FREE LATTICES AND THEIR SUBLATTICES

FREE LATTICES AND THEIR SUBLATTICES

By

CLAIRE WALLACE ZELLER, B.Sc.

A Thesis

Submitted to the Faculty of Graduate Studies

in Partial Fulfillment of the Requirements

for the Degree

Master of Science

McMaster University

October 1965

MASTER OF SCIENCE (1965)
(Mathematics)

McMASTER UNIVERSITY
Hamilton, Ontario

TITLE: Free Lattices and Their Sublattices

AUTHOR: Claire Wallace Zeller, B.Sc. (McMaster University)

SUPERVISOR: Dr. G. Bruns

NUMBER OF PAGES: v, 64

SCOPE AND CONTENTS: This thesis deals with the construction of Free Lattices on various sets of generators, ranging from a set of unordered generators to a countable set of chains. It also considers sublattices of the structures presented.

PREFACE

The purpose of this thesis is to assemble the most important results of Free Lattices into one consolidated form, giving an orderly presentation which is not necessarily chronological, but rather natural in development. Only free lattices without additional relations are considered, that is, free modular, distributive and Boolean lattices are not considered.

In Chapter 0, in the language of Universal Algebra, are found the less familar definitions and theorems which shall be used.

In Chapter 1, we construct FL(n), the free lattice generated by n unordered generators; then CF(P), the free lattice generated by a partially ordered set, P, and preserving only the order of P; then FL(P, $\mathfrak{U}$, $\mathfrak{C}$), the free lattice generated by P, a partially ordered set, preserving the ordering as well as finite meets and joins of P; and finally FL($n_1+\ldots+n_m$), the free lattice generated by a set P consisting of countably many chains of lengths $n_1, n_2, \ldots$ etc., preserving the order of P. This development seems natural since at each step we give the lattice more complex structure by changing the properties of the set of generators.

Chapter 2 considers the sub-lattices of the free lattices constructed in Chapter 1. In particular, all distributive sublattices of a free lattice are characterized.

ACKNOWLEDGMENTS

The author expresses his deep appreciation to Dr. G. Bruns for his helpful suggestions, criticisms and guidance in the preparation of this thesis, and his patient reading and correcting of it.

The author wishes to thank Mrs. Judi Wayman for her typing of the thesis and to Mr. John Stewart Jr. for his work on the technical diagrams.

Finally, the author offers thanks for his Ontario Graduate Fellowship, his National Research Council Fellowship and the financial assistance given him by his University.

TABLE OF CONTENTS

CHAPTER 0

Preliminaries

Many of the notions we shall need in the ensuing chapters can not only be stated in the setting of lattice theory, but can be stated in the more general setting of universal algebra. For this reason we shall state the definitions in terms of a class K of algebras (in our case the algebras will have the same operations, i.e. the same type), but realizing that we shall use the definitions relative to the class of all lattices. We assume definitions of algebra or algebraic system, isomorphism, homomorphism and subalgebra are known.

Definition 1. Let A be a K-algebra (i.e. $A \in K$) and X a subset of A. Then A is said to be freely generated by X and X is said to be a free set of generators of A iff X generates A and every mapping $f:X \longrightarrow B$ $(B \in K)$ can be extended to a homomorphism, $\bar{f}:A \longrightarrow B$. An algebra $A \in K$ is said to be a free K-algebra iff it has a free set of generators.

Definition 2. A K-algebra A is said to be a free K-product of a non-empty family $(A_i)_{i \in I}$ of subalgebras of A iff A is generated by $\bigcup_{i \in I} A_i$ and for every $B \in K$ and every family $(f_i)_{i \in I}$ of homomorphisms $f_i:A_i \longrightarrow B$ there exists a homomorphism $f:A \longrightarrow B$ which extends every f_i.

To insure the existence of free K-algebras and of free K-products we assume that K is a non-trivial equational class in the wider sense, that is, K is the class of all models of some finite or infinite set of equations, and there exists $A \in K$ such that $|A| \geq 2$.

Definition 3. A class K of algebras is said to have the embedding property iff for any $A, B \in K$ there exists $C \in K$ such that A and B are isomorphic to subalgebras of C.

Definition 4. A class K of algebras is said to have the amalgamation property iff the following conditions are satisfied: if A, B_0, $B_1 \in K$ and f_0 and f_1 are isomorphisms of A into B_0 and into B_1, respectively, then there exists $C \in K$ and isomorphisms g_0 and g_1 of B_0 and of B_1 respectively, into C, such that $g_0 \cdot f_0 = g_1 \cdot f_1$.

Theorem 1. Suppose the class K of algebras is non-trivial and equational in the wider sense, and assume that K has the embedding property and the amalgamation property. If A is a free K-product of a family $(A_i)_{i \in I}$, if B_i is a subalgebra of A_i for each $i \in I$, and if B is a subalgebra of A that is generated by the set $\bigcup_{i \in I} B_i$, then B is a free K-product of $(B_i)_{i \in I}$.

Proof. See Jonsson [9] .

Corollary. Take K the class in Theorem 1. If A is a free K-product of $(A_i)_{i \in I}$, and if for each $i \in I$, A_i is isomorphic to a subalgebra of a free K-algebra with m_i generators, then A is isomorphic to a subalgebra of a free K-algebra with $\sum_{i \in I} m_i$ generators.

Proof. See Jonsson [9] .

Remark. If K is taken to be the class of all lattices, that K is non-trivial and equational is clear. The direct product of two lattices being therefore again a lattice, and coupled with the fact that every lattice has a one-element sublattice, it follows K has the embedding property. The proof that K also satisfies the amalgamation property is found in Jonsson [8].

Definition 5. If K is the class of all lattices, and $L \in K$, a lattice, then $a \in L$ is called meet-irreducible ($\wedge$-irreducible) iff $a = x \wedge y$ implies $a = x$ or $a = y$. Dually we define join-irreducible.

CHAPTER I

Free Lattices

1. Construction of FL($\mathfrak{n}$)

In this section a free lattice on $\mathfrak{n}$ (unordered) generators will be constructed. It is pertinent to observe that this structure will depend only on the cardinal $\mathfrak{n}$, since two free lattices on the same number of free generators are isomorphic and any lattice isomorphic to a free lattice is also free with the same number of generators. Hence the free lattice we shall construct on $\mathfrak{n}$ unordered generators will be unique up to isomorphism.

Take G any set whose cardinality is $\mathfrak{n}$.

Definition 1.1: We define inductively sets T_n and S_n $(n = 1,2,\ldots)$ as follows:

$$T_1 = S_1 = G$$

$$T_n = \{(\lambda, A) \mid \lambda = 0,1;\ \emptyset \neq A \subseteq S_{n-1};\ A \text{ finite; and for each } a \in A, \text{ there exists a } k_a \in N \text{ such that } a \in T_{k_a} \text{ and } n = \sum_{a \in A} k_a + 1\}$$

$$S_n = S_{n-1} \cup T_n$$

We can assume without loss of generality that $G \cap T_n = \emptyset$ for all $n > 1$. If this were not the case, some G' of same cardinality could be selected such that $G' \cap T_n = \emptyset$ for $n > 1$, and by the above remarks the same structure would result.

Remark 1. $S_1 \subseteq S_2 \subseteq \ldots \subseteq S_n \subseteq \ldots$ by definition.

Remark 2. $T_n \cap T_m = \emptyset$ for $n \neq m$.

Proof. We prove $T_n \cap T_m = \emptyset$ for all $n \neq m$ by induction over n. For $n = 1$ the claim holds by assumption. Assume the claim to be true for all $n < p$ $(p > 1)$, and take $(\lambda, A) \in T_p \cap T_m$, m arbitrary. For every $a \in A$ there exists, by construction, numbers p_a, m_a such that $a \in T_{p_a} \cap T_{m_a}$, $p = \Sigma\, p_a + 1$, and $m = \Sigma\, m_a + 1$. Since $p_a < p$ we have $p_a = m_a$ by inductive hypothesis, hence $p = m$ and the claim is proved.

Define $S = \bigcup_{n=1}^{\infty} S_n = \bigcup_{n=1}^{\infty} T_n$. By remark 2, every $u \in S$ is contained in exactly one T_n. We define the length, $L(u)$, of u to be this unique n, i.e.

Definition 1.2

$$L(u) = n \quad \text{iff} \quad u \in T_n$$

Definition 1.3. We define $u \leq v$ by induction over $L(u) + L(v) = n$ as follows:

1. $n = 2$, i.e. $u, v \in G$ and $u \leq v$ iff $u = v$
2. $n \geq 3$ $u \leq v$ iff one or more of the following hold
   a. $u = (0, A)$ and $x \leq v$ for all $x \in A$
   b. $u = (1, B)$ and $x \leq v$ for at least one $x \in B$
   c. $v = (0, A)$ and $u \leq x$ for at least one $x \in A$
   d. $v = (1, B)$ and $u \leq x$ for all $x \in B$

Lemma 1.1. $(0,A) \leq v$ iff $x \leq v$ for all $x \in A$. Dually $u \leq (1,B)$ iff $u \leq x$ for all $x \in B$.

Proof (of the first statement). Clearly $x \leq v$ for all $x \in A$ implies $(0,A) \leq v$ by 2a. We show the converse by induction over $L(0,A) + L(v) = n$, clearly $n \geq 3$. If $n = 3$, then $v \in G$ and hence $(0,A) \leq v \in G$ implies $y \leq v$ for all $y \in A$, as required. If $n \geq 4$, we check the four possibilities of Definition 1.3, part 2. If $(0,A) \leq v$ by 2a, then $x \leq v$ for all $x \in A$ follows by definition. Since $(0,A)$ is not of the form $(1,B)$, $(0,A) \leq v$ by 2b, does not occur. Suppose $(0,A) \leq v$ by 2c, i.e. $v = (0,A')$. Then $(0,A) \leq (0,A')$ implies $(0,A) \leq x'$ for some $x' \in A$ by 2c, which implies $x \leq x'$ for all $x \in A$ by induction hypothesis, which in turn implies $x \leq (0,A')$ for all $x \in A$ by 2c. If, finally, $(0,A) \leq v$ holds by 2d, then $v = (1,B)$ and $(0,A) \leq y$ for all $y \in B$. This, by induction hypothesis, gives $x \leq y$ for all $x \in A$, $y \in B$, and, again by 2d: $x \leq (1,B) = v$ for all $x \in A$.

Lemma 1.2. The relation $\leq$ is reflexive on S.

Proof. We show $u \leq u$ for all $u \in S$ by induction over $L(u)$. If $L(u) = 1$, $u \leq u$ since $u = u$ in G. Assume $L(u) > 1$ and $u = (0,A)$. Since $x \leq x$ for all $x \in A$ by induction hypothesis, we have $x \leq (0,A)$ for all $x \in A$ by 2c, and $(0,A) \leq (0,A)$ by 2a, thus $u \leq u$. If $u = (1,B)$, the proof is dual.

Lemma 1.3. The relation $\leq$ is transitive on S.

Proof. We show that $u \leq v$ and $v \leq w$ imply $u \leq w$ for $u,v,w \in S$ by induction over $L(u) + L(v) + L(w) = n$. If $n = 3$, then $u = v = w$, hence $u \leq w$.

Assume $n \geq 4$. If $u \leq v$ or $v \leq w$ by Definition 1.3, 1, then $u \leq w$ trivially. We now consider the possibilities of Definition 1.3, 2. If $u \leq v$ by 2a, then $u = (0,A)$ and $x \leq v$ for all $x \in A$. $v \leq w$ by assumption gives $x \leq w$ for all $x \in A$ by induction hypothesis, and then $u = (0,A) \leq w$ by 2a. If $v \leq w$ holds by 2d, then the proof is dual.

Assume $u \leq v$ by 2b. In this case $u = (1,B)$ and $x \leq v$ for at least one $x \in B$. $v \leq w$ by assumption gives $x \leq w$ for at least one $x \in B$ by induction hypothesis and $u = (1,B) \leq w$ by 2b. If $v \leq w$ by 2c, the proof is dual.

Assume $u \leq v$ by 2c. Thus $v = (0,A)$ and $u \leq x$ for at least one $x \in A$. But $x \leq w$ for all $x \in A$ by Lemma 1.1 implies $u \leq w$ by induction hypothesis. If $v \leq w$ by 2b, then the proof is dual.

Finally, the only case remaining is that $u \leq v$ by 2d, and $v \leq w$ by 2a. For this case $v = (1,B)$ and $v = (0,A)$ which is impossible.

Remark. Lemma 1.2 and lemma 1.3 show that $\leq$ is a quasi-ordering on S. Clearly, if we define the relation R on S by uRv iff $u \leq v$ and $v \leq u$, R is an equivalence relation. For each $u \in S$, let $[u]$ be the equivalence class of u with respect to R. Let $F = S/R$ be the quotient set and define a relation $\leq$ on F as follows: $[u] \leq [v]$ iff $u \leq v$. Clearly, this definition is independent of the representatives. The relation $\leq$ is a partial order on F.

Lemma 1.4. For all $A,B \subseteq S$; $A,B \neq \emptyset$ and A,B finite,

i. $[(0,A)] = [0,A] = \bigvee_{a \in A} [a]$, and

ii. $[(1,B)] = [1,B] = \bigwedge_{b \in B} [b]$.

Proof re i. $a \leq a$ for all $a \in A$, implies $a \leq (0,A)$ by Definition 1.3 (2c) and thus $[a] \leq [0,A]$ for all $a \in A$ making $[0,A]$ an upper bound of $\{[a] \mid a \in A\}$. Suppose $[a] \leq [u]$ for all $a \in A$, and hence $a \leq u$ for all $a \in A$. This gives $(0,A) \leq u$ by 2a and thus $[0,A] \leq [u]$. Therefore $[0,A]$ is the least upper bound of $\{[a] \mid a \in A\}$.

Part ii follows dually.

Remark. Thus F is a lattice, and by the above lemma and a suitable inductive argument is certainly generated by $\{[x] \mid x \in G\}$. We shall write the equivalence classes $[x]$ by simply x whenever $x \in G$.

Theorem 1.1. F is the free lattice generated by G.

Proof. Let L be an arbitrary lattice. Take $f: G \longrightarrow L$ any function and define functions $f_n: T_n \longrightarrow L$ and $\tilde{f}_n: S_n \longrightarrow L$ inductively as follows: $f_1 = f$ and $\tilde{f}_1 = f$.

$$f_{n+1}(0,A) = \bigvee_L \tilde{f}_n(A) \quad \text{and} \quad f_{n+1}(1,B) = \bigwedge_L \tilde{f}_n(B)$$

$$\tilde{f}_{n+1} = \tilde{f}_n \cup f_{n+1}$$

Also, we define $\tilde{f}: S \longrightarrow L$ by $\tilde{f} = \cup \tilde{f}_n$.

Claim 1. For all $u,v \in S$, if $u \leq v$ then $\tilde{f}(u) \leq \tilde{f}(v)$. We prove this by induction over $L(u) + L(v) = n$, considering the conditions for $u \leq v$ in Definition 1.3.

If $n = 2$; $u,v \in G$ $u \leq v \Rightarrow u = v \Rightarrow \tilde{f}(u) = f(u) = f(v) = \tilde{f}(v)$ and $\tilde{f}(u) \leq \tilde{f}(v)$.

Assume $n \geq 3$. If $u \leq v$ by 2a, then $u = (0,A)$ and

$$a \leq v \text{ for all } a \in A \;\Rightarrow\; \tilde{f}(a) \leq \tilde{f}(v) \text{ for all } a \in A \text{ by induction}$$

$$\Rightarrow \bigvee_{a \in A} f(a) = \tilde{f}(0,A) \leq \tilde{f}(v)$$

$$\Rightarrow \tilde{f}(u) \leq \tilde{f}(v)$$

If $u \leq v$ by 2b, then $u = (1,B)$ and $b \leq v$ for at least one $b \in B$

$$\Rightarrow \tilde{f}(b) \leq \tilde{f}(v) \text{ for at least one } b \in B \text{ by induction hypothesis}$$

$$\Rightarrow \bigwedge_{b \in B} f(b) = \tilde{f}(1,B) \leq \tilde{f}(v)$$

$$\Rightarrow \tilde{f}(u) \leq \tilde{f}(v)$$

If $u \leq v$ by 2c, then $v = (0,A)$ and $u \leq a$ for at least one $a \in A$

$$\Rightarrow \tilde{f}(u) \leq \tilde{f}(a) \text{ for at least one } a \in A \text{ by the induction hypothesis}$$

$$\Rightarrow \tilde{f}(u) \leq \bigvee_{a \in A} f(a) = \tilde{f}(0,A)$$

$$\Rightarrow \tilde{f}(u) \leq \tilde{f}(v)$$

If $u \leq v$ by 2d, then $v = (1,B)$ and $u \leq b$ for all $b \in B$

$$\Rightarrow \tilde{f}(u) \leq \tilde{f}(b) \text{ for all } b \in B, \text{ by induction hypothesis}$$

$$\Rightarrow \tilde{f}(u) \leq \bigwedge_{b \in B} f(b) = \tilde{f}(1,B)$$

$$\Rightarrow \tilde{f}(u) \leq \tilde{f}(v)$$

Claim 2. $uRv \Rightarrow \tilde{f}(u) = \tilde{f}(v)$

$$\left.\begin{array}{l} uRv \Rightarrow u \le v \Rightarrow \tilde{f}(u) \le \tilde{f}(v) \\ uRv \Rightarrow v \le u \Rightarrow \tilde{f}(v) \le \tilde{f}(u) \end{array}\right\} \Rightarrow \tilde{f}(u) = \tilde{f}(v)$$

Claim 3. Define $\bar{f}: F \longrightarrow L$ by $\bar{f}([u]) = \tilde{f}(u)$ independent of the representative u, and $\bar{f}|_G = f$; then $\bar{f}$ is a lattice homomorphism.

We observe that $\bar{f}$ is easily a function, then if $[u]$, $[v] \in F$, we see:

$$\bar{f}([u] \vee [v]) = \tilde{f}(0, \{u,v\}) = \tilde{f}(u) \vee \tilde{f}(v)$$
$$= \bar{f}([u]) \vee \bar{f}([v]).$$

and dually $\bar{f}([u] \wedge [v]) = \bar{f}([u]) \wedge \bar{f}([v])$.

Thus besides being generated by G, the lattice F is free.

Clearly, every equivalence class $[u]$ contains an element of shortest length. It is our next aim to show that this element is unique.

Lemma 1.5. If $u = (\lambda, A)$ is of shortest length, then for all $a \in A$, $L(a)$ is minimal.

Proof. Without loss of generality assume $\lambda = 0$. Take $a \in A$ such that $L(a)$ is not minimal. Then, there exists an a' such that aRa' and $L(a') < L(a)$. Consider $u' = (0, (A - \{a\}) \cup a')$.

Claim. $u'Ru$. We know $x \le u$ for all $x \in A - \{a\}$, and also $a' \le a \le u$ since aRa'. Therefore by 2a, $(0, (A-\{a\}) \cup a') \le u$, that is $u \le u'$. Also $u \le u'$ since $x \le u'$ for all $x \in A - \{a\}$ and $a \le a' \le u'$ by the fact that aRa' making $(0, A) \le u'$ by 2a. Thus we have $u'Ru$. Now,

$$L(u) = \sum_{\substack{x \in A \\ x \neq a}} L(x) + L(a) + 1 > \sum_{\substack{x \in A \\ x \neq a}} L(x) + L(a') + 1 = L(u')$$

which contradicts the assumption that u was of shortest length, and thus the lemma is proved.

<u>Lemma 1.6</u>. If u and v are equivalent elements of shortest length, then $u = v$ (i.e. canonical forms are unique).

<u>Proof</u>. We observe first: if $(0,A)$ is a shortest representative of its class, then no $a_0 \in A$ is of the form $a_0 = (0,C)$. Assume contrary to the claim, that there exists an $a_0 \in A$ such that $a_0 = (0,C)$. Analogous to the proof in Lemma 1.5 we get $(0,A)R\,(0,(A-\{a_0\})\cup \bigcup C)$. But,

$$L(0,(A-\{a_0\})\cup \bigcup C) = \sum_{\substack{a \in A \\ a \neq a_0}} L(a) + \sum_{c \in C} L(c) + 1$$

$$= \sum_{\substack{a \in A \\ a \neq a_0}} L(a) + L(a_0)$$

$$< \sum_{\substack{a \in A \\ a \neq a_0}} L(a) + L(a_0) + 1 = L(0,A)$$

which contradicts the shortest length of $(0,A)$.

We show next: if (λ_1,A) and (λ_2,B) are equivalent elements of shortest length, then $\lambda_1 = \lambda_2$. Assume $\lambda_1 = \lambda_2$, say $\lambda_1 = 0$ and $\lambda_2 = 1$. From $(\lambda_1,A)R(\lambda_2,B)$ we get $(0,A) \leq (1,B)$ and $(1,B) \leq (0,A)$. The first inequality implies $a \leq b$ for all $a \in A$ and $b \in B$ by Lemma 1.1. The second implies by definition that either there exists some $b \in B$ with $b \leq (0,A) \leq b$ or some $a \in A$ with $a \leq (1,B) \leq a$. In the first case we get $b\,R(0,A)$ which together with $L(1,B) > L(b)$ and $(1,B)R(0,A)$ contradicts the shortest length of $(0,A)$. Dually, in the second case, and hence $\lambda_1 = \lambda_2$.

We now prove the Lemma by induction over $L(u) = L(v) = n$ where $u = (\lambda_1, A)$ and $v = (\lambda_2, B)$. If $n = 1$, then $u, v \in G$ and uRv then implies $u = v$. For $n \geq 1$, we assume without loss of generality $\lambda_1 = \lambda_2 = 0$. By uRv we know $(0,A) \leq (0,B)$ and $(0,B) \leq (0,A)$, and by our first observation we know no $a \in A$ or $b \in B$ has the form $(0,C)$. We check the remaining possibilities.

Case 1. Assume there exists $a_o = (1, A_o) \in A$ and some $a' \in A_o$ with the property $a' \leq (0,B)$. Then $(1, A_o) \leq a' \leq (0,B) \leq (0,A)$, and also,

$$a \leq (0,B) \leq (0,A) \quad \text{for } a \neq a'.$$

Hence $(0,A) \leq (0, \{a \mid a \neq a_o\} \cup \{a'\}) \leq (0,B) \leq (0,A)$ and

$$(0,A) \; R \; (0, \{a \mid a \neq a_o\} \cup \{a'\}).$$

But,

$$L(0, \{a \mid a \neq a_o\} \cup \{a'\}) = \sum_{\substack{a \in A \\ a \neq a_o}} L(a) + L(a') + 1, \text{and}$$

$$L(a') + 1 \leq L(1, A_o) = L(a_o) \quad \text{implies that}$$

$$\sum_{\substack{a \neq a_o \\ a \in A}} L(a) + L(a') + 1 \leq \sum_{a \in A} L(a) < L(0,A)$$

which is a contradiction of the shortest length of $(0,A)$. Therefore case 1 does not hold, and similarly its dual case does not hold.

Case 2. Assume $(0,A) \leq (0,B)$ and $(0,B) \leq (0,A)$ by Definition 1.3, 2c, which is the only remaining possibility. In this case we have for each $a \in A$ a $b_a \in B$ such that $a \leq b_a$, and also for each $b \in B$ we have $a_b \in A$ such that $b \leq a_b$. Hence for each $a \in A$, $a \leq b_a \leq a_{b_a}$. Suppose for some $a \in A$

we have $a \neq a_{b_a}$. Then $(0,A)R(0,A-\{a\})$ and $L(0,A) > L(0,A-\{a\})$, as before, and we would have a contradiction of the shortest length of (0,A). Thus $a = a_{b_a}$, and hence aRb_a for all $a \in A$. By lemma 1.5 and the inductive hypothesis we get $a = b_a$ for all $a \in A$, and hence $A \subseteq B$. Dually we obtain $B \subseteq A$, hence $A = B$.

2. Sublattices of FL(n)

Having obtained the free lattice on n generators, FL(n), we shall examine an interesting result concerning its sublattices, namely "FL(3) contains FL(n) for finite or countable n .

Definition 2.1. The subset M of a lattice L is called free iff the sublattice L(M) of L, generated by M is freely generated by M.

Lemma 2.1. If $|G| = n$ and n finite, $n \geq 2$, and $u \in S = \bigcup_{n=1}^{\infty} T_n$, $x \in G$, then $x \leq u$ iff $u \nleq (0,G-\{x\})$ and $x \geq u$ iff $u \ngeq (1,G-\{x\})$.

Proof. We show $x \leq u$ iff $u \nleq (0,G-\{x\})$, and the proof of the second statement is dual. Clearly not both of $x \leq u$ and $u \leq (0,G-\{x\})$ hold, since then we have $x \leq (0,G-\{x\})$ which requires $x \leq y$ for some $y \in G-\{x\}$, a contradiction. We show at least one of these conditions must hold, that is, we assume $x \nleq u$ and show $u \leq (0,G-\{x\})$. The proof of this fact is by induction over L(u). If L(u) = 1, then $u \leq (1,G-\{x\})$ is trivial. Let $L(u) = n \geq 2$, and consider the following two cases.

Case 1. If $u = (0,A)$, then $x \leq u$ by assumption implies $x \nleq a$ for all $a \in A$. By induction $a \leq (0,G-\{x\})$ for all $a \in A$, and hence $u = (0,A) \leq (0,G-\{x\})$ by Definition 1.3, part 2a.

Case 2. If $u = (1,B)$, the result follows analogously to Case 1.

Theorem 2.1. In $FL(n)$, a set $M \subseteq FL(n)$ is free iff it satisfies 1) If $u \in M$, $M' \subseteq M$ and M' finite and $u \leq \bigvee M'$, then $u \in M'$ and 2) If $u \in M$, $M' \subseteq M$ and M' finite and $u \geq \bigwedge M'$, then $u \in M'$.

Proof. M free implies the conditions easily, for assume not, then $u \leq \bigvee M'$ for some finite M' and $u \notin M'$. Now, define $f: M \longrightarrow B_o$, B_o the two element Boolean lattice by $f(u) = e$, $f(m) = 0$ for all $m \in M-u$. M free implies there exists a lattice homomorphism $\bar{f}: FL(M) \longrightarrow B_o$ such that $\bar{f}|_M = f$. But $e = \bar{f}(u) \geq \bigvee \bar{f}(M')$ for all $M' \subseteq M$, M' finite implies $u \geq \bigvee M'$, a contradiction. Similarly condition 2) is verified.

For the converse, a class of representatives C of M is taken and the free lattice on C is constructed analogously to $FL(n)$. This free lattice is shown isomorphic to $L(M)$ to complete the proof.

Lemma 2.2. If $n = 3$, then in $FL(3)$ the following $\{u_i\}$ forms a free set.

$$u_1 = [x_1 \wedge (x_2 \vee x_3)] \vee [x_2 \wedge (x_1 \vee x_3)]$$

$$u_2 = [x_1 \wedge (x_2 \vee x_3)] \vee [x_3 \wedge (x_1 \vee x_2)]$$

$$u_3 = [x_1 \vee (x_2 \wedge x_3)] \wedge [x_2 \vee (x_1 \wedge x_3)]$$

$$u_4 = [x_1 \vee (x_2 \wedge x_3)] \wedge [x_3 \vee (x_1 \wedge x_2)]$$

Proof. The proof relies on simple applications of Theorem 2.1.

Theorem 2.2. $FL(3)$ has $FL(n)$ as a sublattice for any finite or countable n.

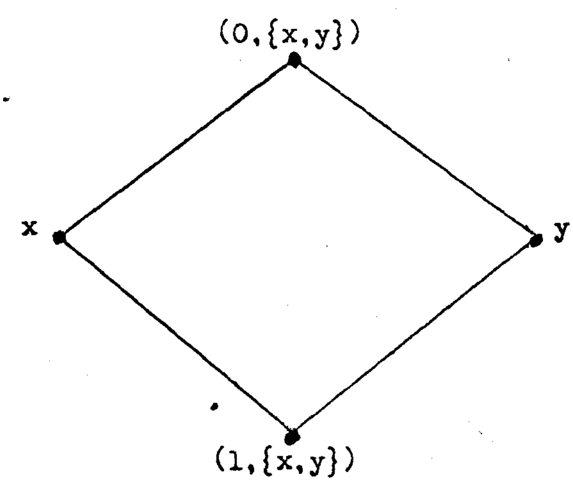
<u>Proof</u>. By Lemma 2.2, $\{u_1,u_2,u_3,u_4\} \subseteq FL(3)$ is free. We prove the Lemma by induction over the number of elements in a free set $\{u_i\}$. Assume $\{u_1,\ldots,u_k\} \subseteq FL(3)$ is free. In particular, by Lemma 2.1 $\{u_{k-2},u_{k-1}u_k\}$ is free, and then there exist v_1,v_2,v_3,v_4 in the lattice generated by $\{u_{k-2},u_{k-1},u_k\}$ which are free by Lemma 2.2. Hence $\{u_1,\ldots,u_{k-3},v_1,v_2,v_3,v_4\}$ is shown free by Theorem 2.1, and by the induction it is clear that $FL(n)$ is contained in $FL(3)$ as a sublattice.

<u>Remark 1</u>. We observe that a sublattice $S \subseteq L$, a free lattice, is free iff there exists a set $X \subseteq S$ with X a free set of generators for S.

<u>Remark 2</u>. $FL(1)$ is trivial, $FL(2)$ is easily finite and is shown in Figure 2.1, and by the above discussion $FL(3)$ is infinite.

<u>Remark 3</u>. It follows from Theorem 2.2, that for $n \geq 3$ and any finite or countable k, $FL(n)$ has $FL(k)$ as a sublattice.

<u>Remark 4</u>. $FL(m)$ is not modular for $m \geq 3$, since the assumption that $m = 3$, and $FL(3)$ is modular yields contradiction.



<u>Figure 2.1: FL(2)</u>

3. Construction of CF(P)

We have considered in the preceding sections the free lattice on a set of unordered generators, which was offered by Whitman in 1940-41. Dilworth in 1945 introduced a free lattice generated by a partially ordered set P preserving meets and joins of pairs of elements in P where they exist, and suggested the possibility of considering the (in a certain sense) weaker structure which would preserve only order. This structure was offered by Dean and is analogous to FL(n), and we shall call it the completely free lattice, CF(P)

Definition 3.1. Let P be any partially ordered set, and analogous to Definition 1.1 take $T_1 = S_1 = P$ and define

$$T_n = \{ (\lambda, A) \mid \lambda = 0,1,\ A \subseteq S_{n-1},\ A \neq \emptyset,\ A \text{ finite and for each } a \in A\ \exists\ k_a \text{ such that } n = \sum_{a \in A} k_a + 1 \}$$

$$S_n = S_{n-1} \cup T_n \qquad \text{and}$$

$$S = \bigcup_{n=1}^{\infty} S_n = \bigcup_{n=1}^{\infty} T_n$$

Definition 3.2. $u \in S \Rightarrow u \in T_n$ for some n, then the length of the element u (L(u) is defined to be n (L(u) = n).

Definition 3.3. On S define the relation $\leq$ as follows:

$$\begin{array}{llll} & u,v \in S, & u \le v & \text{by induction over} \\ L(u) + L(v) = n^{\cdot} & \underline{n=1} & u \le v & \text{iff } u \le v \text{ in } P \\ & \underline{n>1} & u \le v & \text{iff one or more of the following hold} \end{array}$$

a. $u = (0,A)$ and $x \le v$ for all $x \in A$

b. $u = (1,B)$ and $x \le v$ for at least one $x \in B$

c. $v = (0,A)$ and $u \le x$ for at least one $x \in A$

d. $v = (1,B)$ and $u \le x$ for all $x \in B$

It is clear that these basic definitions are directly analogous to those used in constructing FL(n). The results obtained for FL(n) are obtained for CF(P) by precisely the same reasoning and argument and shall therefore be assumed without further proof or statement.

4. <u>Construction of FL(P,$\mathfrak{U}$,$\mathfrak{C}$)</u>

We shall denote by FL(P), the free lattice generated by a partially ordered set <u>P</u> preserving the order in P and the least upper bounds (joins) and greatest lower bounds (meets) of pairs of elements which exist in P, and we shall denote by FL(P, $\mathfrak{U}$, $\mathfrak{C}$), the free lattice generated by the partially ordered set P preserving:

i. the ordering of P;

ii. those least upper bounds of a system $\mathfrak{U}$ of finite non-empty subsets of P which possess least upper bounds in P; and

iii. those greatest lower bounds of a system $\mathfrak{C}$ of finite non-empty subsets of P which possess greatest lower bounds in <u>P</u>.

Certainly FL(P) (originally illustrated by Dilworth) is but a particular case of FL(P, $\mathfrak{U}$, $\mathfrak{C}$) (by Dean). Unfortunately the structure of FL(P, $\mathfrak{U}$, $\mathfrak{C}$) does not follow the reasonably easy format of FL(n) and CF(P). For this reason it is necessary to modify the technique, and rephrase some definitions, but maintaining a similar technique to the one outlined for FL(n).

Let P be a partially ordered set with $\leq$ the relation. Let $\mathfrak{U} \subseteq \mathfrak{P}(P)$ with the properties

i. $p \leq q$ in P implies $\{p,q\} \in \mathfrak{U}$

ii. $S \in \mathfrak{U}$ implies S finite, $S \neq \emptyset$, and $\bigvee S \in P$

and $\mathfrak{C} \subseteq \mathfrak{P}(P)$ with the properties

i'. $p \leq q$ in P implies $\{p,q\} \in \mathfrak{C}$

ii'. $S \in \mathfrak{C}$ implies S finite, $S \neq \emptyset$, and $\bigwedge S \in P$.

For the convenience of the reader, we shall write for $(0, \{A,B\})$ the element $A \vee B$, and for the element $(1, \{A,B\})$, the element $A \wedge B$, which is the association made as before.

<u>Definition 4.1</u>. We define the elements of FL(P, $\mathfrak{U}$, $\mathfrak{C}$) recursively as follows: i. $p,q,r,\ldots \in P$ are elements of FL(P, $\mathfrak{U}$, $\mathfrak{C}$) and ii. if A and B are in FL(P, $\mathfrak{U}$, $\mathfrak{C}$), then $A \vee B$ and $A \wedge B$ are in FL(P, $\mathfrak{U}$, $\mathfrak{C}$).

<u>Definition 4.2</u>. The length of an element will be (intuitively) the number of generators appearing in it (admitting multiple appearances). More particularly $A \in P$ will mean A has length 1 ($L(A) = 1$) and further $L(A \vee B) = L(A \wedge B) = L(A) + L(B)$.

<u>Definition 4.3</u>. In $FL(P, \mathfrak{U}, \mathfrak{C})$ define $A \le B$ if

1. $L(B) = 1$ i.e. $B = p$ for some $p \in P$ in which case $A \le p$ iff for some integer n, $A \le p(n)$ with

   a. $A \le p(0)$ iff $A = q$ and $q \le p$ in P and inductively iff one or more of the following hold

   b. $A = A_1 \vee A_2$ and $A_i \le p(n-1)$ for $i = 1,2$

   c. $A = A_1 \wedge A_2$ and $A_i \le p(n-1)$ for $i = 1$ or 2

   d. $\exists\ S \in \mathfrak{C}\ .\!\ni\!.\ \bigwedge S \le p$ in P and $A \le s(n-1)$ for all $s \in S$.

2. $L(A) = 1$ i.e. $A = p$ for some $p \in P$ in which case $p \le B$ if for some integer n $p \le B(n)$ with

   a. $p \le B(0)$ iff $B = q$ and $p \le q$ in P

and inductively iff one or more of the following hold.

   b. $B = B_1 \wedge B_2$ and $p \le B_i(n-1)$ $i = 1,2$

   c. $B = B_1 \vee B_2$ and $p \le B_i(n-1)$ $i = 1$ or 2

   d. $\exists$ $S \in \mathfrak{U}$ such that $p \le \bigvee S$ in P and $s \le B(n-1)$ for all $s \in S$.

3. $L(A), L(B) > 1$ $A \le B$ iff one or more of the following hold.

   a. $A = A_1 \vee A_2$ and $A_i \le B$ for $i = 1,2$

   b. $A = A_1 \wedge A_2$ and $A_i \le B$ for $i = 1$ or 2

   c. $B = B_1 \wedge B_2$ and $A \le B_i$ for $i = 1,2$

   d. $B = B_1 \vee B_2$ and $A \le B_i$ for $i = 1$ or 2

   e. $A = A_1 \wedge A_2$, $B = B_1 \vee B_2$ and for some $p \in P$, $A \le p$ and $p \le B$.

Remark. We observe that Definition 4.3 will not give us a solution to the so-called word problem of this lattice since parts (1d) and (2d) of the definition are non-constructive.

Lemma 4.1. If $A \leq p(n)$ and $p \leq Q$, then $A \leq q(m)$ when $m \geq n$.

Proof. We show this for $m = n + 1$ which is sufficient. Take $S = \{ p \}$, $S \in \mathfrak{C}$ (by definition) $\bigwedge S = p \leq q$ in P and then by Definition 4.3 (1d) $A \leq q\ (n + 1)$.

Theorem 4.1. If $q \leq p$ in $FL(P, \mathfrak{U}, \mathfrak{C})$, then $q \leq p$ in P.

Proof. Only parts 1 and 2 of Definition 4.3 could hold, in which case it suffices to show "If $q \leq p(n)$, then $q \leq p$ in P". By induction if $n > 0$ apply (1d), that is for some $S \in \mathfrak{C}$, $\bigwedge S \leq p$ and $q \leq s(n-1)$ for all $s \in S$. By induction $q \leq s$ in P for all $s \in S$, therefore $q \leq \bigwedge S \leq p$ $\therefore$ $q \leq p$ by transitivity of $(\leq)$.

The only other possibility is $q \leq p$ by (2d) which follows analogously.

Theorem 4.2. If $A_1 \vee A_2 \leq B$, then $A_i \leq B$ for $i = 1$ and 2.

Proof. By induction on L(B). Assume L(B) = 1. We show if $A_1 \vee A_2 \leq p(n)$ then $A_i \leq p(n-1)$.

If $A_1 \vee A_2 \leq p(n)$ by (1b) of Definition 4.3 the result is trivial. If $A_1 \vee A_2 \leq p(n)$ by Definition 4.3 (1d) we have $A_1 \vee A_2 \leq s(n-1)$ for all $s \in S$. This is impossible for $n = 1$, hence if $n = 1$ $A_1 \vee A_2 \leq p(n) \Rightarrow A_i \leq p(n-1)$. For $n > 1$, by induction $A_i \leq s(n-2)$, and by Definition 4.3 (1d), $A_i \leq p(n-1)$ for $i = 1$ and 2. Assume $L(B) > 1$ and $B = B_1 \vee B_2$. Now either (3a) or (3d) of Definition 4.3 may hold. If (3a) holds the result is trivial. Assume (3d) holds. $B = B_1 \vee B_2$ and $A \leq B_i$ for i = 1 or 2 and by induction $A_i \leq B_1$ for i = 1 and 2 and then from (3d) we get $A_i \leq B$ for i = 1 and 2.

If $B = B_1 \wedge B_2$ then by a dual argument using (3b) or (3c) of Definition 4.3, $A_i \leq B$ for i = 1 and 2.

Theorem 4.3. $\leq$ on $FL(P, \mathfrak{U}, \mathfrak{C})$ is reflexive.

Proof. We show $A \leq A$ for all $A \in FL(P, \mathfrak{U}, \mathfrak{C})$ by induction on $L(A)$. Assume $L(A) = 1$, then $A = p$ for some $p \in P \Rightarrow p \leq P \Rightarrow p \leq p(0)$ hence $p \leq p$ and $A \leq A$. Assume $L(A) > 1$ if $A = A_1 \vee A_2$, by induction $A_i \leq A_i$ and then apply (2c) or (3d) of Definition 4.3 to give $A_1 \vee A_2 \geq A_i$ for i = 1 and 2. This by (3a) gives $A \leq A$. If $A = A_1 \wedge A_2$ a dual argument holds.

Theorem 4.4. $\leq$ on $FL(P, \mathfrak{U}, \mathfrak{C})$ is transitive.

Proof. Assume $C \leq B$ and $B \leq A$ and we show $C \leq A$ by induction on $L = L(A) + L(B) + L(C)$. Take $L = 3 \Rightarrow A = p$, $B = q$ and $C = r$

$\Rightarrow$ by Theorem 4.1 and transitivity of $\leq$ on P

$$r \leq p \Rightarrow C \leq A$$

We observe that if $C = C_1 \vee C_2$ or $A = A_1 \wedge A_2$ we get the transitivity by using Theorem 4.2 or its dual and a suitable induction. Thus we shall assume $L(C) = 1$ or $C = C_1 \wedge C_2$ and similarly $L(A) = 1$ or $A = A_1 \vee A_2$.

Case 1. $L(A) = L(B) = 1$, $L(C) > 1$

We show that if $C \leq q(n)$ and $q \leq p$, then $C \leq p(n)$ by induction on n. If $n = 0$, then $C = r \in P$ and a simple application of Theorem 4.1 and the transitivity of $\leq$ gives $C \leq A$. If $n > 0$ and $C = C_1 \wedge C_2$ and

$C_i \leq q(n-1)$, then by induction $C_i \leq p(n-1)$ and again the statement holds. If $C = C_1 \wedge C_2 \leq q(n)$ holds by (1d) of Definition 4.3 i.e. for some $S \epsilon \mathfrak{C}$, $C \leq S(n-1)$ for all $s \epsilon S$ and $\bigwedge S \leq q$. But since $q \leq p$, and thus $\bigwedge S \leq p$ and we have $C \leq p(n)$ by (1d), proving the Theorem for case 1.

<u>Case 2</u>. $L(B) = 1$ (i.e. $B = p \epsilon P$) $C = C_1 \wedge C_2$ and $A = A_1 \vee A_2$. Transitivity follows here by direct application of Definition 4.3 (3e) Case 1 and its dual and case 2 complete the cases for which $L(B) = 1$.

<u>Case 3</u>. $C = p$ and $B = B_1 \vee B_2$.

We show if $p \leq B_1 \vee B_2$ (n) and $B \leq A$, then $p \leq A(n)$ by induction on n. Clearly $B_i \leq A$ for $i = 1,2$ by Theorem 4.2. For $n = 0$, the statement is vacuously true. Assume $p \leq B_i(n-1)$ then, since $B_j \leq A_j$ by induction $p \leq A(n)$. Finally if $p \leq B_1 \vee B_2$ (n) by Defintion 4.3 (2d), that is, for some $S \epsilon \mathfrak{U}$ $s \leq B(n-1)$ for all $s \epsilon S$ and $p \leq \bigvee S$, we have by induction, $s \leq A(n-1)$ for all $s \epsilon S$, and therefore $p \leq A(n)$ by 2d making case 3 complete.

<u>Case 4</u>. $C = C_1 \wedge C_2$ and $B = B_1 \vee B_2$

If $C \leq B$ holds by Definition 4.3 (3b), then $C_i \leq B$ for $i = 1$ or 2 and by induction on L, $C_i \leq A$ and thus $C \leq A$ by (3b). If $C \leq B$ by Definition 4.3 (3d) then $C \leq B_i$ for $i = 1$ or 2 and since $B_i \leq A$ for $i = 1,2$ by a similar induction $C \leq A$. Finally if $C \leq B$ by (3e), that is there exists $p \epsilon P$ such that $C \leq p$ and $p \leq B$, then by case 3, $p \leq A$. Hence if $A = A_1 \vee A_2$, (3e) $\Rightarrow$ $C \leq A$. It remains that the only other possibility is that $A = q$ in which case $C \leq p$ and $p \leq q$ but then $C \leq q$ by Case 1.

The dual cases for $B = B_1 \wedge B_2$ cover all possibilities and follow in an analogous manner, and complete the proof of Theorem 4.4.

Definition 4.4. For $A,B \in FL(P, \mathfrak{U}, \mathfrak{C})$ $A = B$ iff $A \leq B$ and $B \leq A$.

Remark. As in $FL(n)$ Definition 4.4 is easily an equivalence relation, and the resulting equivalence classes are partially ordered by $[A] \leq [B]$ iff $A \leq B$. This order is independent of class representatives, and thus we may work with only representatives of these classes.

It remains to show that $FL(P, \mathfrak{U}, \mathfrak{C})$ is indeed a lattice and that it is free.

Theorem 4.5. $FL(P, \mathfrak{U}, \mathfrak{C})$ partially ordered by $\leq$ is a lattice in which a meet and join of a finite set of elements S is preserved if $S \in \mathfrak{U}$ or $S \in \mathfrak{C}$ respectively. In particular P is embedded in this lattice.

Proof. It follows from the definition of $\leq$ on $FL(P, \mathfrak{U}, \mathfrak{C})$ that $A \vee B$ and $A \wedge B$ are the least upper bound and greatest lower bound respectively of elements A and B in $FL(P, \mathfrak{U}, \mathfrak{C})$. Also the order is preserved, since by Theorem 4.1, $p \leq q$ in P iff $p \leq q$ in $FL(P, \mathfrak{U}, \mathfrak{C})$.

Claim. S finite, $S \in \mathfrak{U}$ with $\bigvee S$ in P, then $\bigvee S$ is the least upper bound of S in $FL(P, \mathfrak{U}, \mathfrak{C})$. We know $\bigvee S \geq s$ for all $s \in S$ $\therefore \bigvee S$ is an upper bound. Assume $A \in FL(P, \mathfrak{U}, \mathfrak{C})$ and $A \geq s$ for all $s \in S$. Since S is finite, lemma 4.1 permits the assumption $A \geq s(n)$ for all $s \in S$. But now we have condition (2d) of Definition 4.3 with respect to $\bigvee S$ and A, hence $A \geq \bigvee S\ (n+1)$ and thus $A \geq \bigvee S$ in $FL(P, \mathfrak{U}, \mathfrak{C})$ making $\bigvee S$ the least upper bound of S in $FL(P, \mathfrak{U}, \mathfrak{C})$. The preservation of finite meets is a dual result.

Remark. Bounds of infinite sets need not be preserved. To observe this consider the lattice in Figure 1.2. P is a partially ordered set, containing the infinite sequence $\mathfrak{C} = \{c_1, \ldots, c_n, \ldots\}$ possessing a lub, p, and enough other elements such that there is an element W with $c_1 \leq W(i)$, $c_i \nleq W(i-1)$ and yet $p \nleq W$. In this lattice W and p are upper bounds of $\mathfrak{C}$.

We form the free lattice on P preserving meets and joins of sets $\mathfrak{C}$ and $\mathfrak{U}$ respectively, where $\mathfrak{C}$ is arbitrary except for the condition $p \leq q \Rightarrow \{p, q\} \in \mathfrak{C}$ and $\mathfrak{U} = \{\{x\} \mid x \in P\} \cup \{\{s_i, c_j\}, \{c_i, c_j\}, \{s_i, p\},$

$$\{c_i, p\}, \{s_i, b\}, \{c_1, \ldots c_n, \ldots\}\}$$
$$\text{for } i \leq j; i, j = 1, 2, \ldots$$

For the least upper bounds of the sets in $\mathfrak{U}$ we will take those of Figure 1.2. We observe that $c_i \leq c_1 \vee b(i)$, but that $c_i \nleq c_1 \vee b(i-1)$ for $i \geq 1$. The latter comment follows by contradiction as follows: $c_i \nleq b$ and $c_i \nleq c_1$, thus if $c_i \leq c_1 \vee b(i-1)$, we have $c_i \leq \bigvee S$ with $x \leq c_1 \vee b(i-2)$ for all $x \in S$. Now, if $x = p$ or $x = c_k$ for $k > i-2$ by the dual of lemma 4.1 we get $c_{i-1} \leq c_1 \vee b(i-2)$, the contradiction. Thus the members of S must be among the following: b, s_j or c_k with $j = 1, 2, \ldots$ and $k \leq i-2$, but there is no set $S \in \mathfrak{U}$ with these members possessing a least upper bound $\bigvee S$ containing c_i. Similarly it is shown $p \nleq c_1 \vee b$ and hence p is not the least upper bound of $\mathfrak{C}$ in FL(P, $\mathfrak{U}$, $\mathfrak{C}$).

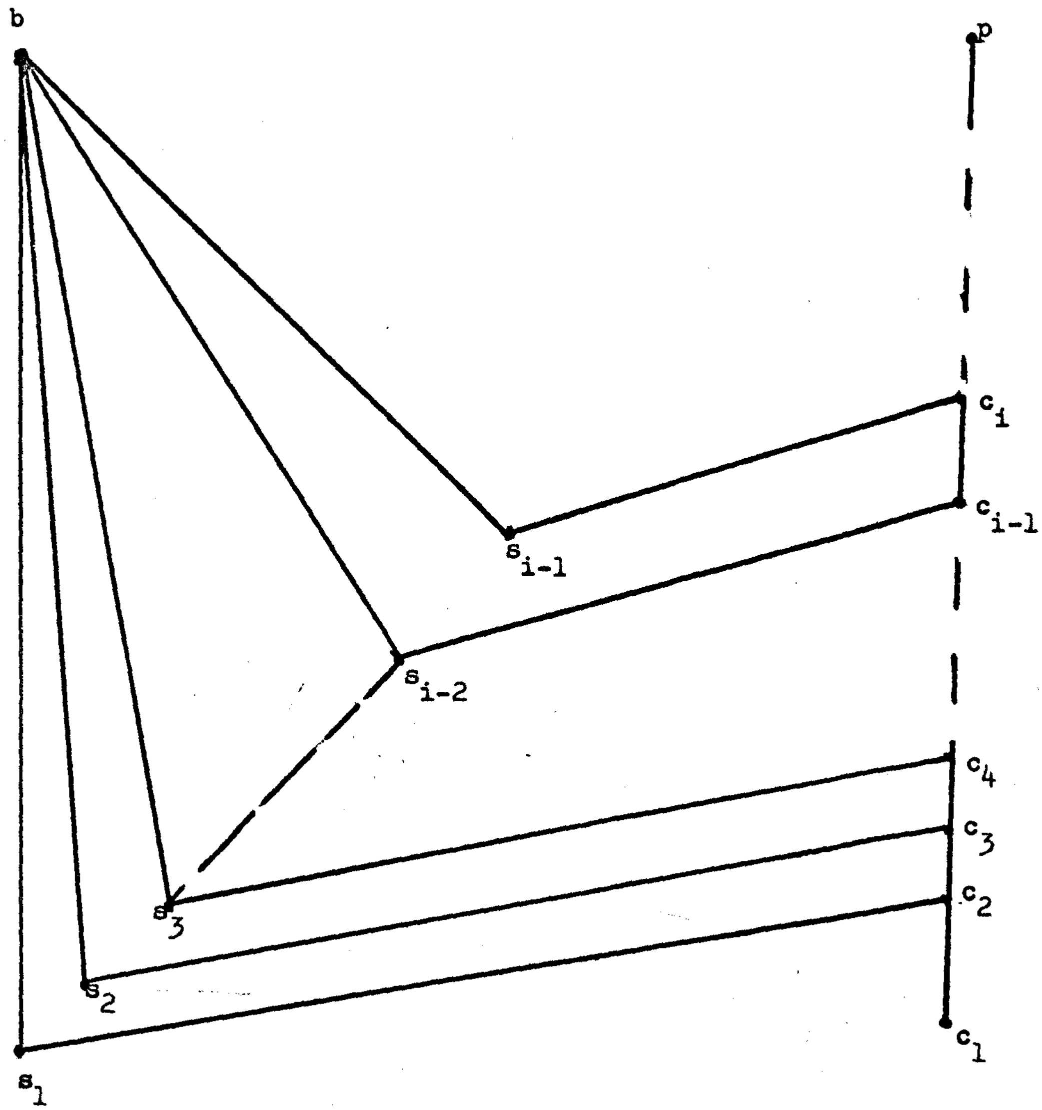


Figure 1.2

<u>Theorem 4.6</u>. Let P be a partially ordered set and let $\mathfrak{U}$ and $\mathfrak{C}$ be families of subsets as defined previously. Let L be a lattice and φ a function, $\varphi : P \longrightarrow L$, such that

i. If $S \in \mathfrak{U}$, then in L $\varphi(\bigvee S) = \bigvee \varphi(S)$

ii. If $S \in \mathfrak{C}$, then in L $\varphi(\bigwedge S) = \bigwedge \varphi(S)$

then φ can be extended to a homomorphism of $FL(P, \mathfrak{U}, \mathfrak{C})$ into L

<u>Proof</u>. Let $P' = \varphi(P)$. The elements generated by P' are clearly a sublattice of L, which by hypotheses must be all of L. We extend φ naturally to $FL(P, \mathfrak{U}, \mathfrak{C})$ by replacing every occurrence of $p \in P$ in the elements of $FL(P, \mathfrak{U}, \mathfrak{C})$ by $\varphi(p) = p' \in P'$, thus giving an element of L. Thus if g(P) is some element of $FL(P, \mathfrak{U}, \mathfrak{C})$ $\varphi(g(P)) = g(\varphi(P)) = g(P')$.

<u>Claim</u>. φ preserves the ordering i.e. if $f(P) \geq g(P)$ then $f(P') \geq g(P')$. Let $L = L(f(P)) + L(g(P))$. If $L = 2$, then $f(P) = p$ and $g(P) = q$ for some $p, q \in P$. Now $p \geq q$ implies $p \geq q$ in P and $p, q \in \mathfrak{U}$ with $p \vee q = p$ and thus by condition (i) $\varphi(p) \geq \varphi(q)$, i.e. $f(P') \geq g(P')$. If $L > 2$, we consider the following three cases.

<u>Case 1</u>. $g(P) = g_1(P) \vee g_2(P)$ implies that if $f(P) \geq g(P)$, then $f(P) \geq g_i(P)$, $i = 1,2$ by Theorem 4.2. Hence by induction $f(P') \geq g(P')$. The dual case also follows.

<u>Case 2</u>. $g(P) = q$. In this case, by Definition 4.3 (2), $f(P) \geq q(n)$ for some n. If $n = 0$, $f(P) = p \in P$ and, as above $\varphi(p) \geq \varphi(q)$ implies $f(P) \geq g(P)$. Assume $n \geq 1$, and take $f(P) = f_1(P) \vee f_2(P)$. If $f_i(P) \geq q(n-1)$

then case 1 implies $f_i(P')\geq \varphi(q)$ in L giving $f(P')\geq \varphi(q)$. Now suppose there exists $S\in \mathfrak{U}$ with the property that $\bigvee S \geq q$ in P and $s\leq f(P)(n-1)$ for all $s\in S$. Here, induction again implies $\varphi(s)\leq f(P')$ in L, for all $s\in S$ and hence $\bigvee\varphi(s)\leq f(P')$. By (i) we have $\bigvee\varphi(S) = \varphi(\bigvee S)$ and therefore $\varphi(\bigvee S)\leq f(P')$. Finally, $\bigvee S\geq q$ implies $\varphi(\bigvee S)\geq \varphi(q)$ and hence $\varphi(q)\leq f(P')$. The dual of this argument concludes the possibilities where $L(f(P))$ or $L(g(P)) = 1$.

<u>Case 3</u>. If $f(P) = f_1(P)\vee f_2(P)$ and $g(P) = g_1(P)\wedge g_2(P)$, then when $f(P)\geq g(P)$ by Definition 4.3 (3a), (3b), (3c) or (3d), the induction hypothesis on L gives $f(P')\geq g(P')$. Assume $f(P)\geq g(P)$ by (3e), that is, for some $p\in P$, $f(P)\geq p$ and $p\geq g(P)$, but then by the previous case and its dual we have $f(P')\geq \varphi(p)\geq g(P')$.

Thus order is preserved by φ, and also the equality and therefore $f(P)\longrightarrow f(P')$ is a well-defined mapping of $FL(P, \mathfrak{U}, \mathfrak{C})$ onto L. By virtue of the order preservation if $u\longrightarrow u'$, $v\longrightarrow v'$ and $w = u\vee v$, then $w' = u'\vee v'$ in L and joins are preserved. Dually, meets are preserved.

<u>Remark 1</u>. If $\mathfrak{U}$ and $\mathfrak{C}$ consist of only the pairs $\{p,q\}$ where $p\leq q$ in P, $FL(P,\mathfrak{U}, \mathfrak{C})$ is simply CF(P).

<u>Remark 2</u>. If $\mathfrak{U}$ consists of all pairs $\{p,q\}$ such that $p\vee q$ exists, and $\mathfrak{C}$ consists of all pairs $\{p,q\}$ such that $p\wedge q$ exists, then $FL(P,\mathfrak{U},\mathfrak{C})$ is the free lattice of Dilworth preserving meets and joins of pairs of elements existing in P.

<u>Remark 3</u>. The lattice $FL(P,\mathfrak{U},\mathfrak{C})$ is different from Dilworth's pair preserving lattice due to the following example. Consider P, as in

Figure 1.3. P is embedded in a lattice L_1 as depicted in Figure 1.4, and L_1 preserves the meets and joins of pairs of elements in P. In Figure 1.5 P is embedded in a lattice L_2 which also preserves the meets and joins of pairs of elements of P. However, if we examine the least upper bound of $\{a,b,c\}$ in P in L_1 it is preserved, but in L_2 it is not. Hence the conclusion that the free lattice preserving meets and joins of pairs of elements need not necessarily preserve the meets and joins of finite subsets of P.

5. Construction of $FL(n_1 + \ldots + n_m)$

We have observed in the preceeding four sections of this chapter free lattices which were generated, first by unordered sets, then partially ordered sets, and in this section, we shall generate a free lattice from a set consisting of countably many unrelated chains of finite length. It is clear that for such a set, P_c, this lattice is essentially $CF(P_c)$, since all we really wish to preserve from P_c is the order within the chains.

Let P_c be the set of unrelated chains: $a_{11} < a_{12} < \cdots < a_{1n_1}$; $a_{21} < a_{22} < \cdots < a_{2n_2}$; $\ldots$ $a_{m1} < a_{m2} < \cdots < a_{mn_m}$. The elements of $FL(P_c)$ are defined analogously to $CF(P)$, and the order in an equivalent fashion. Thus the following theorem results.

Theorem 5.1. Let P_c be a partially ordered set consisting of a set of unrelated chains. In $FL(P_c)$ $A \geq B$ iff one or more of the following hold, provides the necessary lattice structure, and the lattice is free.

P

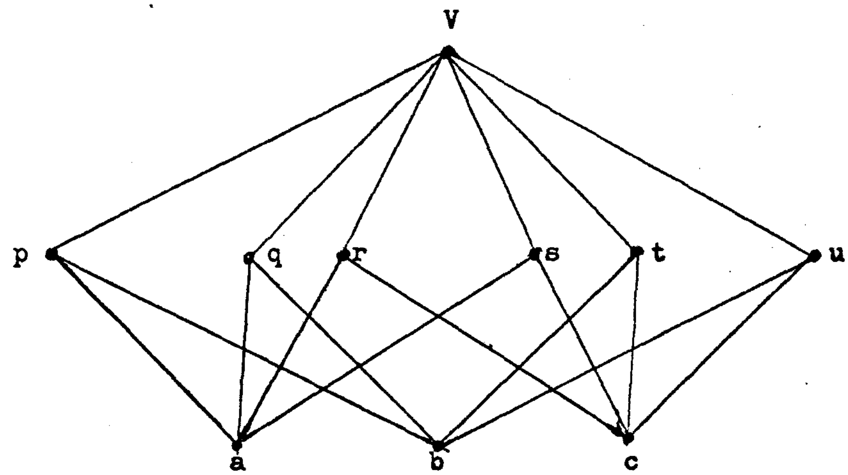


Figure 1.3

L_1

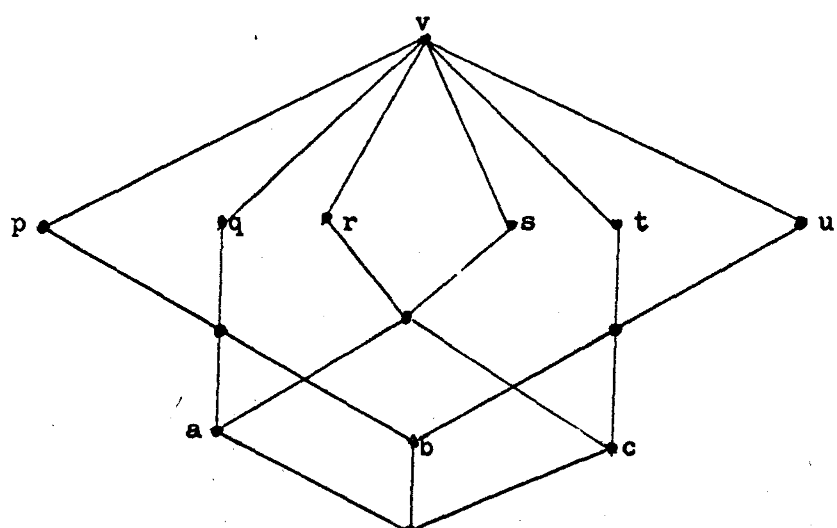


Figure 1.4

L_2

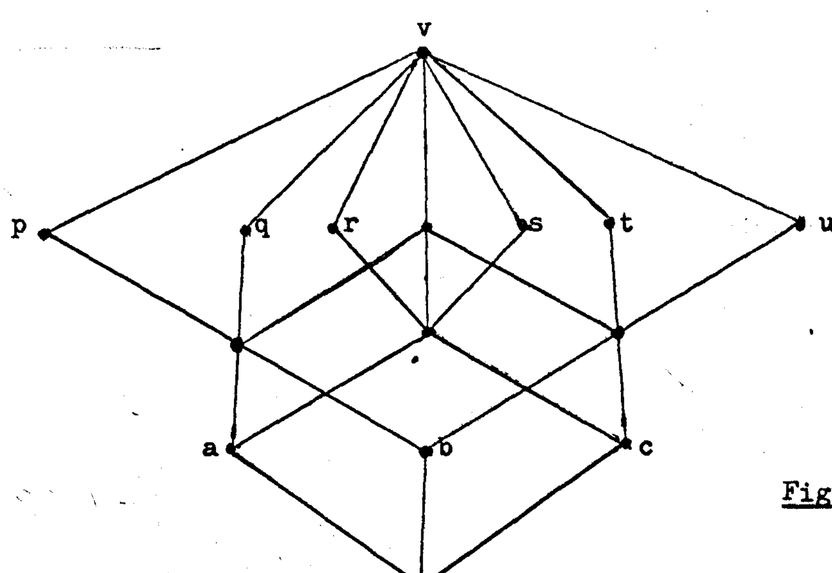


Figure 1.5

i. $p, q \in P_c$ $\quad p \geq q$ iff $p \geq q$ in P_c

ii. $p \geq B_1 \wedge B_2$ iff $p \geq B_1$ or $p \geq B_2$

iii. $A_1 \vee A_2 \geq p$ iff $A_1 \geq p$ or $A_2 \geq p$

iv. $A \geq B_1 \vee B_2$ iff $A \geq B_1$ and $A \geq B_2$

v. $A_1 \wedge A_2 \geq B$ iff $A_1 \geq B$ and $A_2 \geq B$

vi. $A_1 \vee A_2 \geq B_1 \wedge B_2$ iff $A_1 \geq B_1 \wedge B_2$ or $A_2 \geq B_1 \wedge B_2$
or $A_1 \vee A_2 \geq B_1$ or $A_1 \vee A_2 \geq B_2$

Remark. Since the elements of P_c are determined only by the length of the chains and the number of them, P_c is usually written $n_1+n_2+ \ldots +n_m$ and the lattice is written $FL(n_1+n_2+ \ldots +n_m)$.

Definition 5.1. If $A \in FL(n_1+n_2+ \ldots +n_m)$ define A', the dual of A inductively as follows:

1. If $A = a_{i_j}$ $\quad A' = a_{i(n-j+1)}$
2. If $A = A_1 \vee A_2$ $\quad A' = A'_1 \wedge A'_2$
3. If $A = A_1 \wedge A_2$ $\quad A' = A'_1 \vee A'_2$

The details of the structure of some of these free lattices generated by a set of chains are quite interesting, in particular where m is finite. We shall first examine FL(2+2), that is $m = 2$ in $n_1+n_2+\ldots+n_m$ and $n_1 = 2$ and $n_2 = 2$.

We take $a_1 < a_2$ and $b_1 < b_2$ as our two chains and define the elements of FL(2+2) as follows:.

$A_1 = a_2$, $B_1 = b_2$; and for $n > 1$

$A_n = a_2 \wedge (a_1 \vee B_{n-1})$ $B_n = b_2 \wedge (b_1 \vee A_{n-1})$

$C_n = a_1 \vee B_n$

$D_n = b_1 \vee A_n$

$P_n = A_n \vee B_n$

$Q_n = C_n \wedge D_n$

$M_1 = a_1 \vee b_1$

$M_2 = (a_2 \wedge b_2) \vee a_1 \vee b_1$

$V_1 = b_2 \wedge ((a_2 \wedge b_2) \vee a_1 \vee b_1)$

$V_2 = (a_2 \wedge b_2) \vee (b_2 \wedge (a_1 \vee b_1)$

$V_3 = b_2 \wedge (a_1 \vee b_1)$

$W_1 = a_2 \wedge ((a_2 \wedge b_2) \vee a_1 \vee b_1)$

$W_2 = (a_2 \wedge b_2) \vee (a_2 \wedge (a_1 \vee b_1))$

$W_3 = a_2 \wedge (a_1 \vee b_1)$ and the duals of these.

Since the generators of FL(2+2) appear among these elements and their duals, in order to show we have described all elements of FL(2+2), it suffices to exhibit that these elements are closed under meets and joins. Actually, it is quite sufficient to show closure under either meet or join, since it follows that $A \wedge B = (A')' \wedge (B')' = (A' \vee B')'$ and dually. This still entails some lengthy checking of a meet or join table, but for the purpose of this exposition only a diagram of FL(2+2) shall be included (see Figure 1.6).

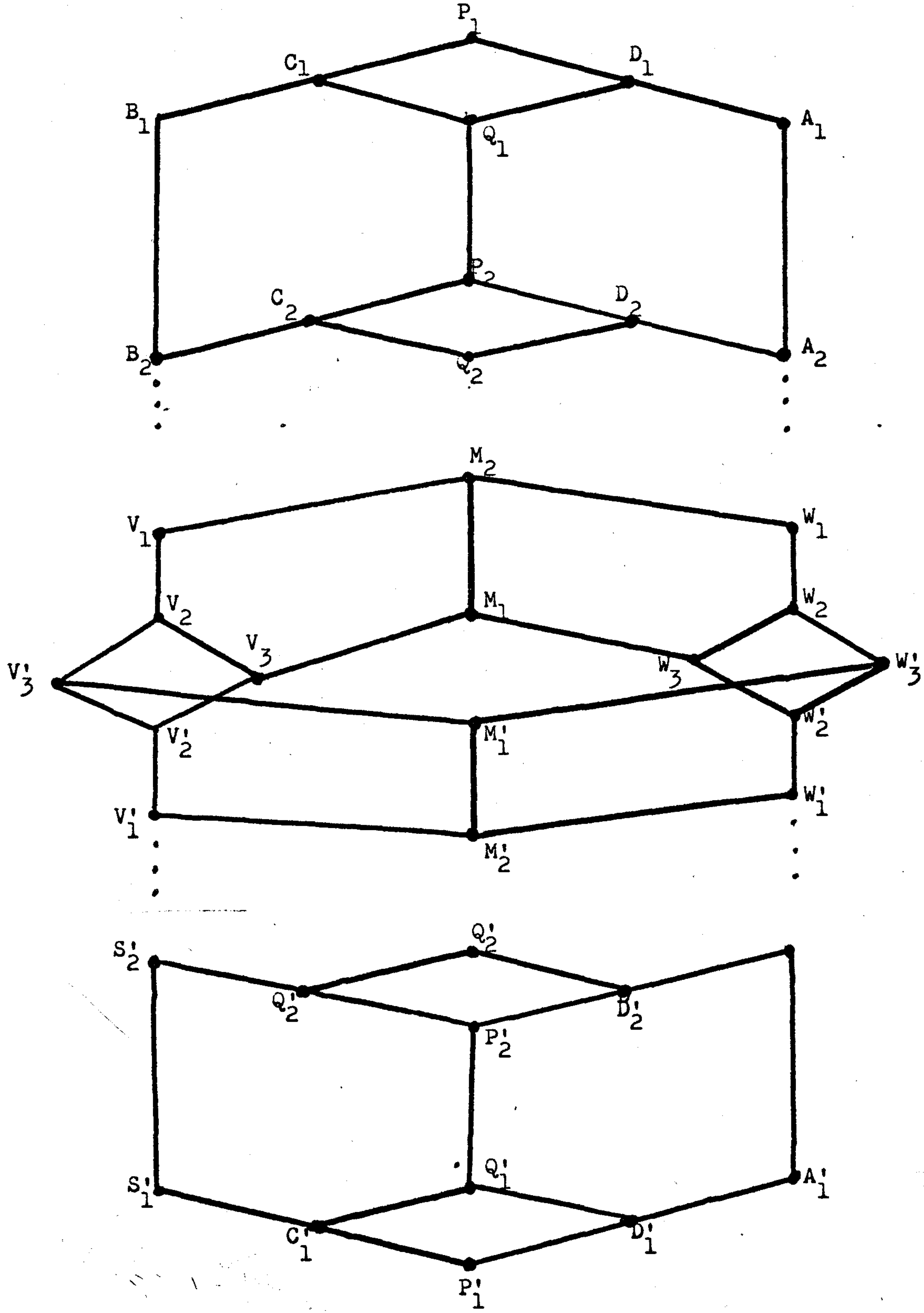

P_1
C_1
D_1
B_1
Q_1
A_1
P_2
C_2
D_2
B_2
Q_2
A_2
M_2
V_1
W_1
V_2
M_1
W_2
V_3
V'_3
W_3
W'_3
M'_1
W'_2
V'_2
V'_1
W'_1
M'_2
Q'_2
S'_2
Q'_2
D'_2
P'_2
S'_1
Q'_1
A'_1
C'_1
D'_1
P'_1

Figure 1.6

Theorem 5.2. In FL(2+2),
$$A_1 > A_2 > \ldots > A_n > \ldots$$
$$B_1 > B_2 > \ldots > B_n > \ldots$$
$$C_1 > C_2 > \ldots > C_n > \ldots$$
$$D_1 > D_2 > \ldots \quad D_n > \ldots$$

Proof. It is clear that $A_1 \geq A_2 \geq \ldots \geq A_n \geq \ldots$ and similarly for the B's and C's and D's from their definition. The equality may be dropped by showing A_n, B_n, C_n, and D_n are canonical for all n, and since the length of these elements are strictly increasing no equality could hold.

Remark. $A_i \vee B_j = \begin{cases} P_i & i=j \\ D_i & i<j \\ C_j & j<i \end{cases}$

Proof. re: i=j implies $A_i \vee B_i = P_i$ by definition

re: $i<j$ we have $B_j \geq b_1$ implies $A_i \vee B_j \geq b_1 \vee A_i = D_i$

Also $b_1 \vee A_i \geq A_i \vee B_j = (b_2 \wedge (b_1 \vee A_{j-1})) \vee A_i$ and $A_n \geq A_m$.

$n \leq m$ implies by (v) $b_1 \vee A_i \geq b_1 \vee A_{j-1}$ for $i<j$ and then by (iv) $b_1 \vee A_i \geq B_j \vee A$ and hence $D_i = b_1 \vee A_i = B_j \vee A_i \quad i<j$.

For $j<i$, $C_j = A_i \vee B_j$ is shown in a similar fashion.

In an analogous manner we consider FL(4+1). Here we take for the chains $a_1 < a_2 < a_3 < a_4$ and b. Then we define:

$A_1 = a_3 \qquad B_1 = a_1 \vee b \qquad A_2 = a_3 \wedge (a_2 \vee B_1)$, and

$B_2 = (a_1 \vee b) \wedge a_4$.

for $n>2$ $A_n = a_3 \wedge (a_2 \vee B_{n-1}) \qquad B_n = B_1 \wedge ((a_4 \wedge b) \vee A_{n-2})$

$C_n = (a_4 \wedge b) \vee A_n$

$D_n = a_2 \vee B_n$

$F_n = A_n \vee B_n$

$G_n = A_n \vee B_{n+1}$

$H_1 = a_4 \wedge D_1$ and, for $n > 1$ $\quad H_n = C_{n-1} \wedge D_n$

$E_1 = a_4$ and, for $n > 1$ $\quad E_n = C_{n-1} \wedge D_{n-1}$

$S_n = A_n \vee H_n$

$T_n = D_n \wedge G_n$

$P_1 = a_4 \wedge F_1$ and, for $n > 1$ $\quad P_n = C_{n-1} \wedge F_n$

$Q_n = B_n \vee E_n$

$V_1 = a_3 \wedge ((a_3 \wedge (a_1 \vee b)) \vee a_2 \vee (a_4 \wedge b))$

$V_2 = (a_3 \wedge (a_1 \vee b)) \vee (a_3 \wedge (a_2 \vee (a_4 \wedge b)))$

$V_3 = a_3 \wedge (a_2 \vee (a_4 \wedge b))$

$W_1 = (a_1 \vee b) \wedge ((a_3 \wedge (a_1 \vee b)) \vee a_2 \vee (a_4 \wedge b))$

$W_2 = (a_3 \wedge (a_1 \vee b)) \vee ((a_1 \vee b) \wedge (a_2 \vee (a_4 \wedge b)))$

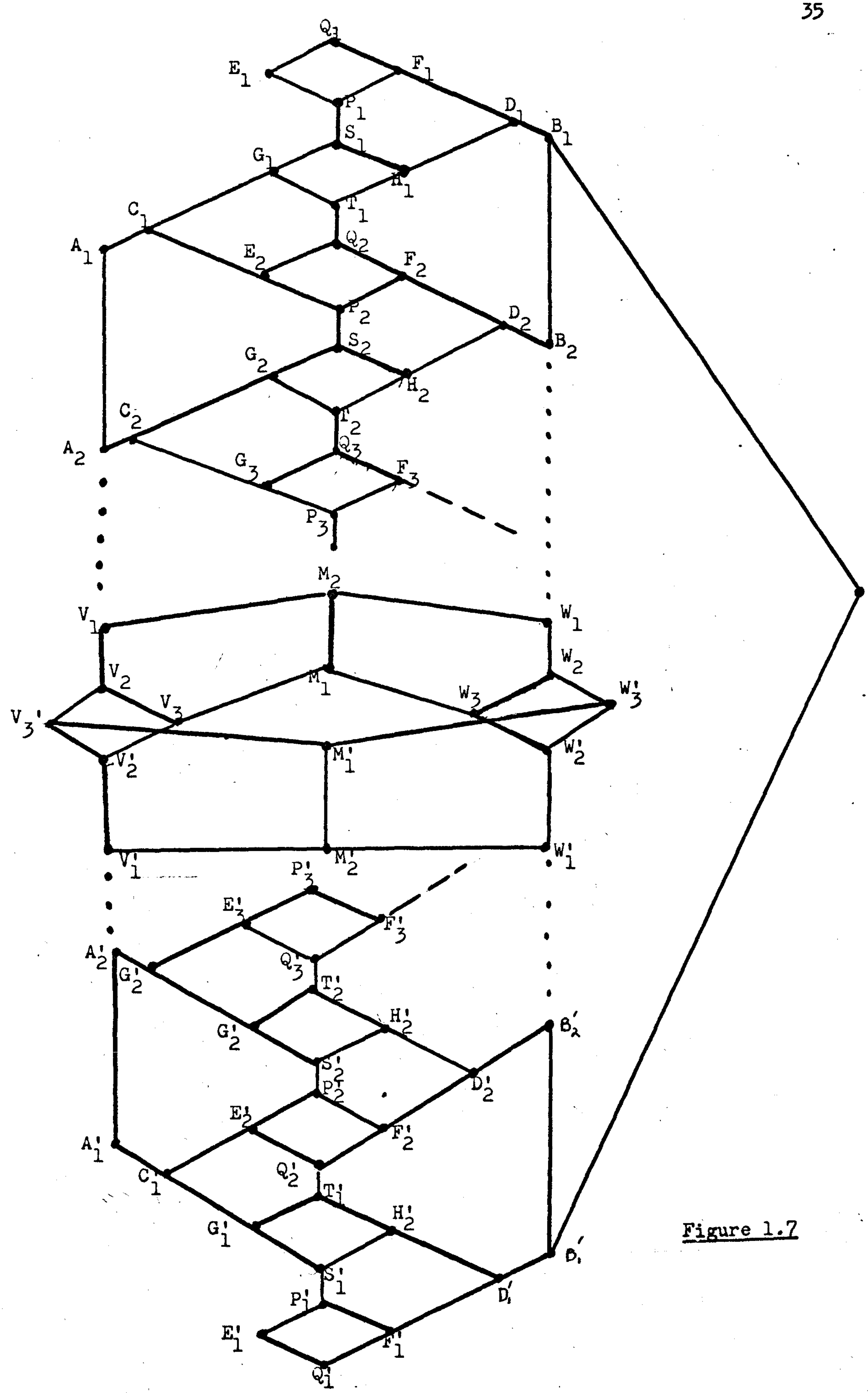
$W_3 = (a_1 \vee b) \wedge (a_2 \vee (a_4 \wedge b))$

$M_1 = a_2 \vee (a_4 \wedge b)$

$M_2 = (a_3 \wedge (a_1 \vee b)) \vee a_2 \vee (a_4 \wedge b)$

These elements together with their duals form all of the elements of FL(4+1), which is, as in the case of FL(2+2) easily verified by showing the closure of meets and joins for these elements. Again we shall rely on a diagram for this fact, rather than check a meet and join table. Figure 1.7 is a diagram for FL(4+1).

<u>Definition 5.2</u>. L, a lattice $\dot{B} = \{ b_i \mid i = 1,2,\ldots \} \subseteq L$ is said to order converge to b iff $\exists \{u_i\}$ and $\{v_i\}$. $\ni$. $v_i \geq v_{i+1} \geq b_{i+1} \geq u_{i+1} \geq u_i$ for all i and $\bigvee_i u_i = \bigwedge_i v_i = b$.



Q1
E1
F1
P1
D1
B1
S1
G1
H1
T1
C1
A1
Q2
E2
F2
P2
D2
B2
S2
G2
H2
T2
C2
A2
Q3
G3
F3
P3
M2
V1
W1
W2
M1
V2
V3
W3
W'3
V3'
V'2
M'1
W'2
V'1
M'2
W'1
P'3
E'3
F'3
A'2
G'2
Q'3
T'2
H'2
G'2
B'2
S'2
P'2
D'2
E'2
F'2
A'1
Q'2
C'1
T'1
H'2
G'1
B'1
S'1
D'1
P'1
E'1
F'1
Q'1
b

Figure 1.7

<u>Theorem 5.4</u>. In FL(2+2) W_1 is the greatest lower bound of $\{A_n\}$, V_1 and M_2 are the greatest lower bounds of $\{B_n\}$ and, $\{C_n\}$ and $\{D_n\}$ respectively.

<u>Proof</u>. Examine Figure 1.6 wherein it is clear that $A_n \geq W_1$, $B_n \geq V_1$ and $C_n \geq M_2$ and $D_n \geq M_2$ for each n. Thus W_1 is a lower bound of $\{A_n\}$, V_1 is a lower bound of $\{B_n\}$ and M_2 is a lower bound for $\{C_n\}$ and $\{D_n\}$. We must now show W_1, V_1, and M_2 are greatest lower bounds. i.e. show

i. If $A_n \geq K$ for each n, then $W_1 \geq K$

ii. If $B_n \geq K$ for each n, then $V_1 \geq K$

iii. If $C_n \geq K$ for each n, then $M_2 \geq K$

iv. If $D_n \geq K$ for each n, then $M_2 \geq K$

This is shown by induction over L(K), where the notion of length for $FL(n_1 + \dots + n_m)$ is defined analogous to that of the previous lattices. Suppose L(K) = 1, then $A_n \geq K$ for each n implies $K = a_1$, but then $W_1 \geq K$. A similar argument is used for V_1 and M_2. Suppose L(K) > 1 and for L(K) < k; i, ii, iii, iv above hold. If L(K) = k and $K = K_1 \wedge K_2$ with $A_n \geq K$ for each n, then $a_2 \geq K$ and $a_1 \vee B_{n-1} \geq K$ for $n > 1$. But, $a_1 \vee B_{n-1} \geq K$ iff one of the following holds:

a. $a_1 \geq K_1 \wedge K_2$

b. $B_{n-1} \geq K_1 \wedge K_2$

c. $C_{n-1} \geq K_1$

d. $C_{n-1} \geq K_2$

Suppose a holds, then $a_1 \geq K_1 \wedge K_2$ implies $W_1 \geq K_1 \wedge K_2$, trivially. If a. does not hold, then one of b,c, or d must hold. We observe $B_m > B_{m+1}$ implies if $B_m \not\geq K_1 \wedge K_2$, then $B_n \not\geq K_1 \wedge K_2$ for $n > m$, for otherwise $B_m > B_n \geq K_1 \wedge K_2$ gives $B_n > K_1 \wedge K_2$. Also $C_m \not\geq K$ implies $C_n \not\geq K$ for $n > m$. Hence b,c,d fail to hold for $n = \max\{i,j,k\}$ where b fails to hold for some i, c fails to hold for some j and d, fails to hold for some k. Thus if a fails to hold , and also b, c, and d, then $a_1 \vee B_{n-1} \not\geq K_1 \wedge K_2$ for each $n > 1$, a contradiction. Therefore one of b, c, and d must hold if a does not. If b holds, then $b_2 \geq K_1 \wedge K_2$ and $a_2 \geq K_1 \wedge K_2$ implies $W_1 \geq K_1 \wedge K_2$. By induction hypothesis if c or d hold, we have $M_2 \geq K_1$ or K_2, hence $M_2 \geq K_1 \wedge K_2$, and this together with $A_2 \geq K_1 \wedge K_2$ gives $W_1 \geq K_1 \wedge K_2$. Thus if $A_n \geq K_1 \wedge K_2$ for each n, and $L(K_1 \wedge K_2 = k$, we get $W_1 \geq K_1 \wedge K_2$. Similarly for $K = K_1 \vee K_2$ and $A_n \geq K_1 \vee K_2$ for each n, we have $W_1 \geq K_1 \vee K_2$ and also $D_n \geq K$ for each n implies $M_2 \geq K$, $B_n \geq K$ for each n implies $V_1 \geq K$, and $C_n \geq K$ for each n implies $M_2 \geq K$ where $L(K) = k$.

<u>Corollary</u>. In FL(2+2) the sequence $\{A_n\}$ order converges to W_1, $\{B_n\}$ order converges to V_1, $\{C_n\}$ and $\{D_n\}$ each order converge to M_2.

<u>Proof</u>. Take $u_n = W_1$, $v_n = A_n$ and apply Definition 5.2 with $\bigvee_n \{u_n\} = \bigwedge_n \{v_n\} = W_1$, we get $\{A_n\}$ order converges to W_1. Similarly by suitable choices the other conclusions follow.

<u>Theorem 5.5</u>. (Generalized Theorem 5.4). In $FL(n_1 + \ldots + n_m)$, where $n_i \geq 2$ and $n_j \geq 2$ for some unequal i,j, the corresponding sequence $\{A_n\}$ order converges to W_1; $\{B_n\}$ order converges to V_1, and $\{C_n\}$ and $\{D_n\}$ each order converge to M_2.

Proof. For the 2 chains $a_{i1} < \ldots < a_{in_i}$ and $a_{j1} < \ldots < a_{jn_j}$ we take the least elements a_{i1}, a_{i2}, a_{j1} and a_{j2}. Now $\{a_{i1}, a_{i2}, a_{j1}, a_{j2}\} = 2+2$. In A_n, B_n, C_n, D_n, W_1, V_1 and M_2 of FL(2+2) replace a_r by a_{ir} and b_r by a_{jr} for $r = 1,2$ giving A_n', B_n', C_n', D_n', W_1', V_1', M_2'. These elements are in the same relation as A_n, B_n, C_n, D_n, W_1, V_1, M_2 of Theorem 5.4, and thus they order converge.

Theorem 5.6. In FL(4+1), $\{A_n\}$ order converges to V_1, $\{B_n\}$ order converges to W_1, $\{C_n\}$ and $\{D_n\}$ each order converge to M_2.

Proof. The proof is analogous to Theorem 5.4.

Theorem 5.7. (Generalized Theorem 5.6). In FL($n_1 + \ldots + n_m$) where $n_i \geq 4$ for some i and $m \geq 2$, $\{A_n\}$ order converges to V_1, $\{B_n\}$ order converges to W_1, $\{C_n\}$ and $\{D_n\}$ each order converge to M_2.

Proof. The proof is analogous to Theorem 5.5.

CHAPTER 2

Sublattices

With respect to structures built from sets with progressively stronger order properties, the situation is settled by the foregoing chapter. The consideration of sublattices is now of distinct interest, for the question arises "Is every sublattice of a free lattice in some sense free?". The result is, of course, false since every chain of length two or greater in FL(n) is not an FL(m) for suitable m. However, certainly any chain in FL(n) is its own completely free lattice, and hence the conjecture that every sublattice in FL(n) is a completely free lattice, CF(P) for suitable P. This is, of ourse, also false because of the following example.

Let a,b,c be generators of FL(3), and consider the sublattice in Figure 21 which is contained in FL(n). It is clear that this sublattice is Boolean and has no elements which are both meet and join irreducible, yet by the definition of ordering on CF(P) every generator must be both meet and join irreducible.

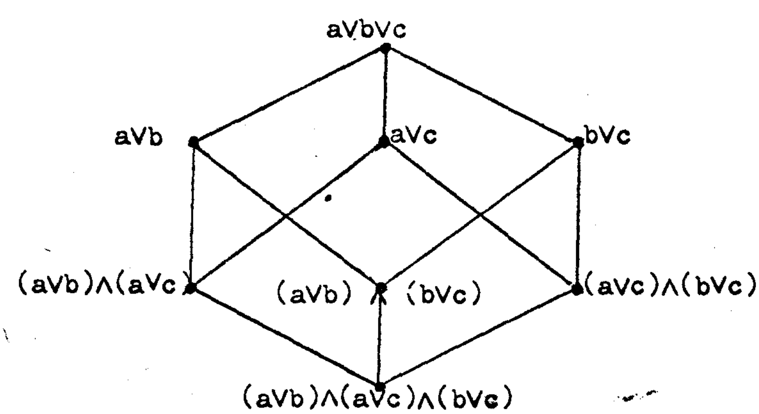


Figure 2.1.

1. Sublattices of $FL(n_1+n_2+\ldots+n_m)$.

In this section we shall examine some interesting results concerning particular free lattices of the type $FL(n_1+\ldots+n_m)$.

Theorem 1.1. Let $T = \{t_i\}$ be any subset of $CF(P)$ with the property that, for any finite subset of indices R, $t_j \leq \bigvee_{i \in R} t_i$ implies there exists $k \in R$ such that $t_j \leq t_k$ and dually. Then, the sublattice $L(T)$ of $CF(P)$ generated by T is lattice isomorphic to $CF(Q)$ where Q is a partially ordered set isomorphic to T.

Proof. Let the elements of Q be denoted by q_i and the correspondence $q_i \longleftrightarrow t_i$. Certainly $L(T)$ provides a minimal embedding for Q; and therefore $L(T)$ is a homomorphic image of $CF(Q)$ through the mapping generated by $q_i \longleftrightarrow t_i$. That this correspondence is one to one is shown by proving that uRv in $L(T)$ for $u, v \in L(T)$ implies $u'Rv'$ in $CF(Q)$ where u' and v' are the images under the $q_i \longleftrightarrow t_i$ correspondence. This is accomplished by induction over the length of u and v, and the result is established.

That the conditions of this theorem are necessary for $L(T)$ to be isomorphic to $CF(Q)$ comes from the observation in $CF(Q)$ that $q_j \leq \bigvee_{i \in R} q_i$ implies $q_j \leq q_k$ for some $k \in R$ and dually, by the definition of $\leq$ in $CF(Q)$.

Applying Theorem 1.1, and Theorem 5.5 of Chapter 1, we get the following theorem, Theorem 1.2.

Theorem 1.2. Let $\mathfrak{U}$ be a subset of $FL(n_1'+n_2'+\ldots n_m')$ and let $\mathfrak{U} = \{u_{ij}\}$ be isomorphic to $n_1+n_2+\ldots+n_m$. Let $u_{ij} \geq u_{pq}$ iff $i = p$ and $j \geq q$. Then, $FL(\mathfrak{U})$ is isomorphic to $FL(n_1+\ldots+n_m)$ iff $\bigvee u_{ij} \geq u_{ab}$ implies $i = a$ and $j \geq b$ for some i and j and the dual property.

Theorem 1.3. $FL(n_1+n_2)$, $n_1 \geq 3$ and $n_2 \geq 2$ contains a sublattice isomorphic to $FL(1+1+1)$, i.e. $FL(3)$.

Proof. Choose $a_1 < a_2 < a_3$ and $b_1 < b_2$ as the chains.

Define $u_{11} = a_2$

$u_{21} = a_3 \wedge (a_1 \vee b_1)$

$u_{31} = a_1 \vee (a_3 \wedge b_2)$

and show they are unrelated. Then, by Theorem 1.2, $FL(u_{11}, u_{21}, u_{31}) \cong FL(1+1+1)$ and the theorem is proved.

Theorem 1.4. $FL(n_1+n_2)$ $n_1 \geq 5$, $n_2 \geq 1$ contains a sublattice isomorphic to $FL(1+1+1)$.

Proof. Choose $a_1 < a_2 < a_3 < a_4 < a_5$, and b_1 as the chains.

Define $u_{11} = a_3$

$u_{21} = a_4 \wedge (a_2 \vee (a_5 \wedge b))$

$u_{31} = a_2 \vee (a_4 \wedge (a_1 \vee b))$

and the remainder of the proof follows as in Theorem 1.3.

Theorem 1.5. The free lattice generated by $3m$ unrelated elements, $FL(1+1+\ldots+1)$ contains a sublattice isomorphic to $FL(n_1+n_2+\ldots+n_m)$.

Proof. Denote the generators by $x_1, \ldots, x_{3m}$. Then, select m sets of elements of $FL(n_1+n_2+\ldots+n_m)$ as follows: for $i = 1,2,\ldots,m$ let $u_{i0} = x_{3i-2}$ and for $j \geq 1$, $u_{ij} = x_{3i-2} \vee (x_{3i} \wedge (x_{3i-1} \vee (x_{3i-2} \wedge (x_{3i} \vee (x_{3i-1}\, u_{i\,j-1})))))$. By a suitable induction $u_{i0} < u_{i1} < \ldots$, and also

$\mathfrak{U} \cong \{u_{ij}\} = n_1+n_2+\ldots+n_m$, then using Theorem 1.2 we get the sublattice generated by $\mathfrak{U}$ isomorphic to $FL(n_1+n_2+\ldots+n_m)$.

Corollary 1. $FL(1+1+1)$ contains a sublattice isomorphic to $FL(n_1+n_2+\ldots+n_m)$.

Proof. $FL(1+1+1)$ contains a sublattice isomorphic to $FL(n)$ for countable n. Let M be a set of $3m$ generators, and hence $FL(M)$ is isomorphic to a sublattice of $FL(1+1+1)$. By Theorem 1.5 $FL(M)$ contains a sublattice isomorphic to $FL(n_1+n_2+\ldots+n_m)$, hence so does $FL(1+1+1)$.

Corollary 2. $FL(m_1+m_2)$ with $m_1 \geq 3$ and $m_2 \geq 2$, or $m_1 \geq 5$ and $m_2 \geq 1$ contains a sublattice isomorphic to $FL(n_1+n_2+\ldots+n_m)$.

Proof. Apply Theorems 1.3, 1.4 and 1.5 directly.

Remark. Theorem 1.5 holds for any cardinal number m, and each chain contains at most countably many elements. In the two corollaries m must be countable, since $FL(1+1+1)$ contains only countably many elements.

2. Sublattices of CF(P)

In this section we shall discuss several additional observations concerning $CF(P)$, the completely free lattice on a partially ordered set P.

Lemma 2.1. If P is any partially ordered set, and if in $CF(P)$, $W = A \vee B$ where $W \neq A$ and $W \neq B$, then $W = C \wedge D$ implies $W = C$ or $W = D$.

Proof. Certainly $C \geq W$, $D \geq W$, $W \geq A$ and $W \geq B$. If $A \vee B \geq C \wedge D$, then $A \geq W$, or $B \geq W$, or $W \geq C$ or $W \geq D$. But $A \geq W$ implies $A = W$, a contradiction, and $B \geq W$ implies $B = W$ a contradiction. $W \geq C$ or $W \geq D$ imply the theorem.

Remark 1. It is clear that the lattice shown in Figure 2.2 is not a sublattice of CF(P) for any P, for there we see $W = A \vee B$ and also $W = C \wedge D$, but $W = C$ and $W = D$.

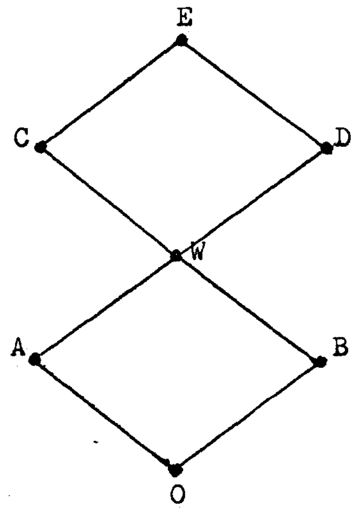


Figure 2.2

Remark 2. The question "Can any countably generated lattice be embedded in a lattice with 3 generators?" is answered affirmatively by the following theorem.

Theorem 2.1. For any countable partially ordered set P, CF(P) is a sublattice of FL(3).

Proof. Since FL($\aleph_0$) is a sublattice of FL(3) it is sufficient to embed CF(P) as a sublattice in FL($\aleph_0$). Let $P = \{p_i \mid i=1,2,\ldots\}$ and the generators of FL($\aleph_0$) be $a_1, a_2, \ldots$. Set up the following correspondence

$$p_1 \longleftrightarrow b_1 = a_1.$$

$$\vdots$$

$$p_r \longleftrightarrow b_r = (a_r \vee \bigvee_{i \in L_r} b_i) \wedge (\bigwedge_{j \in U_r} b_j)$$

where $U_r = \{ j \mid j < r, p_j \geq p_r \}$ and $L_r = \{ i \mid i < r, p_i \leq p_r \}$ and with the convention that if U_r or L_r is empty, that term is omitted. By a suitable induction $p_r \leq p_s$ iff $b_r \leq b_s$ and also Theorem 1.1 is this chapter is satisfied. The embedding is completed by inductive proofs of the following:

i. $a_r \leq b_s$ iff $p_r \leq p_s$

ii. $b_s \leq a_r$ iff $p_s \leq p_r$

iii. $b_r \leq b_s$ iff $p_r \leq p_s$

iv. $b_r \geq \bigwedge_{t \in T} b_t$ for a finite set T implies $b_r \geq b_t$ for some $t \in T$.

v. $b_r \leq \bigvee_{t \in T} b_t$ for a finite set T implies $b_r \leq b_t$ for some $t \in T$.

Theorem 2.2. In CF(P) if x,y and z are three incomparable elements such that $x \wedge y = x \wedge z = y \wedge z = B$, then $x \vee y$, $x \vee z$, $y \vee z$, are incomparable and $(x \vee y) \wedge (x \vee z) \wedge (y \vee z) = B$.

Proof. The proof follows from the dual of Lemma 2.1, and a contradiction of the canonical form of B.

Corollary 1. In CF(P), there is no sublattice isomorphic to the lattice of Figure 2.3.

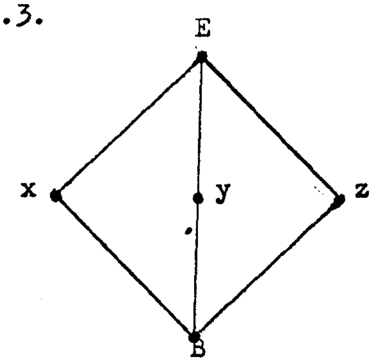


Figure 2.3

Corollary 2. The sublattice generated by x,y and z in Theorem 2.2 is finite, and is the sublattice shown in Figure 2.4.

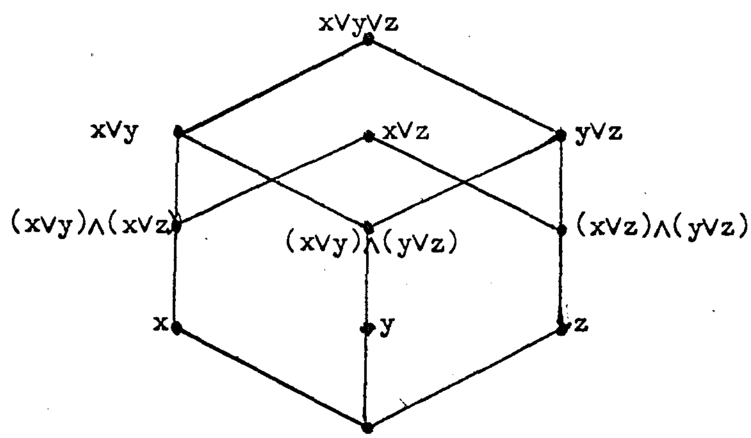


Figure 2.4

3. Some General Theorems

Theorem 3.1. If A is a free-lattice product of $(A_i)_{i \in I}$, and for each $i \in I$, A_i is isomorphic to a sublattice of a free lattice with m_i generators, then A is isomorphic to a sublattice of a free lattice with $\sum_{i \in I} m_i$ generators.

Proof. We have the conditions of Theorem 1, Chapter 0 satisfied, and can thus conclude this result.

Theorem 3.2. Suppose m is an infinite cardinal and F is a free lattice with m generators. If $a, b \in F$ and $b < a$, then $]b,a[$ contains, as a sublattice, a free lattice with m generators.

Proof. Let X be the set of generators of F, i.e. $|X| = m$. There exists a finite subset Y of X such that a and b belong to the sublattice F' of F which is generated by Y. The sublattice F" of F which is generated by Z = X-Y can be mapped homomorphically into $]b,a[$ by f as follows: $f\colon Z \longrightarrow]b,a[$ by $f(z) = b \vee (a \wedge z)$ for $z \in Z$.

Claim. f is an isomorphism.

F is a free lattice-product of F' and F". It then follows from Theorem 1, Chapter 0, that the lattice D generated by $\{a,b\}$ and Z is a free lattice product of F" and the two-element lattice $E = \{a,b\}$. Let $\bar{F}$ be a lattice obtained by adjoining a zero element, 0, and a unit element, 1, to F". We map E and F" into $\bar{F}$ by mapping a into 1 and b into 0, and each element of F" into itself. These isomorphisms have a common extension g which is a homomorphism of Z into $\bar{F}$. Since f is a homomorphism of F" into D, gf is a homomorphism of F" into $\bar{F}$. Furthermore $gf(z) = g(b\vee (a \wedge z)) = 0 \vee (1 \wedge z = Z$ for all $z \in Z$, and therefore $gf(z) = Z$ for all $z \in F''$. Consequently f is an isomorphism of F" into $]b,a[$, and the theorem is proved.

Theorem 3.3. If $m \geq 3$, and if the lattice A is the union of a countable chain $\mathfrak{A}$ of sublattices each of which is isomorphic to a sublattice of a free lattice with m generators, then A is isomorphic to a sublattice of a free lattice with m generators.

Proof. Since a free lattice with 3 generators contains as a sublattice a free lattice with infinitely many generators, we assume m is infinite. Let F be a free lattice with m generators and let $\mathfrak{B}$ be the family of intervals $]b,a[$, with $a,b \in F$ and $b < a$. Then $\mathfrak{B}$ is a partially ordered set. Furthermore, for any two intervals $]b,a[$ and $]d,c[$ in F, $]b,a[<]d,c[$, that is, $b < a < d < c$, then by Theorem 3.2 there exists $x,y \in F$ with $a < y < x < d$, and hence $]b,a[<]y,x[<]d,c[$. Consequently, if C is a maximal chain in $\mathfrak{B}$, then C is dense in itself and $\mathfrak{A}$ is therefore order-isomorphic to a subchain C' of C . Now, Theorem 3.2 and the hypothesis gives each of the lattices $B \in \mathfrak{B}$ isomorphic to a sublattice B' of the corresponding quotient in C', and thus the union A' of these lattices B' is a sublattice of F. Thus A is isomorphic to A'.

Remark. Theorem 3.2 can be used to show the existence of an infinite chain in any free lattice, F, simply by taking $a,b \in F$, $b < a$, placing the free lattice between a and b, hence the existence of x_1 and y_1 such that $b < x_1 < y_1 < a$, then putting the free lattice between x_1 and y_1 and continuing as before.

4. Distributive Sublattices. Our final consideration will be a characterization of those distributive lattices which can be embedded (isomorphically) in free lattices. It is clear from our observations of $CF(P)$ that in a free lattice each element is either meet or join irreducible, and hence every sublattice also has this property. We now examine the class of all distributive lattices in which this condition is satisfied.

Definition 4.1. If A is a partially ordered set, non-empty subsets can be partially ordered (linearly ordered) as follows: $B, C \subseteq A$, $B \leq C$ iff $B = C$ or $b \leq c$ whenever $b \in B$ and $c \in C$. We define a linear decomposition of a lattice L to be a system $\mathfrak{A}$ of sublattices of L such that $L = \bigcup\mathfrak{A}$ and for any two (distinct) $A, B \in \mathfrak{A}$ either $A \leq B$ or $B \leq A$.

Definition 4.2. If $A \in \mathfrak{A}$, then A is called a component of the linear decomposition $\mathfrak{A}$.

Definition 4.3. A lattice, L, is called linearly indecomposable iff every linear decomposition $\mathfrak{A}$ of L has L as a component.

Lemma 4.1. If $\mathfrak{K}$ consists of all those sublattices S of L which satisfy: for all $x \notin S$ either $x \leq s$ for all $s \in S$ or $s \leq x$ for all $s \in S$, then a sublattice S of L is a component of some linear decomposition iff $S \in \mathfrak{K}$.

Proof. Take $\mathfrak{A}$ a linear decomposition of L with $S \in \mathfrak{A}$, and take $x \notin S$. Then, there exists $A \in \mathfrak{A}$ such that $x \in A$, and either $S \leq A$ or $A \leq S$. If $S \leq A$, then $s \leq x$ for all $s \in S$ and if $A \leq S$, then $x \leq s$ for all $s \in S$ which gives $S \in \mathfrak{K}$.

For the converse, assume $S \in \mathfrak{K}$. Define L(S) to be the set of all lower bounds of S, and U(S) the set of all upper bounds of S. Clearly U(S) and L(S) are either empty or sublattices of L, and since $S \in \mathfrak{K}$ we see $L = S \cup L(S) \cup U(S)$. Furthermore $L(S) \leq S$, $L(S) \leq U(S)$, and $S \leq U(S)$. Hence if $\mathfrak{A}$ consists of those of the sets L(S), S, and U(S) which are non-empty, then $\mathfrak{A}$ is a linear decomposition of L, and S is therefore a component of L.

Remark. $\mathcal{K}$ is easily a closure system over L, and thus for every subset M of L there exists a smallest set K(M) in $\mathcal{K}$ containing M. For $a \in L$, write K(a) for $K(\{a\})$.

Lemma 4.2. If $a < b$ and $a \notin K(b)$ or $b \notin K(a)$, then $K(a) \leq K(b)$.

Proof. Assume $a < b$ and $a \notin K(b)$. Since $a \notin K(b)$, $a < b$ and $K(b) \in \mathcal{K}$ we have $a \leq x$ for all $x \in K(b)$, hence $a \in L(K(b))$. Moreover, $L(K(b)) \neq \emptyset$, hence a sublattice of L. We show $L(K(b)) \in \mathcal{K}$. Take $x \notin L(K(b))$. If $x \in K(b)$, then $s \leq x$ for all $s \in L(K(b))$. If $x \notin K(b)$, then $x \not\leq y$ for some $y \in K(b)$, but $K(b) \in \mathcal{K}$ implies $y \leq x$. It again follows $s \leq y \leq x$, i.e. $s \leq x$ for all $s \in L(K(b))$.

We consider $a \in K(a) \cap L(K(b)) \in \mathcal{K}$, i.e. $K(a) \subseteq K(a) \cap L(K(b))$, i.e., $K(a) \subseteq L(K(b))$, i.e. $K(a) \leq K(b)$.

If $b \notin K(a)$, the proof is dual.

Lemma 4.3. If $a, b \in L$, then $K(a) = K(b)$ or $K(a) \leq K(b)$ or $K(b) \leq K(a)$.

Proof. If a and b are incomparable, then clearly $a \in K(b)$ and $b \in K(a)$, hence $K(a) \subseteq K(b) \subseteq K(a)$ and thus $K(a) = K(b)$. If $a < b$ and $a \notin K(b)$ or $b \notin K(a)$, the Lemma 4.2 gives $K(a) \leq K(b)$. If $a < b$, $a \in K(b)$ and $b \in K(a)$, then clearly $K(a) = K(b)$ and if $b < a$ dually. Finally if $a = b$, $K(a) = K(b)$ trivially.

Lemma 4.4. The sets K(a) are linearly indecomposable.

Proof. Take $\mathfrak{A}$ a linear decomposition of K(a). Then there exists $A \in \mathfrak{A}$ such that $a \in A$. Take $x \in L$, $x \notin A$. If $x \in K(a)$, then there exists $B \in \mathfrak{A}$ such that $x \in B$. Either $A \leq B$ or $B \leq A$, hence either $y \leq x$

for all $y \in A$ or $x \leq y$ for all $y \in A$. If $x \notin K(a)$, then clearly either $y \leq x$ (all $y \in A$, even all $y \in K(a)$) or $x \leq y$ (all $y \in A$, even all $y \in K(a)$). Thus $A \in \mathfrak{K}$. Since $a \in A$ we obtain $K(a) \subseteq K(A) = A \subseteq k(a)$, hence $A = K(a)$ and $K(a)$ is linearly indecomposable.

Definition 4.4. A set $K(a)$ is called redundant iff there exists $K(b)$ such that $K(a) \subseteq K(b)$ and $K(a) \neq K(b)$.

Lemma 4.5. If $K(a)$ is redundant, then $K(a)$ is a one element set, i.e. $K(a) = \{a\}$.

Proof. Take $K(b)$ such that $K(a) \subseteq K(b)$ and $K(a) \neq K(b)$. By Lemma 4.3 we have either $K(a) \leq K(b)$ or $K(b) \leq K(a)$. Without loss of generality $K(a) \leq K(b)$. Thus every $x \in K(a)$ is a lower bound of $K(b)$, hence a minimum of $K(b)$. Since minima are unique, we obtain that $K(a)$ is a one-element set.

Theorem 4.1. For every lattice, L, there exists one and only one linear decomposition $\mathfrak{A}$ with the properties:

1. Every $A \in \mathfrak{A}$ is linearly indecomposable.
2. If $\mathfrak{A}' \subset \mathfrak{A}$, then $\mathfrak{A}'$ is not a linear decomposition, i.e. $\bigcup \mathfrak{A}' \neq L$.

Proof. Define $\mathfrak{A} = \{K(a) \mid a \in L, K(a) \text{ not redundant}\}$ Lemma 4.4 gives us that each $K(a)$ is linearly indecomposable. By definition and Lemma 4.5 we get that every redundant $K(a)$ is contained in an irredundant set $K(b)$, so that we have $\bigcup \mathfrak{A} = L$. Lemma 4.3 gives the fact that $\mathfrak{A}$ is indeed a linear decomposition. We show that it satisfies 2. Take $\mathfrak{A}' \subset \mathfrak{A}$, $K(a) \in \mathfrak{A} - \mathfrak{A}'$ and assume $L = \bigcup \mathfrak{A}'$. Then there exists

$K(b) \in \mathfrak{A}'$ such that $a \in K(b)$ and $K(a) \neq K(b)$ which implies $K(a) \subseteq K(b)$, $K(a) \neq K(b)$. It follows that $K(a)$ is redundant which contradicts $K(a) \in \mathfrak{A}$.

It remains to show $\mathfrak{A}$ is unique. Take $\mathfrak{B}$ an arbitrary decomposition satisfying 1 and 2 and take $A \in \mathfrak{B}$ arbitrary. By Lemma 4.1 we have $A \in \mathcal{K}$, hence $K(a) \subseteq A$ for all $a \in A$. It follows that $\{ K(a) \mid a \in A \}$ is a linear decomposition of A. Since A is linearly indecomposable we get $K(a) = A$ for some $a \in A$. Assume $K(a)$ were redundant. This would give a $b \in L$ such that $A = K(a) \subset K(b)$. But $b \in B$ for some $B \in \mathfrak{B}$, hence $A = K(a) \subset K(b) \subseteq B$. Hence $\mathfrak{B} - \{ A \}$ would be a linear decomposition of L which contradicts 2. It follows $A = K(a) \in \mathfrak{A}$, hence $\mathfrak{B} \subseteq \mathfrak{A}$. Next, take $K(a) \in \mathfrak{A}$. Then there exists $A \in \mathfrak{B}$ with $a \in A$. Since $A \in \mathcal{K}$ it follows $K(a) \subseteq A$. Since $\mathfrak{B} \subseteq \mathfrak{A}$ there exists $b \in L$ such that $A = K(b)$, that is $K(a) \subseteq K(b)$. Since $K(a)$ is irredundant we have $K(a) = K(b)$, i.e. $K(a) = A \in \mathfrak{B}$, i.e. $\mathfrak{A} \subseteq \mathfrak{B}$ and therefore $\mathfrak{A} = \mathfrak{B}$.

Remark. Succinctly stated Theorem 4.1 gives us that each lattice L is the union of a unique linearly ordered family $\mathcal{D}$ of linearly indecomposable lattices. Furthermore L is distributive iff each member of $\mathcal{D}$ is distributive and each element of L is meet or join irreducible iff each of the members of $\mathcal{D}$ has this property. Thus, in the following discussion we need only consider the case of linearly indecomposable lattices.

Lemma 4.6. Suppose D is a distributive lattice with the property that every element of D is either meet or join irreducible. If the elements x_1, x_2, $x_3 \in D$ are such that no two of them are comparable, then they generate an eight element Boolean algebra.

Proof. Since the element $(x_2 \vee x_3) \wedge (x_3 \vee x_1) \wedge (x_1 \vee x_3) = (x_2 \wedge x_3) \vee (x_3 \wedge x_1) \vee (x_1 \wedge x_2)$ cannot be both meet and join reducible, either one of the factors on the left must be less equal the other two factors, or else one of the factors on the right must be greater equal the other two factors. By symmetry and duality we may assume that $x_2 \wedge x_3$ and $x_3 \wedge x_1$ are less equal $x_1 \wedge x_2$ so that:

$$x_2 \wedge x_3 = x_3 \wedge x_1 \leq x_1 \wedge x_2 \qquad (\alpha)$$

Considering the element $(x_1 \vee x_3) \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee x_3$, we must have one of the following relations:

i. $x_1 \vee x_3 \leq x_2 \vee x_3$ iii. $x_1 \wedge x_2 \geq x_3$

ii. $x_2 \vee x_3 \leq x_1 \vee x_3$ iv. $x_3 \geq x_1 \wedge x_2$

If i) holds, then $x_1 = (x_1 \wedge x_2) \vee (x_1 \wedge x_3)$ and then it follows by (α) $x_1 = x_1 \wedge x_2 \leq x_2$ which contradicts the hypothesis that x_1, x_2, x_3 are incomparable. If ii) and iii) holds, a similar contradiction arises. Finally if iv) holds, i.e. $x_3 \geq x_1 \wedge x_2$, then $\leq$ in (α) becomes equality, and it follows that x_1, x_2 and x_3 are atoms of a Boolean algebra with eight elements.

Definition 4.5. L, a lattice, the width of L is defined to be the supremum of the cardinals of the sets of pairwise incomparable elements.

Definition 4.6. A subset J of elements of a lattice L is an ideal iff $x \in J$ and $y \in J$ implies $x \vee y \in J$, and $x \in J$ and $t \leq x$ imply $t \in J$. The dual of an ideal is a dual ideal.

Definition 4.7. L a lattice, $a \in L$, an element b is called a cover of a iff $b > a$ and there exists no $x \in L$ such that $b > x > a$.

Lemma 4.7. Suppose D is a linearly indecomposable distributive lattice with the property that every element of D is either meet or join irreducible. Then the width of D is at most 3. Furthermore, if the width of D is 3, then D is a Boolean algebra with eight elements.

Proof. By Lemma 4.6, if the width of D is 3 or more, then D contains as a sublattice a Boolean algebra, B, with eight elements. Let 0 and 1 be the zero and unit of B respectively. We show $d \in D$, $d \notin B$, implies $d < 0$ or $d > 1$.

First observe that if p is an atom of B, then there exists no element $d \in D$ such that $0 < d < p$. In fact, if such an element d exists, and if q and r are the other 2 atoms of B, then the element $q \vee d = (q \vee p) \wedge (q \vee d \vee r)$ is both meet and join reducible. Now take any $d \in D$ and let $p' = 0 \vee (p \wedge d)$, $q' = 0 \vee (q \wedge d)$ and $r' = 0 \vee (r \wedge d)$ where p, q and r are the atoms of B. Then $0 \leq p' \leq p$, hence $p' = 0$ or $p' = p$. Since p is meet reducible (in B and thus also in D), it must be join irreducible. It follows that if $p' = p$, then $p = p \wedge d \leq d$.

Similarly, either $q' = 0$ or $q \leq d$, and either $r' = 0$ or $r \leq d$. By symmetry we need only consider four out of the eight cases that may arise.

If $p' = q' = r' = 0$, then $1 \wedge (0 \vee d) = 0$. Hence $d \leq 0$, for otherwise the element $p \vee d = (p \vee d \vee q) \wedge (p \vee d \vee r)$ would be both meet and join reducible.

If $p \leq d$ and $q' = r' = 0$, then $(q \vee d) \wedge 1 = q \vee (d \wedge 1)$ and $q \vee (d \wedge 1) = q \vee p$. Since $q \vee p$ is meet irreducible it follows that $q \vee p = q \vee d$, $d = d \wedge (q \vee d) = d \wedge (q \vee p) = p$.

If $p \leq d$, $q \leq d$ and $r' = 0$, then $1 \wedge d = p \vee q$, and hence $d = p \vee q$.

If $p \leq d$, $q \leq d$ and $r \leq d$, then $1 \leq d$.

Thus we see if $d \notin B$, then either $d < 0$ or $d > 1$.

The element 0 is meet reducible and must therefore be join irreducible, whence it follows that if there exists elements $d \in D$ with $d < 0$, then the set A of all these elements must be a sublattice of D. The set $C = D-A$ is precisely the set of all elements $d \in D$ with $0 \leq d$, and we therefore have $A < C$. We therefore see that if A were non-empty, then D is linearly decomposable, a contradiction. Similarly the assumption that there exists $d \in D$ with $1 \leq d$ leads to a contradiction and thus $D = B$.

<u>Lemma 4.8</u>. Suppose D is a linearly indecomposable distributive lattice with the property that every element of D is either meet or join irreducible. If the width of D is 2, then D is isomorphic to a direct product of two chains, one of which has exactly two elements.

Proof. Case 1: D has a zero element, 0.

Claim 1. There exists an atom p of D which is meet irreducible.

Proof. 0 must be join reducible, for otherwise the set $D-\{0\}$ would be a sublattice of D, and D would not be linearly indecomposable. Thus, there exists $p,q \in D$ such that $0 = p \wedge q$, $0<p$ and $0<q$. If neither p nor q were an atom, then there would exist $x,y \in D$ such that $0<x<p$ and $0<y<q$, and the elements p,q and $x \vee y$ would be incomparable, which contradicts the width of D is 2. Thus without loss of generality p is an atom.

If p is meet reducible, $p = a \wedge b$ with $p<a$ and $p<b$, then two of the three elements a,b and q must be comparable. Since $a \wedge b < a$, and $a \wedge b < b$, a and b cannot be comparable, and since $p \nleq q$, $a \nleq q$ or $b \nleq q$. Therefore $q \leq a$ or $q \leq b$. We assume $q \leq a$. For any $x \in D$ with $0<x \leq q$ we have $x \vee p = a \wedge (x \vee b)$. Now $x < x \vee p$ and $p < x \vee p$. Also $x \vee p \neq x \vee b$, because equality implies $b \leq x \vee b = x \vee p \leq a$, a contradiction. Therefore, $x \vee p = a$, $q \leq x \vee p$, and $q = x \vee (p \wedge q) = x \vee 0 = x$. Thus q is an atom of D.

If q is also meet reducible, $q = c \wedge d$ with $q<c$ and $q<d$, then $p \leq c$ or $p \leq d$, say $p \leq c$. Observe $q \nleq b$, and therefore $d \nleq b$ and $p \vee q \nleq b$. Similarly $b \nleq d$ and $p \vee q \nleq d$. Furthermore $b \wedge (p \vee q) = p < b$ and $d \wedge (p \vee q) = q < d$, so that $b \nleq p \vee q$ and $d \nleq p \vee q$. Consequently b,d and $p \vee q$ are incomparable. This contradicts the hypothesis, and hence either p or q must be a meet irreducible atom.

Claim 2. If p is a meet irreducible atom of D, then the set $C = \{x \mid x \in D$ and $p \wedge x = 0\}$ is a chain and an ideal of D, and D is the inner product of C and of the 2-element chain $C' = \{0,p\}$.

Proof. Clearly C is an ideal of D, and if $x, y \in C$ then either $x \leq y$ or $y \leq x$, since otherwise x, y, and p would be incomparable, a contradiction. Since C and C' are ideals of D and have only 0 in common, their inner direct product $A = C' \times C$ exists and in an ideal of D. We now show if $B = D - A \neq \emptyset$, then B would be a sublattice of D and $A < B$.

Given $x \in B$ we have $x \notin C$, whence $p \wedge x \neq 0$, and thus $p < x$. For all $y \in C$ we have $x \wedge (y \vee p) = (x \wedge y) \vee p$, whence it follows that $p \leq x \wedge y$ or $x \wedge y \leq p$ or $x \leq y \vee p$ or $y \vee p \leq x$. The first case is excluded because $p \wedge y = 0 < p$, and the third case is ruled out because it would yield $x = (x \wedge y) \vee p \in A$. The case $x \wedge y \leq p$ yields $x \wedge y = 0$, $x \wedge (y \vee p) = p$ and since p is meet irreducible it follows that $y \vee p = p$, $y \leq p$, $y = 0$, $y \vee p = p < x$. Finally, in the last case the equality $y \vee p = x$ is ruled out since $y \vee p \in A$. Thus $p \vee y \lneq x$ whenever $x \in B$ and $y \in C$, whence it follows $A < B$.

Clearly if $x_1 \in B$ and $x_1 \leq x_2$ then $x_2 \in B$. To show B is a sublattice of D it is therefore sufficient to show that if $x_1, x_2 \in B$, then $x_1 \wedge x_2 \in B$. If this fails, then $x_1 \wedge x_2 \in A$. Since every member of B contains p, we have $p \leq x_1 \wedge x_2$, and therefore $x_1 \wedge x_2 = p \vee y$ for some $y \in C$. But since $x_1 \wedge x_2$ is meet reducible, and is therefore join irreducible, it follows that $y = 0$, $p = x_1 \wedge x_2$. However, this is excluded because p is meet irreducible.

Claim 3. The set $C = \{ x \mid x \in D \text{ and } p \wedge x = 0 \}$ consists of all join irreducible elements of D, except the element p.

Proof. Since C is a chain, every element of C is join irreducible in C, and since C is an ideal of D, it follows that every element of C is join irreducible in D. On the other hand, if $a \in D$, $a \notin C$, and $a \nleq p$, then $a = p \vee y$ for some $y \in C$, and therefore a is join reducible.

Case 2. D does not have a zero element.

Claim 4. If $x \in D$ is meet reducible, then the dual ideal generated by x is linearly indecomposable.

Proof. There exists $a, b \in D$ such that $x = a \wedge b$, $x < a$ and $x < b$. Let D_x be the dual ideal generated by x, and suppose there exist sublattices A and B of D_x such that $D_x = A \cup B$ and $A < B$. Clearly $a, b \in A$. If $y \in D$ and $y \notin D_x$, then $y \leq a$ or $y \leq b$, for otherwise the elements a, b, and y would be incomparable. It readily follows that if A' is the set of all those elements $y \in D$ such that $y \leq Z$ for some $z \in A$, then $D = A' \cup B$ and $A' < B$ which contradicts the hypothesis.

Claim 5. If $d \in D$, $c \leq d$ for some meet reducible c.

Proof. If $x \in D$ is not meet reducible, then either x is the largest element of D, or there exists $y \in D$ such that x and y are incomparable, for otherwise D would be the union of the two sublattices $A = \{y \mid x \geq y \in D\}$ and $B = \{y \mid x < y \in D\}$ with $A \leq B$, a contradiction. Thus $x \wedge y \leq x$ and $x \wedge y$ is meet reducible.

Claim 6. The set A consisting of all the join irreducible elements of D is a chain, and every member of A is covered by a unique member of D-A.

Proof. Take $a, b \in A$. Since D does not have a zero element, there exist $x, y \in D$ such that $x < y < a \wedge b$ and by Claim 5 there exists a meet reducible element z with $z \leq x$. Let D_z be the dual ideal generated by z. Claim 4 permits us to apply Claims 1, 2, and 3 with D replaced by D_z. Let p and C be as in Claim 2. Then $a \nleq p \nleq b$ because a and b do not cover z, and it follows by Claim 3 that $a, b \in C$. Since C is a chain we conclude that

$a \leq b$ or $b \leq a$. Thus A is a chain. Finally by Claim 2 a is covered by $p \vee a$ and by no other join reducible element.

Claim 7. Let A be the set consisting of all the join irreducible elements of D, and for each $a \in A$ let a' be the unique member of D-A which covers a. Then the mapping $(0,a) \longrightarrow a$, $(1,a) \longrightarrow a'$ is an isomorphism of $\{0,1\} \times A$ onto D.

Proof. For each meet reducible element z of D let D_z be the dual ideal generated by z and let $A_z = A \cap D_z$. Again by Claim 4, we may apply Claims 1, 2, and 3 with D replaced by D_z. Observe that $p = z'$ satisfies the hypothesis of Claim 2, and denote by C_z the corresponding set defined in Claim 2. Clearly $A_z \subseteq C_z$.

If z_0 and z_1 are meet irreducible elements of D with $z_0 \leq z_1$ then by Claim 5, $z_1' = z_0' \vee z_1$ and $z_0 = z_0' \wedge z_1$, and hence that $C_{z_1} \subseteq C_{z_0}$. Now suppose z is meet reducible, $a \in D_z$ and $a \in A_z$. Then there exist $b,c \in D$ such that $a = b \vee c$, $b < a$, and $c < a$. We can then find a meet reducible element z_0 with $z_0 \leq b \wedge c$. Then a is join reducible in D_{z_0} such that $a \notin C_{z_0}$. Consequently $a \notin C_z$. Thus we see $C_z \subseteq A_z$ hence $A_z = C_z$.

By Claim 2 the mapping $(0,a) \longrightarrow a$, $(1,a) \longrightarrow a' = z' \vee a$ is an isomorphism of $\{0,1\} \times C_z$ onto D_z. The lattices $\{0,1\} \times C_z$ form a chain whose union is $\{0,1\} \times A$, and the lattices D_z form a chain whose union is D. Consequently the indicated mapping is an isomorphism of $\{0,1\} \times A$ onto D.

Theorem 4.2. For any distributive lattice D the following conditions are equivalent:

i. Every element of D is either meet or join irreducible.

ii. D is the union of a linearly ordered family C of sublattices such that each member of C is either a one element lattice, or an eight element Boolean algebra, or else is isomorphic to a direct product of 2 chains, one of which consists of exactly 2 elements.

Proof. Theorem 4.1 gives us that D is the union of a simply ordered family of linearly indecomposable sublattices. i implies ii follows from Lemma 4.7 and 4.8, together with the fact that a lattice of width 1 (a chain) is linearly indecomposable iff it consists of just one element. Conversely, it is easy to show that under the hypothesis ii each member of C has the property that every element of C is either meet or join irreducible, thus D has this property.

Lemma 4.9. Every simply ordered subset of a free lattice is countable.

Proof. Let F be a free lattice generated by a set X. Since X countable gives the result trivially, assume X is not countable. Let X_o be a countably infinite subset of X, and let F_o be the sublattice of F generated by X_o.

For $a, b \in F$ write $a \sim b$ iff there exists an automorphism f of F such that $f(a) = b$. Clearly $\sim$ is an equivalence relation over F. For each $a \in F$ there exists a finite subset $Y \subseteq X$ such that a belongs to the sublattice of F generated by Y. We can find a permutation p of X which maps Y into X_o, and p can be extended to an automorphism f of F. Consequently $a \sim f(a) \in F_o$. Thus every equivalence class modulo $\sim$ contains a member of F_o. The number of equivalence classes must therefore be countable, and the proof will be completed if we show that no simply ordered subset of F contains more than one element from any one equivalence class. That is, it suffices to show that if $a \sim b$ and $a \leq b$, then $a = b$.

Suppose f is an automorphism of F such that $f(a) = b$. There exists a finite subset Y of X such that a belongs to the sublattice of F generated by Y. If Z is the image of Y under f, then there exists a permutation p of X such that $p(x) = f(x)$ whenever $x \in Y$, and $p(x) = x$ whenever $x \in X-(Y \cup Z)$.

If g is the automorphism of F such that $g(x) = p(x)$ whenever $x \in X$, then $g(a) = f(a) = b$, and g is of some finite order n. If now $a \leq b$ then $a \leq g(a) \leq g^2(a) \leq \ldots \leq g^n(a) = a$ hence $a \leq b \leq a$, i.e. $a = b$, and the proof is complete.

Theorem 4.3. For any distributive lattice D the following conditions are equivalent:

i. D is isomorphic to a sublattice of a free lattice with three generators.

ii. D is isomorphic to a sublattice of a free lattice.

iii. D is countable and every element of D is either meet or join irreducible.

iv. D is the union of a countable, linearly ordered family C of sublattices where each member of C is either a one-element lattice or an eight element Boolean algebra, or else is isomorphic to a direct product of a two-element chain and a countable chain.

Proof. i implies ii is trivial.

ii implies by the remarks at the beginning of this section, that every element of D is either meet or join irreducible. By Theorem 4.2 and Lemma 4.9, D is then countable, and we have ii implies iii.

Since Theorem 4.2 exhibits the equivalence of iii and iv we have iii implies iv. It remains to show iv implies i.

If F is a free lattice generated by x,y and z, then it is easy to check that the elements $y \wedge z$, $x \wedge z$, $x \wedge y$ generate an eight element Boolean algebra. Also, F contains as a sublattice a free lattice F' with five generators x_0, x_1, x_2, x_3 and x_4. If C is a countable chain, then there exists an isomorphism f of C into the sublattice generated by x_2, x_3 and x_4. Defining the mapping g: $\{0,1\} \times C \longrightarrow F'$ by the conditions

$$g((1,c)) = x_0 \vee (x_1 \wedge f(c)) \qquad g((0,c)) = (x_0 \vee (x_1 \wedge f(c))) \wedge x_1$$

for all $c \in C$.

Claim. g is an isomorphism of $\{0,1\} \times C \longrightarrow F'$.

Let L be the endomorphism of F' such that $h(x_0) = 0$, $h(x_1) = 1$ and $h(x_i) = x_i$ for i = 2,3,4. Then $hg((1,c)) = f(c) = hg((0,c))$ for all $c \in C$. Thus g is one-to-one one the collection of elements of the form (1,c), and also on the set of elements of the form (0,c). Furthermore, if $c, c' \in C$, then $g((0,c)) \leq x_1$ and $g((1,c')) \nleq x_1$ so that $g((0,c)) \neq g((1,c'))$ and g is one-to-one.

If $c, c' \in C$ and $c \leq c'$, then it is easy to check that $g((1,c)) \vee g((0,c')) = g((1,c'))$ and $g((1,c)) \wedge g((0,c')) = g((0,c'))$ and since g is obviously order-preserving, it follows that g is an isomorphism. Thus iv implies every member of C is isomorphic to a sublattice of a free lattice with three generators, and we conclude by Theorem 1.10, Chapter 2, that i holds. Thus the proof is completed.

BIBLIOGRAPHY

1. Birkhoff, G., Lattice Theory, Amer. Math. Soc. Colloq. Publ. vol. 25, rev. ed., Amer. Math. Soc., Providence, R.I. (1948).

2. Dean, R. A., Completely free lattices generated by partially ordered sets, Trans. Amer. Math. Soc., vol. 83 (1956), 238-249.

3. __________, Component subsets of the free lattice on n generators, Proc. Amer. Math. Soc., vol. 7 (1956), 220-226.

4. __________, Sublattices of free lattices, Proc. Symps. Pure Math., vol. II, 31-42, Amer. Math. Soc., Providence, R.I. (1961).

5. __________, Free lattices generated by partially ordered sets and preserving bounds, Canadian Journal of Math., vol. 16 (1964), 136-148.

6. Dilworth, R. P., Lattices with unique complements, Trans. Amer. Math. Soc., vol. 57 (1945), 64-71.

7. Galvin, F., and Jonsson, B., Distributive sublattices of a free lattice, Canadian Journal of Math., vol. 13 (1961), 265-272.

8. Jonsson, B., Universal relational systems, Math. Scand., vol. 4 (1956), 193-208.

9. __________, Sublattices of a free lattice, Canadian Journal of Math., vol. 13 (1961), 256-264.

10. Rolf, H. L., The free lattice generated by a set of chains, Pacific Journal of Math., vol. 8 (1958), 585-595.

11. Whitman, P. M., Free lattices, Annals of Math., vol. 42, no. 1, (1941), 325-330.

12. ____________, Free lattices II, Annals of Math., vol. 43, no. 1, (1942), 104-115.