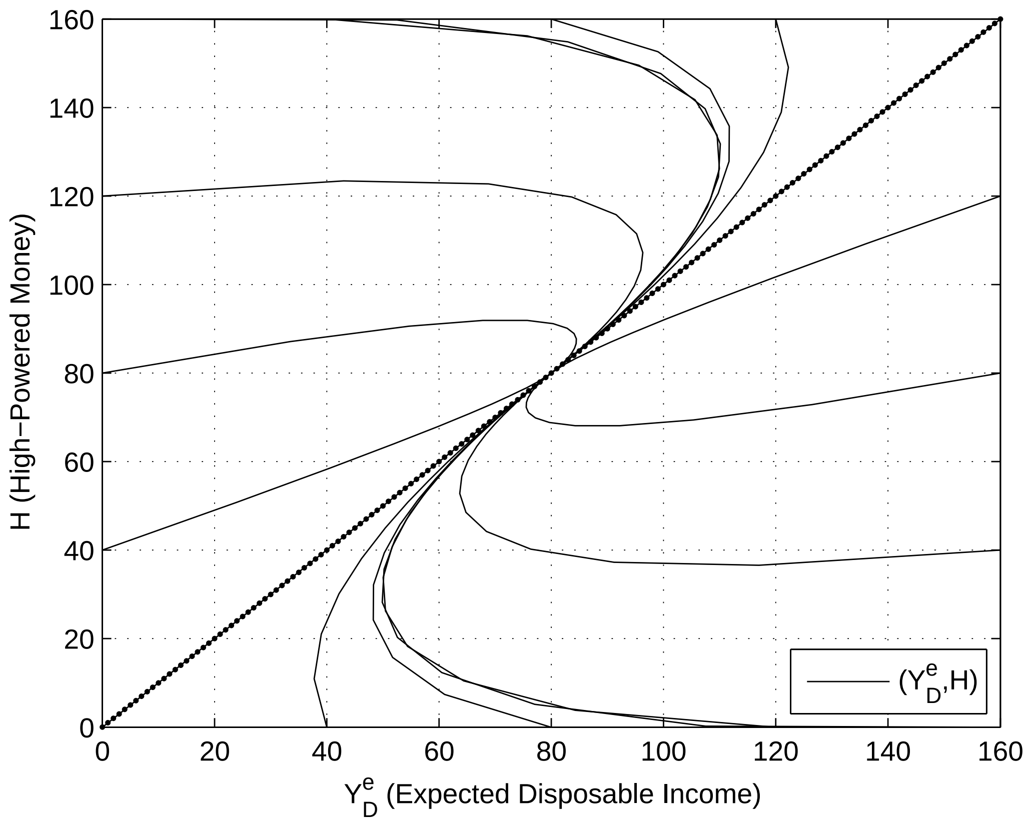

Figure 2: Government Money: Phase graph of original system X with expectations

Figure 2 shows the phase graph of Y_D^e and H for $0 < t < 100$ and different initial conditions. It uses the following values of exogenous variables: $G = 20$, $\alpha_1 = 0.6$, $\alpha_2 = 0.4$, $\theta = 0.2$. The eigenvalues $\Lambda_{1,2}$ of A are negative and real.

Figure 3 shows the graph of the diagonalized system $Y = T^{-1}X$, centered at

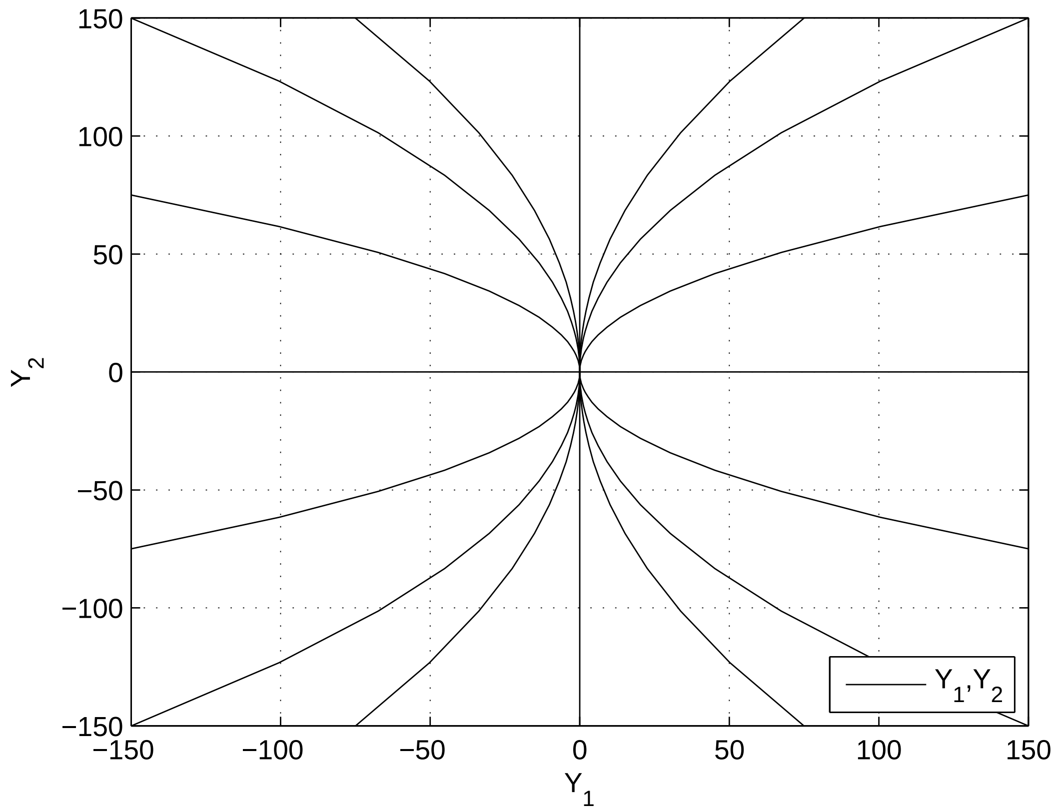

Figure 3: Government Money: Phase graph of the diagonalized system $Y = T^{-1}X$ with expectations

(0,0), where T is the matrix of eigenvectors, X is the original system from Figure 2, and Y is the diagonalized system.

Looking at Figure 2, it is easy to appreciate that closer to the steady state, all solutions tend towards a line of a certain inclination with respect to the expected disposable income Y_D^e. We will define this line as $H = mY_D^e + b*$ where m is the slope and $b*$ is the value of H where the line intersects Y_D^e-axis. For this particular system, that line corresponds to the trajectory of the solution of the system when $H_0 = 0$ and

$Y^e_{D0} = 0$. We can observe from Figure 3, that this line in the diagonalized system is the vertical H*-axis.

Knowing that the line $H = mY^e_D + b*$ corresponds to the vertical H*-axis from the diagonalized system, we may find the values of m and $b*$. For a given system, substituting the vector $Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ from the diagonalized system and obtaining its equivalent vector in the original system X we get

$$X = T \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The new vector X from the given system has the property that its components (denoted by X_1 and X_2) form the tangent of the angle of the slope m, such that $m = arctan\left(\frac{X_2}{X_1}\right)$, which is one of the variables that we wish to obtain.

In order to obtain b, we just substitute the new value of m in the equation $H = mY^e_D + b*$ with the values $H = \bar{H}$ and $Y^e_D = \bar{Y^e_D}$ given by the equilibrium point (steady state) in the original system X.

$b* = \bar{H} - m * \bar{Y^e_D}$

2 Portfolio Choice

2.1 Balance Sheet Matrix, Transactions Matrix and Linear System of Equations

In this model we introduce a new sector into the economy: the central bank. This new sector will be in charge of issuing high-powered money, which will be held by the households only, hence the total stock of money is entered with a minus sign (-) in the central bank column and with a plus sign (+) in the households column from the balance sheet matrix. We also introduce a new financial asset, the government bills or treasury bills, issued by the government (B) and held by the households (B_h) and the central bank (B_{cb}). These government bills carry interest payments from the government. Two extra assumptions needed to explain the balance sheet matrix are the assumptions that production firms do not hold any asset whatsoever, and that the central bank has no net worth. This last assumption implies that the any profit made by the central bank from their holdings on bills is transferred to the government, according to Godley and Lavoie in line with the practice of most central banks in the world (102)[2].

Now that we have set clear all the assumptions regarding the balance sheet matrix, we can derive the following equations

$$V = B_h + H \tag{2.1}$$

Table 2: Porfolio Choice: Matrices

Balance Sheet					Σ
Money stock	$+H$			$-H$	0
Bills	$+B_h$		$-B$	$+B_{cb}$	0
Net worth	$+V$		$-V$	0	0
Transactions					
Consumption	$-C$	$+C$			0
Govt. expenditures		$+G$	$-G$		0
Wages	$+W$	$-W$			0
Interest payments	$+rB_h$		$-rB$	$+rB_{cb}$	0
Central bank profits			$+rB_{cb}$	$-rB_{cb}$	0
Taxes	$-T$		$+T$		0
Σ	S_h	0	S_g	0	0
Flow of Funds					
Chage in money	$+\frac{dH}{dt}$ $(+\Delta H)$			$-\frac{dH}{dt}$ $(-\Delta H)$	0
Chage in bills	$+\frac{dB_h}{dt}$ $(+\Delta B_h)$		$-\frac{dB}{dt}$ $(-\Delta B)$	$+\frac{dB_{cb}}{dt}$ $(+\Delta B_{cb})$	0
Σ	S_h	0	S_g	0	0

$$B = V \tag{2.2}$$

$$H = B_{cb} \tag{2.3}$$

We may dismiss the following equation derived from the system, because it is a linear combination of equations (2.1), (2.2) and (2.3) and provides no new information, following the fact that we are dealing with a stock-flow consistent model:

$$B = B_h + B_{cb}$$

Regarding the transactions matrix, three new rows appear: Interest payments, central bank profits and change in bills. Interest payments are distributed to households and the central bank, depending on their holdings on bills and an exogenously defined interest rate. The central bank profits row only indicates the transactions of all the profits from the central bank to the government. The change in the bills row resembles the row containing the change in money.

From the balance sheet and the flow of funds matrices we may obtain the following three equations:

$$\frac{dH}{dt} + \frac{dB_h}{dt} = -C - T + W + rB_h, \quad t > 0 \tag{2.4}$$

$$W = C + G \tag{2.5}$$

$$\frac{dB}{dt} = -T - r(B_{cb} - B) + G, \quad t > 0$$

The last equation constitutes a linear combination of the previous too, and hence can be dismissed. Equation (2.4) can be defined in discrete time as follows:

$$\Delta H + \Delta B_h = -C - T + W + rB_h \tag{2.6}$$

Again, as in the system from the previous chapter, we need to define the behaviour or of the following endogenous variables:

$$C = \alpha_1 Y_D + \alpha_2 V \tag{2.7}$$

where

$$Y_D = W + rB_h - T \tag{2.8}$$

$$T = \theta(W + rB_h) \tag{2.9}$$

$$W = \text{wNE} \tag{2.10}$$

Equations (2.7), (2.9) and (2.10) are defined in the same way as in the system from the previous chapter in the case of perfect foresight ($Y_D^e = Y_D$), with the only difference being that in equation (2.7) we consider the total net worth of the household V rather than their stock of money H and the actual disposable income (the expectation in the disposable income matches the one realized). The alternative equations in discrete time for equations (2.7) and (2.9) are:

$$C_t = \alpha_1(Y_{Dt}) + \alpha_2 V_{t-1} \tag{2.11}$$

$$T_t = \theta(W_t + rB_{ht-1}) \tag{2.12}$$

$$Y_{Dt} = W_t + rB_{ht-1} - T_t \tag{2.13}$$

Moreover, we need to define how households allocate their wealth between bills and high-powered money based on exogenous parameters. The next two equations describe this behaviour based on the proportion of each asset held by households with respect to their total wealth (or total net worth).

$$\frac{H}{V} = (1 - \lambda_0) - \lambda_1 r + \lambda_2 \frac{Y_D}{V} \tag{2.14}$$

$$\frac{B_h}{V} = \lambda_0 + \lambda_1 r - \lambda_2 \frac{Y_D}{V} \tag{2.15}$$

The alternative equations in discrete time for equations (2.14) and (2.15) are:

$$\frac{H_t}{V_t} = (1 - \lambda_0) - \lambda_1 r + \lambda_2 \frac{Y_{Dt}}{V} \tag{2.16}$$

$$\frac{B_{ht}}{V_t} = \lambda_0 + \lambda_1 r - \lambda_2 \frac{Y_{Dt}}{V_t} \tag{2.17}$$

From equation (2.1), we know that the left hand sides of equations (2.14) and (2.15) must add up to one. The first term on the right hand side of those equations constitutes a default proportion on the amount of each asset held by households. Since there are only two assets in this model, the terms are $(1 - \lambda_0)$ and λ_0 for equations (2.14) and (2.15) respectively, thus making sure that equation (2.1) holds. This proportion is adjusted by a proportion λ_1 on the interest rate r and a proportion λ_2 on the ration between disposable income and total wealth. A higher value of the interest rate r will incentivize households to purchase bills. On the other hand, a higher value of the ratio between disposable income and total wealth will incentivize the households to hold more assets in the form of cash, ready to be used for consumption. Once again, in order for equation (2.1) to hold at all times, the proportional terms in equation 2.13 containing λ_1 and λ_2 must have opposite signs with respect to those that they have in equation (2.14). This constitutes the adding-up constraints described by Tobin (1969) [3]. The reader may find easier to understand this concept in the coming chapters, where the constraints will be presented in matrix form.

Finally, we will consider the government expenditures represented by G as exogenous, as we did in the previous chapter.

2.2 Differential Equation, Steady State Solution and Stability with Perfect Foresight

The setting up of the system just define resembles closely that of Godley and Lavoie "Portfolio Choice", and whose system of difference equations can be found on Appendix 4.1 from their book (126)[2] - they deal with a discrete model time. In a similar fashion as it happened in the previous chapter, it is possible to derive in this case from the linear system of equations presented, a one dimensional linear system of ordinary differential equations of the state variables (V) with exogenous parameters, with all the remaining endogenous variables being defined explicitly in terms of those state variables and exogenous parameters.

Combining equations (2.5), (2.7) and (2.9) and isolating for W, we get

$$W = \frac{\alpha_1 r(1-\theta)B_h + \alpha_2 V + G}{1-\alpha_1(1-\theta)}$$

And from equations (2.1), (2.2) and (2.3) we can express this equation in terms of H and B as follows

$$W = \frac{(\alpha_1 r(1-\theta) + \alpha_2)V - \alpha_1 r(1-\theta)H + G}{1-\alpha_1(1-\theta)} \tag{2.18}$$

Substituting equation (2.18) into equation (2.14), we get

$$H = C_1 V + C_2 \tag{2.19}$$

where

$$C_1 = \left(\frac{(1-\alpha_1(1-\theta))(1-\lambda_0-\lambda_1 r)+1+(1-\theta)(\lambda_2-\alpha_1+\alpha_2)}{1+(1-\theta)(\lambda_2 r-\alpha_1)}\right)$$

$$C_2 = \left(\frac{\lambda_2(1-\theta)G}{1+(1-\theta)(\lambda_2 r-\alpha_1)}\right)$$

From equations (2.4), (2.1), (2.2), (2.3) and (2.9), (2.8) and (2.18) we obtain

$$\dot{V} + AV = b \tag{2.20}$$

where

$$A = -\frac{(1-\alpha_1)(1-\theta)(\alpha_1 r(1-\theta)+\alpha_2-\alpha_1 r(1-\theta)C_1}{1-\alpha_1(1-\theta)} + r(1-C_1)+\alpha_2$$

$$b = \frac{(1-\alpha_1)(1-\theta)(g-\alpha_1 r(1-\theta)C_2)}{1-\alpha_1(1-\theta)} + rC_2$$

Following a similar process but using the alternative discrete time equations, we may derive the following two difference equations:

$$\begin{aligned}\Delta V = &\left(r-\theta\left(r+\frac{\alpha_1 r(1-\theta)+\alpha_2}{1-\alpha_1(1-\theta)}\right)\right)V_{t-1} + \left(\theta\left(\frac{\alpha_1 r(1-\theta)}{1-\alpha_1(1-\theta)}+r\right)-r\right)H_{t-1}\\ &+G\left(1-\frac{\theta}{1-\alpha_1(1-\theta)}\right)\end{aligned} \tag{2.21}$$

$$\Delta H = (1-\lambda_0-\lambda_1 r)\Delta V + \left(\lambda_2(1-\theta)\left(\frac{\alpha_1 r(1-\theta)+\alpha_2}{1-\alpha_1(1-\theta)}\right) + 1-\lambda_0-\lambda_1 r\right) B_{t-1}$$
$$- \left(\lambda_2(1-\theta)\left(\frac{\alpha_1 r(1-\theta)}{1-\alpha_1(1-\theta)} + r\right) + 1\right) H_{t-1} + \frac{\lambda_2(1-\theta)G}{1-\alpha_1(1-\theta)}$$

The reason why we have a two dimensional system in discrete time rather than a one dimensional system like in continuous time is that in the right hand side of equations 2.12 and 2.13 we are considering the instantaneous interest return on the bills B_h, whereas in the alternative discrete time equations, the interest returns take into account the amount of bills held at the previous period of time B_{ht-1}, giving rise to a new state variable in the discrete time model.

Equation 2.20 is a first order non-homogeneous differential equation, and its solution is found considering the integrating factor

$$e^{\int_{s_0}^{t} A ds}$$

and after some manipulation of equation (2.20) together with this integrating factor we will arrive to the following solution:

$$V = e^{-\int_{s_0}^{t} A ds} \int_{s_0}^{t} b e^{\int_{u_0}^{s} A du} ds + V_0 e^{-\int_{s_0}^{t} A ds} \tag{2.22}$$

which yields to

$$V = \frac{b}{A} + (V_0 - \frac{b}{A})e^{-At}$$

This differential equation is asymptotically stable providing that $C_1 + C_2C_4 < 0$. Its steady state, assuming that the system is asymptotically stable, is given by

$$\bar{V} = -\frac{b}{A}$$

Note that for this model, once again the total GDP is equal to the total wage bill W. As pointed in the introduction, the steady state solutions are of special interest in the analysis of stock-flow consistent models. From equation 2.18, it follows that

$$\bar{W} = \frac{(\alpha_1 r(1-\theta) + \alpha_2 - \alpha_1 r(1-\theta)C_1)\bar{V} - \alpha_1 r(1-\theta)C_2 + G}{1 - \alpha_1(1-\theta)}$$

Moreover, the equilibrium for the employment rate is given by

$$\bar{E} = \frac{(\alpha_1 r(1-\theta) + \alpha_2 - \alpha_1 r(1-\theta)C_1)\bar{V} - \alpha_1 r(1-\theta)C_2 + G}{(1 - \alpha_1(1-\theta))\mathrm{wN}} \tag{2.23}$$

For constant exogenous parameters, from equation (2.19), we see that we can express V in terms of H or vice versa.

2.3 System of Differential Equations, Steady State Solutions and Stability with Expectations

Like in the previous chapter, we now will consider the alternative case where there is not perfect foresight for the expected disposable income of the households, that is, their expected disposable income does not necessarily equal their realized disposable income. The expected disposable income will have the following behaviour:

$$\dot{Y}_D^e = \eta(Y_D - Y_D^e) \tag{2.24}$$

Equation (2.24) says that changes in the rate of change of the expected wealth depend on the expected disposable income, unlike, whereas in the case of perfect foresight, it depends on the realized disposable income. Like in the previous chapter, we will consider the case when $\eta = 1$. For the continuous case, equations (2.7), (2.14) and (2.15) will respectively take the following forms:

$$C = \alpha_1 Y_D^e + \alpha_2 V \tag{2.25}$$

$$\frac{H}{V} = (1 - \lambda_0) - \lambda_1 r + \lambda_2 \frac{Y_D^e}{V} \tag{2.26}$$

$$\frac{B_h}{V} = \lambda_0 + \lambda_1 r - \lambda_2 \frac{Y_D^e}{V} \tag{2.27}$$

The analogous in discrete time of the equations (2.24), (2.25), (2.26) and (2.27) previous equations, considering $\eta = 1$, are

$$\Delta Y_D^e = Y_{Dt-1} - Y_{Dt-1}^e \tag{2.28}$$

$$C_t = \alpha_1 Y_{Dt}^e + \alpha_2 V_{t-1} \tag{2.29}$$

$$\frac{H_t}{V_t^e} = (1 - \lambda_0) - \lambda_1 r + \lambda_2 \frac{Y_{Dt}^e}{V_t^e} \tag{2.30}$$

$$\frac{B_{ht}}{V_t^e} = \lambda_0 + \lambda_1 r - \lambda_2 \frac{Y_{Dt}^e}{V_t^e} \tag{2.31}$$

where

$$V_t^e = V_{t-1} + Y_{Dt} \tag{2.32}$$

Equations that were not redefined for the case with expectations will take the same form as in the case with perfect foresight. Combining equations (2.2), (2.1), (2.3), (2.4), (2.8), (2.25) and (2.5), we get

$$\dot{Y}_D^e = ((1-\theta)(\alpha_1 - r\lambda_2) - 1)Y_D^e + ((1-\theta)(\alpha_2 + r(\lambda_0 + \lambda_1 r))V + (1-\theta)G \tag{2.33}$$

$$\dot{V} = ((1-\theta)(\alpha_1 - r\lambda_2) - \alpha_1)Y_D^e + ((1-\theta)(\alpha_2 + r(\lambda_0 + \lambda_1 r)) - \alpha_2)V + (1-\theta)G \tag{2.34}$$

Following a similar process using the alternative discrete time equations, we may derive the following 3 dimensional system of ordinary differential equations:

$$\Delta Y_D = (\alpha_1(1-\theta) - 1)Y_{Dt-1} + (-r(1-\theta))H_{t-1} + (\alpha_2(1-\theta))V_{t-1} + (1-\theta)G \tag{2.35}$$

$$\Delta V = (\lambda_2 - \lambda_0 + \lambda_1 r - \theta\alpha_1)Y_{Dt-1} + (1 - r(1-\theta) - \lambda_0 + \lambda_1 r)H_{t-1} + ((1-\theta)r - \theta r)V_{t-1} \tag{2.36}$$

$$\Delta H = (-\theta\alpha_1)Y_{Dt-1} + (-r(1-\theta))H_{t-1} + ((1-\theta)r - \theta r - 1)V_{t-1} + (1-\theta)G \tag{2.37}$$

which, like in the case of perfect foresight, because of the difference in the definition of equations in continuous and discrete time, it cannot be reduced to a three dimensional model.

Equations (2.34) and (2.33) can be expressed as

$$\dot{X} = AX + b$$

where

$$A = \begin{pmatrix} C_1 & C_2 \\ C_4 & C_5 \end{pmatrix}$$

$$b = \begin{pmatrix} C_3 \\ C_3 \end{pmatrix}$$

$$X = \begin{pmatrix} Y_D^e \\ V \end{pmatrix}$$

where

$$C_1 = ((1-\theta)(\alpha_1 - r\lambda_2) - 1)$$

$$C_2 = (1-\theta)(\alpha_2 + r(\lambda_0 + \lambda_1 r))$$

$$C_3 = (1-\theta)G$$

$$C_4 = ((1-\theta)(\alpha_1 - r\lambda_2) - \alpha_1)$$

$$C_5 = ((1-\theta)(\alpha_2 + r(\lambda_0 + \lambda_1 r)) - \alpha_2)$$

The steady state solutions are given by

$$\bar{Y}_D^e = \frac{C_3(C_5 - C_2)}{C_2 C_4 - C_1 C_5}$$

$$\bar{V} = \frac{C_3(C_1 - C_4)}{C_2 C_4 - C_1 C_5}$$

Equations (2.34) and (2.33) form of a non-homogeneous linear two dimensional system. In general, as we saw in the previous chapter, for systems with not necessarily constant exogenous parameters, the solution of such a non-homogeneous linear system is given by the variation of constants formula. Thus, any solution would be of the following form:

$$X(t) = \Omega(t)\Omega^{-1}(t_0)X_0 + \int_{t_0}^{t} \Omega(t)\Omega^{-1}(s)b(s)ds \tag{2.38}$$

where $\Omega(t)$ is a fundamental matrix of $A(t)$.

The stability of such system is limited by the existence of constants k and $\beta > 0$, such that

$$||\Omega(t)\Omega(s)^{-1}||_{\infty} \leq Ke^{-beta(t-s)} \quad t \geq s \geq 0$$

We are focusing on systems with constant exogenous variables. As we saw in the previous chapter, In the case where $A(t)$ is constant (in which case we will denote it as A) or diagonal, the fundamental matrix $\Omega(t)$ is given by the exponential matrix $e^{tA} = e^{tTJT^{-1}} = Te^{tJ}T^{-1}$, where T is the matrix containing the eigenvectors associated with the matrix A and its eigenvalues. Matrix J is the matrix containing the eigenvalues in diagonal canonical form. The eigenvalues for this constant matrix are given by

$$\Lambda_{1,2} = \frac{(C_1 + C_5) \pm \sqrt{(C_1 + C_5)^2 - 4(C_1C_5 - C_2C_4))}}{2} \tag{2.39}$$

For the stability analysis of a system with constant parameters that is not centered at zero, the following change of variables will create a new system with an equilibrium point located at (0,0) and the stability of that system will guarantee the stability of the original one. Define

$$Y_D^e = y + \bar{Y_D^e}$$

$$V = z + \bar{V}$$

This change of variable gives rise the new system

$$\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ C_4 & C_5 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = A \begin{pmatrix} y \\ z \end{pmatrix}$$

For the point (0,0) to be stable in this new system, and thus the point $(\bar{Y_D^e},\bar{V})$ in the original system to be stable as well, the real part of the eigenvalues of A, given by equation (2.39), must be negative. (By stable, we mean asymptotically stable).

In order to assure asymptotic stability of the system, the real part of the eigenvalues given by eqaution (2.39) must be negative.

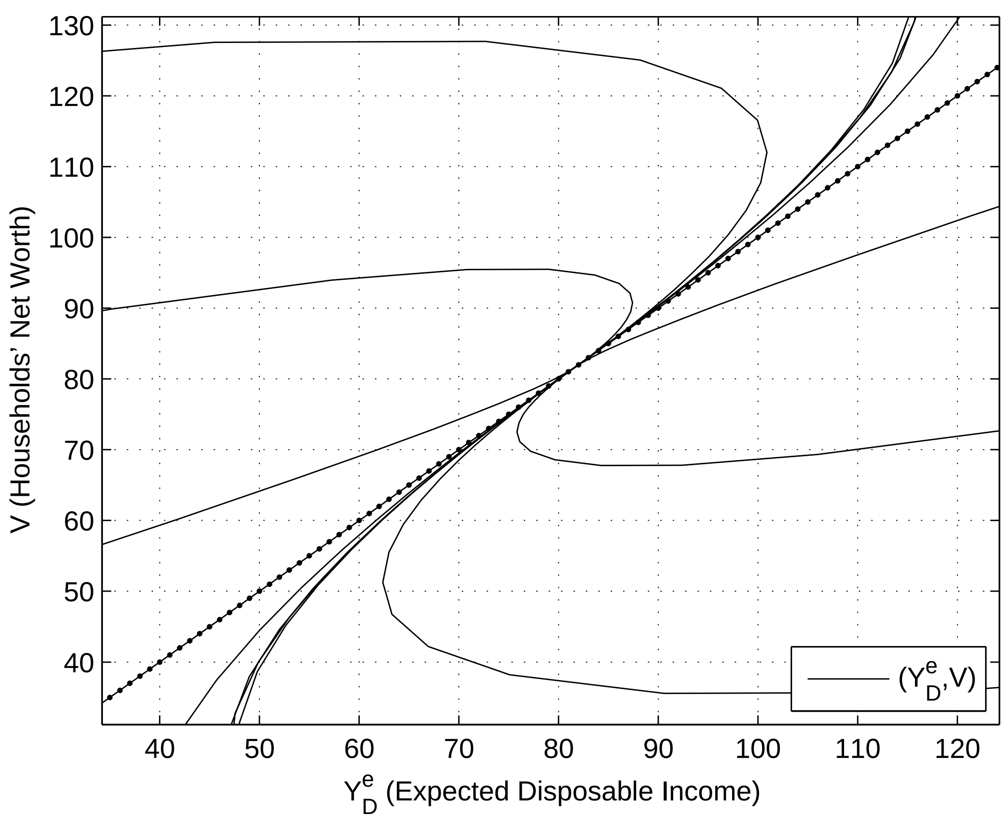

Figure 4: Portfolio Choice: phase Graph of original system X with expectations

Figure 4 shows the phase graph of Y_D^e and V for $0 < t < 100$ and different initial conditions. It uses the following values of exogenous variables: $G = 20$, $\alpha_1 = 0.6$,

$\alpha_2 = 0.4$, $\theta = 0.2$, $r = 0.1$, $\lambda_0 = 0.2$, $\lambda_1 = 0.5$ and $\lambda_2 = 0.2$. The eigenvalues $\Lambda_{1,2}$ of A are negative and real.

Figure 5 shows the graph of the diagonalized system $Y = T^{-1}X$, centered at

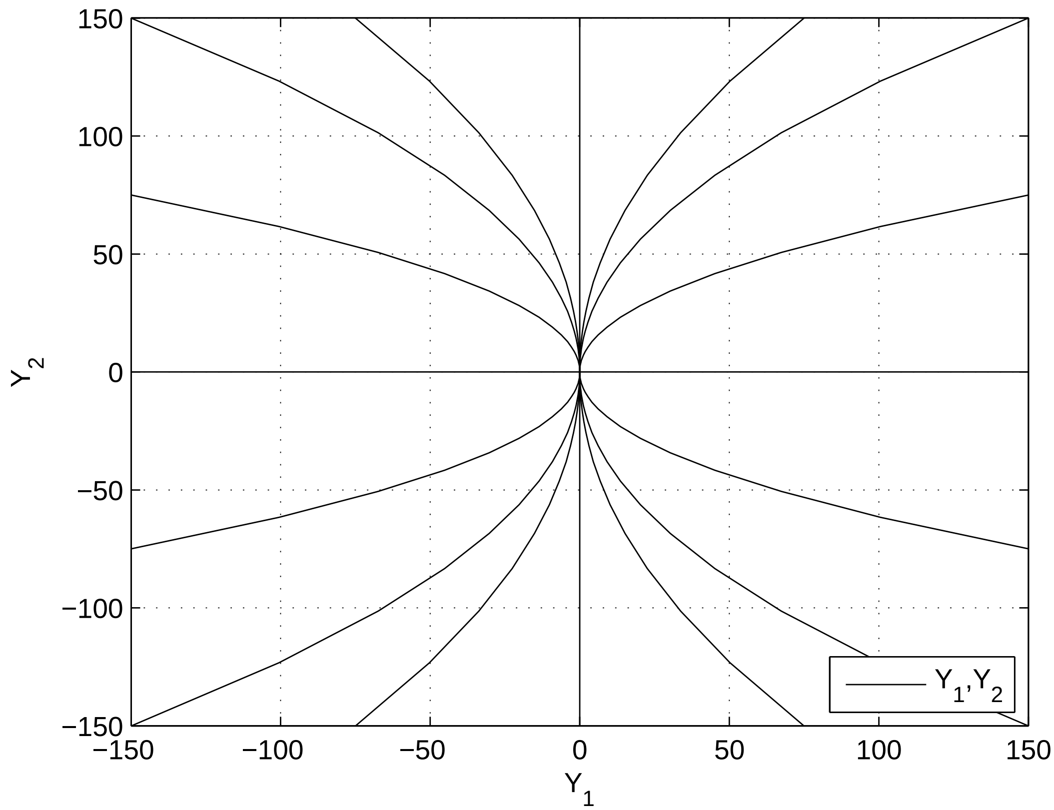

Figure 5: Portfolio Choice: Phase graph of the diagonalized system $Y = T^{-1}X$ with expectations

(0,0), where T is the matrix of eigenvectors, X is the original system from Figure 4, and Y is the diagonalized system.

Looking at Figure 5, it is easy to appreciate that closer to the steady state, all solutions tend towards a line of a certain inclination with respect to the expected disposable income Y_D^e. We will define this line as $H = mY_D^e + b*$ where m is the slope and $b*$ is the value of H where the line intersects Y_D^e-axis. For this particular system, that line corresponds to the trajectory of the solution of the system when $H_0 = 0$ and

$Y_{D0}^e = 0$. We can observe from Figure 3, that this line in the diagonalized system is the vertical H*-axis.

Knowing that the line $H = mY_D^e + b*$ corresponds to the vertical H*-axis from the diagonalized system, we may find the values of m and $b*$. For a given system, substituting the vector $Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ from the diagonalized system and obtaining its equivalent vector in the original system X we get

$$X = T \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The new vector X from the given system has the property that its components (denoted by X_1 and X_2) form the tangent of the angle of the slope m, such that $m = arctan\left(\frac{X_2}{X_1}\right)$, which is one of the variables that we wish to obtain.

In order to obtain b, we just substitute the new value of m in the equation $H = mY_D^e + b*$ with the values $H = \bar{H}$ and $Y_D^e = \bar{Y_D^e}$ given by the equilibrium point (steady state) in the original system X.

$b* = \bar{H} - m * \bar{Y_D^e}$

3 Liquidity Preference

3.1 Balance Sheet Matrix, Transactions Matrix and Linear System of Equations

This model consists basically of an extension to the model introduced in the last chapter with a new class of asset, bonds, as well as the concept of expectations and its role in the consumption function.

Bonds are defined here as perpetuities, which are never redeemed and pay the holder one dollar at the end of each period of time as a coupon. We will denote the bonds by B^L, where the superscript L stands "long", given the long term nature of these securities. Bills, which were denoted by B in the last chapter, will be denoted by B^S, where the superscript S stands for "short", given the short term nature of these securities in comparison to the bonds.

High-powered money and bills are distributed in the balance sheet matrix exactly in the same way as in chapter 2, following the same assumptions. Again, in this chapter, the production firms are assumed not to hold any asset, under the hypothesis that we are dealing with a "pure service economy". The central bank is assumed to have no net worth, under the assumption that all the profits are automatically transferred to the government.

Bonds are assumed to be issued directly by the government and held by the house-

Table 3: Liquidity Preference: Matrices

Balance Sheet	Households	Production	Govenrment	Central Bank	Σ
Money Stock	$+H$			$-H$	0
Bills	$+B_h^S$		$-B^S$	$+B_{cb}^S$	0
Bonds	$+B^L p_{BL}$		$-B^L p_{BL}$		0
Net worth	$+V$		$-V$	0	0
Transactions					
Consumption	$-C$	$+C$			0
Govt. expenditures		$+G$	$-G$		0
Wages	$+W$	$-W$			0
Interest on bills	$+r_S B_h^S$		$-r_S B^S$	$+r_S B_{cb}^S$	0
Interest on bonds	$+B^L$		$-B^L$		0
Central bank profits			$+r_S B_{cb}$	$-r_S B_{cb}$	0
Taxes	$-T$		$+T$		0
Σ	S_h	0	S_g	0	0
Flow of Funds					
Chage in money	$+\frac{dH}{dt}$ $(+\Delta H)$			$-\frac{dH}{dt}$ $(-\Delta H)$	0
Chage in bills	$+\frac{dB_h^S}{dt}$ $(+\Delta B_h^S)$		$-\frac{dB^S}{dt}$ $(-\Delta B^s)$	$+\frac{dB_{cb}^S}{dt}$ $(+\Delta B_{cb})$	0
Chage in bonds	$+\frac{dB^L}{dt} p_{BL}$ $(+\Delta B_L)$		$-\frac{dB^L}{dt} p_{BL}$ $(-\Delta B_L)$		0
Σ	S_h	0	S_g	0	0

holds only. Like any perpetuity, the price of these bonds p_{BL} is inversely proportional to the interest rate of their coupons, which is denoted by r_L. The interest rate for bills is denoted by r_S.

$$p_{BL} = \frac{1}{r_L} \tag{3.1}$$

Hence, in the balance sheet matrix, the total value of the stock of bonds is $B^L p_{BL}$, and B_L is in this case the number of bonds in circulation.

Having set clear all the assumptions regarding the balance sheet matrix, we can derive the following equations

$$V = H + B_h^S + B^L p_{BL} \tag{3.2}$$

$$V = B^S + B^L p_{BL} \tag{3.3}$$

$$H = B_{cb}^S \tag{3.4}$$

We may dismiss the following equation derived from the system, because it is a linear combination of equations (3.2), (3.2) and (3.3) and provides no new information, following the fact that we are dealing with a stock-flow consistent model:

$$B^S = B_h^S + B_{cb}^S$$

A new row appears in the transactions matrix with respect to the balance sheet of the last chapter: interest on bonds. Because each bond pays one dollar at the end of each period of time, the total interest coupon payments equals the quantity of bonds in circulation. Payments are issued by the government and received by households.

From the columns of the transactions and the flow of funds matrices we may obtain the following three equations:

$$\frac{dH}{dt} + \frac{dB_h^S}{dt} + \frac{dB^L}{dt} p_{BL} = -C - T + W + r_S B_h + B^L, \quad t > 0 \tag{3.5}$$

$$W = C + G \tag{3.6}$$

The following equation that arises from the Government column can be dismissed, since it can be obtained through a combination of the rest of the equations, and thus offers no new information:

$$\frac{dB^S}{dt} + \frac{dB^L p_{BL}}{dt} + T + r_s(B_{cb} - B^S) - B^L - G = 0, \quad t > 0$$

Equation (3.5) and the equation dismissed can be represented respectively in discrete time as follows:

$$\Delta H + \Delta B_h^S + \Delta dB^L p_{BL} = -C_t - T_t + W_t + r_S B_{ht-1} + B_{t-1}^L \tag{3.7}$$

$$\Delta B^S + \Delta B^L p_{BL} + T_t + r_s(B_{cbt-1} - B_{t-1}^S) - B_{t-1}^L - G = 0$$

Let us define the behaviour or of the following endogenous variables:

$$C = \alpha_1 Y_D^e + \alpha_2 V \tag{3.8}$$

$$T = \theta(W + r_S B_h^S + B^L) \tag{3.9}$$

$$W = \text{wNE} \tag{3.10}$$

Equations (3.8) and (3.9) can be alternatively defined in discrete time as

$$C_t = \alpha_1 Y_{Dt}^e + \alpha_2 V_{t-1} \tag{3.11}$$

$$T_t = \theta(W_t + r_S B_{ht-1}^S + B_{t-1}^L) \tag{3.12}$$

We will, again, generalize the behaviour of the expected disposable income Y_D^e with the following equation:

$$\dot{Y_D^e} = \eta(Y_D - Y_D^e), \quad t > 0$$

where

$$Y_D = W - T + r_S B_h^S + B^L \tag{3.13}$$

whose analogous discrete equation is

$$Y_{Dt} = W_t - T_T + r_S B^S_{ht-1} + B^L_{t-1} \tag{3.14}$$

We will consider the case where $\eta = 1$, so that the alternative definition in discrete time matches the one used by Godley and Lavoie, where that the expected disposable income equals the disposable income from the last period of time (166)[1]:

$$\dot{Y^e_D} = Y_D - Y^e_D, \quad t > 0 \tag{3.15}$$

whose analogous discrete equation is

$$\Delta Y^e_D = Y_{Dt-1} - Y^e_{Dt-1} \tag{3.16}$$

We will now define how the households allocate their wealth between bills, high-powered money and the newly introduced bonds. The next three equations describe this behaviour based on the proportion of each asset held by households with respect to their total wealth:

$$\frac{H}{V} = \lambda_{10} + \lambda_{12} r_S + \lambda_{13} r_L + \lambda_{14} \frac{Y^e_D}{V} \tag{3.17}$$

$$\frac{B^S_h}{V} = \lambda_{20} + \lambda_{22} r_S + \lambda_{23} r_L + \lambda_{24} \frac{Y^e_D}{V} \tag{3.18}$$

$$\frac{B^L p_{BL}}{V} = \lambda_{30} + \lambda_{32} r_S + \lambda_{33} r_L + \lambda_{34} \frac{Y^e_D}{V} \tag{3.19}$$

Equations (3.17), (3.18) and (3.19) will be represented in discrete time as

$$\frac{H^e}{V^e} = \lambda_{10} + \lambda_{12} r_S + \lambda_{13} r_L + \lambda_{14} \frac{Y_D^e}{V^e} \tag{3.20}$$

$$\frac{B_h^S}{V^e} = \lambda_{20} + \lambda_{22} r_S + \lambda_{23} r_L + \lambda_{24} \frac{Y_D^e}{V^e} \tag{3.21}$$

$$\frac{B^L p_{BL}}{V^e} = \lambda_{30} + \lambda_{32} r_S + \lambda_{33} r_L + \lambda_{34} \frac{Y_D^e}{V^e} \tag{3.22}$$

where

$$V_t^e = V_{t-1} + Y_{Dt}^e \tag{3.23}$$

Equations (3.17), (3.18) and (3.19) (and in a similar fashion, their discrete time alternative definition) can be expressed in the following matrix form, which makes their joint analysis easier to understand:

$$\begin{pmatrix} H \\ B_h^S \\ B^L p_{BL} \end{pmatrix} = \begin{pmatrix} \lambda_{10} \\ \lambda_{20} \\ \lambda_{30} \end{pmatrix} V + \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \begin{pmatrix} 0 \\ r_S \\ r_L \end{pmatrix} V + \begin{pmatrix} \lambda_{14} \\ \lambda_{24} \\ \lambda_{34} \end{pmatrix} Y_D^e$$

It is better to use the matrix form for an easier understanding of the adding constraints underlined by Tobin which must hold in order for the system to be stock-flow consistent (1969) [3]. This implies the following 5 conditions:

$$\lambda_{10} + \lambda_{20} + \lambda_{30} = 1$$

$$\lambda_{11} + \lambda_{21} + \lambda_{31} = 0$$

$$\lambda_{12} + \lambda_{22} + \lambda_{32} = 0$$

$$\lambda_{13} + \lambda_{23} + \lambda_{33} = 0$$

$$\lambda_{14} + \lambda_{24} + \lambda_{34} = 0$$

The first condition implies that the total share of each asset must sum to unity. The conditions 2-4 imply that the sum over all assets' response to each interest rate is zero. Condition 5 implies that sum over all assets' response to expectations in the disposable income is zero.

Note that λ_{11}, λ_{21} and λ_{31} are zero in this case because they represent sensitivity over the interest rate on cash, which is zero.

Another constraint proposed by Godley is that the sum of all the coefficients on rates of return, reading horizontally, should also sum to zero, more precisely, the coefficient on each positive "own" rate of return should equal the (negative of) the sum of all the other coefficients in the row (18)[1]. The grounds for introducing this constraint are that the effect on demand for the asset in question of an increase in the own rate of interest, with all the other rates remaining constant, should not be any different from that of a fall from the same size, in all the other rates, with the own rate staying put (144-5)[2]. This is represented with the three next constraints:

$$\lambda_{11} + \lambda_{12} + \lambda_{13} = 0$$

$$\lambda_{21} + \lambda_{22} + \lambda_{23} = 0$$

$$\lambda_{31} + \lambda_{32} + \lambda_{33} = 0$$

Finally, we will consider the government expenditures represented by G as exogenous, as we did in the previous chapter.

3.2 System of Differential Equations, Steady State Solutions and Stability of the Continuous Case

From equations (3.15), (3.9), (3.6), (3.8), (3.18) and (3.17), we obtain

$$\dot{Y}_D^e = C_1 Y_D^e + C_2 V + C_3 \tag{3.24}$$

where

$$C_1 = (1-\theta)(\alpha_1 + r_S\lambda_{24} + r_L\lambda_{34}) - 1$$

$$C_2 = (1-\theta)(\alpha_2 + r_S(\lambda_{20} + \lambda_{22}r_S + \lambda_{23}r_L) + r_L(\lambda_{30} + \lambda_{32}r_S + \lambda_{33}r_L))$$

$$C_3 = (1-\theta)G$$

From equations (3.24), (3.15) and (3.8), we obtain

$$\dot{V} = (C_1 + 1 - \alpha_1)Y_D^e + (C_2 - \alpha_2)V + C_3 \tag{3.25}$$

Equations (3.24) and (3.25) can be represented in matrix form as

$$\begin{pmatrix} \dot{Y_D^e} \\ \dot{V} \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ C_1 + 1 - \alpha_1 & C_2 - \alpha_2 \end{pmatrix} \begin{pmatrix} Y_D^e \\ V \end{pmatrix} + \begin{pmatrix} C_3 \\ C_3 \end{pmatrix} \tag{3.26}$$

The steady state solutions for these two state variables when G is constant depend on the initial conditions H_0, Y_0^e and V_0, and are given by

$$\bar{Y_D^e} = \frac{C_3\alpha_2}{C_2(\alpha_1 - 1) - C_1\alpha_2}$$

$$\bar{V} = \frac{C_3(1-\alpha_1)}{C_2(\alpha_1 - 1) - C_1\alpha_2}$$

Equations (3.24) and (3.25) form of a non-homogeneous linear two dimensional system. In general, as we saw in the previous chapter, for systems with not necessarily constant exogenous parameters, the solution of such a non-homogeneous linear system is given by the variation of constants formula. Thus, any solution would be of the following form:

$$X(t) = \Omega(t)\Omega^{-1}(t_0)\Omega_0 + \int_{t_0}^{t} \Omega(t)\Omega^{-1}(s)b(s)ds \tag{3.27}$$

where $\Omega(t)$ is a fundamental matrix of $A(t)$.

The stability of such system is limited by the existence of constants k and $\alpha* > 0$, such that

$$||\Omega(t)\Omega(s)^{-1}||_{\infty} \leq Ke^{-\alpha(t-s)} \quad t \geq s \geq 0$$

We are focusing on systems with constant exogenous variables. Again, as in the previous chapter, in the case where $A(t)$ is constant with respect to time or diagonal, the fundamental matrix $\Omega(t)$ is given by the exponential matrix $e^{tA} = e^{tTJT^{-1}} = Te^{tJ}T^{-1}$, where T is the matrix containing the eigenvectors associated with the matrix A and its eigenvalues. Matrix J is the matrix containing the eigenvalues in diagonal canonical form. The eigenvalues for this constant matrix are given by

$$\Lambda_{1,2} = \frac{(C_1 + C_2 - \alpha_2) \pm \sqrt{(C_1 + C_2 - \alpha_2)^2 + 4(C_1\alpha_2 + C_2(1 - \alpha_1))}}{2} \tag{3.28}$$

For the stability analysis of a system with constant parameters that is not centered at zero, the following change of variables will create a new system with an equilibrium point located at (0,0) and the stability of that system will guarantee the stability of the original one. Define

$$Y_D^e = y + \bar{Y_D^e}$$

$$V = z + \bar{V}$$

This change of variable gives rise the new system

$$\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ C_1 + 1 - \alpha_1 & C_2 - \alpha_2 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = A \begin{pmatrix} y \\ z \end{pmatrix}$$

For the point (0,0) to be stable in this new system, and thus the point ($\bar{Y_D^e}$,$\bar{V}$) in the original system to be stable as well, the real part of the eigenvalues of A, given by equation (3.28), must be negative. (By stable, we mean asymptotically stable).

As in the previous chapters, looking at figure 6, we can see that because all solutions followa spiraling trajectory. This means that the exogenous parameters were such that the eigenvalues of A came out to be complex.

Figure 6 shows the phase graph of Y_D^e and V for $0 < t < 100$ and different initial conditions. It uses the following values of exogenous variables: $G = 20$, $\alpha_1 = 0.6$, $\alpha_2 = 0.4$, $\theta = 0.2$, $r_S = 0.03$, $r_L = 0.7$, $\lambda_{20} = .44196$, $\lambda_{22} = 1.1$, $\lambda_{23} = 1$, $\lambda_{24} = .03$, $\lambda_{30} = .3997$, $\lambda_{32} = 1$, $\lambda_{33} = 1.1$ and $\lambda_{34} = .03$. The eigenvalues $\Lambda_{1,2}$ of A are complex in this case.

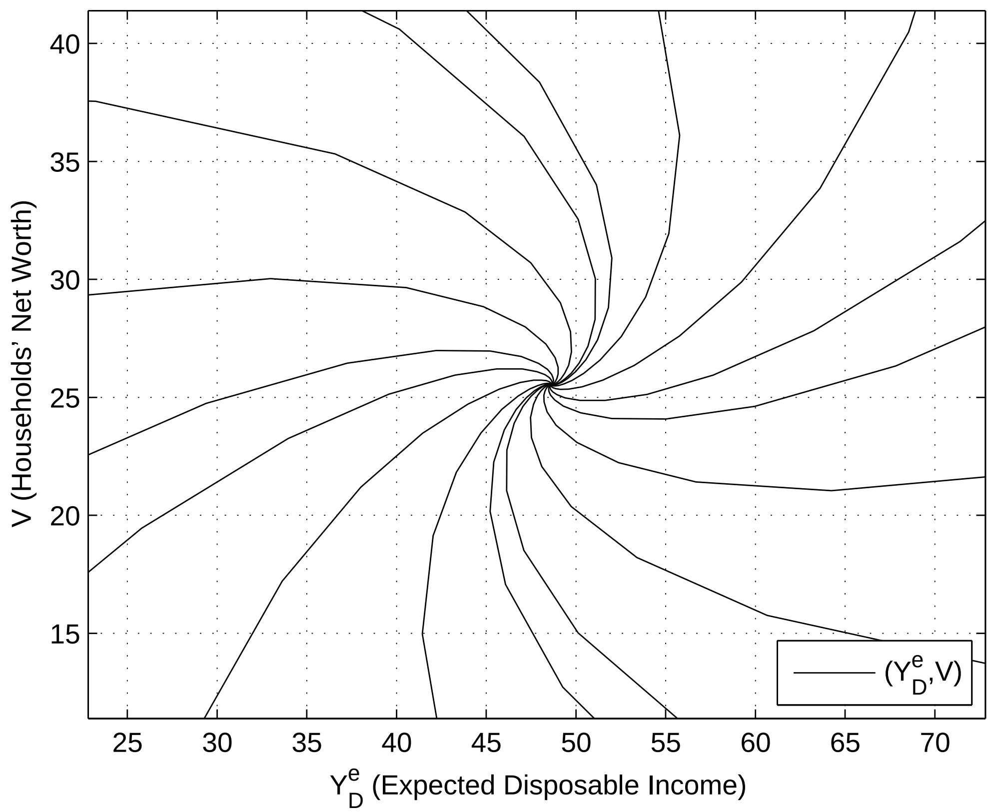

Figure 6: Liquidity Preference: phase Graph of original system X with expectations.

For the case of the perfect foresight, where $Y_D^e = Y_D$, equation (3.24) equals zero, and from that and equation (3.25), we obtain the following ordinary differential equation:

$$\dot{V} = \left(\frac{(C_1 + 1 - \alpha_1)(-C_2\alpha_2)}{C_1} + C_2 - \alpha_2 \right) V - \frac{C_3(1-\alpha_1)}{C_1} \tag{3.29}$$

It will converge monotonically to its equilibrium point, given by

$$\bar{V} = \frac{C_3(1-\alpha_1)}{(C_1 + 1 - \alpha_1) + C_1(C_2 - \alpha_2))} \tag{3.30}$$

and its solution is given by

$$V = -\frac{C_3(1-\alpha_1)}{(C_1+1-\alpha_1)+C_1(C_2-\alpha_2))} + \left(V_0 + \frac{C_3(1-\alpha_1)}{(C_1+1-\alpha_1)+C_1(C_2-\alpha_2))}\right) exp\left(\frac{(C_1+1-\alpha_1)+C_1(C_2-\alpha_2)}{C_1}t\right) \tag{3.31}$$

and whose stablity is conditioned by

$$0 > \frac{(C_1+1-\alpha_1)+C_1(C_2-\alpha_2)}{C_1} \tag{3.32}$$

3.3 System of Differential Equations and Steady State Solutions for the Discrete Case

The system in discrete time for expectations resembles closely that of Godley and Lavoie "Liquidity Preference" in continuous time. Its system of difference equations can be found on Appendix 5.1 from their book (165)[2] and it was used as a reference to define the alternative discrete time equations for our model. An important difference of our model is that we are not dealing with capital gains, since they can be computed at the beginning of the analysis adjusting the initial conditions of the system.

It is important to mention that unlike the system in continuous time, this cannot be reduced to a two dimensional system of ordinary differential equations, because in the discrete equations, we have the values of the previous and present period of B_h^S and B^L, creating a state variable. In continuous time, we only consider their instantaneous values, which eliminates dimensions from the system.

We can reduce the system of discrete equations to following system of difference equations:

$$\begin{pmatrix} \Delta Y_D \\ \Delta V \\ \Delta H \\ \Delta B^S \end{pmatrix} = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 \\ C_6 & C_7 & C_8 & C_9 \\ C_{11} & C_{12} & C_{13} & C_{14} \\ C_{16} & C_{17} & C_{18} & C_{19} \end{pmatrix} \begin{pmatrix} Y_{Dt-1} \\ V_{t-1} \\ H_{t-1} \\ B^S_{t-1} \end{pmatrix} + \begin{pmatrix} C_5 \\ C_{10} \\ C_{15} \\ C_{20} \end{pmatrix} \tag{3.33}$$

where

$$C_1 = (1-\theta)\alpha_1 - 1 \tag{3.34}$$

$$C_2 = (1-\theta)(\alpha_2 + r_L) \tag{3.35}$$

$$C_3 = -(1-\theta)r_S \tag{3.36}$$

$$C_4 = (1-\theta)(r_S - r_L) \tag{3.37}$$

$$C_5 = (1-\theta)G \tag{3.38}$$

$$C_6 = -\theta\alpha_1 \tag{3.39}$$

$$C_7 = (1-\theta)r_L - \theta\alpha_2 \tag{3.40}$$

$$C_8 = -(1-\theta)r_S \tag{3.41}$$

$$C_9 = (1-\theta)(r_S - R_L) \tag{3.42}$$

$$C_{10} = (1-\theta)G \tag{3.43}$$

$$C_{11} = (\lambda_{10} + \lambda_{12}r_S + \lambda_{13}r_L)(1-\alpha_1) + (1+\lambda_{14})C_1 \tag{3.44}$$

$$C_{12} = -(\lambda_{10} + \lambda_{12}r_S + \lambda_{13}r_L)\alpha_2 + (1+\lambda_{14})C_2 \tag{3.45}$$

$$C_{13} = (1+\lambda_{14})C_3 \tag{3.46}$$

$$C_{14} = (1+\lambda_{14})C_4 \tag{3.47}$$

$$C_{15} = (1+\lambda_{14})C_5 \tag{3.48}$$

$$C_{16} = -\theta\alpha_1 - (\lambda_{30} - \lambda_{32}r_S + \lambda_{33}r_L)(1-\alpha_1) - \lambda_{34}C_1 \tag{3.49}$$

$$C_{17} = r_L - \theta(\alpha_2 + r_L) + (\lambda_{30} + \lambda_{32}r_S + \lambda_{33}r_L)\alpha_2 - \lambda_{34}C_2 \tag{3.50}$$

$$C_{18} = -(1+\theta)r_S - \lambda_{34}C_3 \tag{3.51}$$

$$C_{19} = r_S - \theta(r_S - r_L) - \lambda_{34}C4 \tag{3.52}$$

$$C_{20} = (1-\theta)G - \lambda_{34}C_5 \tag{3.53}$$

Providing that the system is stable, its steady state solutions are given by equation (3.33) are found solving the following linear system of equations:

$$\begin{pmatrix} \bar{Y_D} \\ \bar{V} \\ \bar{H} \\ \bar{B^S} \end{pmatrix} = -\begin{pmatrix} C_1 & C_2 & C_3 & C_4 \\ C_6 & C_7 & C_8 & C_9 \\ C_{11} & C_{12} & C_{13} & C_{14} \\ C_{16} & C_{17} & C_{18} & C_{19} \end{pmatrix}^{-1} \begin{pmatrix} C_5 \\ C_{10} \\ C_{15} \\ C_{20} \end{pmatrix}$$

For the case of perfect foresight, dimensions of the system are reduced to three dimensional system

$$\begin{pmatrix} \Delta V \\ \Delta H \\ \Delta B^S \end{pmatrix} = \begin{pmatrix} C_{21} & C_{22} & C_{23} \\ C_{25} & C_{26} & C_{27} \\ C_{29} & C_{30} & C_{31} \end{pmatrix} \begin{pmatrix} V_{t-1} \\ H_{t-1} \\ B^S_{t-1} \end{pmatrix} + \begin{pmatrix} C_{24} \\ C_{25} \\ C_{32} \end{pmatrix} \tag{3.54}$$

$$C_{21} = C_7 - \frac{C_6 C_2}{C_1} \tag{3.55}$$

$$C_{22} = C_8 - \frac{C_6 C_3}{C_1} \tag{3.56}$$

$$C_{23} = C_9 - \frac{C_6 C_4}{C_1} \tag{3.57}$$

$$C_{24} = C_{10} - \frac{C_6 C_5}{C_1} \tag{3.58}$$

$$C_{25} = C_{12} - \frac{C_{11} C_2}{C_1} \tag{3.59}$$

$$C_{26} = C_{13} - \frac{C_{11}C_3}{C_1} \tag{3.60}$$

$$C_{27} = C_{14} - \frac{C_{11}C_4}{C_1} \tag{3.61}$$

$$C_{28} = C_{15} - \frac{C_{11}C_5}{C_1} \tag{3.62}$$

$$C_{29} = C - 17 - \frac{C_{16}C_2}{C_1} \tag{3.63}$$

$$C_{30} = C_{18} - \frac{C_{16}C_3}{C_1} \tag{3.64}$$

$$C_{31} = C_{19} - \frac{C_{16}C_4}{C_1} \tag{3.65}$$

$$C_{32} = C_{20} - \frac{C_{16}C_5}{C_1} \tag{3.66}$$

Providing that the system is asymptotically stable, its equilibrium point is given by

$$\begin{pmatrix} \bar{V} \\ \bar{H} \\ \bar{B}^S \end{pmatrix} = - \begin{pmatrix} C_{21} & C_{22} & C_{23} \\ C_{25} & C_{26} & C_{27} \\ C_{29} & C_{30} & C_{31} \end{pmatrix}^{-1} \begin{pmatrix} C_{24} \\ C_{28} \\ C_{32} \end{pmatrix}$$

4 Private Money

4.1 Balance Sheet Matrix, Transactions Matrix and Linear System of Equations

In this chapter, we will consider a model not adding more assumptions to our previous models, but with new rules on its own. From the balance sheet we may see that this model will consider the economy to be divided into three sectors: households, production and private banks. Hence there will not be a government sector nor a central bank.

The three assets in the economy are money deposits M, loans and fixed capital. Money deposits are held by households in banks. Therefore they are entered in the households columns with a plus (+) sign and in the banks columns with a minus (-) sign. Firms are assumed to hold no deposits.

Loans L are issued by the banks and held by the firms only. Therefore they are entered in the banks column with a plus (+) and in the firms column with a minus (-) sign.

There is a new asset in the economy: fixed capital K. Fixed capital comprises the assets and goods that a firm owns in order to meet its production needs. These goods are long-term in nature and face depreciation. Fixed capital assets don't have a liability counterpart because they are not financial assets. They will add up to the real value of the economy.

An extra assumption needed to be made with respect to the balance sheet matrix is that both firms and banks have zero net worth. With this assumption, because

Table 4: Private Money: Matrices

Balance Sheet	Households	Firms		Banks	Σ
Money deposits	$+M$			$-M$	0
Loans		$-L$		$+L$	0
Fixed capital		$-K$			$+K$
Net worth	V_h	0		0	V_h
Transactions	Households	Firms		Banks	
		Current	Capital		
Consumption	$-C$	$+C$			0
Wages	$+W$	$-W$			0
Investment		$+I$	$-I$		0
Depreciation		$-\delta K$	$+\delta K$		0
Interest on loans		$-r_l L$		$+r_l L$	0
Interest on deposits	$+r_m M$			$-r_m M$	0
Σ	S_h	0	$\delta K - I$	0	0
Flow of Funds					
Chage in loans			$-\frac{dL}{dt}$ $(-\Delta L)$	$+\frac{dL}{dt}$ $(+\Delta L)$	0
Chage in deposits	$+\frac{dM}{dt}$ $(+\Delta M)$			$-\frac{dM}{dt}$ $(-\Delta M)$	0
Σ	S_h		$\delta K - I$	0	0

we are dealing with a stock-flow consistent model, we may obtain the single equation arising from the balance sheet:

$$M = L = K = V_h \tag{4.1}$$

From the columns of the transactions and flow of funds matrices, we may obtain the following equations:

$$\frac{dM}{dt} = W + r_m M - C, \quad t > 0 \tag{4.2}$$

$$C - W - r_l L = \delta K - I \tag{4.3}$$

$$r_l = r_m \tag{4.4}$$

The following equation may be dismissed, since it can be obtained by a linear combination of equations (4.2), (4.3) and (4.4):

$$\frac{dL}{dt} + \delta K - I = 0, \quad t > 0$$

Equations (4.2), (4.3) and the dismissed equation have the following equivalent difference equations representation in discrete time respectively:

$$\Delta M = W_t + r_m M_{t-1} - C_t \tag{4.5}$$

$$C_t - W_t - r_l L_{t-1} = \delta K_{t-1} - I_t \tag{4.6}$$

$$\Delta L + \delta K_{t-1} - I_t = 0$$

Because of the assumptions we made for this model, the interest rate on loans and deposits must be equal. The term δK constitutes the depreciation, which is assumed to be a constant proportion δ of the value of the stock of tangible capital (fixed capital).

Like in the previous chapters, the wage bill is defined by

$$W = \text{wNE} \tag{4.7}$$

Consumption is defined by the following equation

$$C = \alpha_0 + \alpha_1(W + r_m M) + \alpha_2 M \tag{4.8}$$

where α_0 is a new constant proportion while α_1 and α_2 are the former α_0 and α_1 from the previous models. This exogenous new α_0 parameter will make the system solvable.

Equation (4.8) has the following analogous discrete time representation:

$$C_t = \alpha_0 + \alpha_1(W_t + r_m M_{t-1}) + \alpha_2 M_{t-1} \tag{4.9}$$

We will also define the rate at which firms accumulate capital:

$$\dot{K} = I - \delta K, \quad t > 0 \tag{4.10}$$

whose alternative representation in discrete time is given by

$$\Delta K = I_t - \delta K_{t-1} \tag{4.11}$$

Equation (4.10) says that the rate at which firms accumulate capital equal investment minus depreciation costs, where δ is an exogenous parameter. The next equation describes how firms invest:

$$I = \gamma(\kappa Y - K) + \delta K \tag{4.12}$$

where κ is an exogenous constant target level of capital to output ratio and γ is also exogenous. Its alternative representation in discrete time is given by

$$I_t = \gamma(\kappa Y_{t-1} - K_{t-1}) + \delta K_{t-1} \tag{4.13}$$

Equation (4.12) says that it is assumed that investment is composed by the sum of depreciation allowances, plus a constant exogenous proportion of the difference of a proportion of the GDP that the firms aim spend in order to maintain a normal rate of utilization of their capacity, minus the existing capital.

Finally, the GDP for this model will be defined as

$$Y = C + I \tag{4.14}$$

4.2 Differential Equation and Steady State Solution of the continuous time model and System of Difference Equations for the discrete model

The setting up of this continuous time system resembles closely that of Godley and Lavoie "Simple Model with Private Bank Money", and whose system of difference equations can be found on Appendix 7.1 from their book (248)[2] and it was used as a reference to define the alternative discrete time equations for our model.

Combining equations (4.1), (4.3), (4.8), (4.12) and (4.14), we obtain

$$Y = \frac{(\alpha_2 + \delta(1-\alpha_1) - \gamma)K + \alpha_0}{1 - \alpha_1 - \gamma\kappa} \tag{4.15}$$

Plugging in equations (4.12) and (4.15) back into equation (4.10), we obtain

$$\dot{K} = \left(\frac{\gamma\kappa(\alpha_2 + \delta(1-\alpha_1) - \gamma) - \gamma(1 - \alpha_1 - \gamma\kappa)}{1 - \alpha_1 - \gamma\kappa}\right) K + \left(\frac{\alpha_0\gamma\kappa}{(1 - \alpha_1 - \gamma\kappa)}\right) \tag{4.16}$$

Differential equation (4.16) encompasses all the information need in order to know the value of all the other exogenous variables.

Substituting the discrete time equations (4.1), (4.6), (4.5), (4.9) and (4.11), and isolating $\Delta Y = Y_t - Y_{t-1}$, we get

$$\Delta Y = \frac{\alpha_0 + (\alpha_1 + \gamma\kappa - 1)Y_{t-1} + ((1-\alpha_1)\delta + \alpha_2 - \gamma)\, K_{t-1}}{1-\alpha_1}$$

On the other hand, substituting equation (4.13) in (4.11), we get

$$\Delta K = \gamma\kappa Y_{t-1} - \gamma K_{t-1}$$

It interesting to see that the alternative discrete time model does not consist of a model with a single state variable, but two, which can be represented as follows:

$$\begin{pmatrix} \Delta K \\ \Delta Y \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \begin{pmatrix} K_{t-1} \\ Y_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ C_5 \end{pmatrix}$$

where

$$C_1 = -\gamma$$

$$C_2 = \gamma\kappa$$

$$C_3 = \frac{(1-\alpha_1)\delta + \alpha_2 - \gamma}{1-\alpha_1}$$

$$C_4 = \frac{\gamma\kappa}{1-\alpha_1} - 1$$

$$C_5 = \frac{\alpha_0}{1-\alpha_1}$$

The reason behind it comes from equation 4.9, where the instantaneous GDP Y is considered, while in its alternative equation, Y_{t-1} is taken into account.

Focusing our attention in the continuous time model, equation 4.13 is a differential equation of the form

$$\dot{K} + AK = b \tag{4.17}$$

where

$$A = -\left(\frac{\gamma\kappa(\alpha_2 + \delta(1-\alpha_1) - \gamma) - \gamma(1-\alpha_1-\gamma\kappa)}{1-\alpha_1-\gamma\kappa}\right)$$

$$b = \left(\frac{\alpha_0\gamma\kappa}{(1-\alpha_1-\gamma\kappa)}\right)$$

This differential equation is of first order non-homogeneous and its solution is found considering the integrating factor

$$e^{\int_{s_0}^{t} A ds}$$

and after some manipulation of equation 4.13 together with this integrating factor we will arrive to the following solution:

$$K = e^{-\int_{s_0}^{t} A ds} \int_{s_0}^{t} b e^{\int_{u_0}^{s} A du} ds + K_0 e^{-\int_{s_0}^{t} A ds} \tag{4.18}$$

which yields to

$$K = \frac{b}{A} + (K_0 - \frac{b}{A})e^{-At}$$

Differential equation 4.15 is asymptotically stable providing that the following condition holds:

$$-\left(\frac{\gamma\kappa(\alpha_2 + \delta(1-\alpha_1) - \gamma) - \gamma(1-\alpha_1-\gamma\kappa)}{1-\alpha_1-\gamma\kappa}\right) > 0$$

Its steady state, assuming that the system is asymptotically stable, is given by

$$\bar{K} = \frac{b}{A}$$

Figure 7 shows the evolution over time of K and h_Y for $0 < t < 100$ for inicial condition $H_0 = 0$ and the following all constant exogenous parameters: $\alpha_0 = 25$, $\alpha_1 = 0.75$, $\alpha_2 = 0.1$, $r_l = r_m = 0.04$, $\gamma = 0.15$, $\kappa = 1$, $\delta = 0.1$.

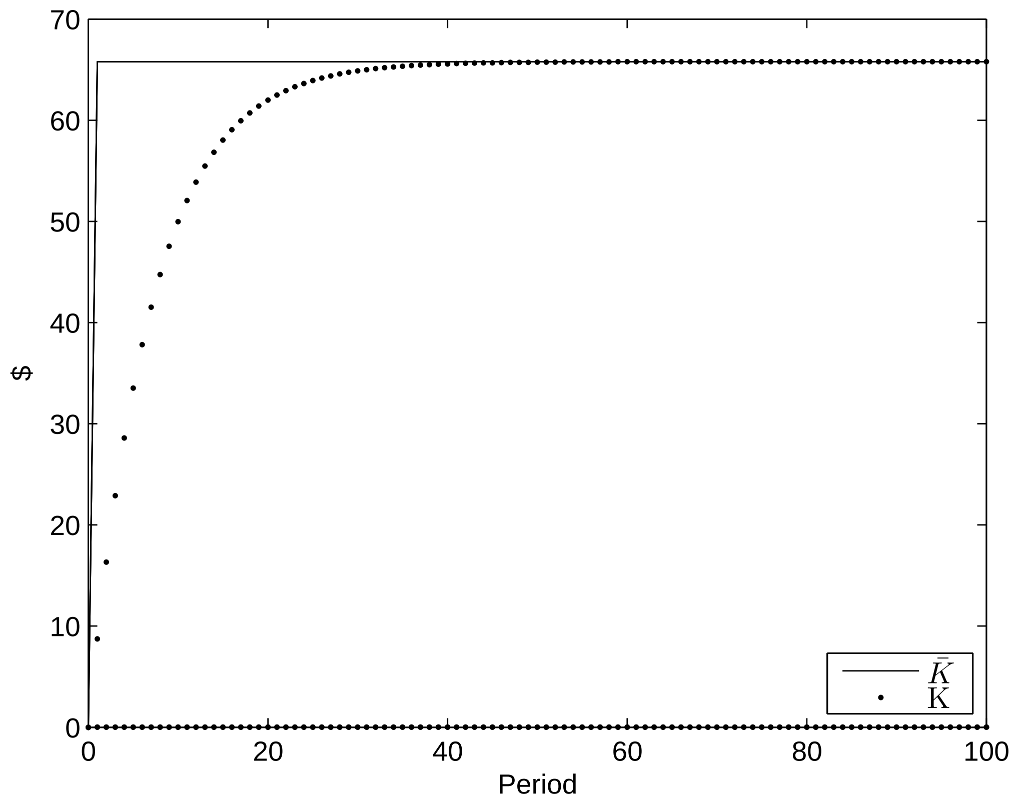

Figure 7: Private Money: Evolution of K over time and steady state $\bar{K}$.

5 Government and Private Money with Liquitidy Preference

5.1 Balance Sheet Matrix, Transactions Matrix and Linear System of Equations

In this chapter we will combine the models from chapters 3 and 4 as well as their assumptions regarding the different sectors and the distribution of the assets.

In summary, the economy will be divided into 5 sectors: Households, production firms, private banks, the government sector and the central bank. Out of these, only the households and the government will be considered to have a net worth. There will be also 6 different types of assets. High-powered money H will be held by households and issued by the central bank. Money deposits M will be held by households and issued by private banks. Loans L will be issued by banks and used by the production firms only. Fixed capital K, as in chapter 4 belongs to firms and since it is not a financial asset, there will be no liability counterpart and it will add up to the net worth of the economy. Bills B^S are issued by the government and held by the households and the central bank (hence the subindices). Bonds B^L issued by the government and held by the households only. There is an extra asset, bank advancements A, which will act as a buffer between the difference in supply and demand of loans and deposits. Like in chapter 4, the nature of the bills is short term and the nature of the bonds is long term. Bonds are treated as perpetuities which pay one currency unit each period of time, and whose price is denoted by p_{BL}. Hence, the interest rate concerning to them (the long term interest rate) is given by,

Table 5: Government and Private Money with Liquidity Preference: Matrices

B. Sheet	HH	Firms		P. banks	Govt.	C. Bank	Σ
H. money	$+H$					$-H$	0
M. deposits	$+M$			$-M$			0
Loans		$-L$		$+L$		0	0
Advances				$-A$		$+A$	0
Fixed Cap.		$+K$					$+K$
Bills	$+B_h^S$				$-B^S$	$+B_{cb}^S$	0
Bonds	$+B^L p_{BL}$				$-B^L p_{BL}$		0
Net worth	V_h	0		0	V_g	0	V
Trans.	HH	Firms		P. banks	Govt.	C. Bank	
		Curr.	Cap.				
Consumption	$-C$	$+C$					0
Govt. exp.		$+G$			$-G$		0
Investment		$+I$	$-I$				0
Wages	$+W$	$-W$					0
Depreciation		$-\delta K$	$+\delta\kappa$				0
Int. on bills	$+r_S B_h^S$				$-r_S B^S$	$+r_S B_{cb}^S$	0
Int. on bonds	$+B^L$				$-B^L$		0
Int. on dep.	$+r_m M$			$-r_m M$			0
Int. on loans		$-r_l L$		$+r_l L$			0
C. Bank prof.					$+r_S B_{cb}^S$	$-r_S B_{cb}^S$	0
Taxes	$-T$				$+T$		0
Σ	S_h	0	$\delta K - I$	0	S_g	0	0
Flow of F.							
Ch. in money	$+\frac{dH}{dt}$					$-\frac{dH}{dt}$	0
Ch. in bills	$+\frac{dB_h^S}{dt}$				$-\frac{dB^S}{dt}$	$+\frac{dB_{cb}^S}{dt}$	0
Ch. in bonds	$+\frac{dB^L}{dt} p_{BL}$				$-\frac{dB^L}{dt} p_{BL}$		0
Ch. in dep.	$+\frac{dM}{dt}$			$-\frac{dM}{dt}$			0
Ch.			$-\frac{dL}{dt}$	$+\frac{dL}{dt}$			0
Σ	S_h	0	$\delta K - I$	0	S_g	0	0

$$p_{BL} = \frac{1}{r_L} \tag{5.1}$$

Having summarized the assumptions made in the balance sheet, we may derive five equations from its columns and two relevant equations from its rows.

$$H + M + B^S + B^L p_{BL} = V_h \tag{5.2}$$

$$L = K = M + A \tag{5.3}$$

$$-\left(B^S + B^L p_{BL}\right) = V_g \tag{5.4}$$

$$H = B^S_{cb} + A \tag{5.5}$$

$$K = V = V_h + V_g \tag{5.6}$$

$$B^S = B^S_h + B^S_{cb} \tag{5.7}$$

In a similar fashion, we may derive the following relevant equations from the transactions and flow of funds matrices:

$$\frac{dH}{dt} + \frac{dB_h^S}{dt} + \frac{dB^L}{dt}p_{BL} + \frac{dM}{dt} = W + r_S B_h^S + B^L + r_m M - T - C, \quad t > 0 \tag{5.8}$$

$$C + I + G - W - r_l L - \delta K = 0 \tag{5.9}$$

$$\delta K - I + \frac{dL}{dt} = 0 \tag{5.10}$$

$$r_m M = r_l L = r_m (L - A) \tag{5.11}$$

$$H = A + B_{cb}^S \tag{5.12}$$

$$\frac{dB^S}{dt} + \frac{dB^L}{dt}p_{BL} = -T - r_S B_{cb}^S + B^L + r_S B^S + G, \quad t > 0 \tag{5.13}$$

We still need to define the following equations in order for the system to be solvable.

$$C = \alpha_1 Y_D^e + \alpha_2 V_h \tag{5.14}$$

In the case of the consumption, we won't be using the extra parameter α_0 because it will not be necessary in this case for the system to be solvable. Let us define the taxes collected by the government as a constant proportion of the disposable income of the households:

$$T = \theta(W + r_S B_h^S + B^L + r_m M) \tag{5.15}$$

The wage bill will be defined as in the previous models as

$$W = \text{wNE} \tag{5.16}$$

where as in every model exposed here, (w) is the wage rate, N is the total population of the model and E is the employment rate. Next, define investment as

$$I = \gamma(\kappa Y - K) + \delta K \tag{5.17}$$

and the corresponding change in capital as

$$\dot{K} = I - \delta K, \quad t > 0 \tag{5.18}$$

The rate of accumulation of fixed capital by the firms $\dot{K}$ is defined as in the previous chapter.

The GDP Y for this chapter will also include the government spending:

$$Y = C + I + G \tag{5.19}$$

We may also define the expected income in a similar fashion as in the previous chapters:

$$\dot{Y_D^e} = \eta(Y_D - Y_D^e), \quad t > 0$$

which for $\eta = 1$ reduces to

$$\dot{Y_D^e} = Y_D - Y_D^e, \quad t > 0 \tag{5.20}$$

where

$$Y_D = W + r_S B_h^S + B^L + r_m M - T \tag{5.21}$$

Now we have define the equations of the distribution of households' assets.

$$H = (\lambda_{10} + \lambda_{12} r_m + \lambda_{13} r_S + \lambda_{14} r_L) V_h^e + \lambda_{15} Y_D^e \tag{5.22}$$

$$M = (\lambda_{20} + \lambda_{22} r_m + \lambda_{23} r_S + \lambda_{24} r_L) V_h^e + \lambda_{25} Y_D^e \tag{5.23}$$

$$B_h^S = (\lambda_{30} + \lambda_{32} r_m + \lambda_{33} r_S + \lambda_{34} r_L) V_h^e + \lambda_{35} Y_D^e \tag{5.24}$$

$$B^L p_{bL} = (\lambda_{40} + \lambda_{42} r_m + \lambda_{43} r_S + \lambda_{44} r_L) V_h^e + \lambda_{45} Y_D^e \tag{5.25}$$

The adding up constraints underlined by Tobin (1969)[3] must hold. Representing equations (5.22), (5.23), (5.24), and (5.25) can in matrix form will give us a better visualization:

$$\begin{pmatrix} H^e \\ M \\ B_h \\ BLp_{bL} \end{pmatrix} = \begin{pmatrix} \lambda_{10} \\ \lambda_{20} \\ \lambda_{30} \\ \lambda_{40} \end{pmatrix} V_h^e + \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \lambda_{34} \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} \end{pmatrix} \begin{pmatrix} 0 \\ r_m \\ r_S \\ r_L \end{pmatrix} V_h^e + \begin{pmatrix} \lambda_{15} \\ \lambda_{25} \\ \lambda_{35} \\ \lambda_{45} \end{pmatrix} Y_D^e$$

Finally, in order to make the system solvable, we need to define G as exogenous and for now we will set it as constant.

5.2 System of Differential Equations, Steady State Solutions and Stability

This model does not represent any particular model from the book of Godley and Lavoie, but it is a combination of those of Chapters 3 and 4 in this thesis, while each of these emulates models from the book just previously mentioned.

Combining equations (5.19), (5.14), and (5.17), we get

$$Y = \frac{\alpha_1 Y_D^e + \alpha_2 V_h + (\delta - \gamma)K + G}{1 - \gamma\kappa} \tag{5.26}$$

Combining equations (5.26), (5.18) and (5.17), we get

$$\dot{K} = \gamma\kappa \left(\frac{\alpha_1 Y_D^e + \alpha_2 V_h + (\delta - \gamma)K + G}{1 - \gamma\kappa} \right) - \gamma K \tag{5.27}$$

Combining equations (5.20), (5.21), (5.9), (5.24)and (5.25), we get

$$\begin{aligned}\dot{Y_D^e} &= (1-\theta)\left(\frac{\alpha_1 Y_D^e + \alpha_2 V_h + (\delta-\gamma)K + G}{1-\gamma\kappa}\right) - (1-\theta)\delta K \\ &+ (1-\theta)r_S((\lambda_{30} + \lambda_{32}r_m + \lambda_{33}r_S + \lambda_{34}r_L)V_h + \lambda_{35}Y_D^e) \\ &+ (1-\theta)r_L((\lambda_{40} + \lambda_{42}r_m + \lambda_{43}r_S + \lambda_{44}r_L)V_h + \lambda_{45}Y_D^e) - Y_D^e\end{aligned} \tag{5.28}$$

Combining equations (5.28), (5.8) and (5.14), we get

$$\dot{V_h} = \dot{Y_D^e} + (1-\alpha_1)Y_D^e + \alpha_2 V_h \tag{5.29}$$

$$\begin{pmatrix} \dot{K} \\ \dot{Y_D^e} \\ \dot{V_h} \end{pmatrix} = \begin{pmatrix} C_1 & C_2 & C_3 \\ C_5 & C_6 & C_7 \\ C_9 & C_{10} & C_{11} \end{pmatrix} \begin{pmatrix} K \\ Y_D^e \\ V_h \end{pmatrix} + \begin{pmatrix} C_4 \\ C_8 \\ C_{12} \end{pmatrix} \tag{5.30}$$

where

$$C_1 = \gamma\kappa\left(\frac{(\delta-\gamma)}{1-\gamma\kappa}\right) - \gamma$$

$$C_2 = \gamma\kappa\left(\frac{\alpha_1}{1-\gamma\kappa}\right)$$

$$C_3 = \gamma\kappa\left(\frac{\alpha_2}{1-\gamma\kappa}\right)$$

$$C_4 = \gamma\kappa \left(\frac{G}{1-\gamma\kappa} \right)$$

$$C_5 = (1-\theta) \left(\frac{\delta-\gamma}{1-\gamma\kappa} - \delta \right)$$

$$C_6 = (1-\theta) \left(\frac{\alpha_1}{1-\gamma\kappa} + r_S\lambda_{35} + r_L\lambda_{45} \right) - 1$$

$$\begin{aligned} C_7 = (1-\theta) &\left(\frac{\alpha_2}{1-\gamma\kappa} \right) \\ &+ (1-\theta) r_S((\lambda_{30} + \lambda_{32} r_m + \lambda_{33} r_S + \lambda_{34} r_L) \\ &+ (1-\theta) r_L((\lambda_{40} + \lambda_{42} r_m + \lambda_{43} r_S + \lambda_{44} r_L) \end{aligned}$$

$$C_8 = (1-\theta) \left(\frac{G}{1-\gamma\kappa} \right)$$

$$C_9 = (1-\theta) \left(\frac{\delta-\gamma}{1-\gamma\kappa} - \delta \right)$$

$$C_{10} = (1-\theta) \left(\frac{\alpha_1}{1-\gamma\kappa} + r_S\lambda_{35} + r_L\lambda_{45} \right) - \alpha_1$$

$$C_{11} = (1-\theta)\left(\frac{\alpha_2}{1-\gamma\kappa}\right)$$
$$+ (1-\theta)r_S((\lambda_{30} + \lambda_{32}r_m + \lambda_{33}r_S + \lambda_{34}r_L)$$
$$+ (1-\theta)r_L((\lambda_{40} + \lambda_{42}r_m + \lambda_{43}r_S + \lambda_{44}r_L) + \alpha_2$$

$$C_{12} = (1-\theta)\left(\frac{G}{1-\gamma\kappa}\right)$$

Equilibrium point is given by

$$\begin{pmatrix} K \\ Y_D^e \\ V_h \end{pmatrix} = -\begin{pmatrix} C_1 & C_2 & C_3 \\ C_5 & C_6 & C_7 \\ C_9 & C_{10} & C_{11} \end{pmatrix}^{-1} \begin{pmatrix} C_4 \\ C_8 \\ C_{12} \end{pmatrix} \tag{5.31}$$

Asymptotic stability is assured as long as the real part of the eigenvalues Λ is negative

$$\Lambda = \det\left(\begin{pmatrix} C_1 & C_2 & C_3 \\ C_5 & C_6 & C_7 \\ C_9 & C_{10} & C_{11} \end{pmatrix} - \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{pmatrix}\right) \tag{5.32}$$

For the case with perfect foresight, where $Y_D^e = Y_D$, the three dimensional system given by (5.30) reduces to the two dimensional system

$$\begin{pmatrix} \dot{K} \\ \dot{V}_h \end{pmatrix} = \begin{pmatrix} C_{21} & C_{22} \\ C_{24} & C_{25} \end{pmatrix} \begin{pmatrix} K \\ V_h \end{pmatrix} + \begin{pmatrix} C_{23} \\ C_{26} \end{pmatrix}$$

where the constants are now given by

$$C_{21} = C_1 - \frac{C_2 C_5}{C_6}$$

$$C_{22} = C_3 - \frac{C_2 C_7}{C_6}$$

$$C_{23} = C_4 - \frac{C_2 C_8}{C_6}$$

$$C_{24} = C_9 - \frac{C_{10} C_5}{C_6}$$

$$C_{25} = C_{11} - \frac{C_{10} C_7}{C_6}$$

$$C_{26} = C_{12} - \frac{C_{10} C_8}{C_6}$$

The steady states for $\dot{K}$ and $\dot{V}_h$ for this system is given by

$$\bar{Y_D^e} = \frac{C_{23} C_{25} - C_{22} C_{26}}{C_{22} C_{24} - C_{21} C_{23}} \tag{5.33}$$

$$\bar{V_h} = \frac{C_{21}C_{26} - C_{23}C_{24}}{C_{22}C_{24} - C_{21}C_{23}} \tag{5.34}$$

And stability is assured as long as the real part of the eigenvalues Λ, which in this case are given by

$$\Lambda_{1,2} = \frac{(C_{21} + C_{25}) \pm \sqrt{(C_{21} + C_{25})^2 - 4(C_{21}C_{25} - C_{22}C_{24})}}{2} \tag{5.35}$$

6 Conclusion

The main achievement of this thesis was to derive simple linear systems of differential equations whose state variables are sufficient to know the values of all the endogenous variables for some stock-flow consistent models proposed by Godley and Lavoie, and knowing some key features of the state variable, such as stability and its steady state, let us know how the whole system behaves.

For the continuous time models, we saw how the systems derived were much simpler in terms of the number of dimensions than their discrete counterparts, with very little difference in assumptions. We saw that even though they became more complex in nature (involving more varied basket of assets to be held by households), the systems of ordinary differential equations for the case with expectations ended up being of two dimensions and of just one dimension for the case with perfect foresight, for the chapters 2, 3 and 5. For the discrete time models, the dimensions of the models were increased by the number of assets included, and further increased and extra dimension for the case with expectations, due the interaction of the value of the asset involving its values of current and previous period within the same system. In this sense, continuous time models offer much simpler models with very little variation in their approach and dynamics arising from their construction.

Also, even though in general, discrete time models proposed by Godley and Lavoie are much more complex than the continuous time models proposed in this thesis in terms of the dimensions of the systems derived from them, the systems of difference equations derived here are much simpler than in their original presentation by Godley and Lavoie. Their steady states are easier to compute and even for systems with non-constant parameters, the numerical analysis is much more intuitive and efficient.

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