MODEL REDUCTION METHODS APPLIED TO POWER SYSTEMS

by

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ABSTRACT

This thesis presents a continuation in the process of rationalizing, unifying and improving existing model reduction techniques. Thus a method of reduction is developed which combines the method of aggregation and partial Padé approximation in such a way as to maintain their separate advantages while simultaneously removing their disadvantages. The important aspects associated with the reduced-order models obtained are: guaranteeing the stability of the reduced-order models, saving computation time, retaining the invariance property under state variable feedback conditions and matching some of the original system time moments.

Also, a criterion is proposed for selecting the state variables of the original system to be retained in the reduced-order model. This criterion leads to developing a reduction technique which can be regarded as a combination of the methods of aggregation and singular perturbation. Therefore, the reduced-order model obtained retains the physical significance of the state variables and the dominant eigenvalues of the original system.

Furthermore, a procedure is developed for obtaining dynamic equivalents of multimachine systems. This procedure utilizes the

concept of component cost analysis for identifying the coherent groups of generators.

Verification of the methods developed in the thesis is established using a variety of realistic power system models including a single synchronous machine connected to an infinite bus, a three-machine system and a 10-machine system. These applications include simulation, analysis and simple controller design.

LIST OF PRINCIPAL SYMBOLS

Generating Unit Model

v_d, v_q - stator voltages in direct- and quadrature-axis circuits, respectively.

v_t - stator voltage.

i_d, i_q - stator currents in direct- and quadrature-axis circuits, respectively.

ψ_d, ψ_q - stator flux linkages in direct- and quadrature-axis circuits, respectively.

x_d, x_q - synchronous reactances in direct- and quadrature axis circuits, respectively.

x_f, x_{kd1} - self reactances of field and direct-axis damper windings.

x_{kq1}, x_{kq2} - self reactances of quadrature-axis damper windings.

x_{af} - stator field mutual reactance.

x_{ad}, x_{aq1}, x_{aq2} - stator-rotor mutual reactances with damper windings.

R_a - stator resistance.

$R_f, R_{kd1}, R_{kq1}, R_{kq2}$ - field and damper winding resistances.

$i_f, i_{kd1}, i_{kq1}, i_{kq2}$ - currents in field and damper windings.

ψ_f ψ_{kd1} ψ_{kq1} ψ_{kq2} - flux linkages with field and damper windings.

E_{fd} - field voltage.

x_e - total reactance between generator terminal and busbar.

R_e - total resistance between generator terminal and busbar.

δ - rotor angle.

ω_o - angular frequency of infinite bus.

H or M - inertia constant.

T_m - input torque to generator shaft.

ω - angular speed of rotor.

P,Q - active and reactive power.

E - voltage behind synchronous impedance.

E'_d - voltage proportional to quadrature-axis flux linkage.

E'_q - voltage proportional to direct-axis flux linkage.

x' - stator transient reactance.

τ'_{qo} - quadrature-axis transient open-circuit time constant.

τ'_{do} - direct-axis transient open-circuit time constant.

D - damping coefficient.

Excitation System

V_R - voltage sensor output.

V_1 - amplifier output voltage.

V_3	- stabilizer output voltage.
τ_R	- voltage sensor time constant.
τ_A	- amplifier time constant.
τ_F	- stabilizing loop time constant.
τ_E	- exciter time constant.
K_A	- amplifier gain.
K_F	- stablizing loop gain
K_E	- exciter gain.
V_{ref}	- reference voltage.

Miscellaneous

Δ	- prescript denoting incremental change.
$\cdot$	- superscript denoting differentiation with respect to time.
$\underline{\ }$	- subscript denoting vector quantity.
T or t	- superscript denoting matrix or vector transpose.
-1	- superscript denoting matrix inverse.
s	- Laplace operator.

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CHAPTER 1

INTRODUCTION

1.1 The Basic Problem

The study of power system disturbances for the purpose of planning or security assessment requires the solution of potentially over a thousand coupled differential equations. To obtain the time domain response of these equations, even in linearized form, for as little as one second, is a very expensive computational task. Therefore, there has been a considerable effort over the last ten years to reduce the number of equations required to perform a satisfactory analysis of a power system's response to a disturbance. Besides reducing the computational burden associated with simulation, analysis and control system design, the reduction of the order of the large-scale system provides a better understanding of the phenomena being studied and facilitates the design of controllers with a simpler structure that are therefore more easily implemented.

Historically, there have been two major approaches to reducing the order of power system models. In coherency methods, the full set of equations is solved for a test disturbance. The accelerations of all the generators are then checked, to see if a group of generators accelerate at the same rate, thereby maintaining their initial angle

differences with respect to each other. Such a group of generators is called coherent. Each coherent group is then replaced by one equivalent, or "aggregated" generator, thus reducing the number of equations by reducing the number of generators. This reduced-order model is then used to analyze the power system response for all disturbances that occur in areas outside the coherent group.

The other approach to forming dynamic equivalents for power systems is the modal approach. In modal analysis methods, the interconnected power system is assumed to consist of an internal (study) system and an external system. A detailed model of the internal system is retained while the external system is reduced. This is done by finding a linear model of the external system, transforming it to canonical form, and then eliminating the modes that have no impact on the internal system using controllability, observability and fast eigenvalue concepts.

Both methods have certain advantages and certain disadvantages. The coherency method has great intuitive appeal because it yields a reduced-order model composed of equivalent lines and generators. Further, the models for the equivalent lines and generators can be either linear or non-linear, and of any degree of detail desired.

The major shortcoming of the coherency method is that its theoretical foundations are incomplete. This causes no operational problems with using the method in its present form, but a more complete understanding of the theory of coherency equivalents would undoubtedly lead to a better utilization of this method.

The modal analysis method of producing reduced-order equivalents has no shortcoming theoretically, but it has some drawbacks from the functional point of view.

First of all, the reduced-order model is not formulated in terms of equivalent lines and generators, but in terms of retained canonical states. This robs the equivalent of much of its physical insight. The second drawback is that the mode elimination procedure requires the calculation of eigenvalues and eigenvectors. This is a computationally expensive step. However, one important advantage of the modal approach is that a reduced-order model can be found for a group of generators that are not necessarily coherent.

In addition to the previous methods, the singular perturbation approach has been applied recently to the power system. This approach is basically of interest where a system has two groups of time constants, one relatively fast compared with the other. The solution technique is to set the derivatives of these fast modes to zero, thus changing a subset of the dynamic equations for the system from differential to algebraic form. These algebraic equations are then solved for the fast modes and back substituted into the remaining equations to eliminate the fast variables. Thus, the singular perturbation approach provides a means of decoupling the fast and slow dynamics of a system through an approximation method.

Among the model reduction techniques which have no prior application in power systems is the Padé approximation (moment matching)

method. Although this method leads to satisfactory reduced-order models and is computationally simple, it has several shortcomings that have limited its application.

First, it may lead to an unstable reduced model for certain stable large-scale systems or vice-versa; and second, the method loses its computational appeal in the multivariable case.

The major goal of this thesis is to continue rationalizing, improving and combining some of the existing model reduction techniques. A method of reduction is developed which unifies the method of aggregation (modal techniques are a special case of aggregation methods) and that of moment matching, in such a way as to secure the separate advantages of each technique while simultaneously removing their disadvantages. A second reduction technique is developed which unifies the two well known techniques of singular perturbation and aggregation.

1.2 Thesis Structure

The thesis has a total of eight chapters. The seven chapters following this introductory chapter are divided into two groups. The first group includes Chapters 2, 3 and 4 and may be viewed as background information. The remaining chapters develop, describe and apply the proposed reduction techniques.

Chapter 2, defines the approximation problem to be studied in this thesis. The problem formulation is divided into two categories -

frequency domain and time domain. Also, performance criteria that provide a measure of the approximation accuracy are defined.

Chapter 3 reviews four different types of reduction methods which have appeared in the literature, including the Padé approximation (moment matching) method, reduction methods based on error minimization, reduction methods based on the aggregation principle and the singular perturbation approach.

Chapter 4 reviews, in some detail, the two primary methods of forming power system dynamic equivalents, namely coherency and modal analysis. Also, a linearized model for a power system is established.

In Chapter 5, an algorithm for obtaining reduced-order models for scalar systems as well as multivariable systems has been developed. This algorithm combines the method of aggregation with that of moment matching in such a way as to secure the separate advantages of each technique. This technique is computationally simple and can be used to obtain a variety of reduced-order models. Two examples of a synchronous machine connected to an infinite bus (with and without excitation system) have been considered to illustrate the application of this algorithm.

Chapter 6 is divided into two main sections. In the first section, a criterion is proposed for selecting the most important state variables of the original system to be retained in the reduced-order model. Also, an algorithm for obtaining reduced-order models which retain the physical signficance of the state variables has been developed.

In the second section, a procedure for obtaining dynamic equivalents of multimachine systems is proposed. This procedure utilizes the idea of component cost analysis for identifying the coherent generators.

The procedure has been applied to a system of 46 transmission lines, 39 buses and 10 generators. This system is a well known and documented model of the 345-kV system of the New England area.

In Chapter 7, several model reduction techniques have been utilized for designing PID controllers so as to minimize the integral square error as well as designing compensators satisfying frequency domain specifications. The responses of a sample system equipped with various controllers has been compared in order to illustrate the usefulness of the reduced-order model for the design of suboptimal control of the original system.

Chapter 8 summarizes the main conclusions of the thesis. Also, the specific contributions of the research and suggestions for future work are outlined.

CHAPTER 2

FORMULATION OF THE APPROXIMATION PROBLEM

2.1 Introduction

This chapter defines the approximation problem to be studied in this thesis. The problem, simply stated, is that given a linear system find a reduced-order methematical model or system which approximates the given system in some specified sense. This problem is often referred to as order reduction of dynamic systems. Such model reduction has several advantages, including (i) reducing the computational complexity and burden, (ii) providing better understanding of the system, whether the problem is analysis, synthesis, or identification, (iii) facilitating the design of controllers with simpler structure that are therefore more easily implemented. Often, these model reductions are justified on the basis of physical reasoning related to the phenomena and processes concerned.

The approximation problem may be performed in either the time domain or the frequency domain. In the frequency domain, it is assumed that the transfer function of the original n^{th} order system is known and is in the form of a rational function of the complex frequency s. The objective is to compute an r^{th} order transfer function ($r<n$) which approximates the given n^{th} order transfer function.

In contrast to the frequency domain approach, the time domain formulation of the approximation problem starts from differential equations. It is assumed that the state space representation of the given n^{th} order system is known. The objective is to compute an r^{th} state space representation ($r<n$) which approximates the given n^{th} order original system. One way to use the time domain approach to solve the approximation problem is to first transform the problem to the frequency domain.

In order to compare various model reduction techniques and different reduced-order models for a given system, it is desirable to define an error criterion as a measure of the quality of a reduced model. In other words, we must have a criterion to determine how closely and in what sense a particular aspect of the behaviour of a given large system has been approximated by a particular reduced-order model. The question of accuracy is related to the problem of determining the most suitable order for the reduced model. This will be discussed in Section 2.4.

2.2 Frequency Domain Formulation

The problem of reduced-order modeling of large linear systems in the frequency domain can be stated as follows:

Given the transfer function of a large system as

$$G(s) = \frac{B_1 s^{n-1} + B_2 s^{n-2} + \cdots + B_n}{s^n + a_1 s^{n-1} + \cdots + a_n} + B_o \qquad (2.1)$$

obtain the transfer function of a reduced model as

$$G_r(s) = \frac{\hat{B}_1 s^{r-1} + \quad - - - - - - \quad + \hat{B}_r}{s^r + \hat{a}_1 s^{r-1} + \quad - - - - - - \quad + \hat{a}_r} + \hat{B}_o \quad , \; r \; n \qquad (2.2)$$

For a linear system with ℓ outputs and m inputs, the numerator coefficients in equation (2.1) and (2.2) are ℓ x m matrices. In some of the reduced models the matrix $\hat{B}_o$ may be zero, but in some others it may be non-zero even though the transfer function in (2.1) is strictly proper so that B_o is zero.

The frequency domain expression of the approximation problem may be stated as the computation of (2.2) from (2.1) so that $G_r(s)$ is a good approximation of G(s). The question of what is meant by a "good" approximation will be delayed until Section 2.4 when accuracy criteria are discussed. The objective, therefore, is to find an algorithm for computing the polynomial coefficients in (2.2) from the polynomial coefficients in (2.1).

Approximation methods may be classified in two groups. For some methods, the denominator of $G_r(s)$ is computed from both the numerator and denominator of G(s). For other methods, the denominator of $G_r(s)$ is computed from only the denominator of G(s). For both groups, the numerator of the reduced-order model is computed from both the numerator and denominator of the given transfer functon. An algorithm for computing the approximation represented by the second group will be discussed in Chapter 5.

2.3 Time Domain Formulation

Consider a linear system described by a set of first order differential equations where the number of equations (n) may be high. Writing the differential equations in matrix form gives the state space formulation

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} \end{aligned} \tag{2.3}$$

where x, y, and u are the state, output, and input vectors of dimension n, ℓ, and m, respectively. A, B, and C are matrices of order n x n, n x m, and ℓ x n, respectively. The problem of finding a new state space representation

$$\begin{aligned} \dot{\underline{z}} &= F\underline{z} + G\underline{u} \\ \underline{y}_r &= H\underline{z} + D\underline{u} \end{aligned} \tag{2.4}$$

of reduced-order (dimension of z is r<n) such that (2.4) approximates (2.3) shall be referred to as a time domain formulation of the approximation problem.

In (2.3) it is assumed there is no direct path from the input vector u to the output vector y. The direct path in the approximate system denoted by D is a function of the matrices A, B, and C. If the

given system does happen to possess a direct path whose gain matrix is D_o of dimension $\ell \times m$, then replace D in (2.4) by $D + D_o$.

The time domain formulation of the approximation problem may be viewed as developing algorithms for computing the matrices in (2.4) from the matrices in (2.3) as depicted by the solid arrow in Figure 2.1. Another method, is to first compute the exact transfer function G(s). Finally, compute the matrices F, G, H, and D which realize $G_r(s)$. This indirect approach reduces the time domain formulation to the frequency domain formulation of the problem and is indicated by the dashed arrows in Figure 2.1. The difficulty with the indirect approach is the task of computing the coefficients of the transfer function G(s). The objective of the time domain approximation is to find a direct method for computing the matrices F, G, H, and D from A, B, and C which avoid the computational problems inherent in first finding the transfer function.

Another reason for studying the time domain formulation is that it is desirable that a relationship between the given system and reduced model states be known. This will help in utilizing the reduced order model for designing suboptimal controllers and for studying the dynamic behavior of the large system.

2.4 Accuracy Criteria

Model reduction involves a trade off between model order and the degree to which the characteristics of the plant are reflected by the model. Because the relative importance of various plant

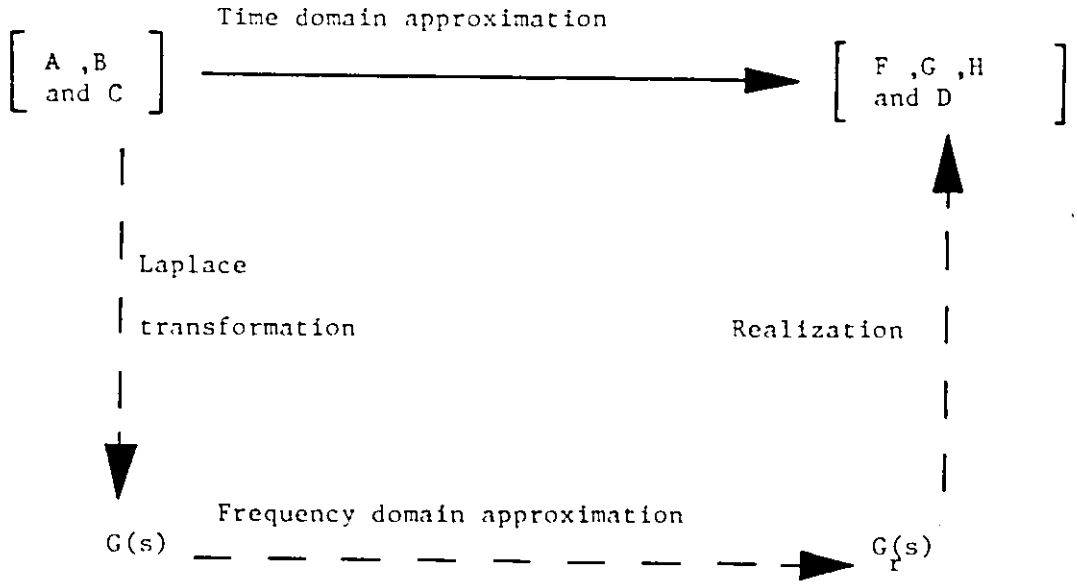


Figure 2.1 Time domain and frequency domain formulation of the approximation problem.

characteristics is highly dependent upon the application, there can be no universal model reduction algorithm. The statement that an r^{th} order linear system is a good approximation of an n^{th} order linear system ($n>r$) refers to the accuracy of the approximation relative to the exact n^{th} order linear system. In order to compare various model reduction techniques and different reduced models, it is desirable to define an error criterion as a measure of the quality of a reduced model. A meaningful error criterion in the time domain would be

$$||\varepsilon|| \; / \; ||y|| \tag{2.5}$$

where $||\varepsilon||$ and $||y||$ are the norm of the function $\varepsilon(t)$ and $y(t)$, respectively, and $\varepsilon(t) = y(t) - y_r(t)$. This ratio defines an accuracy criterion that depends on the choice of the norm and the choice of the input $u(t)$ used to determine $y(t)$ and $y_r(t)$. Another criterion is the component cost analysis [43], which identifies the critical components (physical or mathematical) in a linear system.

2.4.1 Relative Impulse Error

In this subsection a reduced-order model (F, G, H) will be judged by the impulse response matrix. The error impulse response matrix

$$E(t) = Ce^{At}B - He^{Ft}G \tag{2.6}$$

characterizes the error. Moore [30] proved that a reduced-order model is good if the largest principal component of E(t) over (o, ∞) is "small" compared to the smallest principal component of $Ce^{At}B$.

A different way for evaluating the quality of any reduced-order model, regardless of the means used to obtain it, is defined by the model error index (ME)

$$ME = \frac{\delta V}{V} \tag{2.7}$$

and $$\delta V = \lim_{t \to \infty} \left|\left|y - y_r\right|\right|^2, \qquad V = \lim_{t \to \infty} \left|\left|y\right|\right|^2$$

where y is the output of the original model (2.3) and y_r is the output of reduced model (2.4). To calculate the ME it is convenient to define a model whose output is $(y-y_r)$. Such a model is obtained as

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\underline{z}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{z} \end{bmatrix} + \begin{bmatrix} B \\ G \end{bmatrix} [u]$$
$$\begin{bmatrix} y - y_r \end{bmatrix} = \begin{bmatrix} C & H \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{z} \end{bmatrix} \tag{2.8}$$

The cost associated with model (2.8) is

$$\delta V = \lim_{t \to \infty} \left|\left|y - y_r\right|\right|^2 = tr[MP] \tag{2.9}$$

where [P] satisfies

$$SP + PS^T + W = 0 \tag{2.10}$$

and where

$$S = \left[\begin{array}{c|c} A & O \\ \hline O & F \end{array}\right] \qquad W = \left[\begin{array}{c|c} BB^T & BG^T \\ \hline GB^T & GG^T \end{array}\right]$$

and

$$M = \left[\begin{array}{c|c} C^TC & -C^TH \\ \hline -H^TC & H^TH \end{array}\right]$$

The ME may now be expressed as:

$$ME = \frac{\delta V}{V} = tr(MP) \times (tr\ C^T C\hat{P}) \tag{2.11}$$

where $[\hat{P}]$ satisfies

$$A\hat{P} + \hat{P}A^T + BB^T = 0 \tag{2.12}$$

2.4.2 Relative Step Error

If the norm is defined by (2.9) and the input is a unit step

$$u(t) = \mu(t) = \begin{cases} 1 & t>0 \\ \\ 0 & t<0 \end{cases} \tag{2.13}$$

then the step response is

$$\underline{y}(t) = CA^{-1} (e^{At} - I)B \tag{2.14}$$

From equation (2.14) for an asymptotically stable system, the step response will eventually settle to the steady state value

$$\underline{y}(\infty) = CA^{-1}B \tag{2.15}$$

which is usually different from zero. Consequently, the integral

$$\int_0^\infty y^2(t)dt \tag{2.16}$$

is likely to be unbounded and cannot be used to define the step response energy. This difficulty is overcome by defining a new function g(t) which is

$$g(t) = \underline{y}(\infty)\, \mu(t) - \underline{y}(t) \tag{2.17}$$

The new function eliminates the steady state bias, i.e., $g(\infty) = 0$.

The step response energy is then defined by

$$V_S = \int_0^\infty g^2(t)dt \tag{2.18}$$

The step response energy V_S is related to the impulse response energy V as given below:

Consider the solution of equation (2.12). response energy is

$$V = tr(C^T\hat{C}P) \tag{2.19}$$

From (2.3), (2.15), and (2.17)

$$g(t) = \bar{C}\, x(t) \tag{2.20}$$

where

$$\bar{C} = -CA^{-1}$$

Substituting (2.20) in (2.18) gives the step response energy

$$V_S = tr(\bar{C}^T\bar{C}P) \tag{2.21}$$

The choice of the step response energy or the impulse response energy as a criterion for accuracy should depend upon the type of inputs that are likely to occur. Also, it has been shown [18,41] that the usefulness of the reduced-order model for the design of a suboptimal controller for the original system depends very much upon the criteria used for designing the controller. This will be discused in detail in Chapter 7.

2.4.3 Modal Cost Analysis

In this subsection we choose the modal equations as component models for the system to obtain the component costs. The choice will be based on the selection of the modes having the highest input-output interaction for a given input-output pair (u_i, y_j). Only highly observable and or controllable modes are presented. In this case, A might be in Jordan form or its real counterpart

$$\hat{A} = \text{block diag}\{\ldots J_i \ldots\}, \quad i = 1, \ldots, n \tag{2.22}$$

where the eigenvalues of A are $\lambda_i = \sigma_i + j\omega_i$, $i = 1, \ldots, n$ and

$$J_i = \sigma_i \text{ if } \omega_i = 0 \text{ and } J_i = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \text{ if } |\omega_i| > 0$$

which can be obtained by the following choice for a real matrix U

$$U = [U_{1R}, U_{1I}, U_{3R}, U_{3I}, U_{5R}, U_{5I}, \ldots], \quad (A - \lambda_i)(U_{iR} + jU_{iI}) = 0$$

where

$$i = 1, \ldots, N, \quad N = \sum_{i=1}^{n} n_i$$

and the complex part of the i^{th} eigenvector, U_{iI}, is suppressed in U if λ_i is real.

After a coordinate transformation $\hat{x} = Ux$ is applied to the system (2.3), the new system will be

$$\begin{aligned} \dot{\hat{\underline{x}}} &= \hat{A}\hat{\underline{x}} + \hat{B}\underline{u} \\ \underline{y} &= \hat{C}\hat{\underline{x}} \end{aligned} \tag{2.23}$$

where $\hat{A} = U^{-1}AU$, $\hat{B} = U^{-1}B$, and $\hat{C} = CU$

The matrices $\hat{A}$, $\hat{B}$ and $\hat{C}$ will all be real, and for convenience this notation will be used.

Rewriting (2.23) to be in the form

$$\dot{\hat{\underline{x}}}_i = J_i \hat{\underline{x}}_i + \hat{B}_i \underline{u}, \qquad i = 1, \ldots, N$$

$$\underline{y}_i = \hat{C}_i \hat{\underline{x}}_i, \quad \underline{y} = \sum_{j=1}^{N} \hat{C}_j \hat{\underline{x}}_j \tag{2.24}$$

where J_i has been defined before, and

$$\hat{C}_i = \hat{c}_i \text{ if } \omega_i = 0, \quad \hat{C}_i = [c_i^{(1)}, c_i^{(2)}] \text{ if } |\omega_i| > 0$$

$$\hat{B}_i = b_i^T \text{ if } \omega_i = 0, \quad \hat{B}_i = [b_i^{T(1)}, b_i^{T(2)}] \text{ if } |\omega_i| > 0$$

The component costs V_i [44] are given by

$$V_i = \sum_{j=1}^{n} V_{ij}, \qquad i = 1, \ldots, n \tag{2.25}$$

where, $V_{ij} = \text{tr}(P_{ij} \hat{B}_j \hat{B}_i^T)$, and P_{ij} is the solution of

$$P_{ij} J_j + J_i^T P_{ij} + \hat{C}_j^T C_i = 0 \tag{2.26}$$

Note, V_{ij} is the correlation between the excitation of the i^{th} and j^{th} modes, and V_i is the effect of all excitation of the i^{th} mode. We therefore select the r largest values of V_i. The quality of the reduced-order model will be evaluated by

$$ME = \frac{\delta V}{V_R + \delta V} \tag{2.27}$$

where

$$\delta V = \sum_{i,j=r+1}^{n} V_{ij}, \quad V_R = \sum_{i=1}^{r} \left(\sum_{j=1}^{n} V_{ij}\right) + \sum_{i=r+1}^{n} \left(\sum_{j=1}^{r} V_{ij}\right) \tag{2.28}$$

However, if, in addition to the component uncoupling enjoyed by the modal components, the inputs for the modal components are also uncoupled, then $\hat{B}_j \hat{B}_i^T = 0$ for all $i \neq j$, then δV and V_R in equation (2.27) will be given by

$$V_R = \sum_{i=1}^{r} V_i, \quad \delta V = \sum_{i=r+1}^{n} V_i \tag{2.29}$$

CHAPTER 3

A REVIEW OF EXISTING METHODS OF MODEL REDUCTION

3.1 Introduction

This chapter is a review of existing work in four major approximation methods described in the literature for reducing the order of large-scale systems. It is not intended to be an exhaustive discussion of all possible methods. The four methods described in Section 3.2 through 3.5, respectively, were chosen because they are considered the most prominent methods and because they are in some way related to what has been done in Chapters 5 and 6, respectively.

The first approximation method, described in Section 3.2, is based on the well known Padé approximation. It is shown how continued fractions can be used to facilitate the computation of the Padé approximation. The Padé approximation possesses two major limitations in connection with approximation of linear systems. First, it is possible for the Padé method to result in an unstable reduced-order model to a stable system. Second, the method is not designed for approximating multiple input -multiple output systems. A modification of the Padé approximation method for removing these limitations is developed in Chapter 5.

In Section 3.3, the method of computing the reduced-order model is based on minimizing an accuracy criterion chosen a priori.

In Section 3.4, aggregation methods are described. These methods substitute the complex system representation by one of much smaller dimension which retains key features of the system for the given problem. It has been shown that most of the modal and projection techniques are subsets of the more general aggregation technique.

In Section 3.5, principles of the singular perturbation method have been discussed. This method provides a means of decoupling the fast and slow dynamics of the original system, and can be viewed as an approximate aggregation technique. However, it is applicable to both linear and non-linear problems.

3.2 Reduction Methods Based on Padé Approximation

Consider the approximation problem in the frequency domain as defined in Section 2.2 where, given the exact transfer function G(s), the objective is to compute an approximate transfer function $G_r(s)$ of reduced-order. For the time being the problem is restricted to single input - single output systems, subsequently the discussion will be extended to multivariable systems.

3.2.1 Definition of the Padé Approximation Method

Starting from the exact transfer function, expand G(s) in a Taylor series about s = 0 to give

$$G(s) = c_o + c_1 s + c_2 s^2 + \ldots\ldots\ldots = \sum_{i=0}^{\infty} c_i s^i \qquad (3.1)$$

where

$$c_i = \frac{1}{i!} \frac{d^i G(s)}{ds^i}\bigg|_{s=0} = (-1)^i \int_0^{\infty} t^i g(t)dt = -CA^{-i-1}B$$

$$i = 0, 1, 2, \ldots\ldots \qquad (3.2)$$

[A], [B], [C] are the matrices of the state model corresponding to the given G(s), and g(t) represents the impulse response of the system.

The objective is to find a strictly proper rational function $G_r(s)$ which is identical to (3.1) for as many low order terms as possible. The maximum number of terms is 2r.

To compute $G_r(s)$, form a polynomial of degree 2r-1 by truncating (3.1) after the first 2r terms. Setting this polynomial equal to $G_r(s)$ in the standard form results in the equation

$$\underbrace{\hat{b}_1 s^{r-1} + \ldots + \hat{b}_r}_{\text{numerator}} = \underbrace{(s^r + \hat{a}_1 s^{r-1} + \ldots + a_r)}_{\text{denominator}} \times$$

$$(c_o + c_1 s + \ldots + c_{2r-1} s^{2r-1}) \qquad (3.3)$$

Performing the multiplication and equating the coefficients of like powers in s leads to the following set of 2r simultaneous equations

$$\hat{b}_{r-i} = \sum_{j=0}^{i} \hat{a}_{r-j}\, c_{i-j} \qquad (i = 0, 1, \ldots, 2r-1) \qquad (3.4)$$

where $\hat{b}_{r-i} = 0$, $i>r$ and $\hat{a}_{r-j} = 0$ for $j > r$

Equation (3.4) can be solved uniquely for the unknowns

$$\hat{b}_i \ (i = 1, 2, \ldots r) \text{ and } \hat{a}_i (i = 1, 2, \ldots, r)$$

This method of finding a rational function approximation based on a truncation of the Taylor series is not restricted to transfer functions and was developed by the French mathematician Padé [31]. In the development given above, the degree of the numerator is one less than the degree of the denominator. However, in the development of the general Padé approximation method the degree of the numerator and denominator may be arbitrarily chosen.

3.2.2 Continued Fractions

The use of the Padé approximation as described above requires the computation of the Taylor series coefficients in (3.1). Next (3.4) must be solved for the unknown coefficients of the approximate rational function. This tedious computation can be simplified by introducing the use of continued fractions.

The procedure for utilizing continued fractions to compute the Padé approximation consists of the following steps. Starting from the exact transfer function G(s) in the standard form (2.1), expand G(s) into continued fraction in Cauer second form

$$G(s) = \cfrac{1}{h_1 + \cfrac{1}{\cfrac{h_2}{s} + \cfrac{1}{h_3 + \cfrac{1}{\cfrac{h_4}{s} + \cdots\cdots + \cfrac{1}{\cfrac{h_{2n}}{s}}}}}}$$

$$\triangleq \cfrac{1}{h_1 + \cfrac{1}{\cfrac{h_2}{s} + \cfrac{1}{h_3 + \cdots\cdots \cfrac{1}{\cfrac{h_{2n}}{s}}}}} \tag{3.5}$$

Then truncate it as

$$G_r(s) = \cfrac{1}{h_1 + \cfrac{1}{\cfrac{h_2}{s} + \cdots\cdots \cfrac{1}{\cfrac{h_{2r}}{s}}}}, \quad r<n \tag{3.6}$$

It has been shown [28,56] that the $G_r(s)$ obtained from (3.6) has the same first 2r coefficients of the Taylor series expansion as G(s) and hence $G_r(s)$ is the Padé approximation of G(s).

Padé approximation about $s = \infty$ may also be accomplished. In this case, the Taylor series (3.1) is replaced by a Laurent series,

$$G(s) = d_1 s^{-1} + d_2 s^{-2} + \cdots\cdots\cdots = \sum_{i=1}^{\infty} d_i s^{-1} \tag{3.7}$$

where

$$d_i = \frac{1}{(i-1)!} \frac{d^{i-1}}{dt^{i-1}} g(t) \Big|_{t=0} = \frac{1}{2nj} \int s^{i-1} G(s) ds$$

$$= CA^{i-1}B, \quad i = 1, 2, 3, \cdots\cdots \tag{3.8}$$

It can be easily shown as in the case of low frequency Padé approximations (expanding G(s) about $s = 0$) that an r^{th} order high frequency reduced model (expanding G(s) about $s = \infty$) can be simply obtained by expanding G(s) into continued-fraction in Cauer first form and truncating it, i.e.

$$G(s) = \frac{1}{H_1 s +} \frac{1}{H_2 +} \frac{1}{H_3 s +} \cdots\cdots \frac{1}{H_{2n}} \tag{3.9}$$

$$G_r(s) = \frac{1}{H_1 s +} \frac{1}{H_2 +} \cdots \frac{1}{H_{2r}} \tag{3.10}$$

Similarly, a Padé approximation $G_r(s)$ which approximates low as well as high frequency response, can either be obtained by expanding the

given G(s) about two points (s = 0 and s = ∞) and obtaining an approximation $G_r(s)$ which matches the first suitable number of terms of the expansion of G(s) about s = 0 and s = ∞.

Such a reduced model can equivalently be obtained by expanding G(s) into continued fraction in mixed or third Cauer form and truncating it. For G(s), the mixed form is

$$G(s) = \cfrac{1}{(h_1' + H_1' s) + \cfrac{1}{(\frac{h_2'}{s} + H_2') + \cdots\cdots}} \qquad (3.11)$$

In fact, Padé approximation provides as with a method for model reduction for the general case when the poles of G(s) are not clustered close to or far from the origin (s = 0) but are scattered in the left half s-plane.

3.2.3 Limitations of the Padé Approximation

Padé approximation leads to satisfactory reduced order models for many large scale systems and it is computationally a simple method. However, it has several shortcomings that limit its application. First, it might lead to an unstable reduced model for certain stable large-scale systems or vice-versa, and second, the method loses its computational appeal in the multivariable case where polynomial matrix inversion becomes necessary. Finally, when applied to the reduction of multivariable systems it may lead to nominally reduced models that are,

in fact, higher in order than the original system. As an example the first order Padé approximation of

$$G(s) = \frac{8s + 8}{s^2 + 6s + 8} \tag{3.12}$$

is

$$G_1(s) = \frac{-4}{s - 4} \tag{3.13}$$

which is clearly unstable.

The reason for the instability is attributed to the fact that the poles of the Padé approximation are a function of both the poles and zeros of the exact transfer function.

Another example considers the first order Padé approximation of

$$G(s) = \frac{\begin{bmatrix} 8 \\ --- \\ 2 \end{bmatrix} s + \begin{bmatrix} 8 \\ --- \\ -8 \end{bmatrix}}{s^2 + 6s + 8} \tag{3.14}$$

which is

$$G_1(s) = \left[\begin{array}{c} \dfrac{-4}{s - 4} \\ \hline \dfrac{-1}{s - 1} \end{array} \right] = \frac{\begin{bmatrix} -4 \\ --- \\ -1 \end{bmatrix} s + \begin{bmatrix} -4 \\ --- \\ 4 \end{bmatrix}}{s^2 - 3s - 4} \tag{3.15}$$

The approximation of (3.14) results in a different first order denominator for each element of $G_1(s)$. Thus the characteristic polynomial of $G_1(s)$ has degree 2. In general, for a system with p outputs and m inputs, the degree of $G_r(s)$ will be equal or less than rxpxm.

Some solutions have been suggested to overcome the stability problem [24,39]. For example, we may choose the r dominant poles of G(s) as the denominator of the low frequency reduced model $G_r(s)$, and determine the r numerator coefficients of $G_r(s)$ by matching the first r terms in the Taylor series expansion of G(s) and $G_r(s)$ about $s = 0$. This can be done by solving the first r equations in the set of equations (3.4), which may be called a partial-Padé approximation. However, if such clustering of poles does not exist for a given system, we may determine the denominator of $G_r(s)$ from only the denominator of G(s) by Routh approximation, as proposed by Hutton [24]. Once the denominator of the reduced model is obtained, the numerator can be obtained by partial-Padé approximation. It will be shown in Chapter 5 that the proposed Routh approximation method in the time domain overcomes these shortcomings.

3.3 Reduction Methods Based on Error Minimization

In this approach, a criterion such as mean square error for the error between the large system dynamics and the reduced order model is defined. Assuming a suitable form for the dynamics of the reduced model, the unknown elements of the reduced model are determined by minimizing the error function. The error criterion may be defined in

time or frequency domain. The criterion commonly used is the square root of the integral squared error

$$||\varepsilon|| = ||y(t) - y_r(t)|| \tag{3.16}$$

$$||E|| = \sqrt{\int_0^\infty \varepsilon^2(t)dt} \tag{3.17}$$

where $y(t)$ and $y_r(t)$ denote the output of the exact and approximate systems, respectively.

When the approximation problem is expressed in the frequency domain, (3.17) can be written, using Parseval's theorem, as

$$||e||^2 = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} |E(s)|^2 \, ds \tag{3.18}$$

with $|E(s)|^2 = E(s)\,E(-s)$ and where E denotes the Laplace transform of ε.

It should be noted that the minimization problem is not restricted to deterministic inputs. By replacing (3.17) with

$$||e||^2 = E\{\int_0^\infty \varepsilon^2(t)dt\} \tag{3.19}$$

where the operator E { } denotes the expected value, it is possible to identify a reduced-order model from noisy input-output data. Also, the reduced-order model obtained depends on the choice of the norm in (3.16) and the input signal. This approach may also be used for obtaining an optimal aggregation matrix corresponding to an optimality criterion.

Advantages of this method include the possibility of guaranteed cost control, and its application to nonlinear distributed parameter systems. However, apart from the numerical difficulties and complicated computations involved, there is no guarantee that these models will be useful for obtaining a near-optimal design of the controller. Furthermore, the states of the reduced model are not directly related to the states of the original system [42].

3.4 Reduction Methods Based on Aggregation Principle

The intuitive idea behind the notion of aggregation is quite simple. Suppose that S_1 is a mathematical description of a physical system using a given set of variables, and S_2 is a consistent description of the same system using a smaller set of variables. Then S_2 is termed an aggregate model for S_1, and the variables of the system S_2 are termed aggregate variables. The phrase "S_2 is an aggregated model of S_1" has been introduced in the control theoretic literature and made precise by Aoki [3] for linear systems.

3.4.1 Aggregation Principle

Suppose that the system S_1 is described by the state equation

$$\dot{\underline{x}}(t) = [A]\ \underline{x}\ (t) + [B]\ \underline{u}\ (t) \tag{3.20}$$

where $x \varepsilon R^n$ and $u \varepsilon R^m$, and the aggregated system S_2 is described by the state equation

$$\dot{\underline{z}}\ (t) = [F]\ \underline{z}\ (t) + [G]\ \underline{u}\ (t) \tag{3.21}$$

where $z \varepsilon R^r$, $r < n$

In order for S_2 to be an aggregated model of S_1 the state vectors of these two systems, denoted by x and z, respectively, are to satisfy the relationship

$$\underline{z}\ (t) = [K]\ \underline{x}\ (t) \tag{3.22}$$

where K is an r x n constant matrix. This requirement is termed dynamic exactness. From (3.20) to (3.22) it follows that dynamic exactness is achieved if and only if the maxtrix equations

$$FK = KA \tag{3.23}$$

$$G = KB \tag{3.24}$$

are satisfied. Analysis of these conditions reveals that dynamic exactness is achieved only when the aggregate state vector $z(t)$ is a linear combination of certain modes of $x(t)$, i.e., the aggregation matrix K exists when the matrices F and A have common eigenvalues [23]. To see this simply, assume that A has n distinct eigenvalues denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$ and let $v_1, v_2, \ldots, v_n$ be the associated normalized eigenvectors. Then it follows from equation (3.23) that

$$FKv_i = KAv_i \tag{3.25}$$

as

$$Av_i = \lambda_i v_i \tag{3.26}$$

then

$$FKv_i = \lambda_i Kv_i \tag{3.27}$$

The above equation shows that if $Kv_i \neq 0$, then it is an eigenvector of F with the same eigenvalue λ_i. Thus, the notion of aggregation for linear systems is, in fact, a generalization of the familiar idea of simplifying linear systems by retaining the dominant modes [10].

Further insight into the nature of the class of matrices for which dynamic exactness can be achieved is obtained by realizing that the aggregation problem as posed for linear systems is in fact a problem of minimal realization. Notice that (3.20) and (3.22)

determine a dynamic relationship between u(t) and z(t) that can be described by the transfer function matrix

$$G(s) = K(sI - A)^{-1}B \tag{3.28}$$

In order for the dynamic exactness condition to hold, we must also have

$$G(s) = (sI - F)^{-1}G \tag{3.29}$$

Hence, if z(t) has lower dimension than x(t), the state description defined by (3.20) and (3.22) is nonminimal. But this is possible if and only if there are pole-zero cancelations. Thus, the class of aggregation matrices K is restricted to those that create zeros in the input-output relationship between u(t) and z(t) that cancel poles of the relationship [35].

Although dynamic exactness restricts the class of possible aggregation matrices for a given system of the form (3.20), there are still many ways to choose the matrix K. Various model reduction techniques proposed by several authors can be viewed as aggregation methods with a particular choice of K implied. For example, a natural approach [10] is to require that a given mode of S_1 that is retained in S_2 be represetned in z(t) in the same proportions that is represented in x(t), i.e., that the mode shapes be preserved. Another criterion that has been discussed by several authors [7,13] is that z(t) should have the same steady state response to a step input as x(t). Yet

another approach is to minimize some measure of the difference $z(t) - x(t)$ for a given class of inputs [8,43,55]. Any of these techniques may be of interest depending upon the intended use of the aggregate model.

3.4.2 Determination of the Aggregation Matrix

3.4.2.1 General Case

The system (3.20) can be put into the modal representation

$$\dot{\hat{x}} = [\Lambda]\,\hat{x} + [\hat{B}]\,u \tag{3.30}$$

where $$[\Lambda] = [V]^{-1}\,[A]\,[V] = \text{diag}\,(\lambda_1, \lambda_2 \ldots, \lambda_n)$$

$$[\hat{B}] = [V]^{-1}\,[B]$$

The dynamic system given by (3.30) can be partitioned as follows

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \hline \dot{\hat{x}}_2 \end{bmatrix} = \left[\begin{array}{c|c} \Lambda_1 & 0 \\ \hline 0 & \Lambda_2 \end{array}\right] \begin{bmatrix} \hat{x}_1 \\ \hline \hat{x}_2 \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ \hline \hat{B}_2 \end{bmatrix} [u] \tag{3.30a}$$

where subscripts 1 and 2 refer to 'dominant' and 'non-dominant' quantities of dimension r and (n-r), respectively.

Thus the modal representation of the reduced model should be of the form

$$\dot{\hat{x}}_1 = [\Lambda_1]\,\hat{x}_1 + \hat{B}_1 u \tag{3.31}$$

All reduced models, represented by (3.21) and with [F] similar to $[\Lambda_1]$, are defined by the transformation

$$\underline{z} = [T]\,\hat{x}_1 \tag{3.32}$$

where T is any nonsingular but arbitrary matrix of dimension r. Therefore the most general aggregated model is defined by the matrices

$$[K] = [T]\,[K_o] = [T]\,[I_r : 0]\,[V]^{-1}$$

$$[F] = [T]\,[\Lambda_1]\,[T]^{-1},\quad [\hat{B}_1] = [T]\,[K_o]\,[B] \tag{3.33}$$

In the particular case when T is an identity matrix, F will be obtained in the diagonal or Jordan form.

Note that by choosing the matrix T as the upper left partition $[V_1]$ of the modal matrix V, the aggregation matrix takes the form

$$[K] = [V_1]\,[I_r : 0]\,[V]^{-1} \tag{3.34}$$

Thus the reduced model is defined by the matrices

$$[F] = [V_1]\,[\Lambda_1][V_1]^{-1},\quad [G] = [V_1]\,[I_r : 0]\,[V]^{-1}B \tag{3.35}$$

which is the model proposed by Davison [12].

3.4.2.2 Aggregation Matrix with a Steady-State Constraint

In the class of aggregation matrices defined by (3.32), those which give the same steady-state response for the system and the reduced model, when excited by step inputs, are of special interest. Assuming that the system is stable, this implies that

$$\underline{z}^i(\infty) = [P]\underline{x}^i(\infty) \quad ; \quad i = 1, \ldots, m \tag{3.36}$$

where P is the projection matrix defining the r components of x to be retained in the reduced model; $x^i(\infty)$ and $z^i(\infty)$ represent the steady-state values of the system and the reduced model, respectively, when a step function is applied to the i^{th} input, all other inputs not being excited. Equation (3.36) implies that

$$[T]\ [K_o]\ [A]^{-1}\ [B] = [P]\ [A^{-1}]\ [B] \tag{3.37}$$

In this last equation the only unknown is the square matrix T which has to be nonsingular. For a solution to exist, the matrices $K_oA^{-1}B$ and $PA^{-1}B$ must have the same rank and the general solution is given by Rao and Mitra [33]

$$[T] = [R] + [N]\ [S] \tag{3.38}$$

with

$$[R] = [PA^{-1}B]\,[K_oA^{-1}B]^+$$

and (3.39)

$$S = I_r - [K_oA^{-1}B]\,[K_oA^{-1}B]^+$$

Here $[\quad]^+$ represents the pseudo inverse of the matrix $[\quad]$, I_r the identity matrix of order r and N an arbitrary square matrix of dimension r.

3.4.2.3 Optimal Aggregation Matrix

The choice of the aggregation matrices mentioned above is quite arbitrary; thus it seems logical to introduce some measure for the quality of reduction. Now, a quadratic criterion function of the error resulting from the reduction processes, is introduced. The reduction error is defined as

$$\begin{aligned} \underline{e}^j(t) &= [P]\,\underline{x}^j(t) - \underline{z}^j(t) \\ &= (P - TK_o)\,\underline{x}^j(t) \end{aligned} \qquad (3.40)$$

where P is the projection matrix which specifies the r components of x(t) to be retained; $x^j(t)$ and $z^j(t)$ represent the states x(t) and

$z(t)$ when only the j^{th} input is excited. Thus a quality measure of the aggregation can be chosen as

$$J = \sum_{j=1}^{m} \int_0^{\infty} ||e^j(t)||_Q^2 \, dt \tag{3.41}$$

with Q a symmetric positive definite matrix of order r. Thus equations (3.40) and (3.41) yield

$$TK_o = P \tag{3.42}$$

which has the solution

$$T = R + NS \tag{3.43}$$

with

$$R = PK_o^+ \quad , \quad S = I_r - K_o K_o^+ \tag{3.44}$$

To solve for T, J is written in the following form

$$J = \text{trace}\,(P^T Q P W) - 2\,\text{trace}\,(KQPW) + \text{trace}\,(K^T Q K W) \tag{3.45}$$

where

$$W = \sum_{j=1}^{m} \int_0^{\infty} \underline{x}^j(t)\, \underline{x}^{j^T}(t)\, dt \quad , \quad K = TK_o \tag{3.46}$$

Since the matrix T is a function of an unknown matrix N, a necessary condition for J to be minimum is obtained by expressing the nullity of

the gradient matrix of J with respect to N. Using the results concerning the derivation of a trace with respect to a matrix [4] and taking into account the constraints (3.42) and (3.43), the following relation is obtained

$$[T][K_0 W K_0^T S^T] = [P W K_0^T S^T]' \tag{3.47}$$

The solution of (3.47), when it exists, leads to the optimal aggregation matrix and thus to the best reduced model. Indeed, since aggregation is a generalization of projection [21] the reduced model given by optimal projection is an optimal aggregated model.

3.4.2.3.1 **Determination of the Optimal Output Matrix**

In many instances, however, one is interested in closely approximating the output of the large-scale system. The addition of an output equation to the reduced model adds a new dimension to the problem, for the choice of eigenvalues to be retained may become critical [22]. The following aggregated model of order $p \geq r$ is chosen:

$$\dot{\underline{z}}(t) = [F]\underline{z}(t) + [G]\underline{u}(t), \quad \hat{\underline{y}}(t) = [H]\underline{z}(t) \tag{3.48}$$

where $z \in R^p$, $u \in R^m$, $\hat{y} \in R^r$; and, without loss of generality, it is assumed that $\underline{z}(t) = [K_0]\underline{x}(t)$.

The reduction error is obtained by substitution of $z(t)$ by $\hat{y}(t)$ in equation (3.40):

$$e^j(t) = Px^j(t) - \hat{y}^j(t),$$

and the minimizatiion of (3.41), with respect to the output matrix H, leads to similar results as those previously obtained.

3.4.3. Control of a System via the Aggregated Model

Control of a large scale system via control policies based on S_2 with an assumed aggregation matrix K is now considered. The criterion funtion for S_1 is assumed to be given by

$$J = \int_0^\infty (x^T Q\, x + u^T Ru)\, dt \qquad (3.49)$$

where $Q \geq 0$ and $R > 0$.

Consider the behaviour of S_2 where u is generated by

$$\underline{u} = -[L]\underline{z} \qquad (3.50)$$

and where L is a m x r gain matrix given by equation (3.52) below. Now, if the system S_2 has a criterion function

$$J_m = \int_0^\infty (z^T Q_m z + u^T Ru)\, dt \qquad (3.51)$$

then, for a controllable pair (F, G) the control law (3.50) results in a stable closed-loop system and is optimal when the control gain for S_2 is given by

$$L = R^{-1}G^{T}P \tag{3.52}$$

where [P] is the solution of

$$0 = F^{T}P + PF - PGR^{-1}G^{T}P + Q_m \tag{3.53}$$

and the minimal value of J_m is given by

$$J_m^* = z_o^{T}Pz_o \tag{3.54}$$

where [P]>0.

The optimal control for S_1 is

$$u' = -[L^*]x$$

where

$$[L^*] = [R^{-1}]\,[B]^{T}[T] \tag{3.55}$$

and where [T] satisfies

$$0 = A^T T + TA - TBR^{-1}B^T T + Q \tag{3.56}$$

when K is such that (3.22) is satisfied. Then from equations (3.23) and (3.24) and by pre- and post-multiplication of equation (3.53) by K^T and K, respectively,

$$0 = A^T K^T PK + K^T PKA - K^T PKBR^{-1}B^T K^T PK + K^T Q_m K \tag{3.57}$$

Comparing equation (3.57) with equation (3.56), it is seen that $K^T PK$ corresponds to T if $K^T Q_m K$ is made to correspond to Q. Of course, they cannot be equated because T is of rank n while $K^T PK$ is at most of rank r. The above argument indicates that the control law given by equation (3.50) is a suboptional control for S_1 with a suitably chosen Q_m, for example, where Q_m is chosen as

$$[Q_m] = [KK^T]^{-1}[KQK^T][KK^T]^{-1} \tag{3.58}$$

3.4.3.1 Stability Analysis

The closed-loop stability of S_1 when it is controlled by equation (3.50) is now investigated. From equations (3.21), (3.50) and (3.53)

$$[F - GL]^T[P] + [P][F - GL] = -[L^T RL + Q_m]$$

Let $[L^T RL + Q_m]$ be positive definite. Then it is well known that $[F - GL]$ is a stable matrix if and only if P is positive definite. The use of the control law (3.50) in S_1 with Q_m given by equation (3.58) results in the value of the criterion function (3.49)

$$J = \underline{x}_o^T [N] \underline{x}_o \tag{3.59}$$

where N satisfies the matrix equation

$$[A - BLK]^T [N] + [N][A - BLK] = [K^T L^T RLK + Q] \tag{3.60}$$

Aoki [3] has shown that

$$[N] \geq [K^T PK] \tag{3.61}$$

where $K^T PK$ is only positive semidefinite.

Thus, a stable [F-GL] does not necessarily imply that [A-BLK] is stable. This can be also seen from the fact that with an rxn aggregation matrix K, at most r of the characteristic values of A can be moved by the feedback control. The remaining n-r eigenvalues of A are unaffected by the feedback and remain the same, i.e., these are the uncontrollable modes of [A].

3.4.4 Approximate Aggregation

Now, consider the case where for a given K, (3.22) and (3.23) are not satisfied. As an attempt to apply a similar approach to the

general case, assume that a relationship

$$z(t) \approx [K]x(t) + e(t) \tag{3.62}$$

exists where it is desirable that the error term e(t) should either die out or remain bounded as time evolves. A further restriction is that K should have full rank. Using equations (3.20), (3.21) and (3.62) it is evident that

$$\dot{e} = Fe + (FK - KA)x \tag{3.63}$$

Thus, the discrepancy between z and Kx evolves with time as

$$e(t) = e^{F(t)}e(o) + \int_0^t e^{F(t-\tau)}[FK - KA]\, x(\tau)\, d\tau \tag{3.64}$$

Bounds on e(t) can be obtained from this. Sometimes K can be chosen in such a way that some components of e are zero. If K can be chosen such that the modes of F associated with nonzero components of e die out very quickly, then even if FK ≠ KA, the suboptional controls for S_1 derived from S_2 may still be satisfactory.

3.5 Reduction Methods Based on Singular Perturbations

3.5.1 Introduction

Realistic models of large scale systems involve interacting dynamic phenomena of widely different speeds. In a power system model,

for example, voltage and frequency transients range from intervals of seconds, corresponding to generator voltage regulator, speed governor action and shaft dynamics, to several minutes, corresponding to load voltage regulator action, prime mover and thermal energy storage. Since such models are of high order and numerically stiff, order reduction and separation of time scales are often made using reduction techniques based on retention of dominant eigenvalues (poles). The underlying assumption is that during the fast transients the slow variables remain constant and that by the time their changes become noticeable, the fast transients have already reached their quasi-steady-states (qss). Based on this qss assumption and experience, the state variables are divided into n_s "slow" states $\underline{x}_s$ and n_f "fast" states $\underline{x}_f$. Thus the full scale model is

$$\dot{x}_s = f(x_s, x_f, t) \qquad x_s(t_o) = x_s^o \tag{3.65}$$

$$\dot{x}_f = g(x_s, x_f, t) \qquad x_f(t_o) = x_s^o \tag{3.66}$$

Then the only states used for short term studies are x_f, disregarding (3.65) and considering the states x_s as constant parameters. In long term studies the only states are x_s and equation (3.66) is reduced to algebraic equation by formally setting $\dot{x}_f = 0$. The qss model is thus

$$\dot{\bar{x}}_s = f(\bar{x}_s, \bar{x}_f, t), \qquad \bar{x}_s(t_o) = \bar{x}_s^o \tag{3.67}$$

$$0 = g(\bar{x}_s, \bar{x}_f, t) \tag{3.68}$$

An inconsistency of this classical qss approach is the requirement that $\bar{x}_f$ equals a constant as implied by

$$\dot{x}_f = 0$$

is violated by (3.68), which defines $\bar{x}_f$ as a time varying quantity. If a qss model fails to provide a good approximation of the actual solution $x_s(t)$ and $x_f(t)$, there is no provision for improving the approximation.

Singular perturbation is a method for the separation of slow and fast models which removes the inconsistencies of the classical qss approach and systematically improves the accuracy of the lower order models.

If it is known that the states x_f are $1/\varepsilon$ times faster than x_s, then $\dot{x}_f$ is about $1/\varepsilon$ times larger than $\dot{x}_s$ and equations (3.65), (3.66) then becomes

$$\dot{x}_s = f(x_s, x_f, t) \qquad x_s(t_o) = x_s^o \tag{3.69}$$

$$\varepsilon\dot{x}_f = G(x_s, x_f, t) \qquad x_f(t_o) = x_f^o \tag{3.70}$$

For $\varepsilon = 0$, the model (3.69), (3.70), defines the qss model as

$$\frac{d\bar{x}_f}{dt} = f(\bar{x}_s, \bar{x}_f, t) \qquad x_s(t_o) = x_s^o \tag{3.71}$$

$$0 = G(\bar{x}_s, \bar{x}_f, t) \tag{3.72}$$

Although this is the same qss model (3.67), (3.68), its origin is different. The difference is that

$$\dot{x}_f = 0$$

as required by (3.72) is not contradicted by

$$\varepsilon \cdot \dot{x}_f = 0$$

which is now due to $\varepsilon = 0$, and not

$$\dot{x}_f = 0$$

Although in principle applicable to both linear and non-linear problems, we shall for simplicity restrict attention to the linear case. For a linear system

$$\dot{\underline{x}}_s = [A_{11}]\, \underline{x}_s + [A_{12}]\, \underline{x}_f + [B_1]\underline{u} \tag{3.73}$$

$$\epsilon \dot{\underline{x}}_f = [A_{21}]\underline{x}_s + [A_{22}]\underline{x}_f + [B_2]\underline{u} \tag{3.74}$$

where ϵ is a small positive parameter and $[A_{22}]$ is nonsingular, setting $\epsilon = o$ results in

$$\dot{\bar{\underline{x}}}_s = [A_{11} - A_{12} A_{22}^{-1} A_{21}]\bar{\underline{x}}_s + [B_1 - A_{12} A_{22}^{-1} B_2]\underline{u} \tag{3.75}$$

$$\bar{\underline{x}}_f = -[A_{22}^{-1} A_{21}]\bar{\underline{x}}_s - [A_{22}^{-1} B_2]\underline{u} \tag{3.76}$$

The major shortcoming of the singular perturbation theory is that the mathematical model of a system is rarely given in the form (3.73), (3.74). The model is usually written in the form

$$\dot{\underline{x}} = [A]\underline{x} \tag{3.77}$$

where $\underline{x}$ is n-dimensional state vector and $n = n_s + n_f$.

3.5.2 Transformation of Physical Models into Singularly Perturbed Form

The main difficulty with the method of singular perturbation is the problem of deciding the proper partitioning of the state vector. The problem is complicated by the fact that, in general, the states are

not decoupled, and therefore, it is difficult to relate a particular state to a particular mode.

Now, in order to convert the state equation (3.77) into the form (3.73 to 3.74), one needs to examine three aspects:

(i) determination of the number of states belonging to the fast or slow state group.

(ii) classification of the original state variables as slow states x_s or fast states x_f if such classification exists.

(iii) selection of a small parameter ε.

3.5.2.1 A Two-Time-Scale Property

Let the eigenvalues of A in model (3.77) be arranged in increasing order of magnitude and be divided into two distinct sets

$$\Lambda_s = \{\lambda_{s1}, \ldots\ldots, \lambda_{sn_s}\}$$
$$\Lambda_f = \{\lambda_{f1}, \ldots\ldots, \lambda_{fn_f}\} \qquad (3.78)$$

such that $|\lambda_{si}| < |\lambda_{fj}|$ $i = 1, \ldots, n_s$, $j = 1, \ldots, n_f$.

Model (3.77) is said to possess a two-time-scale property if the largest absolute eigenvalue of Λ_s is much smaller than the smallest absolute value of Λ_f, that is when

$$|\lambda_{sn1}| \;/\; |\lambda_{fn}| \ll 1 \qquad (3.79)$$

When (3.79) is satisfied the number of slow states to be selected from x is n_s and the number of fast states is n_f. Once n_s and n_f are known it is necessary to group the given state variables into slow and fast states x_s and x_f. The set x_s includes the states exhibiting primarily slow modes whereas x_f includes those exhibiting fast modes. Now, if we rewrite (3.77) into the partitioned form

$$\begin{bmatrix} \dot{\underline{x}}_s \\ \hline \dot{\underline{x}}_f \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{bmatrix} \underline{x}_s \\ \hline \underline{x}_f \end{bmatrix} \tag{3.80}$$

where the matrix A is appropriately rearranged and partitioned. A sufficient condition that (3.80) is in singularly perturbed form is given in [9], that is

$$||A_{22}^{-1}|| << (||A_o|| + ||A_{12}||||L_o||)^{-1} \tag{3.81}$$

where $$L_o^{-1} = -A_{22}A_{21}^{-1}, \quad A_o = A_{11} + A_{12} L_o$$

It should be emphasized that as mentioned in [1,27] the criterion is somewhat restrictive and is not satified by some typical two time-scale problems. Once the sufficient condition (3.81) is satisfied the small parameter ε, can be shown as in [1],

$$\varepsilon = ||A_{22}^{-1}|| * ||A_o|| \tag{3.82}$$

or as in [1],

$$\varepsilon = |\lambda_{sn1}| \; / \; |\lambda_{fn2}| \tag{3.83}$$

Thus the major problem in converting a given system to singularly perturbed form is in grouping the state variables into slow and fast states such that (3.81) is satisfied. If the original system possesses the two time-scale property (3.79), but the sufficient condition is not satisfied, it may still be possible to satisfy (3.81) by an appropriate scaling and regrouping of the state variables.

Different methodologies for regrouping the states are given in [1,9,32]. In [1] state grouping is pursued by comparing row norms of the matrix A while [9] compares row norms of normalized eigenvectors corresponding to slow modes. In recent work [34], the choice of the states is made on the basis of an energy integral participation matrix.

3.5.3 Control Design Using Simplified Models

It is recognized from the last section that the aggregation approach to model simplification neglects the fast modes. In the singular perturbation approach both slow modes and fast modes are retained, but control analysis and design are performed in two stages. Consider equations (3.73) and (3.74) together with an output vector described by

$$\underline{y} = C_1 \underline{x}_s + C_2 \underline{x}_f \tag{3.84}$$

Using the quasi-steady state assumption the slow subsystem is given by

$$\begin{aligned} \dot{\underline{x}}_s &= A_s \underline{x}_s + B_s \underline{u} \quad , \quad \underline{x}_s(t) = \underline{x}_s^o \\ \underline{y}_s &= C_s \underline{x}_s + D_s \underline{u} \end{aligned} \tag{3.85}$$

where $A_s = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $B_s = B_1 - A_{12}A_{22}^{-1}B_2$

$C_s = C_1 - C_2A_{22}^{-1}A_{21}$, $D_s = - C_2A_{22}^{-1}B_2$

and the fast subsystem is described by

$$\begin{aligned} \frac{d\hat{\underline{x}}_f}{d\tau} &= A_{22}\hat{\underline{x}}_f + B_2 u \quad , \quad \hat{x}_f(o) = \underline{x}_f^o(o) - \bar{\underline{x}}_f(o) \\ \underline{y}_f &= C_2 \hat{\underline{x}}_f \end{aligned} \tag{3.86}$$

where $\tau = \frac{t}{\varepsilon}$

We now present the main results of using the simplified models in the control design of linear dynamical systems. These results are divided into:

(i) state feedback by eigenvalue assignment,

(ii) linear quadratic control design.

3.5.3.1 State Feedback by Eigenvalue Assignment

The control design problem is that of finding a state feedback gain matrix so that a set of eigenvalues is placed at desired

locations. A two stage control design can be obtained in the form

$$u = [(I + K_f A_{22}^{-1} Bs)K_s + K_f A_{22}^{-1} A_{21}]\, \underline{x}_s + K_f \underline{x}_f \qquad (3.87)$$

where K_s places the eigenvalues of the pair (A_s, B_s) at n_s desired locations and K_f places the eigenvalues of the pair (A_{22}, B_2) at n_f desired locations. The control law (3.87) when applied to the system (3.73, 3.74) gives an order of ϵ approximation to the n desired locations [9]. In deriving (3.87) both the slow and fast subsystems are assumed completely controllable.

3.5.3.2 Linear Quadratic Control Deisgn

Consider that the performance of the system (3.73, 3.74 and 3.84) is regulated by the minimization of the performance measure

$$J = \int_0^\infty (y^T y + u^t R u)dt \quad , \quad R > 0$$

Chow and Kokotovic [9] have shown that a composite near-optimal control is given by

$$u_c = R^{-1} B^T M_c \underline{x} \qquad \underline{x}^t = [\underline{x}_s, \underline{x}_f] \qquad (3.88a)$$

where

$$B^T = [B_1^T, B_2^T/\epsilon] \qquad (3.88b)$$

$$M_c = \left[\begin{array}{c|c} K_s & 0 \\ \hline \epsilon K_m^T & \epsilon K_f \end{array}\right] \tag{3.88c}$$

with

$$K_m = [K_s(B_1 R^{-1} B_2^T K_f - A_{12}) - (A_{21}^T K_f + C_1^T C_2)]^* (A_{22} - B_2 R^{-1} B_2^T K_f)^{-1}, \tag{3.88d}$$

$$0 = -K_s(A_s - B_s R^{-1} D_s^T C_s) - (A_s - B_s R_o^{-1} D_s^T C_s)^T K_s + K_s B_s R_o^{-1} B_s K_s - C^T(I - D_s R_o^{-1} D_s^T) C_s, \tag{3.88e}$$

and

$$0 = -K_f A_{22} - A_{22}^T K_f + K_f B_2 R^{-1} B_2^T K_f - C_2^T C_2 \tag{3.88f}$$

and

$$R_o = R + D_S^T D_S$$

The Riccati equations (3.88e and 3.88f) are completely independent under the assumption that (A_s, B_s and C_s) and (A_{22}, B_2, and C_2) are stabilizable-detectable. If an optimal control u* is defined as

$$u^* = -R^{-1} B^T P x \quad , \quad J^* = \frac{1}{2} x_o^T P x_o \tag{3.89}$$

and $J_c = \frac{1}{2} x_o^T P_c x_o$ where P_c is the solution of

$$P_c(A - BR^{-1}B^T M_c) + (A - BR^{-1}B^T M_c)^T P_c + M_c^T B R^{-1} B^T M_c + c^T c = 0$$

then it has been shown that [10]

$$u_c = u^* + 0(\varepsilon) \quad , \quad J_c = J^* + 0(\varepsilon^2)$$

which means that the composite control u of (3.88a) is a first order approximation of u* and yields a second order approximation to the optimal solution of the system (3.73, 3.74 and 3.84).

Hence, the near-optimal design consists of the following steps:

1. Solving $\frac{1}{2}n_s(n_s + 1)$ algebraic equations (see 3.88e).
2. Solving $\frac{1}{2}n_f(n_f + 1)$ algebraic equations (see 3.88f).
3. Inverting the $n_f \times n_f$ matrix in (3.88d).

The computational simplicity of the above procedure compared with solving the Riccati equation of the large system

$$\left(\frac{1}{2}(n_s + n_f)\ (n_s + n_f + 1) \text{ algebraic equations}\right)$$

is threefold. First, it represents $n_f(n_s - 1)$ less algebraic equations than solving the large-scale Riccati equation. Second, steps 1 through 3 can be performed consecutively.

3.6 Conclusions

A review of four major methods of model reduction has been presented. The Padé approximation methods require the least amount of computation but suffer from the serious drawback that the stability of

the reduced model cannot be guaranteed. Another shortcoming is the limitation in approximating multi input - multi output systems. A modification of Padé approximation to alleviate this difficulty, based upon Routh approximation, will be discussed in Chapter 5.

The reduction methods based on error minimization give a measure of the quality of the reduced model. However, apart from the numerical difficulties and complicated computations involved, a reasonably good fit can not always be obtained and a reduced-order model may turn out to be unstable even for a stable system. Also, there is no guarantee that these models will be useful for obtaining a near-optimal design of the controller. Furthermore, the states of the reduced model are not directly related to the states of the original system.

The reduction methods which use the aggregation principle appear to be highly interesting since they retain some modal and structural properties of the original system. Moreover, the possibility of stabilizing the original system by a control law based on the simplified model holds for every aggregated model. For a given linear-time-invariant system defined by its state, input-output matrices [A], [B] and [C] the determination of an aggregated system consists of two problems:

(i) the choice of the eigenvalues of A to be retained,

(ii) the choice of the ouput matrix C.

In the singular perturbation approach, the reduced model is obtained by first neglecting the fast phenomena, i.e., $\varepsilon=0$ in equation

(3.74) and substituting for x_f in equation (3.73), a reduced-order model is obtained for studying the slow phenomena. Although this method suffers from the disadvantage that identification of the proper partitioning form of the state vector may be very difficult, it remains the only method which allows partial recovery of the information lost upon passage to the reduced model. It has been successfully applied in obtaining a control law based on the simplified model. It may be pointed out that unlike other methods, singular perturbation preserves the physical nature of the problem.

CHAPTER 4

A REVIEW OF METHODS OF PRODUCING SIMPLIFIED POWER SYSTEM DYNAMIC MODELS

4.1 Introduction

The increasing size and complexity of modern power systems has stimulated the search for new methods of analysis, modeling and control. Power systems are characterized by certain features that play a primary role in determining the class of solution methods to be used, and these include the following.

(i) The spatial distribution of power system covers a sizable area.

(ii) Power system phenomena are often characterized by a variety of different frequencies and time constants that extend over a wide range.

(ii) Power systems often use an overall control structure consisting of a number of controllers at the local level which seek to meet local and, it is hoped, global, objectives.

(iv) Power systems use an information structure such that often only a small subset of total system information is available to each "local controller".

(v) The problem specification for power systems is highly complex.

(vi) The problems involved are often of very high dimensionality involving large numbers of variables and constraints.

Traditionally, the analysis of the dynamic behaviour of power systems has been reduced to a manageable level by finding some simplified or lumped model for a large part of the system. This approach has great appeal in the sense that it is consistent with the usual questions asked about the stability of power systems. That is, stability is normally analyzed from the perspective of a particular utility. That utility is interested in the dynamic stability of its own generation and transmission network to disturbances, and primarily to disturbances that occur within its own network. A particular utility's interest is how disturbances impact the remainder of the system is quite secondary to its interest in how its own equipment is affected.

The natural approach to analyzing dynamic behaviour has been to divide the overall system into a study system and one or more external systems, as illustrated in Figure 4.1. The study system is that area where disturbances are to be applied and where responses are to be observed (i.e., whose detailed behavior is of interest). The detailed behavior of the external system is of no interest. The external systems are important only insofar as they affect the response of the study system to disturbances. It is therefore desirable to represent the external systems by equivalents that faithfully preserve the

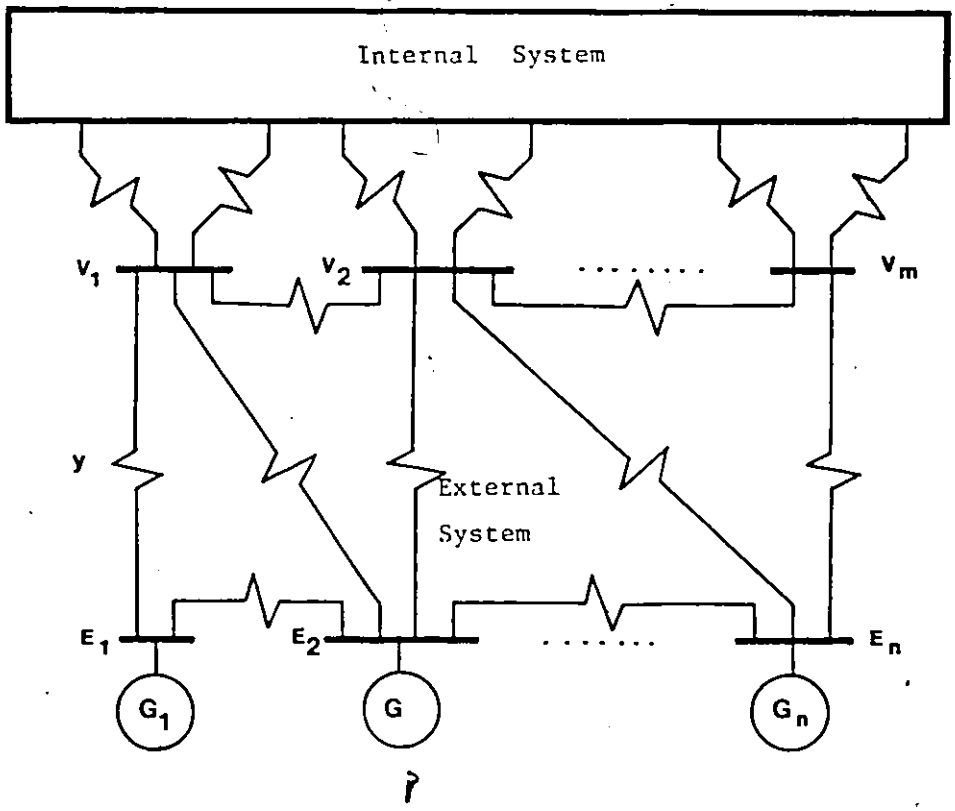


Figure 4.1 Interconnected Power System Structure .

V_k, k=1, -----, m are boundary bus voltages .

E_i, i=1, -----, n are generator internal voltages.

interaction between the study and external systems. Thus the simplification of the analysis is done at the expense of losing information about the detailed behavior of the external systems.

Historically, there have been two distinct approaches to finding simplified models of power systems. One approach, called the coherency method, is based on the empirical observation that when a power system is disturbed, groups of generators tend to accelerate together, maintaining the same relative angles with respect to each other. This is particularly true for generators electrically distant from the disturbance. The overall procedure for getting the coherency based equivalent is as follows:

(i) identification of groups of coherent generators,

(ii) network reduction and equivalencing of the coherent groups with appropriate power generation and inertia,

(iii) combination of the equivalent with the system retained.

The two major limitations of the coherency based dynamic equivalents method are:

(i) it requires the numerical integration of the differential equations of the entire system, which is precisely the problem one wishes to avoid,

(ii) there is no theoretical justification for the coherency based method of producing dynamic equivalents when coherency is defined based strictly on the difference in angles.

The recent work by Systems Control Incorporated (SCI) on coherency based dynamic equivalents has shown that coherency can be evaluated using a model based on the following assumptions:

(i) The coherent groups of generators are independent of the size of the disturbance. Therefore, coherency can be determined by considering a linearized system model,

(ii) The coherent groups are independent of the amount of detail in the generating unit model. Therefore, a classical synchronous machine model is considered and the excitation and turbine-governor systems are ignored. This assumption is based upon the observation that although the amount of detail in the generating unit models has a significant effect upon the swing curves, particularly the damping, it does not radically affect the more basic characteristics such as the natural frequencies and mode shapes. These above assumptions and their justification are quoted from [31].

The other approach to forming dynamic equivalents for power systems is to obtain a linearized model of the external system and then reduce the order of this linear model by applying the results of linear system theory, in particular the concepts of aggregation [3,12]. There is a considerable wealth of literature on the use of aggregation to generate reduced-order models for power systems. Undrill et al. [52] suggested aggregation of a part of the dynamics associated with a power system using Davison's method [13]. Van Ness et al. [53] developed a

simplified model of the total system retaining the dominant modes and then used a change of coordinates to produce a state representation with satisfactory structure. Altalib and Krause [2] proposed decomposing the system and simplifying each subsystem using Davison's method, i.e., keeping only the slowest modes and then choosing appropriate interconnection variables. A criticism that can be made of several of these approaches is that the reduced-order system often loses physical meaning. Furthermore, the effects of designing regulators based on an aggregated model being applied to the original power system model needs to be investigated in greater detail, as modes not represented in the reduced model could possibly interact with the feedback loops.

The two methods of constructing power system equivalents outlined above represent the two main lines of research.

They have developed in almost complete isolation, and appear to be unrelated. However, the fact that they both yield good equivalents might lead one to speculate that the two methods are related. In fact, some recent work [36] indicates that, for the proper choice of coherency measure, the two methods are indeed closely related.

4.2 The Form of the Linearized Model of Multi-Machine Power Systems

A linearized power system model is introduced at this point. This model will be used throughout the subsequent analysis. It represents the power system in terms of a set of ordinary, linear

differential equations for the electromechanical motion of the generators plus a set of algebraic equations for the power flows among the generators and load buses of the system. The differential equations are:

$$M_i \Delta\dot{\omega}_i = \Delta P_{mi} - \Delta P_{gi} - D_i \Delta\omega_i \tag{4.1}$$

$$\Delta\dot{\delta}_i = 2\pi f_o \Delta\omega_i, \quad i = 1, 2, \ldots N \tag{4.2}$$

where i subscript for generator i.

Δ indicates that this variable represent a deviation from a specified steady state operating point.

M_i inertia constant of generator i - pu

$\Delta\omega_i$ speed deviation of generator i - pu

$\Delta\delta_i$ rotor angle deviation of generator i - radians

D_i damping constant of generator i - pu

ΔP_{mi} change in mechanical input power at generator i - pu

ΔP_{gi} change in electrical output power at generator i - pu

The network equations in polar form are linearized with the real power equations decoupled from the reactive power equations to obtain:

$$\begin{bmatrix} \Delta\underline{P}_g \\ \hline \Delta\underline{P}_\ell \end{bmatrix} = \left[\begin{array}{c|c} \frac{\partial \underline{P}_g}{\partial \delta} & \frac{\partial \underline{P}_g}{\partial \theta} \\ \hline \frac{\partial \underline{P}_\ell}{\partial \delta} & \frac{\partial \underline{P}_\ell}{\partial \theta} \end{array}\right] \begin{bmatrix} \Delta\underline{\delta} \\ \hline \Delta\underline{\theta} \end{bmatrix} \tag{4.3}$$

where

$$\Delta \underline{P_g}^T = [\Delta P_{g1}, \Delta P_{g2} \cdots , \Delta P_{gN}]$$

$$\Delta \underline{P_\ell}^T = [\Delta P_{\ell 1}, \Delta P_{\ell 2} \cdots , \Delta P_{\ell K}]$$

$$\Delta \underline{\delta}^T = [\Delta \delta_1, \Delta \delta_2 \cdots , \Delta \delta_N]$$

$$\Delta \underline{\theta}^T = [\Delta \theta_1, \Delta \theta_2 \cdots , \Delta \theta_K]$$

Some comments are in order about these equations. First, there are two first order differential equations required to represent the dynamics of one generator. This is essentially the simplest dynamic model. The behaviour of the generator can be modeled in much greater detail to account for all the electrical and mechanical phenomena at work (see Appendix I). Even a modest attempt at accounting for these phenomena results in a seventh or eighth order differential equation set for each generator. In dealing with large systems this obviously results in an enormous number of differential equations. Thus for the analysis of large power systems there is really no choice but to use the simplest representation possible. The simpler representation used here neglects the effects of exciter and turbine governor, at least in detail. The damping constant D_i serves to represent in a general way the overall effect of these control systems. In a power system the various control systems tend to dampen the response of the power system without greatly affecting its natural frequencies [32].

The second observation is that the real power equations can be decoupled from the reactive power equations. This is a commonly made approximation, based on the following reasoning. The real power flows are largely dependent on the voltage angles at the generator and load buses, not the voltage magnitudes; further, the generator voltages behind the transient reactances are constant.

With this background information, and assuming uniform damping ($\frac{D_i}{M_i} = \beta$ for all i), the equations for the power system can now be put in state space model form as follows.

First, a reference frame is chosen for the angles and speeds of the generators. We can have the state space model based on machine angle as a reference (MAR) or centre of angle as a reference (COA). We discuss both types of models in this subsection and point out the connection between them.

4.2.1. State Space Model Using Machine Angle as Reference

The reference frame chosen is the generator angle of the n^{th} machine, $\Delta\delta_N$. That is, one establishes N-1 angle difference:

$$\Delta\hat{\delta}_i = \Delta\delta_i - \Delta\delta_N \quad , \quad i = 1, 2, \ldots , N-1 \tag{4.4}$$

and N-1 speed differences

$$\Delta\hat{\omega}_i = \Delta\omega_i - \Delta\omega_N \quad , \quad i = 1, 2 \ldots , N-1 \tag{4.5}$$

Next consider the N equations of the form represented by (4.1), and subtract equation N from equation i to get

$$\dot{\Delta\omega}_i - \dot{\Delta\omega}_N = \left(\frac{\Delta P_{mi}}{M_i} - \frac{\Delta P_{gi}}{M_i}\right) - \left(\frac{\Delta P_{mN}}{M_N} - \frac{\Delta P_{gN}}{M_N}\right) - \left(\frac{D_i}{M_i}\Delta\omega_i - \frac{D_N}{M_N}\Delta\omega_N\right) \quad (4.6)$$

Now the left hand side of equation (4.6) is simply $\dot{\Delta\hat{\omega}}_i$. However, the right hand side contains all N speeds. Let us consider the case of uniform damping, the last term on the right hand side will be written as

$$\beta(\Delta\omega_i - \Delta\omega_N) = \beta\Delta\hat{\omega}_i.$$

The state variable will now be

$$\underline{x} = [\omega_1 - \omega_N,\ \omega_2 - \omega_N,\ \ldots,\ \omega_{N-1} - \omega_N,\ \delta_1 - \delta_N,\ \delta_2 - \delta_N,\ \ldots,\ \delta_{N-1} - \delta_N]^T$$

The state equations will be

$$\dot{\hat{\delta}}_i = 2\pi f_o \Delta\hat{\omega}_i$$

$$\dot{\hat{\omega}}_i = \frac{1}{M_i}(\Delta P_{mi} - P_{gi}) - \frac{1}{M_N}(\Delta P_{mN} - \Delta P_{gN}) - \beta\Delta\hat{\omega}_i \quad (4.7)$$

$$i = 1, 2, \ldots, n$$

Now, equation (4.3) must be written in terms of the new variables, i.e.,

$$\begin{bmatrix} \Delta \underline{P}_g \\ \hline \Delta \underline{P}_\ell \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \underline{P}_g}{\partial \hat{\underline{\delta}}} & \vdots & \dfrac{\partial \underline{P}_g}{\partial \hat{\underline{\theta}}} \\ \hline \dfrac{\partial \underline{P}_\ell}{\partial \hat{\underline{\delta}}} & \vdots & \dfrac{\partial \underline{P}_\ell}{\partial \hat{\underline{\theta}}} \end{bmatrix} \begin{bmatrix} \Delta \hat{\underline{\delta}} \\ \hline \Delta \hat{\underline{\theta}} \end{bmatrix} \tag{4.8}$$

where $\hat{\theta}_i = \theta_i - \delta_N \quad , \quad i = 1, 2, \ldots, K$

and

$$\hat{\underline{\delta}}^T = [\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_{N-1}], \quad \hat{\underline{\theta}}^T = [\hat{\theta}_1, \hat{\theta}_2, \ldots \hat{\theta}_K]$$

The next step is to express $\Delta \underline{P}_g$ in terms of $\Delta \hat{\underline{\delta}}$ and $\Delta \underline{P}_\ell$. This can be written as

$$\Delta \underline{P}_g = \left\{ \frac{\partial \underline{P}_g}{\partial \hat{\underline{\delta}}} - \frac{\partial \underline{P}_g}{\partial \hat{\underline{\theta}}} \left[\frac{\partial \underline{P}_\ell}{\partial \hat{\underline{\theta}}} \right]^{-1} \frac{\partial \underline{P}_\ell}{\partial \hat{\underline{\delta}}} \right\} \Delta \hat{\underline{\delta}} + \frac{\partial \underline{P}_g}{\partial \hat{\underline{\theta}}} \left[\frac{\partial \underline{P}_\ell}{\partial \hat{\underline{\theta}}} \right]^{-1} \Delta \underline{P}_\ell$$

or

$$\Delta \underline{P}_g = [T] \Delta \hat{\underline{\delta}} + [L] \Delta \underline{P}_\ell \tag{4.9}$$

The model becomes

$$\dot{\underline{x}}(t) = [A]\underline{x}(t) + [B]\underline{u}(t) \tag{4.10}$$

where

$$\underline{x} = \begin{bmatrix} \Delta\hat{\underline{\delta}} \\ \hline \Delta\hat{\underline{\omega}} \end{bmatrix}, \qquad \underline{u} = \begin{bmatrix} \Delta\underline{P}_m \\ \hline \Delta\underline{P}_l \end{bmatrix}$$

$$A = \left[\begin{array}{c|c} 0 & 2\pi f_o I \\ \hline -\hat{M}T & -\beta I \end{array}\right], \qquad B = \left[\begin{array}{c|c} 0 & 0 \\ \hline \hat{M} & \hat{M}L \end{array}\right]$$

and

$$\hat{M} = \begin{bmatrix} \frac{1}{M_1} & 0 & 0 & -\frac{1}{M_N} \\ 0 & \frac{1}{M_2} & 0 & -\frac{1}{M_N} \\ & & & \\ 0 & 0 & \frac{1}{M_{N-1}} & -\frac{1}{M_N} \end{bmatrix}$$

4.2.2 State Space Model in Center of Angle Reference Frame

An alternative way of viewing the model is through what is called the center of inertia or center of angle (COA) formulation. This was proposed by Stanton [46]. As we shall see, this formulation is particularly suited for the aggregation of generators used in coherency equivalents.

Assume that, there are N = m+n generators in the power system, the first m being the study group and the last n = N − m being the external group.

Define the center of angle δ_o as

$$\delta_o = \frac{1}{M_T} \sum_{i=m+1}^{N} M_i \delta_i \tag{4.11}$$

where $$M_T = \sum_{i=m+1}^{N} M_i$$

Also define the new rotor angle with respect to δ_o as

$$\tilde{\delta}_i = \delta_i - \delta_o.$$

Let $$\dot{\tilde{\delta}}_i = \tilde{\omega}_i = \omega_i - \omega_o,$$

where $\tilde{\delta}_i$'s and $\tilde{\omega}_i$'s are linearly independent since from the definition of $\tilde{\delta}_i$ it follows that

$$\sum_{i=m+1}^{N} M_i \tilde{\delta}_i = 0 \quad \text{and} \quad \sum_{i=m+1}^{N} M_i \tilde{\omega}_i = 0 \tag{4.12}$$

Equations in the new reference frame are obtained after some manipulations and for assuming uniform damping case as

$$\dot{\tilde{\delta}}_i = 2\pi f_o \tilde{\omega}_i \tag{4.13}$$

$$\dot{\tilde{\omega}}_i = \frac{1}{M_i}(\Delta P_{mi} - \Delta P_{gi}) - \frac{1}{M_T}(\Delta P_{mT} - \Delta P_{gT}) - \beta \tilde{\omega}_i \tag{4.14}$$

$$\begin{bmatrix} \Delta \underline{P}_g \\ \hline \Delta \underline{P}_\ell \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \underline{P}_g}{\partial \tilde{\underline{\delta}}} & \vdots & \dfrac{\partial \underline{P}_g}{\partial \tilde{\underline{\theta}}} \\ \hline \dfrac{\partial \underline{P}_\ell}{\partial \tilde{\underline{\delta}}} & \vdots & \dfrac{\partial \underline{P}_\ell}{\partial \tilde{\underline{\theta}}} \end{bmatrix} \begin{bmatrix} \Delta \tilde{\underline{\delta}} \\ \hline \Delta \tilde{\underline{\theta}} \end{bmatrix} \tag{4.15}$$

where

$$\left.\begin{aligned} \tilde{\delta}_i &= \delta_i - \delta_o \\ \tilde{\omega}_i &= \omega_i - \omega_o \end{aligned}\right\} \quad i = 1, 2, \ldots, N-1$$

$$\tilde{\theta}_J = \theta_J - \delta_o, \quad J = 1, 2, \ldots, K$$

$$\tilde{\underline{\delta}} = [\tilde{\delta}_1, \tilde{\delta}_2, \ldots, \tilde{\delta}_{N-1}]^T$$

$$\tilde{\underline{\theta}} = [\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_K]^T$$

$$\underline{P}_g = [P_{g1}, P_{g2}, \cdots, P_{gn}]^T$$

$$P_\ell = [P_{\ell 1}, P_{\ell 2}, \cdots, P_{\ell K}]^T$$

$$P_{mT} = \sum_{i=m+1}^{N} P_{mi} \quad , \quad P_{gT} = \sum_{i=m+1}^{N} P_{gi}$$

These equations can be put in the state space form as

$$\dot{\tilde{\underline{x}}} = \tilde{A}\,\tilde{\underline{x}} + \tilde{B}\,\underline{u} \tag{4.16}$$

where

$$\tilde{A} = \overbrace{\left[\begin{array}{c|c} 0 & 2\pi f_o I_{N-1} \\ \hline -\tilde{M}T & -\sigma I_{N-1} \end{array}\right]}^{N-1 \qquad\qquad N-1}$$

$$\tilde{B} = \left[\begin{array}{c|c} 0 & 0 \\ \hline \tilde{M} & \tilde{M}L \end{array}\right]$$

with,

$$\tilde{M} = \begin{bmatrix} \frac{1}{M_1} & 0 & \cdots & -\frac{1}{M_T} & \cdots & -\frac{1}{M_T} \\ 0 & \frac{1}{M_2} & & -\frac{1}{M_T} & & \vdots \\ & & & \left(\frac{1}{M_{m+1}} - \frac{1}{M_T}\right) & \cdots & -\frac{1}{M_T} \\ & & & -\frac{1}{M_T} & & \\ \vdots & & & \vdots & & \vdots \\ & & & -\frac{1}{M_T} & & \\ 0 & \cdots & & -\frac{1}{M_T} & \cdots & \left(\frac{1}{M_{N-1}} - \frac{1}{M_T}\right) - \frac{1}{M_T} \end{bmatrix}$$

$$T = \left\{ \frac{\partial P_g}{\partial \underline{\delta}} - \frac{\partial P_g}{\partial \underline{\theta}} \left[\frac{\partial P_\ell}{\partial \underline{\theta}} \right]^{-1} \frac{\partial P_\ell}{\partial \underline{\delta}} \right\}$$

$$L = \frac{\partial P_g}{\partial \underline{\theta}} \left[\frac{\partial P_\ell}{\partial \underline{\theta}} \right]^{-1}$$

Note that equation (4.16) is identical to equation (4.10) with $\tilde{M}$ substituted for $\hat{M}$. Thus, the state space representation for the uniform damping case is the same whether we take the machine angle or center of angle as reference angle. However, one should remember that in the case of non-uniform damping, the COA reference equations do not lend themselves to a neat representation.

The linear model plays an important part in both coherency and modal analysis, although the role in the two methods is quite different. If the dynamic equivalent is produced using the modal analysis technique, the internal system is represented in detail, with each generator being described by high order (seven or more) linear or non-linear differential equations. The external system is divided into subsections and each subsection is represented by a linear model. The principle of aggregation described in Chapter 3 is then used to reduce the order of each linearized subsection of the external system. In the coherency method, a linearized model of the entire power system, internal system and external system, is subjected to a disturbance and the coherent generator groups are determined. Those generators that are coherent are then replaced by a single "equivalent" generator, and a linear or non-linear model can be produced. Thus in the coherency approach the linearized model is used only to determine which generators swing together in response to a disturbance. This coherent behavior could just as well be determined using a non-linear model of the system, but this would, of course, be computationally much more expensive. The justification for assuming that the linear model of the power system captures the coherency behaviour of the system is given in reference [32].

4.3 Coherency-Based Dynamic Equivalents

The concept of forming reduced order dynamic equivalents using coherency was first introduced by Chang and Adibi [5]. They defined

two machines to be coherent if there exists a constant c_{ij} such that

$$\delta_i(t) - \delta_j(t) = c_{ij} \text{ for } o < t < t_o$$

A group of generators is said to be coherent if each pair of generators in the group is coherent. The principal work in coherency has been done by Podmore and Germond [32]. The overall procedure for forming coherency-based dynamic equivalents can be divided into four basic steps:

(i) The study system is defined.

(ii) A disturbance is applied to the power system and the coherent groups are determined. The coherency of both generator internal and terminal buses is of interest. The coherency of the generator terminal buses forms the basis for the network reduction step. The coherency of the generator internal buses is assumed in the dynamic aggregation step.

(iii) The static network equations are reduced in order by first replacing all the coherent generator terminal buses by a single equivalent bus. The coherency-based reduction of the generator buses can be rigorously and mathematically expressed as a power invariant transformation of the network admittance equations on the basis that certain voltages are linearly dependent. Second, node elimination techniques are used to remove as many load buses as possible.

The network representing the original power system is naturally very sparse. When a network reduction is performed

and buses are eliminated, additional equivalent lines are introduced into the network. Thus a point is reached at which the elimination of additional load buses actually increases the number of non-zero terms in the admittance matrix, i.e., the equivalent network may have many more branches than the original one. Since the computation time is roughly proportional to the number of non-zero terms in the admittance matrix, load bus elimination is ended at the point where the number of non-zero terms in the admittance matrix begins to increase. Recently, sparsity techniques have been applied successfully to the network reduction problem in order to minimize the number of branches which are introduced into the equivalent network [ ].

(iv) The generators at the coherent buses are replaced by one or a small number of equivalent machines at the equivalent terminal bus. One equivalent machine will be used at the bus if the set of coherent generators are similar enough in response characteristics. A "small number" of generators will be used if the coherent generators are of very different response characteristics. For instance, if the set of coherent generators include both steam and hydro units, then two equivalent generators will be used at the bus since it has been found empirically that a satisfactory single machine equivalent for a group of generators that include both steam and hydro cannot usually be found. The criteria for an acceptable equivalent model of generating units, from the dynamic viewpoint, is that its

electric power output response matches the total electric power output of the unit it replaces, and that the voltage response at its terminal bus matches the voltage response of the bus with the individual unit models. The aggregation method relies upon the consideration that the units to be aggregated, being attached to the same bus, have the same terminal voltage, and the assumption that these units, being coherent, have the same speed. Thus, every excitation system among a group of coherent units measures the same input voltage signal. Also, every governor-turbine system among a group of coherent units measures the same input speed signal. The equivalencing procedure assumes that each generator in the group to be equivalenced can be represented by a block diagram of transfer functions of the same form. The equivalent machine is assumed to have a block diagram of the same form; and an identification procedure, based on least square error, is used to identify the parameters that best match the frequency response (or time response) of the equivalent machine to the cummulative frequency response (or time response) of the coherent group. Also, any of the model reduction approaches mentioned in Chapter 3 can be used to obtain the aggregate model. This will be illustrated in Chapter 6.

4.3.1 Structural Conditions Under Which a Group of Machines Behaves as a Single Machine

Some more recent work by Dicaprio [14,15] and Dorset et al. [16] defines a structural condition under which a group of generators responds to a disturbance outside the group as if it were a single generator. These conditions are fundamental to an understanding of the relationship between coherency and modal dynamic equivalents.

Consider the power system network of Figure 4.1. Dicaprio shows that if certain structural conditions exist in the power system at time $t = 0^-$, then no matter what disturbance occurs in the internal system at time $t \geq 0$, generators G_i, $i = 1, \ldots n$ of the external system will remain strictly coherent, for all $t \geq 0$. The conditions that must exist in the system of Figure 4.1 at $t = 0^-$ are:

$$\frac{E_i}{M_i} Y_{ik} e^{-j(\delta_i^o - \delta_n^o)} = \frac{E_n}{M_n} Y_{nk} \qquad (4.17)$$

for any $i = 1, 2, \ldots, n-1$

any $k = 1, 2, \ldots, m$

Dicaprio calls the conditions specified in (4.17) the conditions for "theoretical coherency in the large", with "large" meaning that the conditions imply coherency for the nonlinear representation of the system used to derive (4.17), namely the algebraic equations

$$\begin{bmatrix} \underline{I}_K \\ \hline \underline{I}_n \end{bmatrix} = \left[\begin{array}{c|c} Y_{KK} & Y_{Kn} \\ \hline Y_{nK} & Y_{nn} \end{array}\right] \begin{bmatrix} \underline{V}_K \\ \hline \underline{E}_n \end{bmatrix} \qquad (4.18)$$

plus a second order differential equation of the form

$$P_{mi} = P_{gi} + M_i \ddot{\delta}_i \qquad i = 1, 2, \ldots, n$$

for each generator in the system with

$\underline{I}_K$ a K x 1 vector of the currents injected at the boundary buses

I_n a n x 1 vector of the currents injected at the internal buses of the n generators of the coherent group.

$\underline{V}_K$ a K x 1 vector of the voltages at the k boundary buses

$\underline{E}_n$ a n x 1 vector of the voltages at the internal buses of the n generators of the coherent groups each with magnitude E_i, phase angle

$$\delta_i^o, \; i = 1, 2, \ldots, n \text{ at } t = 0^-$$

Y_{KK} a K x K matrix of the admittances between the k boundary buses.

Y_{Kn} a K x n matrix of the admittances between the k boundary buses and the n internal generator buses of the coherent group

Y_{nn} an n x n matrix of the admittances between the n internal generator buses of the coherent group.

$Y_{Kn} = Y_{nk}^T$

P_{mi} the constant mechanical input power of the generator i of the coherent group

P_{gi} the electrical output power of the generator i of the coherent groups

M_i the inertia constant of generator i of the coherent groups.

The coherency that results from the satisfaction of condition (4.17) will be referred to as strict geometric coherency (SGC) because it results from the structural geometry of the network and load flow conditions. These conditions are purely hypothetical in the sense that they could probably never be satisfied exactly in any real power system. The utility of an approximate satisfaction of these conditions can only be answered empirically. It is interesting to note that the most important use of the conditions (4.17) comes from their trivial satisfaction when the Y_{IK} are very small. That is, conditions (4.17) explain conceptually the well known empirical fact that generators a long electrical distance from a disturbance accelerate together, even if their inertias are different.

The second type of coherency, will be called strict synchronizing coherency (SSC). This type of coherency results from a group of n machines being very tightly interconnected. Its utility when approximately satisfied has been well established by Podmore and others.

Dorsey has shown that those conditions yield the same result if the equations for the power system are linearized about some stable

point. Consider the state space representation of (4.10) and assume that load buses can be in the following locations:

(1) In the part of the power system outside the external group.

(2) On the boundary between the external system and the study system.

That is, that all the load buses and generator terminal buses of the external system have been eliminated and only the generator internal buses remain. Consider now, the N-1 differential equations

$$\dot{\Delta\hat{\underline{\omega}}} = -[\hat{MT}]\Delta\hat{\underline{\delta}} - [\beta I]\Delta\hat{\underline{\omega}} + [\hat{M} : \hat{ML}] \begin{bmatrix} \Delta\underline{P}_m \\ \Delta\underline{P}_\ell \end{bmatrix} \tag{4.19}$$

of equation (4.10). Then partition the matrices of equation (4.19) as:

$$\begin{bmatrix} \dot{\Delta\hat{\underline{\omega}}}_m \\ \dot{\Delta\hat{\omega}}_{n-1} \end{bmatrix} = \begin{bmatrix} -(\hat{MT})_{11} & -(\hat{MT})_{12} \\ -(\hat{MT})_{21} & -(\hat{MT})_{22} \end{bmatrix} \begin{bmatrix} \Delta\hat{\underline{\delta}}_m \\ \Delta\hat{\underline{\delta}}_{n-1} \end{bmatrix} + \begin{bmatrix} -\beta I_m & 0 \\ 0 & -\beta I_n \end{bmatrix} \begin{bmatrix} \Delta\hat{\underline{\omega}}_n \\ \Delta\hat{\underline{\omega}}_{n-1} \end{bmatrix}$$

$$+ \begin{bmatrix} \hat{M}_1 & (\hat{ML})_1 \\ \hat{M}_2 & (\hat{ML})_2 \end{bmatrix} \begin{bmatrix} \Delta\underline{P}_m \\ \Delta\underline{P}_\ell \end{bmatrix} \tag{4.20}$$

where m is the number of generators in the study system, n is the number of generators in the external system. Considering the assumption mentioned above concerning the locations of the load buses, Dorsey has shown that SGC and SSC are associated with structural conditions on one

of submatrices of $-(MT)$. He proves that SGC causes $-(MT)_{21} = 0$. Also, that SSC causes $-(MT)_{22}^{-1} \to 0$ in the limit as n-1 interconnections among group of n generators are progressively stiffened. Then, SGC can be considered as a loss of controllability condition between generators in the study system that can be disturbed and those in the external group that swing together. This loss of controllability permits eliminating eigenvalues by modal analysis rules for forming dynamic equivalents or aggregation of the group of generators satisfying SGC by coherency based methods. Also, it was shown that if the system can be subdivided into groups of generators that are stiffly bound and thus satisfy the SSC condition, the magnitude of imaginary parts of eigenvalue pairs that describe the oscillations within such groups are large compared to the eigenvalues that describe the intergroup oscillations. As the power system model has no modes that are highly damped and decay rapidly to zero the singular perturbation approach is not directly applicable to the power system model. However, Chow et al. [11] have shown that this same approach can be applied to the case where a system contains a set of lightly damped high frequency modes. Thus, this SSC condition causes the two time scale condition required for singular perturbation dynamic equivalents. Also, Dorsey has identified a third type of coherency called Pseudo coherency (PC). This is reflected in the linear model by the fact that the condition for PC is $(-MT)_{12} = 0$. This condition causes the external group to appear to swing together (even though it doesn't) based on the observed effect of such oscillation, in the external group on the study system. The conditon

can be applied to eliminate eigenvalues in the external system based on the loss of observability mode elimination rule for producing dynamic equivalents. In [16] an algorithm has been developed for detecting coherent groups based on SSC and based on SSC/SGC using the rms coherency measure defined as

$$C_{K\ell} = \sqrt{\frac{1}{T^n} E \left\{ \int_0^T [\Delta\delta_K(t) - \Delta\delta_\ell(t)]^2 \, dt \right\}} \qquad (4.21)$$

where E is the expectation operator. The expectation operator appears, because as shown in [36], the optimum disturbance for detecting coherent groups that depend on the power system structure is not deterministic.

The method of forming equivalents by the use of coherency has been reviewed in detail in this section. The motivation for doing so is that this equivalent is the one most widely accepted and used.

4.4 Modal Dynamic Equivalents Technique

The main work in forming power system equivalents by modal analysis has been done by Undrill [52].

For purposes of applying the modal analysis technique, the power system is divided into a study system and one or more external systems. The study system is that area where disturbances are to be applied and where the response of machines is to be observed. It is modeled in detail.

The external systems are not of direct interest in stablity studies and are important only insofar as they affect the response of the study system to disturbances within the study system. It is therefore desirable to represent the external systems by equivalents.

The key step in the modal analysis approach is to define the buses that connect the external system to the study system and the other external systems. A basic set of assumptions is needed in order to make the problem mathematically tractable. The first assumption is that the buses that connect the study system with the external systems are far enough removed from the site of the disturbance, for the loads of the external system to be modeled by linearized current-voltage characteristics. The second assumption is that the disturbances propogated into the external system are sufficiently small to allow its generators to be modelled by linearized algebraic and differential equations. Within the assumptions concerning linearity, each generating unit in the external system may be modelled to any desired degree of detail.

Implementation of the modal analysis dynamic equivalents consists of:

(i) The construction of a comprehensive linear differential equation describing the external system.

(ii) Transforming the equations into a simplified canonical form.

(iii) Reduction of the order of the canonical form equations by the deletion of selected natural modes.

(iv) The integration of the reduced linear differential equations, in conjunction with the nonlinear differential equations of the study system, to give a simulation of the study system as it is affected by the external system.

The comprehensive linear differential equation takes the form:

$$\dot{\underline{x}} = A\,\underline{x} + B\underline{\Delta v}_T$$

$$\underline{\Delta I}_T = C\,\underline{x} + D\underline{\Delta v}_T \qquad (4.22)$$

$$\underline{\omega}_R = E\,\underline{x}$$

where

$\underline{x}$ is a vector of state variables sufficient to describe the behaviour of the external system.

$\underline{\Delta v}_T$ and $\underline{\Delta I}_T$ are vectors of the current and voltage changes at the boundary nodes between the study system and external system.

ω_R the speed deviation of reference generator in the external system.

The diagonalized state equations are obtained from the linearized state equations (4.22) by obtaining the eigenvalue, and eigenvectors of the A matrix and performing a canonical transformation to decouple the modes. Using the transformation,

$$\underline{x} = [V]\underline{y} \qquad (4.23)$$

the resulting equations have the following form:

$$\begin{aligned} \dot{\underline{y}} &= \Lambda \underline{y} + V^{-1} B \, \Delta \underline{v}_T \\ \Delta \underline{I}_T &= CV \, \underline{y} + D \, \Delta \underline{v}_T \\ \omega_R &= EV \, \underline{y} \end{aligned} \tag{4.24}$$

where $\Lambda = V^{-1}AV$

To avoid handling complex quantities, the following algorithm is useful for determining V. This algorithm was mentioned in Chapter 2, and is presented in this section for clarity.

(1) Perform eigenanalysis on A, i.e., find U such that $AU = \text{diag}\,[\lambda_1, \lambda_2, \ldots \lambda_n]U$.

(2) Suppose λ_i is real and λ_j and λ_{j+1} are a complex pair for some i and j. Then the i^{th} column of V is taken as the ith column of U and the j^{th} and $j+1^{st}$ columns of V are taken as the real and imaginary parts of the j^{th} column of U.

The reduced state equations are obtained from the diagonalized state equation (4.24) by eliminating selected modes.

Some of the grounds for deleting specific modes are:

1. The negative real part of any eigenvalues of the system matrix for the linearized state model, corresponding to a particular mode, is very large.

2. The corresponding column of [C][V] contains such small numbers in relation to other columns that the mode may be assumed unobservable.

3. The corresponding rows of $[V^{-1}][B]$ contains such small numbers in relation to other rows that the mode may be assumed uncontrollable.

In Chapter 6 the component cost analysis presented in Chapter 2 will be considered as a criterion for deleting the unwanted modes.

Once the required selection of modes to be retained has been made, equation (4.24) can be rearranged in the form:

$$\underline{y} = \begin{bmatrix} \underline{z} \\ \underline{y}_1 \\ \underline{y}_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\underline{z}} \\ \dot{\underline{y}}_1 \\ \dot{\underline{y}}_2 \end{bmatrix} = \begin{bmatrix} F & 0 & 0 \\ 0 & \Lambda_1 & 0 \\ 0 & 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} \underline{z} \\ \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} + \begin{bmatrix} G \\ \theta_1 \\ \theta_2 \end{bmatrix} \Delta \underline{v}_T \qquad (4.25)$$

$$\begin{bmatrix} \Delta \underline{I}_T \\ \omega_R \end{bmatrix} = \begin{bmatrix} -H & \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} \underline{z} \\ \underline{y}_1 \\ \underline{y}_2 \end{bmatrix}$$

where

$\underline{z}$ subvector of y to be retained

$\underline{y}_1$ subvector of $\underline{y}$ assumed to have fast response and jump immediately to steady state.

$\underline{y}_2$ subvector of $\underline{y}$ assumed to be zero.

Now, assuming $y_1 = 0$ and $y_2 = 0$, the form of the reduced-order model is as follows:

$$\dot{\underline{z}} = F\underline{z} + G\Delta\underline{v}_T$$

$$\begin{bmatrix} \Delta\underline{I}_T \\ \omega_R \end{bmatrix} = \begin{bmatrix} H \end{bmatrix} \begin{bmatrix} \underline{z} \end{bmatrix} + \left[\begin{bmatrix} D \\ 0 \end{bmatrix} - \begin{bmatrix} \phi_1 \Lambda_1^{-1} \theta_1 \end{bmatrix} \right] \begin{bmatrix} \Delta\underline{v}_T \end{bmatrix} \tag{4.26}$$

There are several points worth noting about the dynamic equivalent. First, the dynamic simulation of the power system requires the simultaneous integration of the differential equations of the study system and the external system. The integration of these equations is straightforward, once the input $\Delta\underline{v}_T$ has been determined. The determination of the $\Delta\underline{v}_T$ requires the combined solution of the algebraic network equation for the study system and the external system.

Second, the modal analysis technique determines a linear equivalent for the external system. In the coherency method, by contrast, the equivalent can be either linear or non-linear. In one respect this is a drawback. However, an advantage is that a reduced

order model can be found for a group of generators that are not necessarily coherent.

Third, the input to the modal equivalent model are the voltage difference Δv_T at the boundary.

There are some significant disadvantages to the modal approach that also deserve some attention. First, the order of the external system to be equivalenced must be limited due to the cost of computing eigenvalues and eigenvectors. Second, it may be difficult to give the aggregated variables $\underline{z}(t)$ a physical interpretation, furthermore, in aggregation some of the original structure of the problem may have been destroyed, leading, for instance, to the introduction of nonphysical coupling. Third, the form of the equivalent is a linearized state vector model and cannot be represented in terms of equivalent power system components. These equivalents cannot be used without modifying present transient stability programs. Fourth, the step disturbance of voltages on boundary buses does not describe physical disturbance in the internal system and thus the modes eliminated may be quite different if based on some general deterministic or probabilistic model of actual power system disturbances.

4.5 Summary

This chapter has established a linearized model for a power system and then described, in some detail, the two primary methods of forming power system dynamic equivalents, namely coherency and modal analysis. It has provided the basic understanding necessary to

discover the connection between these two approaches. An algorithm for obtaining power system dynamic equivalents will be presented in Chapter 6. First, the coherent generators will be identified using component cost analysis. Second, dynamic equivalents of each coherent group will be obtained using a proposed aggregation procedure.

CHAPTER 5

A UNIFIED ALGORITHM FOR MODEL REDUCTION OF LINEAR TIME INVARIANT DYNAMICAL SYSTEMS

5.1 Introduction

The goal of this chapter is to combine the method of aggregation, discussed in Chapter 3, with that of moment matching, in such a way as to secure the separate advantage of each technique while simultaneously removing their disadvantages. It has become increasingly clear that there are common features between the different model reduction techniques. For example, most of the modal and projection techniques are subsets of the more general aggregation technique. Similarly, most of the series expansion techniques can now be classified as variations of the Padé approximation. A large class of continued fraction techniques have been shown to fall within the larger class of Padé approximation [31]. The Padé methods, however, initially suffered from a serious drawback in that the stability of the simplified model could not be guaranteed. This drawback was removed by Hutton and Friedland [24] who introduced the Routh approximation method. This method has been generalized and extended to multivariable system in a recent paper [41]. A brief discussion of the Routh approximation technique will be presented in Section 2.

An algorithm for obtaining reduced-order models for single-input - single-output systems will be presented in Section 3. This procedure does not require the calculation of the system eigenvectors. Furthermore, it gives one the opportunity to obtain reduced-order models which retain some specified eigenvalues while matching the first r time moments of the original system (r is the order of the simplified model).

A numerical example of a synchronous machine connected to an infinite bus is presented in Section 4 to illustrate the application of this reduction technique.

5.2 Routh Approximation Method

5.2.1 Frequency Domain Routh Approximation

Consider a scalar linear system represented by

$$G(s) = \frac{b_n s^{n-1} + b_{n-1} s^{n-2} + \ldots\ldots + b_2 s + b_1}{s^n + a_n s^{n-1} + \ldots\ldots\ldots + a_2 s + a_1} \qquad (5.1)$$

The transfer function can be expanded as

$$G(s) = \frac{\delta_1 W_1(s)}{s} + \frac{\delta_2 W_1(s) W_2(s)}{s} + \ldots\ldots + \frac{\delta_n W_1(s) \ldots W_n(s)}{s} \qquad (5.2)$$

where δ_i, $i = 1,2, \ldots n$ are constants and

$$W_i(s) = \cfrac{1}{\cfrac{\gamma_i}{s} + \cfrac{1}{\cfrac{\gamma_{i+1}}{s} + \cfrac{1}{\cfrac{\gamma_n}{s}}}} \qquad (5.3)$$

for $i = 2,3, \ldots n$; however, for $i = 1$ the first term in the expansion is $(1 + \frac{\gamma_1}{s})$ instead of $\frac{\gamma_1}{s}$. If an r^{th} order model is desired ($r < n$) the γ-δ expansion (5.2) is truncated after r terms.

The γ and δ coefficients for the expression (5.2) are obtained by employing the procedure of Hutton and Friedland [24].

5.2.2 State-Space Routh Approximation

Let a phase variable representation of (5.1) be

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} \end{aligned} \qquad (5.4)$$

A Routh canonical form of (5.4) corresponding to (5.2) can be realized and in the following form

$$\begin{aligned} \dot{\underline{v}} &= R\underline{v} + M\underline{u} \\ \underline{y} &= E\underline{v} \end{aligned} \qquad (5.5)$$

where the system matrices are given by

$$R = \begin{bmatrix} -\gamma_1 & 0 & -\gamma_3 & 0 & -\gamma_5 & \cdots & -\gamma_n \\ 0 & 0 & \gamma_3 & 0 & \gamma_5 & \cdots & \gamma_n \\ -\gamma_1 & -\gamma_2 & -\gamma_3 & 0 & -\gamma_5 & \cdots & -\gamma_n \\ 0 & 0 & 0 & 0 & \gamma_5 & \cdots & \gamma_n \\ & & & & & & \\ & & & & & & \\ -\gamma_1 & -\gamma_2 & -\gamma_3 & -\gamma_4 & -\gamma_5 & \cdots & -\gamma_n \end{bmatrix}, \quad M = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \\ \\ 1 \end{bmatrix} \quad \text{for n odd} \tag{5.6}$$

or

$$R = \begin{bmatrix} 0 & \gamma_2 & 0 & \gamma_4 & 0 & \cdots & \gamma_n \\ -\gamma_1 & -\gamma_2 & 0 & -\gamma_4 & & \cdots & -\gamma_n \\ 0 & 0 & 0 & \gamma_4 & & \cdots & \gamma_n \\ -\gamma_1 & -\gamma_2 & -\gamma_3 & -\gamma_4 & & \cdots & -\gamma_n \\ & & & & & & \\ & & & & & & \\ -\gamma_1 & -\gamma_2 & -\gamma_3 & -\gamma_4 & -\gamma_5 & \cdots & -\gamma_n \end{bmatrix}, \quad M = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \\ \\ 1 \end{bmatrix} \quad \text{for n even} \tag{5.7}$$

and

$$E = [\delta_1 \ \delta_2 \ \cdots \ \delta_n] \tag{5.8}$$

The required linear transformation $\underline{v} = P\underline{x}$ can be derived from the Routh table, as shown below. The γ coefficients are also

determined from the Routh table. Let the characteristic equation of system (5.4) be given by

$$f(s) = s^n + a_n s^{n-1} + \ldots\ldots\ldots + a_2 s + a_1$$

Routh Table

a_1	a_3	a_5	$\ldots\ldots\ldots$ a_{n-1}	1	
a_2	a_4	a_6	a_n	0	
a_0^2	a_1^2	a_2^2	1	0	for n even
a_0^{n-1}	0	0			
a_0^n	0				

The γ coefficients can be determined from Routh table as

$$\gamma_1 = \frac{a_1}{a_2}, \quad \gamma_2 = \frac{a_2}{a_0^2}, \quad \gamma_i = \frac{a_0^{i-1}}{a_0^i}, \quad i = 3,4, \ldots, n$$

The linear transformation matrix P can be developed from Routh table as

$$P = \begin{bmatrix} a_2 & 0 & a_4 & 0 & a_6 & 0 & \cdots & 0 \\ 0 & a_0^{(2)} & 0 & a_1^{(2)} & 0 & a_2^{(2)} & \cdots & 1 \\ 0 & 0 & a_0^{(3)} & 0 & a_1^{(3)} & 0 & \cdots & 0 \\ & & & & & & & \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix} \quad \text{for n even} \tag{5.9}$$

For an odd order n the last column of P is replaced by $[1\ 0\ 1\ \ldots\ 1]^T$. If an r^{th} order model is desired it can be obtained by discarding $v_{r+1}, v_{r+2}, \ldots, v_n$ state variables in (5.5). The value of r should be even for an even order system and odd for an odd order system. Let the state variable representation of the reduced-order model derived by a truncation of (5.5) be given by the following equation

$$\begin{aligned} \dot{\underline{z}} &= F\underline{z} + G\underline{u} \\ \hat{\underline{y}} &= H\underline{z} \end{aligned} \tag{5.10}$$

where $F = KRK^+,\ G = KM,\ H = EK^+$

(5.11)

and $K = [I_r : 0]_{rxn}$

Thus

$$\underline{z} = K\underline{v} = KP\underline{x} = L\underline{x}$$

For the Routh approximation procedure an evaluation of the system eigenvalues and eigenvectors is not necessary, the reduced model is guaranteed to be stable if the original model is stable, models of all orders can be computed recursively, and the impulse response energy of the model increases monotonically with an increase of the model order r.

5.3 A Proposed Procedure for Obtaining Reduced-Order Models [20]

Consider a linear time-invariant system described by the equations

$$\dot{\hat{\underline{x}}} = \hat{A}\hat{\underline{x}} + \hat{B}\underline{u} \tag{5.12}$$

$$\underline{y} = \hat{C}\hat{\underline{x}} \tag{5.13}$$

where $x \varepsilon R^n$, $y \varepsilon R^m$ and $u \varepsilon R^\ell$ are the state, output and input vectors, respectively.

The first step is to transform equations (5.12) and (5.13) to the controllable canonical form or the observable canonical form [34].

In this chapter the procedure will be described through the controllable canonical form.

5.3.1 Case of Single-Input Systems

We shall first consider the case of a single-input system defined by equations (5.12) and (5.13) and completely controllable from its only input.

Since the system is controllable from its sole input, we know that there exists a transformation matrix [M] ($\hat{x} = Mx$) which converts the matrix $\hat{A}$ into a companion matrix A, and the matrix $\hat{B}$ into B, with

$$A = M^{-1}\hat{A}M, \quad B = M^{-1}\hat{B} \quad \text{and} \quad C = \hat{C}M \tag{5.14}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & 1 \\ -a_1 & -a_2 & \cdots & & -a_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = [c_1, c_2 \cdots c_n]$$

In this form, the characteristic equation manifests itself directly as

$$s^n + a_n s^{n-1} + \dots\dots\dots\dots + a_2 s + a_1$$

Now, apply the following linear transformation

$$\underline{x} = P\underline{z} \qquad (5.15)$$

where

$$[P] = [P_1][P_2]$$

and P, P_1 and P_2 are matrices of order n x n, respectively. Also, the structure of the matrices P, P_1 and P_2 is dependent on the order of the reduced order model chosen. If the order of the reduced-order model is r, then,

$$P_1 = \begin{array}{c} \uparrow \\ r \\ \downarrow \\ \uparrow \\ n-r \\ \downarrow \end{array} \left[\begin{array}{cccccc|c} 1 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{r-1} & \\ 0 & 1 & \beta_1 & \beta_2 & \cdots & \beta_{r-2} & 0 \\ \vdots & & \ddots & \ddots & & \vdots & \\ 0 & & & & 1 & \beta_1 & \\ 0 & & & & & 1 & \\ \hline & & 0 & & & & I_{n-r,\ n-r} \end{array}\right] \qquad (5.15a)$$

and

$$P_2 = \begin{array}{c} \overleftarrow{\quad r \quad} \quad \overleftarrow{\qquad n-r \qquad} \\ \left[\begin{array}{cc:ccc} & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & I_{r,r} & & & \\ & & 0 & 0 & -\beta_r \\ \hdashline 0 & & \multicolumn{3}{c}{I_{n-r,n-r}} \end{array}\right] \end{array} \tag{5.15b}$$

Using the above transformation (5.15), we get

$$\begin{aligned} \dot{\underline{z}} &= \bar{A}\underline{z} + \bar{B}\underline{u} \\ \underline{y} &= \bar{C}\underline{z} \end{aligned} \tag{5.16}$$

with

$$\bar{A} = P^{-1}AP, \quad \bar{B} = P^{-1}B \quad \text{and} \quad \bar{C} = CP \tag{5.17}$$

where,

$$\bar{A} = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & & 0 & \\ 0 & 0 & 1 & & 0 & \\ & & & & & \bar{A}_{12} \\ 0 & & & 0 & 1 & \\ -\bar{a}_1 & -\bar{a}_2 & & & -\bar{a}_r & \\ \hline & & \bar{A}_{21} & & & \bar{A}_{22} \end{array}\right], \quad \bar{B} = \left[\begin{array}{c} 0 \\ 0 \\ \\ \\ \beta_r \\ \hline 0 \\ 1 \end{array}\right]$$

(the first r columns form the left block and the last $n-r$ columns the right block)

and

$$\bar{C} = [\bar{c}_1, \bar{c}_2 \; \ldots \; \bar{c}_r, \; \ldots \; \bar{c}_n]$$

Using the following aggregation matrix

$$\underline{z}_1 = [I_r : 0] \, \underline{z} \tag{5.18}$$

the r^{th} reduced-order model will be

$$\begin{aligned} \dot{\underline{z}}_1 &= F\underline{z}_1 + G\underline{u} \\ \hat{\underline{y}} &= H\underline{z}_1 \end{aligned} \tag{5.19}$$

where

$$F = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & \\ & & & & \\ & & & & \\ & & & & 0 \\ \cdot & \cdot & \cdot & 0 & \cdot \quad 1 \\ -\bar{a}_1 & -\bar{a}_2 & \cdot & \cdot & -\bar{a}_r \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \beta_r \end{bmatrix}$$

and

$$H = [\bar{c}_1, \bar{c}_2, \bar{c}_3, \ldots\ldots \bar{c}_r]$$

with

$$\bar{a}_i = (a_i + \sum_{j=1}^{i-1} \beta_{i-j}\, a_j)\, \beta_r \qquad (5.20)$$

$$\bar{c}_i = (c_i + \sum_{j=1}^{i-1} \beta_{i-j}\, c_j), \quad i = 1,2, \ldots r, \text{ and } \beta_o = 0 \qquad (5.21)$$

The reduced order model represented by equation (5.19) always matches the first r time moments of the original system.

Also, the first r time moments of the reduced-order model are independent of the values of β's in $\bar{a}_i$ and $\bar{c}_i$

The proof of this result is given below:

Definition: Consider a linear time invariant system described by equation (5.14), the time moments are defined by

$$T_i = -CA^{-(i+1)}B, \quad i = 0,1, ..,\infty$$

We can prove our result by induction. First, assume that we have an original fifth-order model, with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 & -a_5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$C = [c_1, c_2, c_3, c_4, c_5]$$

and it is intended to obtain a third-order model. Following the procedure presented above, we get

$$\dot{\underline{z}}_1 = F\underline{z}_1 + G\underline{u}$$

$$\hat{\underline{y}} = H\underline{z}_1$$

with

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\bar{a}_1 & -\bar{a}_2 & -\bar{a}_3 \end{bmatrix} \quad , \quad G = \begin{bmatrix} 0 \\ 0 \\ \beta_3 \end{bmatrix}$$

and

$$H = [\bar{c}_1, \bar{c}_2, \bar{c}_3]$$

where $\bar{a}$ and $\bar{c}$ are defined by equations (5.20) and (5.21) as

$$\bar{a}_1 = a_1\beta_3, \quad \bar{a}_2 = (a_2 + \beta_1 a_1)\beta_3 \quad \text{and} \quad \bar{a}_3 = (a_3 + \beta_1 a_2 + \beta_2 a_1)\beta_3$$

$$\bar{c}_1 = c_1, \quad \bar{c}_2 = c_2 + \beta_1 c_1 \quad \text{and} \quad \bar{c}_3 = c_3 + \beta_1 c_2 + \beta_2 c_1$$

The first three time moments of the original system are

$$* T_1^0 = -\frac{c_1}{a_1}$$

$$* T_2^0 = c_1 \frac{a_1}{a_1^2} - \frac{c_2}{a_1}$$

and

$$* T_3^0 = c_1 \left[-\frac{a_2^2}{a_1^3} + \frac{a_3}{a_1^2}\right] + c_2 \frac{a_2}{a_1^2} - \frac{c_3}{a_1}$$

The first three time moments of the reduced-order model are

$$* T_1^3 = -\frac{\bar{c}_1}{\bar{a}_1} \beta_3 = -\frac{c_1}{a_1} = T_1^0$$

$$* T_2^3 = \bar{c}_1 \frac{\bar{a}_2}{\bar{a}_1^2} \beta_3 - \frac{\bar{c}_2}{\bar{a}_1} \beta_3$$

$$= \frac{c_1(a_1 + \beta_1 a_1)}{a_1^2 \beta_3^2} \beta_3^2 - \frac{(c_2 + \beta_1 c_1)}{a_1 \beta_3} \beta_3$$

$$= \frac{c_1 a_2}{a_1^2} - \frac{c_2}{a_2} = T_2^0$$

$$* \quad T_3^3 = c_1 - \frac{\bar{a}_2^2}{\bar{a}_1^3} + \frac{\bar{a}_3}{\bar{a}_1^2}\beta_3 + \bar{c}_2 \frac{\bar{a}_2}{\bar{a}_1^2}\beta_3 - \frac{\bar{c}_3}{\bar{a}_1}\beta_3$$

$$= c_1 - \frac{(a_2 + \beta_1 a_1)^2}{a_1^3 \beta_3^3}\beta_3^3 + \frac{a_3 + \beta_1 a_2 + \beta_2 a_1}{a_1^2 \beta_3^2}\beta_3^2$$

$$+ (c_2 + \beta_1 c_1)\frac{(a_2 + \beta_1 a_1)}{a_1^2 \beta_3^2}\beta_3^2$$

$$- \frac{c_3 + \beta_1 c_2 + \beta_2 c_1}{a_1 \beta_3}\beta_3$$

$$T_3^3 = c_1 - \frac{a_2^2}{a_1^3} + \frac{a_3}{a_1^2} + c_2 \frac{a_2}{a_1^2} - \frac{c_3}{a_1} = T_3^0$$

Thus the first three time moments of the original system and the reduced-order model are the same, independent of the β_i coefficients.

If a second-order model is needed, F, G and H matrices will have the following form

$$F = \begin{bmatrix} 0 & 1 \\ -\bar{a}_1 & -\bar{a}_2 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} \text{ and } \quad H = \begin{bmatrix} \bar{c}_1 \bar{c}_2 \end{bmatrix}$$

where

$$\bar{a}_1 = a_1\beta_2 \quad \text{and} \quad \bar{a}_2 = (a_2 + \beta_1 a_1)\beta_2$$

$$\bar{c}_1 = c_1 \quad \text{and} \quad \bar{c}_2 = (c_2 + \beta_1 c_1)$$

The first two time moments of the reduced-order model are

$$T_1^2 = -\frac{\bar{c}_1}{\bar{a}_2}\beta_2 = -\frac{c_1}{a_1} = T_1^0$$

$$T_2^2 = \bar{c}_1 \frac{\bar{a}_2}{\bar{a}_1^2}\beta_2 - \frac{\bar{c}_2}{\bar{a}_1}\beta_2$$

$$= c_1 \frac{(a_2 + \beta_1 a_1)\beta_2^2}{a_1^2 \beta_2^2} - \frac{(c_2 + \beta_1 c_1)}{a_1 \beta_2}\beta_2$$

$$T_2^2 = \frac{c_1 a_2}{a_1^2} - \frac{c_2}{a_1} = T_2^0$$

Thus the first two time moments of the original system and the second-order model are the same, independent of the value of β_i coefficients. This proof can now be generalized through induction.

Following the above result, one is able to solve r algebraic equations in $r \times \beta$ parameters to obtain a reduced-order model which satisfies one of the following conditions:

(1) The first r time moments and the first r Markov parameters of the reduced-order model are matched to those of the original system

Definition: Consider a linear time invariant system described by equation (5.14), the Markov parameters are defined by

$$J_i = CA^i B \quad , \quad i = 0,1 \ldots\ldots, \infty$$

To obtain a solution for the β_i coefficients which satisfy the above condition, one must solve the following equation

$$L \underline{\beta} = \underline{J} \tag{5.22}$$

where L is rxr matrix with the following entries:

$$L(i,j) = c_{r+1-j} \quad , \quad j = 1,2, \ldots, r$$

$$L(i,j) = c_{r+2-(i+j)} - \sum_{k=1}^{i-1} J_{K-1}\, a_{r+k+2-(i+j)}$$

$$i = 2,3 \ldots, r \text{ and } j = 1,2 \ldots, r$$

$$\beta = [\beta_r : \beta_1\beta_r : \beta_2\beta_r : \ldots\ldots\ldots\ldots \beta_{r-1}\,\beta_r]^T$$

and $$J = [J_0 : J_1 : \ldots\ldots\ldots : J_r]^T$$

It may be noted while the reduced-order model obtained by satisfying the above condition has the best fit about $s = 0$ (matching time moments makes the steady state response of the reduced model closer to that of the original system) and $s = \infty$ (matching Markov parameters improved the transient response), it does not guarantee the stability of the reduced-order model if the original system is stable.

(2) The first r time moments of the reduced-order model are matched to those of the original system. Moreover, the stability of the reduced-order model is guaranteed if the original system is stable.

This can be achieved by solving equation (5.20), provided that the characteristic polynomial of the reduced-order model is known, i.e., $\bar{a}_i$, $i = 1,\ldots,r$ in equation (5.20) are known. The characteristic polynomial of the reduced model is obtained by one of the following:

(i) We start with the continued fraction expansion, about the origin, of the ratio of the even and odd parts of the characteristic polynomial of the original system. Truncation of this expansion after r terms gives the characteristic polynomial of the reduced model, and guarantees stability through the Hurwitz criterion provided that the original system is stable.

(ii) We obtain the zeros of the characteristic polynomial of the original system (eigenvalues of the system). We form the characteristic polynomial of the reduced model from the dominant eigenvalues.

(3) The first r time moments and one Markov parameter of the reduced-order model are matched to those of the original system, provided that the reduced-order model is stable if the original system is stable. The following procedure is proposed.

First, we obtain the r^{th}-1 characteristic polynomial as in (2). Second, we assign the r^{th} zero (eigenvalue) so that the first

Markov parameter of the reduced model matches that of the original system. If the r^{th}-1 characteristic polynomial is

$$s^{r-1} + \bar{a}_{r-1}\, s^{r-2} + \ldots\ldots\ldots \bar{a}_2\, s + \bar{a},$$

the r^{th} characteristic polynomial will be

$$s^{r} + (\bar{a}_{r-1} + \lambda)\, s^{r-1} + \ldots\ldots\ldots (\bar{a}_1 + \bar{a}_2\lambda)\, s + \bar{a}_1\lambda$$

where $-\lambda$ is the value of the assigned zero. Solving equation (5.20), one is able to obtain the value of β_i coefficients in terms of λ.

Equating the first Markov parameter of the reduced model and the original system we obtain the value of λ. Thus, we obtain the value of β_i coefficients. Finally, we form the transformation matrix in equation (5.15).

The first 2r time moments of the reduced-order model can also be matched to those of the original system, but in this case, we have to solve r nonlinear algebraic equations.

5.3.2 Case of Multi-Input System

Consider the general case of a system having several inputs. The control matrix B is then of dimension (n,ℓ), ℓ being the number of

inputs, and the system is assumed ot be controllable from the complete set of its inputs, i.e.,

$$\text{rank } [B, AB, \ldots, A^{n-1}B] = n$$

Two cases can arise:

(1) The system is completely controllable from a single input u_i, i.e., if ^iB is the i^{th} column of B (corresponding to the input u_i), the matrix $[^iB, A^iB, \ldots, A^{n-1}\ ^iB]$ is of rank n.

The matrices A and B of (5.14) will appear in the form

$$A = \begin{bmatrix} 0 & 1 & 0 & & \\ 1 & 0 & 1 & & \\ & & & 0 & 1 \\ -a_1 & -a_2 & \cdots\cdots\cdots & & -a_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} & & & 0 & \\ {}^1B & {}^2B & {}^{i-1}B & 0 & {}^{(i+1)}B \; {}^{\ell}B \\ & & & 1 & \end{bmatrix}$$

Now, applying the linear transformation

$$\underline{x} = P\underline{z} \tag{5.23}$$

where $$[P] = [P_1][P_2][P_3]$$

and $[P_1]$, $[P_2]$ and $[P_3]$ are matrices of order nxn, respectively. Also $[P_1]$ and $[P_2]$ have the same structure as in (5.15a) and (5.15b), respectively, while $[P_3]$ has the following structure:

$$P_3 = \begin{array}{c} \overset{\leftarrow r \rightarrow}{} \quad \overset{\leftarrow n-r-1 \rightarrow}{} \quad \overset{\leftarrow 1 \rightarrow}{} \\ \left[\begin{array}{c:c:c} I_{rxr} & P_3^{12} & 0 \\ \hdashline 0 & I_{(nr)x(r-r)} & \end{array}\right] \end{array} \qquad (5.23a)$$

where, r is the order of the reduced model,

$$r \leq n - \ell$$

P_3^{12} is matrix of order rxn-r-1. This matrix has (ℓ-1) non-zero columns and the rest of the columns are null. For example, if n = 10, ℓ = 2, r is $\leq$ 8.

Now, if we choose r to be = 5

$$P_3^{12} = \begin{array}{c} \overset{\longleftarrow 4 \longrightarrow}{} \\ \left[\begin{array}{c:c:c} & \gamma_{11} & \\ & \gamma_{21} & \\ 0 & & 0 \\ & \gamma_{r1} & \end{array}\right] \end{array} \quad \begin{array}{c} \\ \\ \\ \\ \\ \uparrow \\ 5 \\ \downarrow \end{array}$$

Consider that the original system is completely controllable from the first input. Applying the transformation (5.23) together with the aggregation matrix (5.18), one is able to obtain a reduced-order model which has the following form

$$\dot{\underline{z}}_1 = FA + G\underline{u}$$

$$\hat{\underline{y}} = H\underline{z}_1$$

where F and H have the same structure as in equation (5.19)

$$G = \begin{bmatrix} 0 & \bar{B}_{21} & \cdots\cdots\cdots & \bar{B}_{\ell 1} \\ 0 & \bar{B}_{22} & & \\ \beta_r & \bar{B}_{2r} & & \bar{B}_{\ell r} \end{bmatrix}$$

It is important to know that,

- The value of β_i coefficients will be obtained as in single-input - single-output case.
- The first r time moments, with respect to the first input, of the reduced-order model are matched to those of the original system.
- The columns of matrix G, except the first column, are linear function in γ_i coefficients.

- One is able to obtain a solution for the vlaue of γ_i coefficient so that, the first r time moment, with respect to the second, the third, ..., and the ℓ^{th} input, respectively, match those of the original system.

(2) The system is controllable from the whole set of inputs. In general, the system will not be completely controllable except through the action of several inputs. If a system is controllable, it is always possible to partially decompose it into k subsystems (k is less than or equal to the number of inputs ℓ) such that

S_k reacts on subsystems S_{k-1}, ..., S_1, S_1

S_2 reacts only on S_1, where S_k denotes the k^{th} system

The decomposed matrices A and B will have the following forms

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{kk} \end{bmatrix}, \quad B = \begin{bmatrix} {}^{1}B & 0 & 0 & X \\ 0 & {}^{2}B & 0 & X \\ 0 & 0 & {}^{k}B & X \end{bmatrix}$$

where the matrices A_{ii}, A_{ij} and ${}^{i}B$ have the forms

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ -a_1^i & -a_2^i & & & -a_{n_i}^i \end{bmatrix}, \quad A_{ij,j/i} = \begin{bmatrix} & & 0 & & \\ & & 0 & & \\ & & 0 & & \\ & & 0 & & \\ x & x & x & x & x & x \end{bmatrix}$$

and

$$i_B = \begin{bmatrix} 0 \\ 0 \\ \\ 1 \end{bmatrix}$$

In particular, it can happen that the above structural decomposition leads to the number of subsystems being equal to the number of inputs ℓ. In this case each column of B contains only one non-zero element, which is unity.

Having decomposed the original system into controllable subsystem, we apply the following linear transformation

$$\underline{x} = P\underline{z} \tag{5.24}$$

with

$$P = \begin{bmatrix} {}^1P & 0 & 0 & 0 \\ 0 & {}^2P & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & & {}^kP \end{bmatrix} \tag{5.25}$$

where the number of subsystems k is less than or equal the number of inputs ℓ.

In case of k is less than ℓ, the sub-matrices ${}^{i}P$ in equation (5.25) has the same structure as that of P in equation (5.23). However, if k is equal to the number of inputs, the sub-matrices ${}^{i}P$ in equation (5.25) has the same structure as that of P in equation (5.15). At this point, each subsystem is decomposed into two smaller subsystems, i.e., the state vector $\underline{z}$ of the transformed system has the following form

$$\underline{z} = [{}^{1}\underline{z}_1 \quad {}^{1}\underline{z}_2 \quad {}^{2}\underline{z}_1 \quad {}^{2}\underline{z}_2 \quad \cdots \quad {}^{k}\underline{z}_1 \quad {}^{k}\underline{z}_2] \qquad (5.26)$$

where ${}^{i}\underline{z}_1$ are the state variables of the i^{th} subsystem to be retained in the reduced-order model, i = 1,2,...,k.

By using a nxn permutation matrix, we rearrange the state variables in equation (5.26). Thus, $\underline{z}$ can be rewritten as

$$\underline{z} = [{}^{1}\underline{z}_1 \quad {}^{2}\underline{z}_1 \quad \cdots \quad {}^{k}\underline{z}_1 \quad {}^{1}\underline{z}_2 \quad {}^{2}\underline{z}_2 \quad \cdots \quad {}^{k}\underline{z}_2] \qquad (5.27)$$

Finally, we apply the aggregation matrix given by equation (5.18) to the system described by the states in equation (5.27).

Some observations about the reduced-order model obtained are

(1) The first r_1 time moments, with respect to the first input of the reduced-order model are matched to those of the original system (r_1 is the order of the reduced-order model of the first subsystem).

(2) A number of time moments (equal to the minimum of r_1 and r_2, r_2 is the order of the reduced-order model of the second subsystem), with respect to the second input, of the reduced-order model are matched to those of the original system.

(3) A number of time moments (equal to the minimum of r_1, r_2,... and r_k), with respect to the k^{th} input, of the reduced-order model are matched to those of the original system.

5.4 Application to Power Systems

5.4.1 A Synchronous Machine Connected to an Infinite Bus

The system to be studied is that of one machine connected to an infinite bus through a transmission line. The model used for this synchronous machine is given in Appendix 1.

The synchronous machine to be studied is a 60 Hz synchronous machine with the following parameters:

x_d = 1.79 pu x_q = 1.74 pu

x_ℓ = 0.165 pu x_{kq1} = 2.025 pu

x_f = 1.796 pu x_{kq2} = 1.679 pu

x_{kd1} = 1.79 pu		R_f = 0.001 pu	
R_a = 0.004 pu		R_{kq1} = 0.011 pu	
R_{kd1} = 0.027 pu		D = 2.0 pu	
R_{kq2} = 0.023 pu		M = 2.8 sec	
Rated MVA = 675		Power Factor = 0.85	
Rated Voltage = 20 KV		Speed = 3600 rpm	

Note: Reactance and resistance values are in pu based on 675 MVA and 20 kV.

This machine is connected to an infinite bus through a transmission line having R_e = 0.02 pu and x_e = 0.4 pu. The infinite bus voltage is 1.0 pu. The machine loading is P = 1 pu at 0.85 Pf).

The linearized state space representation for this system is given by:

$$\underline{\dot{x}} = A\underline{x} + B\underline{u}$$

$$\underline{y} = C\underline{x} + D\underline{u}$$

where

$$\underline{x}^t = [i_d,\ i_f,\ i_{kd},\ i_q,\ i_{kq1},\ i_{kq2}\ \delta,\ \omega]^t$$

$$\underline{u}^t = [E_{fd},\ T_m]^t$$

$$\underline{y}^t = [V_t\ \ \delta,\ \omega]^t$$

and the matrices $\hat{A}$, B, C and D evaluated for this specific example and operating conditions are on the following page.

The procedure discussed previously in this chapter is used to obtain two reduced-order models (two variations within the procedure). The first reduced model is a third-order model with a characteristic equation obtained using the Routh table. The second reduced model is a third-order model with a characteristic equation obtained by assigning some of the original model eigenvalues. The reduced-order models matrices are:

(1) The first reduced model is

$$F = \begin{bmatrix} 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ -0.27365\times10^{-6} & -0.36326\times10^{-3} & -0.43992\times10^{-1} \end{bmatrix}$$

$$G = \begin{bmatrix} 0.0 & -0.3154204\times10^{6} \\ 0.0 & 0.5923547\times10^{3} \\ 0.5684\times10^{4} & 0.617799\times10^{2} \end{bmatrix}$$

$$A = \begin{bmatrix} -0.03722 & 0.000724 & 0.020262 & -3.3186 & -2.442439 & -2.442439 & 1.55108 & -2.26265 \\ 0.0173787 & -0.003455 & 0.066947 & 1.5496 & 1.14047 & 1.14047 & -0.7246 & 1.05185 \\ 0.0180106 & 0.002479 & -0.09425 & 1.60594 & 1.18195 & 1.18195 & -0.7506 & 1.0901 \\ 3.39442 & 2.51869 & 2.51869 & -0.037199 & 0.0030377 & 0.0274828 & 2.1917 & 1.5755 \\ -0.60478 & -0.44875 & -0.44875 & 0.006627 & -0.02063 & 0.034506 & -0.3905 & -0.280712 \\ 2.62684 & -1.94172 & -1.94172 & 0.028677 & 0.016503 & -0.071848 & -1.6896 & -1.2146 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ -0.000005 & -0.000194 & -0.000174 & -0.000703 & -0.000399 & -0.000394 & -1.6896 & -1.2146 \end{bmatrix}$$

$$B^T = \begin{bmatrix} -0.72411 & 3.455417 & -2.4795 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.00047366 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.74933 & 0.47435 & 0.471739 & 0.29335 & 0.32634 & 0.3315 & -0.79438 & 0.857617 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.09674599 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}$$

$$H = \begin{bmatrix} 0.311035\times10^{-7} & 0.38929\times10^{-5} & -0.3647\times10^{-3} \\ -0.25662\times10^{-7} & -0.2788115\times10^{-5} & 0.327606\times10^{-3} \\ 0.0 & -0.2566\times10^{-7} & -0.278\times10^{-5} \end{bmatrix}$$

$$D = \begin{bmatrix} 0.09674599 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}$$

(2) The second reduced model is

$$F = \begin{bmatrix} 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ -0.645018\times10^{-6} & -0.7747\times10^{-3} & -0.52628^{-2} \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0 & 0.9390718\times10^{6} \\ 0.0 & 0.1038811610^{6} \\ 0.134424\times10^{5} & -0.121047\times10^{4} \end{bmatrix}$$

$$C = \begin{bmatrix} 0.311035\times10^{-7} & 0.003517\times10^{-5} & 0.13472\times10^{-4} \\ 0.25662\times10^{-7} & 0.037705\times10^{-5} & 0.37666\times10^{-4} \\ 0.0 & 0.32409\times10^{-7} & 0.0469\times10^{-5} \end{bmatrix}$$

$$D = \begin{bmatrix} 0.09674594 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}$$

The eigenvalues of the original as well as the reduced models are shown in Table 5.1. Also, the torque angle time response to a 10% step change in T_m and the terminal voltage time response to a 10% step change in E_{fd} are shown in Fig. (5.1) and Fig. (5.2), respectively.

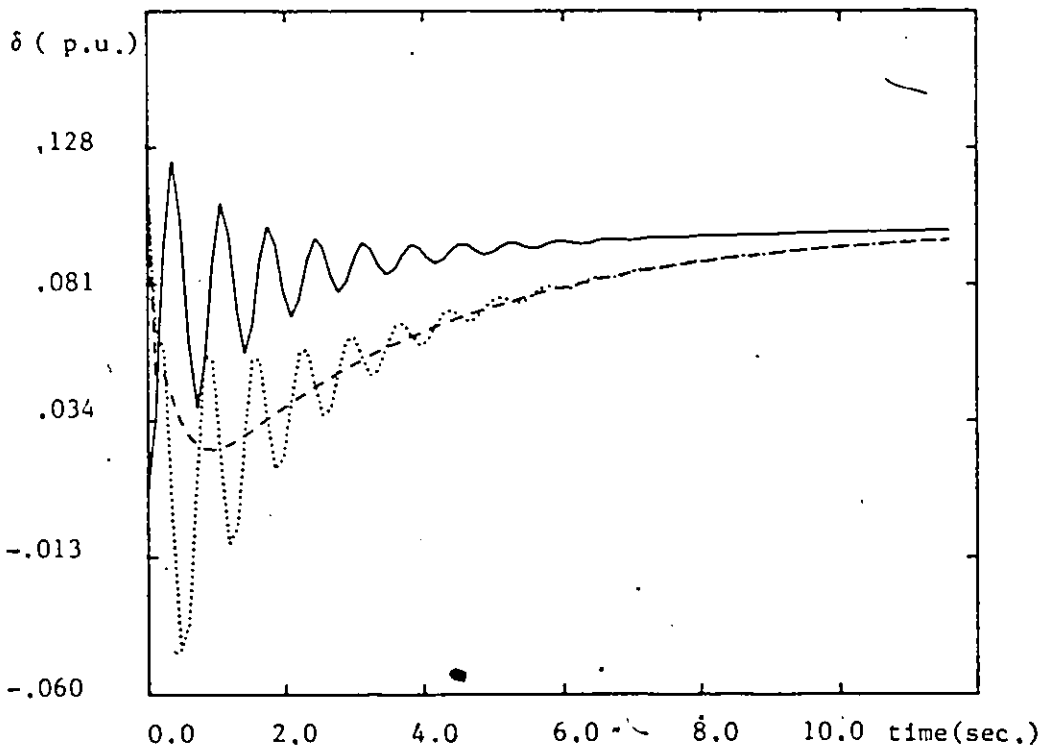


Figure 5.1 Torque angle (p.u) response following a 10% step change in mechanical torque .

—— original system

- - - reduced order model-1

.....reduced order model -2

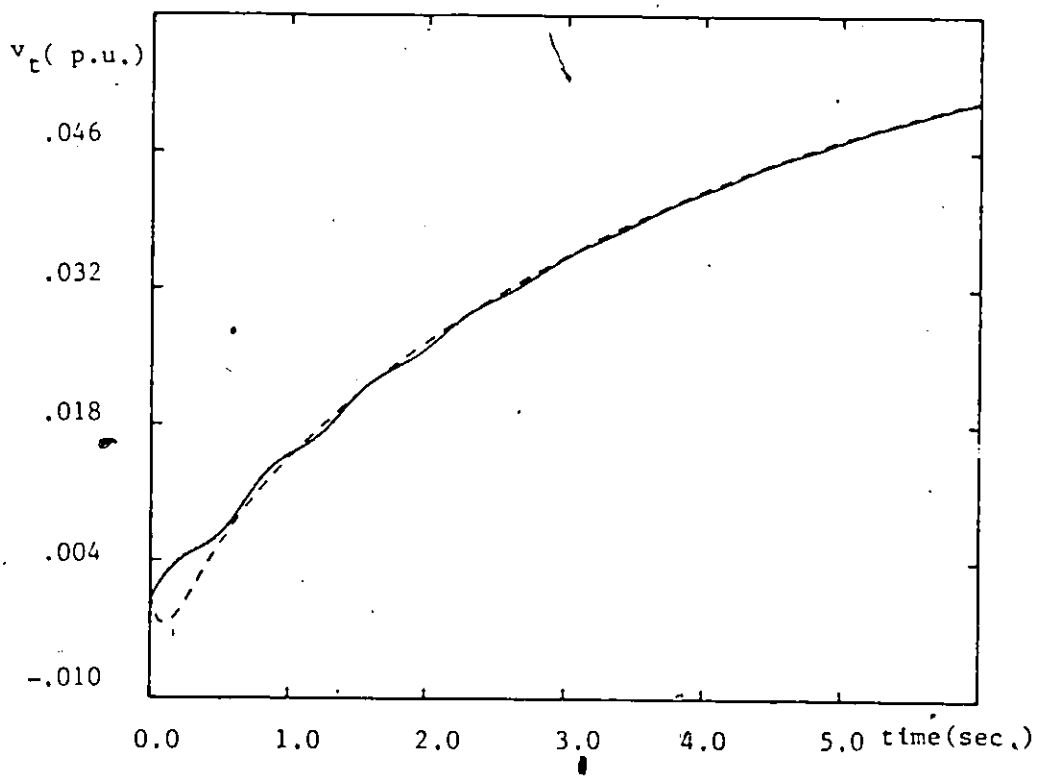


Figure 5.2 Terminal voltage response to a 10% step change in field voltage.

——— original system

- - - reduced order model - 1

..... reduced order model —2

Table 5.1 Eigenvalues of Original System as Well as Reduced Models

Original System	Reduced Model 1	Reduced Model 2
-0.00216 ± j0.02768	-0.033343	-0.00216 ± j0.02768
-0.000836	-0.000836	-0.000836
-0.00967	-0.00091	
-0.09559		
-0.0372 ± j0.999		
-0.08069		

All the eigenvalues are given in rad/rad. Note that there are two pairs of complex eigenvalues in the first column. The first pair correspond to frequency of approximately 1.66 Hz; they are damped with a time constant of 1/(0.00216x377) or 1.228 sec. This complex pair and the real eigenvalue -0.000836 dominate the transient response of the system. The other complex pair corresponds to a very fast transient of about 60 Hz, which is damped at a much faster rate. This is the 60 Hz component injected into the rotor circuits to balance the MMF caused by the stator dc currents. Note also that the real parts of all eigenvalues are negative, which means that the system is stable under the conditions assumed in development of this model, small perturbation about a quiescent operating condition.

In the second column, the pair of complex eigenvalues (-0.00216 ± j0.0276) has been approximated by the real eigenvalue -0.03334, while the other two real eigenvalues are almost the same as the two smallest real eigenvalues of the original system. Thus, the

eigenvalues of the reduced-order model obtained by Routh approximation approximates those of the original system which have the smallest absolute values.

In the third column, those eigenvalues of the original system which dominate the transient response have been retained. As the second reduced-order model retains the dominant eigenvalues and the first three time moments of this model are matched to those of the original system, we expect that its response to a step disturbance will be better than the first reduced-order model. This is confirmed from Figs. (5.1) and (5.2) which show that the second reduced-order model has an oscillatory response as that of the original system. Also, its steady state response approximates that of the original system.

5.4.2 A Synchronous Machine Connected to an Infinite Bus Through a Tansmission Line with an Exciter

In example 5.4.1, we were not able to decompose the original system into number of blocks equal to the number of inputs because it was controllable from each input alone. Thus, we present here another example of a system which is decomposed into number of blocks equal to the number of inputs. In this example the excitation system is included. The model used for the excitation is given in Appendix I. In this example another synchronous machine has been modeled with the following parameters [58]

$$x_d = 1.7 \text{ pu} \qquad x_f = 1.65 \text{ pu}$$

x_ℓ	= 0.15 pu	x_{kq1}	= 1.526 pu
x_q	= 1.64 pu	R_{kd1}	= 0 0131 pu
x_{kd1}	= 1.605 pu	R_f	= 0.000742 pu
R_a	= 0.001096 pu	D	= 0.0 pu
R_{kq1}	= 0.054 pu	M	= 2.37 sec

Rated MVA = 160 Power Factor = 0.85 Rated kV = 15 kV

Frequency = 60 Hz Speed = 3600 rpm

This machine is also connected to an infinite bus through a transmission line having R_e = 0.02 pu and x_e = 0.4 pu. The infinite bus voltage is 1 pu. The machine loading is (P = 1.0 pu at 0.85 Pf). Note: Reactance and resistance values are in pu based on 160 MVA and 15 kV.

The excitation system parameters are given by:

τ_R	= 0.01 s	τ_E	= 0.5 s
K_R	= 1.0	K_E	= -0.05

$\tau_A = 0.05$ s $\qquad\qquad$ $\tau_F = 0.715$ s

$K_A = 40$ $\qquad\qquad$ $K_F = 0.04$

The exciter saturation is represented by the following nonlinear function

$$S_E = 0.0039 \exp (1.555\ E_{fD})$$

The linearlized state space representation for that system is given by

$$\dot{\underline{x}} = A\underline{x} + \underline{u}$$

$$\underline{y} = C\underline{x}$$

where

$$\underline{x}^t = [i_d, i_f, i_{kd}, i_q, i_{kq}, \delta, \omega, V_1, V_3, V_R, E_{fD}]^t$$

$$\underline{u}^t = [T_m, V_{ref}]$$

$$\underline{y}^t = [V_r, \delta, \omega]$$

Two reduced-order models are obtained using the same procedure as before. The first reduced model is a fifth-order model with a characteristic equation obtained using the Routh table. The second reduced model is a fifth-order model with a characteristic equation obtained by assigning some of the original model eigenvalues.

The eigenvalues of the original model as well as the reduced models are shown in Table 5.2. Also, the terminal voltage time response to a 10% step change in V_{ref} is shown in Fig. 5.3.

Table 5.2 Eigenvalues of Original System as Well as Reduced Models

Block No.	Original System	Reduced Model 1	Reduced Model 2
	-0.0015 ± j0.029	-0.0329 ± j0.0203	-0.0015 ± j0.029
	-0.0002 ± j0.0064	-0.0002 ± j0.006903	-0.0002 ± j0.0064
1	-0.0986	-	
	-0.09559	-	
	-0.0359 ± j0.9983	-	
	-0.1217		
	-0.0037	-0.00369	-
2	-0.0548	-	-0.0548
	-0.2653	-	

All the eigenvalues are given in rad/rad. Note that there are two pairs of complex eigenvalues in the first column which dominate the

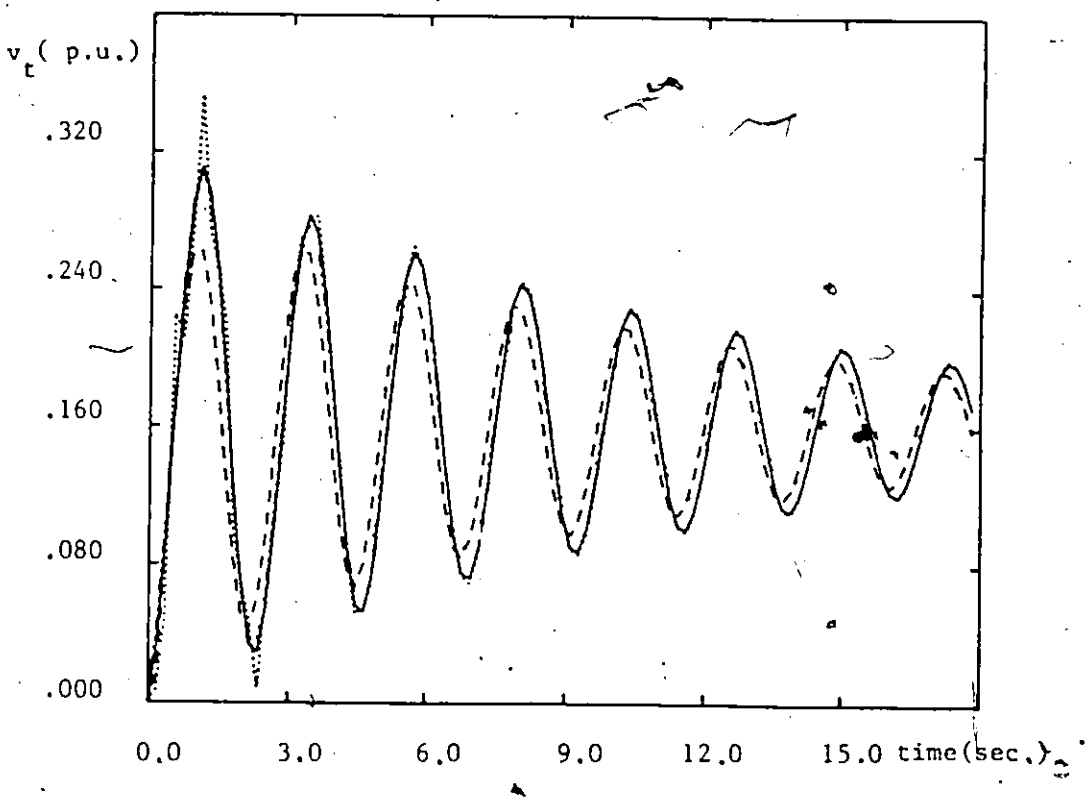


Figure 5.3 Terminal voltage response to a 10% step change in reference voltage.

——— original system

– – – reduced order model -1

..... reduced order model -2

transient response. These pairs are -0.0015 ± j0.029 and -0.0002 ± j0.064. The first pair of complex eigenvalues correspond to the shaft dynamics while the second pair respresents the interaction between the field circuit and the exciter. The real eigenvalues -0.0037, -0.0548 and -0.2653 are related to the exciter parameters. These eigenvalues correspond to the stabilizing circuit, the regulator amplifier and the regulator input filter, respectively. Although the stabilizing circuit has a small time constant it has a small gain. This makes us consider the eigenvalue corresponding to the regulator amplifier to be dominant.

In the second column, the pair of complex eigenvalue -0.0002 ± j0.006403 and the real eigenvalue -0.00369 are essentially present in the first column. Note that the reduced-order models obtained by Routh approximation may have non-dominant eigenvalues (-0.00369). The reason is they approximate the eigenvalues which have the smallest absolute value.

In the third column, those eigenvalues of the original system which dominate the transient response have been retained. Thus the second reduced-order model retains the dominant eigenvalues and its steady state response is matched to that of the original system. Figure (5.3) illustrates that the second reduced-order model responds to a step disturbance better than the first reduced-order model. Based on the results shown in Figs. (5.1), (5.2) and (5.3) and the characteristics of the reduced-order models obtained using the procedure presented in this chapter, we may say that a reduced-order model which

retains the domiannt eigenvlaues of the original system will respond to a step disturbance better than the one which has a reduced-order characteristic equation obtained from the Routh table.

5.5 Conclusions

An algorithm for obtaining reduced order models for single-input - single-output systems, as well as multi-input - multi-output systems has been presented in this chapter. This algorithm combines the method of aggregation with that of moment matching, discussed in Chapter 3, in such a way as to secure the separate advantages of each technique while simultaneously removing their disadvantages as discussed in the following paragraph.

The method of aggregation suffers from the drawback that an eigenvalue/eigenvector calculation must be accomplished for the A matrix, which will generally be asymmetric. After transformation to Hessenberg form, several iterations of the QR algorithm, requiring $4n^2$ operation per iteration, may be needed to determine each eigenvalue [21,22,23]. Calculation of the eigenvectors will take at least as many operations and thus the computer time involved may become prohibitive for a large matrix ($n > 50$). It is worth noting that the procedure presented in this chapter does not require the calculation of the system eigenvectors. However, it requires that the state equations of the system be transformed to either the controllable or observable companion form. This transformation is quite straightforward and can be performed in a routine manner. Moreover, with this transformation

we may decompose the original system (multivariable system) into number of blocks less than or equal to the number of inputs. This means that in order to obtain the system eigenvalues, it is possible to solve for the eigenvalues of subsystems with smaller order, i.e., it has the advantage of reducing the computation of the system eigenvectors and reducing the computation of the system eigenvalues.

The reduced-order models obtained using this procedure retain all the good features of the method of aggregation, e.g.,

- A relationship between the states of the original and reduced model.
- The invariance property under linear state variable feedback.
- A stable (unstable) reduced model of a stable (unstable) system.

Furthermore, the first r time moments (r is the order of the simplified model) of the reduced-order model are matched to those of the original system. In the case of single-input - single-output systems, one is able to obtain reduced models such that its first r Markov parameters are matched to those of the original system.

Two examples of a synchronous machine connected to an infinite bus (with and without excitation system) have been considered to illustrate the application of this procedure. The time responses of the reduced-order models obtained to a step disturbance indicate that they are almost the same as those of the original system, when the reduced models retain the dominant eigenvalues of the original system. In case of reduced-order models obtained by using Routh approximation, the time

responses to a step disturbance differ slightly from those of the original system. This may be due to the fact that these reduced models may have non-dominant eigenvalues of the original system.

In general, we may say that such reduced-order models combine the excellent qualitities of both aggregation principle and moment matching approaches.

CHAPTER 6

ALGORITHMS FOR DETERMINING DYNAMIC EQUIVALENTS OF MULTIMACHINE SYSTEMS

6.1 Introduction

This chapter is divided into two main sections. In the first section, a new procedure is presented to obtain a reduced order model computed by aggregation techniques which is a good approximation to the original model. Moreover, a measure is proposed permitting the recognition of the dependence of the state variables upon the dominant eigenvalues. Those state variables which strongly correspond to dominant eigenvalues are chosen to be state vector components of the reduced-order model. The proposed method of reduction retains the physical significance of the state, while retaining the important eigenvalues of the original system. An example of a synchronous machine connected to an infinite bus through a transmission line is used to illustrate the results.

In the second section, a procedure for obtaining dynamic equivalents of multimachine systems is proposed. This procedure utilizes the idea of component cost analysis, [44,45] for identifying the coherent generators. It has the advantage of relating the coherency measure to the parameters of the system and the statistics

of the disturbance such that simulation is not required to determine coherent groups.

Moreover, one is able to obtain a measure of quality of the dynamic equivalence with respect to the original system. The procedure has been applied to a system of 46 transmission lines, 39 buses and 10 generators. This system is a well documented model of the 345-KV system of the New England area [29].

6.2 A New Algorithm for Model Reduction

6.2.1 The Proposed Algorithm

One approach to determine a reduced-order model is to decompose a linear time invariant system

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} \end{aligned} \tag{6.1}$$

into two parts as follows

$$\begin{aligned} \begin{bmatrix} \dot{\underline{x}}^1 \\ \dot{\underline{x}}^2 \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \underline{x}^1 \\ \underline{x}^2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} \underline{u} \end{bmatrix} \\ \underline{y} &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \underline{x}^1 \\ \underline{x}^2 \end{bmatrix} \end{aligned} \tag{6.2}$$

such that the state variables $\underline{x}^1$ are the most significant states of the original system, i.e., the states to be retained in the reduced-order model.

The first step is to determine the dominant eigenvalues for the system. The dominant eigenvalues have been defined before in Chapter 4. They can be reliably found using the modal cost analysis discussed earlier in Chapter 2.

In the second step, those state variables which are heavily influenced by the dominant eigenvalues are found.

This can be done by employing an appropriately defined measure as:

$$P_k(\lambda_i) = \frac{1}{m} \left| u_{Ki} \underline{v}_i^T \sum_{j=1}^{m} A^{-1} \underline{b}_j \right| \qquad (6.3)$$

where u_{ki} is the k^{th} entry of the i^{th} right eivenvector

$\underline{v}_i^T$ is the i^{th} left eigenvector transpose

$\underline{b}_j$ is the j^{th} vector of the B matrix corresponding to the i^{th} input

m is the number of inputs.

The measure $P_k(\lambda_i)$ accounts for the influence of the eigenvalue λ_i on the k^{th} state variable. This measure is derived from the step response of equation (6.1) when exciting the j^{th} input only,

$$\underline{x}(t) = U \exp(\Lambda t) V^T (x(o) + A^{-1}\underline{b}_j) - A^{-1}\underline{b}_j \qquad (6.4)$$

where U is the right eigenvector matrix, $AU = U\Lambda$

V^T is the left eigenvector matrix transpose, $A^TV = V\Lambda$

V is the left eigenvector matrix transpose, $A^T V = V\Lambda$

Λ is diagonal $[\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n]$, assuming that the eigenvalues λ_i are distinct (not repeated)

A^{-1} is equal to $U\Lambda^{-1}V^T$.

It may be noted that in this measure both the A and B matrices have been utilized, unlike the measure in [54] which utilizes the A matrix alone.

After re-ordering the eigenvalues λ_i in accordance with decreasing modal costs, Table (6.1) can be formed to study the dependence of the state variables upon the eigenvalues.

For convenience $P_k(\lambda_i)$ is shortened to P_{ki}.

Table 6.1 Dependence of the States Upon the Eigenvalues

States \ Eigenvalues	λ_1	λ_2	λ_r	λ_n
x_1	P_{11}	P_{12}	P_{1r}	P_{1n}
x_j	P_{j1}	P_{j2}	P_{jr}	P_{jn}
x_n	P_{n1}	P_{n2}	P_{nr}	P_{nn}

To obtain an assessment which is easier to interpret, the value P_{ki} is replaced by

$$Q_{ki} = \sum_{j=1}^{i} P_{kj} \qquad (6.5)$$

This measure will show how the k^{th} state variable will participate in the time response of the first i dominant eigenvalues.

with Q_{kj} from (6.5), Table (6.2) can be obtained as

Table 6.2. Dependence of the States Upon a Subset of Eigenvalues

States	Eigenvalues λ_1	λ_r	λ_n
x_1	Q_1	Q_{1r}	Q_{1n}
x_k	Q_{k1}	Q_{kr}	Q_{kn}
x_n	Q_{n1}	Q_{nr}	Q_{nn}

Assume for the present that the reduced model order is r. Then, the r^{th} column in Table (6.2) is used to choose the components of $\underline{x}^1$ in equation (6.2) as those which have the largest measure Q_{ir}.

After we have selected the most significant states of the original system, we need to obtain the reduced model. Let us consider that, by a permutation of the states, they have been so arranged that the first r state variables are the most significant. We now partition the state vector $\underline{x}$ as

$$\underline{x}^T = [\underline{x}^1, \underline{x}^2]^T \tag{6.6}$$

so that we obtain equation (6.2).

Introducing the following similarity transformation

$$\begin{bmatrix} \underline{x}^1 \\ \hline \underline{\xi} \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ \hline V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \underline{x}^1 \\ \hline \underline{x}^2 \end{bmatrix} \tag{6.7}$$

where V_{22} is a non-singular matrix, and

$$V = \begin{matrix} \\ n-r \end{matrix} \overset{\begin{matrix} \leftarrow r \rightarrow & \leftarrow n-r \rightarrow \end{matrix}}{\begin{bmatrix} V_{21} & V_{22} \end{bmatrix}}$$

is the n-r rows of the left eigenvector matrix transpose corresponding to the n - r non-dominant eigenvalues.

This will transform equation (6.2) to

$$\begin{bmatrix} \dot{\underline{x}}^1 \\ \dot{\underline{\xi}} \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}V_{22}^{-1}V_{21} & A_{12}V_{22}^{-1} \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} \underline{x}^1 \\ \underline{\xi} \end{bmatrix} + \begin{bmatrix} B_1 \\ V_{21}B_1 + V_{22}B_2 \end{bmatrix} [\underline{u}] \tag{6.8}$$

$$\underline{y} = [C_1 - C_2V_{22}^{-1}V_{21} \;\vdots\; C_2V_{22}^{-1}] \begin{bmatrix} \underline{x}^1 \\ \underline{\xi} \end{bmatrix}$$

where Λ_2 is diag $[\lambda_{r+1}, \lambda_{r+2} \cdots, \lambda_n]$, λ_i, $i = r+1, \ldots, n$ are the non-dominant eigenvalues.

Since the state variables ξ in Equation (6.8) correspond to the non-dominant eigenvalues, they can be set equal to zero. This will lead to the following reduced order model

$$\begin{aligned} \dot{\underline{x}}^1 &= [A_{11} - A_{12}V_{22}^{-1}V_{12}]\,\underline{x}^1 + [B_1]\,\underline{u} \\ \hat{\underline{y}} &= [C_1 - C_2V_{22}^{-1}V_{21}]\,\underline{x}^1 \end{aligned} \tag{6.9}$$

which has the same dominant eigenvalues as the original system. Moreover, its state variables $\underline{x}^1$ retain the physical significance of the states of the original system. Also, it preserves the physical coupling through the matrix B_1 unlike that reported by Van Ness et al. [53].

One possible criticism of the proposed method is the requirement of calculating the eigenvalues and eigenvectors of a large matrix. For instance, when we are trying to understand the structure of a nominal model of a system that is going to operate for a large set of slightly different operating conditions, one may permit a single computation of the complete eigenstructure of the nominal model; structural insights obtained from this case may then be applied to slightly different operating conditions.

Sometimes the information concerning the dependence of the state variable upon a subset of eigenvalues (Table 6.2) is not available. In this case, the criterion to be used in the selection of r significant state variables will strongly depend on a priori information about the system, since knowledge regarding the existence of critical modes, or their approximate natural frequencies, or regarding the most significant state variables might be available. A fact that has been observed in all the power system models studied so far is that the set of significant state variables always includes the speed and rotor angle of all the machines and, sometimes, a few more state variables. Also, in dynamic stability studies we are often interested

in those lightly damped, oscillatory modes in the frequency range of 0.1 to a 2 Hz, that are associated with power oscillations. Based on such a priori information, one is able to obtain a partial eigensolution for the original system [26,47,48].

This will provide us with the set of eigenvalues of largest absolute magnitude for the matrix together with the corresponding left or right egenvectors.

In the next section two examples are considered to test the model reduction technique developed above.

6.2.2 Simplified Models of Power Systems

The algorithm presented in the previous section has been utilized for obtaining simplified models of a synchronous machine connected to an infinite bus, as well as a nine-bus power system that has three generators and three loads.

6.2.2.1 Single Machine Infinite Bus System

Figure (6.1) is a single line diagram of a synchronous machine connected to an infinite bus through a transmission line having resistance R_e and reactance x_e. The synchronous machine is represented by eight state variables which represent the shaft dynamics, the stator circuits, and four rotor circuits. The rotor circuits are the field circuit, one amortisseur circuit in the direct axis, and two amortisseur circuits in the quadrature axis. The system data is the same as in example (5.4.1).

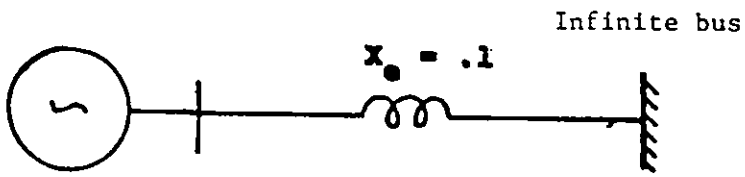


Figure 6.1 Single machine-infinite bus configuration.

In Table (6.3) the eigenvalues of the original system and their modal cost indices (the information concerning the modal cost indices has been discussed in Chapter 2) based on the output

$$\underline{y} = [\Delta v_t, \Delta\omega, \Delta\delta]^T$$

are listed. Examination of Table (6.3) indicates that the complex pair of eignvalues $-0.00216 \pm j0.0276$ and the real eigenvalue -0.00083 should be retained in the reduced order model. These eigenvalues are the dominant eigenvalues in the output vector $\underline{y}$.

The most significant state variables are obtained from Table (6.4). This table illustrates the dependence of the state variables upon the eigenvalues of the original system, whereby the eigenvalues are arranged with respect to decreasing modal cost indices. The results given in this table are calculated based on the information provided by Table (6.2). As only three eigenvalues dominate the transient response, we decided to obtain a third order

Table 6.3. Eigenvalues of the Original System and the Respective Modal Cost indices

Eigenvalues	Dominance Indices
-0.00216 ± j0.02768	3075.4
-0.00083	294.8
-0.00967	1.4826
-0.09559	0.10318
-0.0372 ± j0.999	0.00072
-0.080697	0.00025

Table 6.4. Dependence of the State Variables Upon the Eigenvalues

States	Eigenvalues					
	-0.00216 ± j0.0276	-0.00083	-0.0096	-0.095	-0.0372 ± j0.999	-0.08069
Δi_d	25.3	50.7	59.5	59.9	99.9	100.0
Δi_f	41.6	99.3	99.6	99.9	99.91	100.0
Δi_{kd}	35.8	66.5	78.4	88.2	91.3	100.0
Δi_q	23.8	47.5	55.4	65.1	99.8	100.0
Δi_{kq1}	41.2	69.4	85.4	93.7	95.6	100.0
Δi_{kq2}	39.5	58.2	81.4	91.2	93.9	100.0
$\Delta\delta$	80.8	99.8	99.84	99.9	100.0	100.0
$\Delta\omega$	99.6	99.9	99.92	100.0	100.0	100.0

model. The three state variables representing this reduced order model are selected according to the second column of Table (6.4). These are the state variables which influenced heavily by the dominant eigenvalues. Thus, the reduced order model is represented by the following state variables.

$\Delta\delta$: the change in torque angle

$\Delta\omega$: the change in rotor speed

Δi_f : the change in field current

Applying the proposed algorithm discussed previously in this chapter we obtain the following reduced order model

$$\dot{\underline{z}} = F\underline{z} + G\underline{u}$$

$$\underline{y} = H\underline{z} + D\underline{u}$$

where $\underline{z} = [\Delta\delta, \Delta\omega, \Delta i_f]^T$

$\underline{u} = [\Delta E_{fd}, \Delta T_m]^T$

$\underline{y} = [\Delta v_t, \Delta\omega, \Delta\delta]^T$

$$F = \begin{bmatrix} 0.0 & 1.0 & 0.0 \\ -0.00039 & -0.0052 & -0.0023 \\ 0.00295 & 1.688 & -0.005 \end{bmatrix}, \quad G = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.00047 \\ 3.4554 & 0.0 \end{bmatrix}$$

$$H = \begin{bmatrix} -0.9918 & -0.0298 & 0.1151 \\ 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 0.0967 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}$$

The above reduced order model is obtained by setting the values of the state variables $\underline{\xi}$ in equation (6.8) equal to zero. If, instead, we set the values of the derivitive of these state variables equal to zero we obtain a reduced order model which has F and H matrices similar to the above F and H matrices, but different G and D matrices which they are:

$$G = \begin{bmatrix} 0.0 & 0.0 \\ 0.00118 & -0.00087 \\ 1.53509 & -0.0004 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2.3136 & 0.007 \\ 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}$$

Figures (6.2) through (6.4) are simulations comparing the response of the original system to those of the first ($\xi = 0$) and the second ($\xi = 0$) reduced order models. The disturbances used in the simulations are 10% step change in the mechanical torque and the field voltage, respectively. These simulations illustrate that although the first reduced model responds to a step disturbance the same way as the original system (has the same shape), it provides a poor steady state response. However, the second reduced order has almost the same responses as the original system. These differences in the response

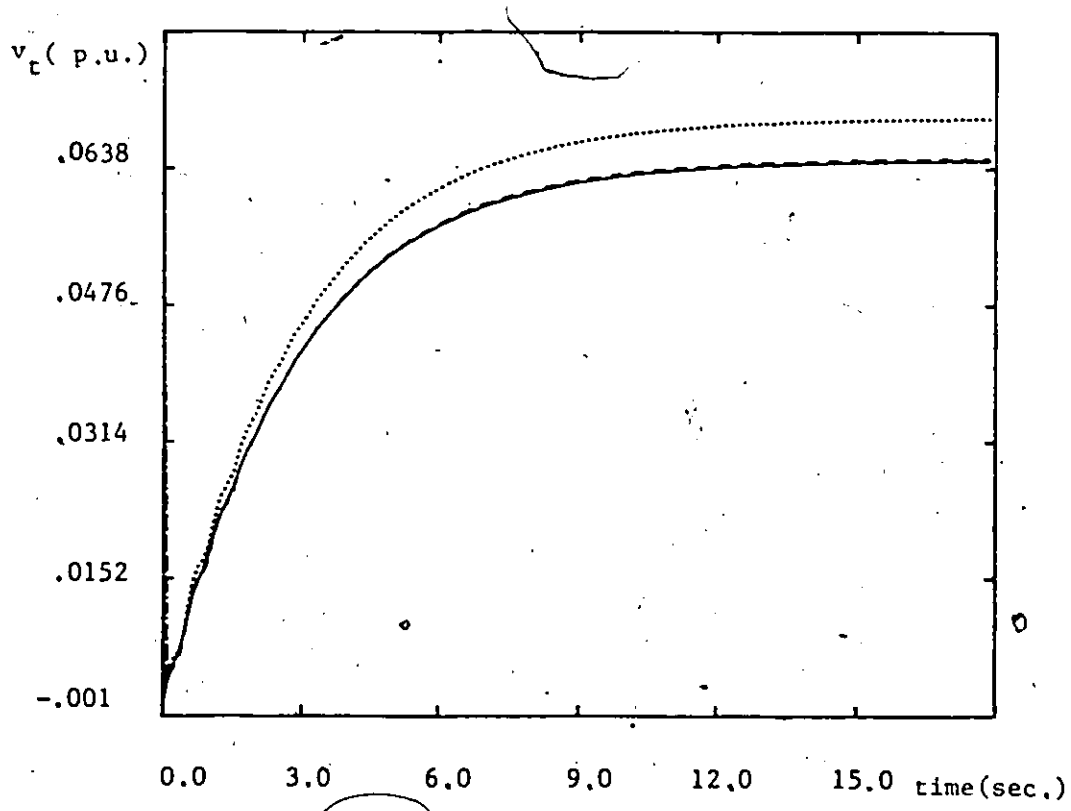


Figure 6.2 Terminal voltage response following a 10% step change in field voltage .

——— original system
- - - - reduced order model - 2
..... reduced order model - 1

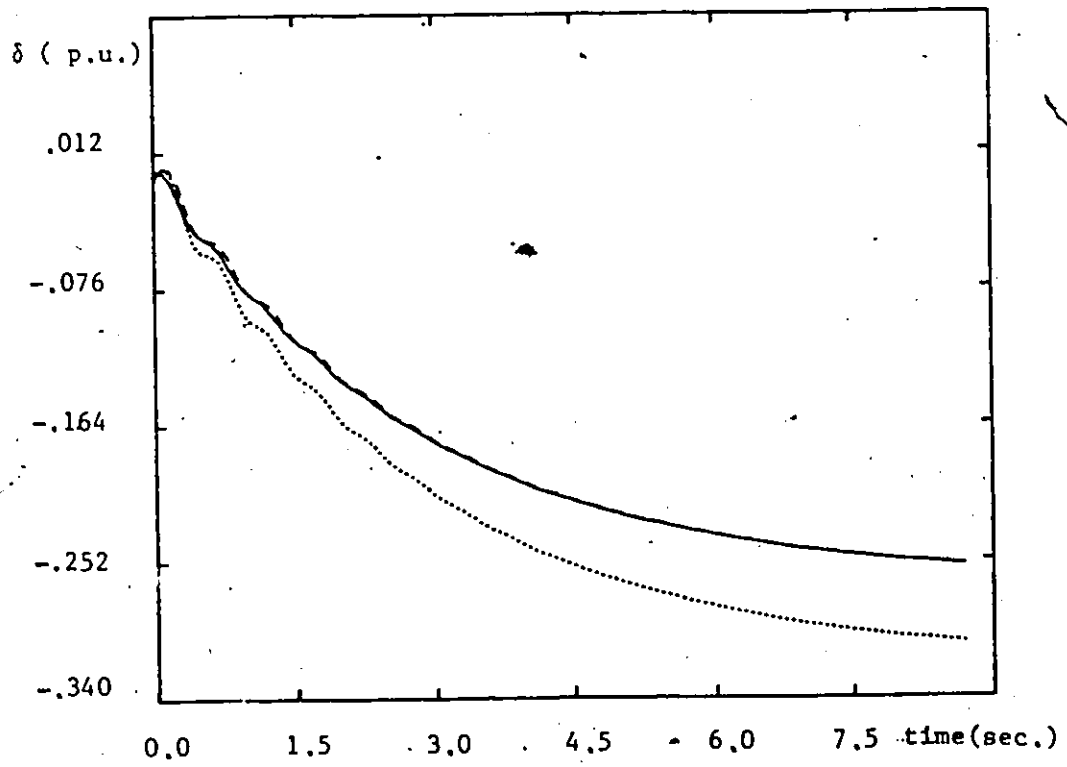


Figure 6.3 Torque angle (p.u.) response following a 10% step change in field voltage.

——— original system
– – – reduced order model – 2
..... reduced order model – 1

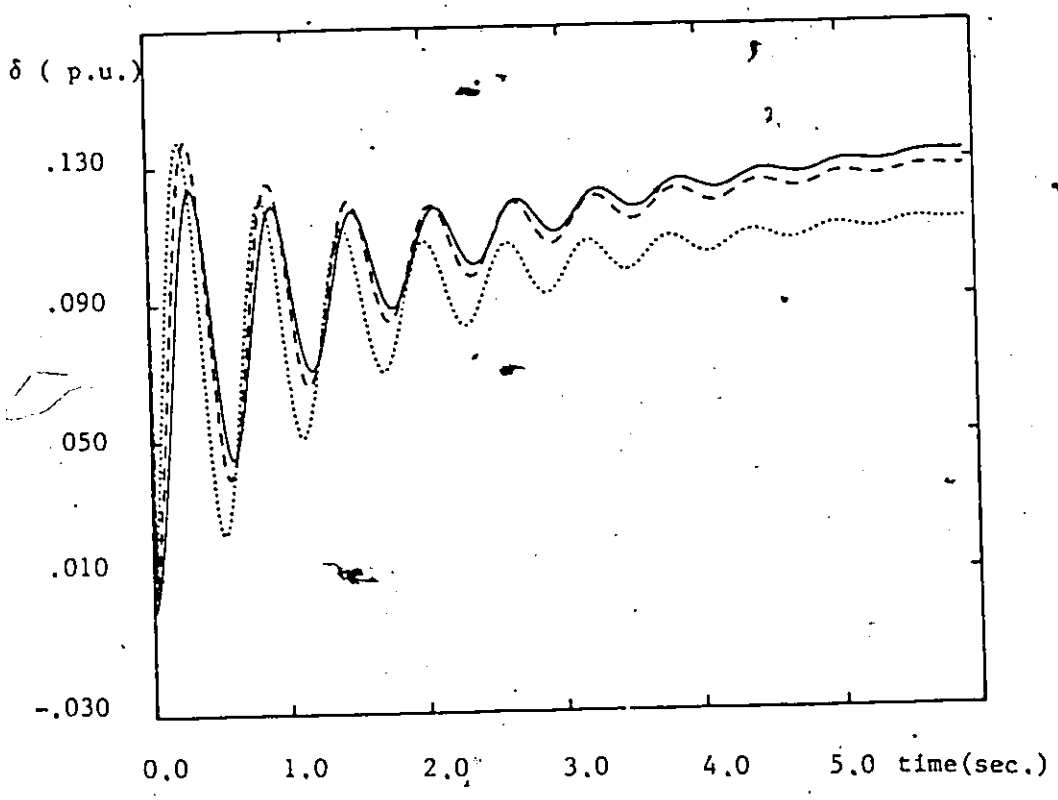


Figure 6.4 Torque angle (p.u.) response following a 10% step change in mechanical power.

——— original system
- - - reduced order model - 2
.... reduced order model - 1

of the first reduced order model occurred because we neglected the effect of the non-dominant eigenvalues, while in the second reduced order model, their effect had been approximated by their steady state values.

6.2.2 Multimachine System

The system under study comprises three generators and three loads [58]. A one-line impedence diagram is given in Figure (6.5). The initial condition indicating the power flows and bus voltages, is given in Figure (6.6). Data for the three generators are given in Table (6.5). The synchronous machine model to be used are as follows: a classical model for generator 1, and two-axis model for generators 2 and 3. The gnerator equations are as follows (the prefix Δ is omitted).

- Generator 1 (classical)

$$2M_1\omega_0\dot{\omega}_1 = T_{m1} - E_1 I_{q1} - D_1\omega_1$$

$$\dot{\delta}_1 = \omega_1$$

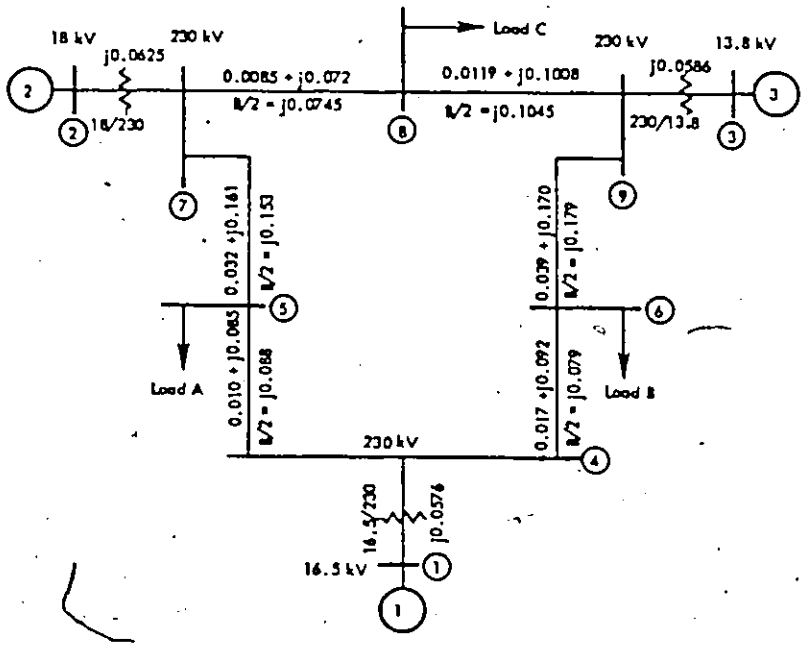


Figure 6.5 Nine-bus system impedance diagram;all imedances in pu on a 100-MVA base *

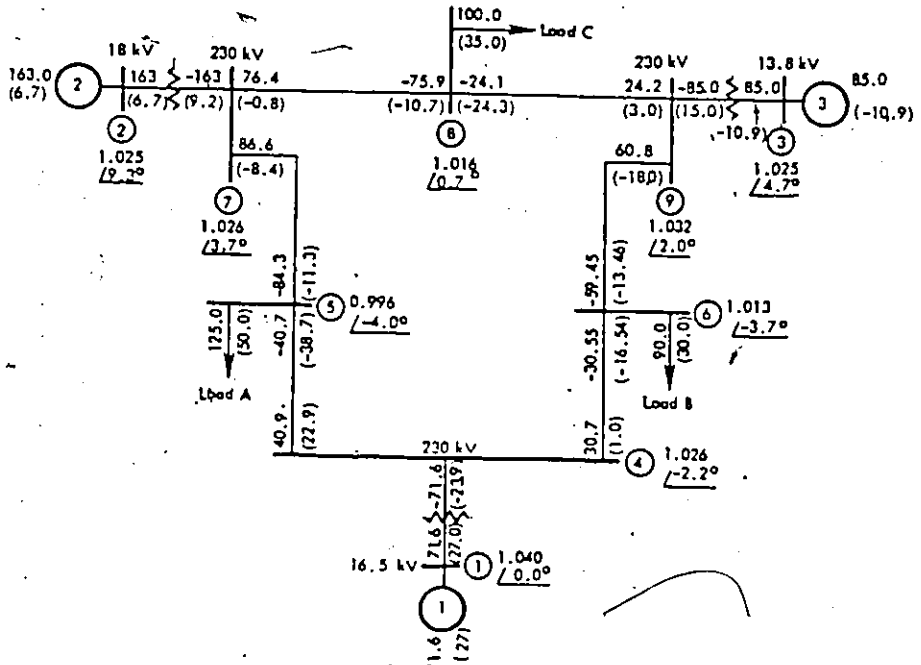


Figure 6.6 Nine-bus system load-flow diagram showing prefault conditions;all flows are in MW and MVAR *.

* Reference [58] .

- Generators 2 and 3 (two-axis model)

$$\tau'_{qoi}\dot{E}'_{di} = -E'_{di} - (x_{qi} - x'_i) I_{qi}$$

$$\tau'_{qoi}\dot{E}'_{qi} = -E_{fdi} - E'_{qi} + (x_{di} - x'_i) I_{di}$$

$$2M_i\omega_0\dot{\omega}_i = T_{mi} - D_i\omega_i - I_{doi}E'_{di} - I_{qoi}E'_{qi} - E'_{doi}I_{di} - E'_{qoi}I_{qi}$$

$$\dot{\delta}_i = \omega_i \quad , \quad i = 1,2$$

To obtain an independent set of equations, we define δ_r as

$$\frac{M_2\delta_2 + M_3\delta_3}{M_2 + M_3} - \delta_1$$

and δ_{23} as $\delta_2 - \delta_3$, where δ_r describes the motion of the centre of inertia of machines 2 and 3 with respect to machine 1, while δ_{23} describes the motion of machines 2 and 3 relative to each other.

Thus the system under study comprises nine first-order differentail equations and is represented by the following state variables

E'_{q2}, E'_{d2}, E'_{q3}, E'_{d3}, ω_1, ω_r, ω_{23}, δ_r and δ_{23} (generator 1 is the reference generator).

This sytem is subjected to the following forcing variables E_{fd2}, E_{fd3}, T_{m1}, T_{m2} and T_{m3}.

Table 6.5 Generator Data [58]

Generator	1	2	3
Rated MVA	247.5	192	128
KV	16.5	18.0	13.8
Power factor	1.0	0.85	0.85
Type	Hydro	Steam	Steam
Speed	180 rpm	3600 rpm	3600 rpm
x_d	0.146	0.8958	1.3125
x'_d	0.0608	0.1198	0.1813
x_q	0.0969	0.8645	1.2578
x'_q	0.0969	0.1969	0.25
x_ℓ	0.0336	0.0521	0.0742
τ'_{do}	8.96	6.00	5.89
τ'_{qo}	0.0	0.535	0.6
E'_{qo}	-1.0558	0.7882	0.7679
E_{do}	0.0419	0.6940	-0.6668
I_{qo}	-0.678	0.932	0.6194
I_{do}	0.2872	-1.2902	-0.5615
M	23.64	6.4	3.01
D	1	1	1

Note: Reactance values are in pu on a 100-MVA base.
All time constants are in sec.
The inertia constants are in sec.

As in the case of the single machine infinite bus system we calculate the eigenvalues of the original system and their modal cost indices as well as the contribution of their step responses to the step responses of the state variables (the reader is referred to Table (6.2). These results are given in Tables (6.6) and (6.7), respectively. Examination of Table (6.6) indicates that two pairs of complex eigenvalues ($-0.000198 \pm j0.000129$ and $-0.000622 \pm j0.02298$) and one real eigenvalue (-0.00045) have the largest modal cost indices. Consequently, these eigenvalues will be retained in the reduced order model. Examination of the third column in Table (6.7) indicates that the state variables (ω_1, ω_r, δ_r, E'_{q2} and E'_{q3} are influenced heavily by the dominant eigenvalues. Thus, we select these state variables to represent the reduced order model.

Here the results have been obtained in a purely formal and routine way, but they are consistent with the a priori information about the candidates for the most significant state variables. These candidates are ω_1, ω_r, δ_r, E'_{q2} and E'_{q3}. A physical interpretation of this is that the motion of the two smaller machines together against the largest machine (large inertia constant), which is analogous to the motion of the single machine against an infinite bus is slower than the motion of the two smaller machines relative to each other. Also, the time constants associated with the quadrature axis component of the voltages behind the transient reactance are larger than those associated with the direct axis components.

Table (6.6) Eigenvalues of the original system and the respective modal cost measures

Eigenvalues	Dominance Indices
-0.000198 ± j0.000124	0.8133
-0.000622 ± j0.022484	0.0993
-0.00045	0.0408
-0.00266± j0.03464	0.00748
-0.01662	0.0000012
-0.010373	0.0000004

Table 6.7 Dependence of the State Variables Upon the Eigenvalues

States	Eigenvalues					
	-0.000198 ± j0.000129	-0.000622 ± j0.0229	-0.00045	-0.00266 ± j0.0346	-0.01664	-0.010373
ω_1	51.1	92.4	94.1	98.9	99.2	100.0
ω_r	17.7	89.8	90.2	94.6	100.0	100.0
δ_r	18.5	88.1	93.9	99.9	100.0	100.0
E'_{q2}	73.1	76.2	92.1	93.9	100.0	100.0
E'_{d2}	46.2	55.2	59.6	70.4	100.0	100.0
E'_{q3}	78.3	89.4	95.9	96.2	97.3	100.0
E'_{d3}	42.1	58.3	65.7	67.2	79.1	100.0
ω_{23}	29.3	50.4	59.1	89.3	95.1	100.0
δ_{23}	25.6	41.5	56.7	86.6	95.4	100.0

Applying the proposed reduction technique, we obtain the reduced order model matrices as,

$$F = \begin{array}{c} \begin{matrix} \omega_1 & \omega_r & \delta_r & E'_{q2} & E'_{q3} \end{matrix} \\ \begin{bmatrix} -0.5159 & 3.996 & 1.5868 & 0.7951 & 0.3429 \\ -2.092 & -5.5349 & -5.278 & -4.716 & -2.397 \\ 0.0 & 10000. & 0.0 & 0.0 & 0.0 \\ -1.346 & 15.432 & -7.156 & -9.727 & -2.648 \\ 2.856 & -6.6908 & -2.898 & -1.493 & -5.368 \end{bmatrix} \end{array} \times 10^{-4}$$

$$G = \begin{array}{c} \begin{matrix} T_{m1} & T_{m2} & T_{m3} & E_{fD2} & E_{fD3} \end{matrix} \\ \begin{bmatrix} 0.561 & 0.0 & 0.0 & 0.0 & 0.0 \\ -0.561 & 1.409 & 1.409 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 4.421 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 4.503 \end{bmatrix} \end{array} \times 10^{-4}$$

By setting the values of the derivatives of the state variables ξ in equation (6.8) equal to zero, we obtain the same F but different G matrices, which is:

$$G = \begin{bmatrix} 0.5781 & -0.067 & 0.005 & 0.0015 & -0.0036 \\ -0.5579 & 1.538 & 1.111 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -0.051 & -0.8818 & 2.279 & 4.399 & 0.0326 \\ 0.1969 & 1.309 & -4.368 & 0.0537 & 4.413 \end{bmatrix} \times 10^{-4}$$

Figures (6.7) through (6.12) are simulations comparing the transient response of the original system to those of the first ($\xi = 0$) and the second ($\xi = 0$) reduced order models. The disturbances used in the simulation are 10% step changes in the mechanical torque and field voltage at the third generator. All the responses indicate that the second reduced order model ($\xi = 0$) approximate the original system better than the first one ($\xi = 0$) does. Although the above simulations indicate that the second reduced-order model is probably the more accurate method of reducing the original system, it may lead to a reduced system structure that cannot be interpreted easily in terms of the physical system as the first reduced model does. This is clearly indicated by examining the entries of the G matrices for the reduced models. For example, in the case of the single machine connected to an infinite bus we notice that in the second reduced

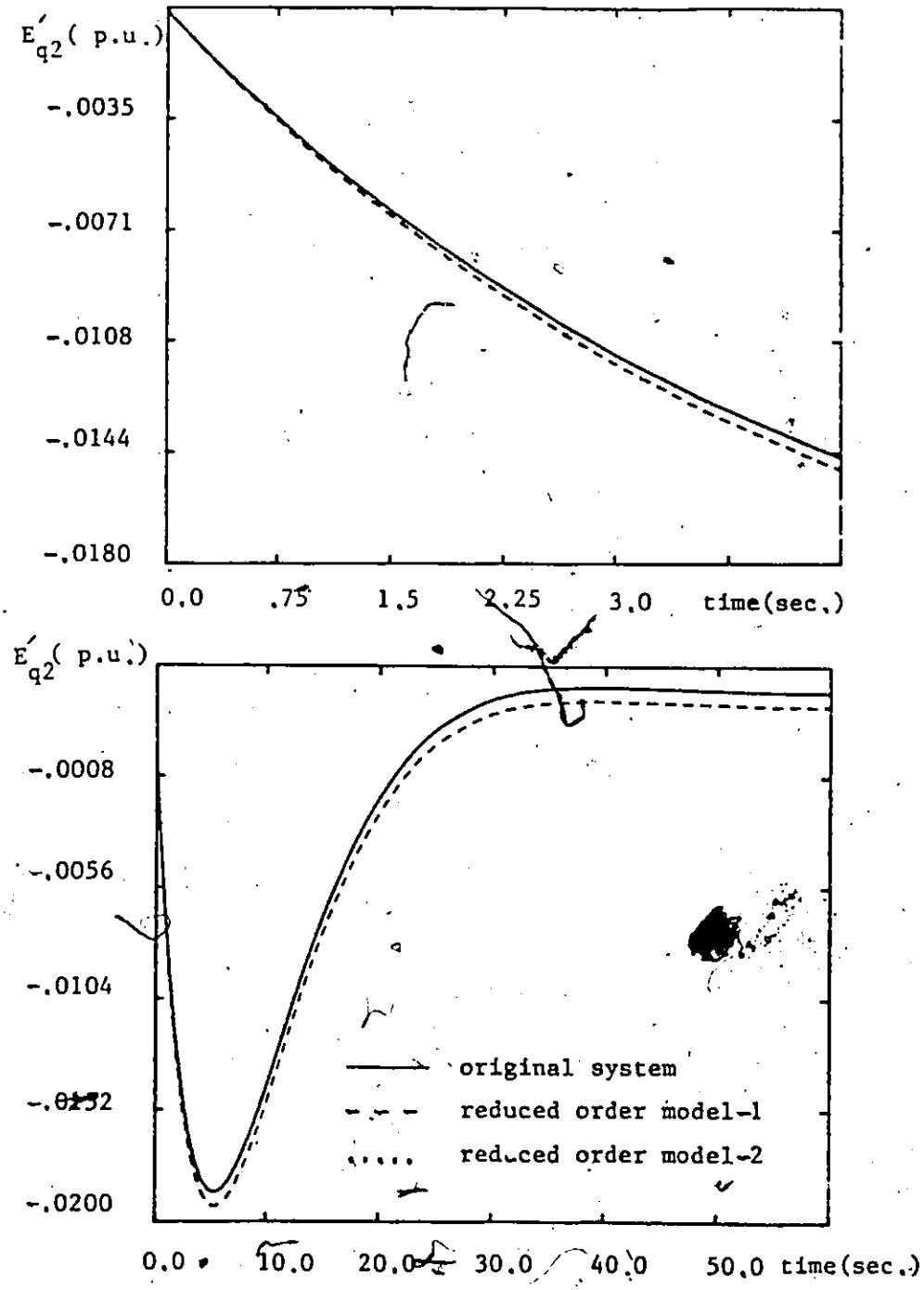


Figure 6.7 E'_{q2} response following a 10% step change in E_{fd3}.

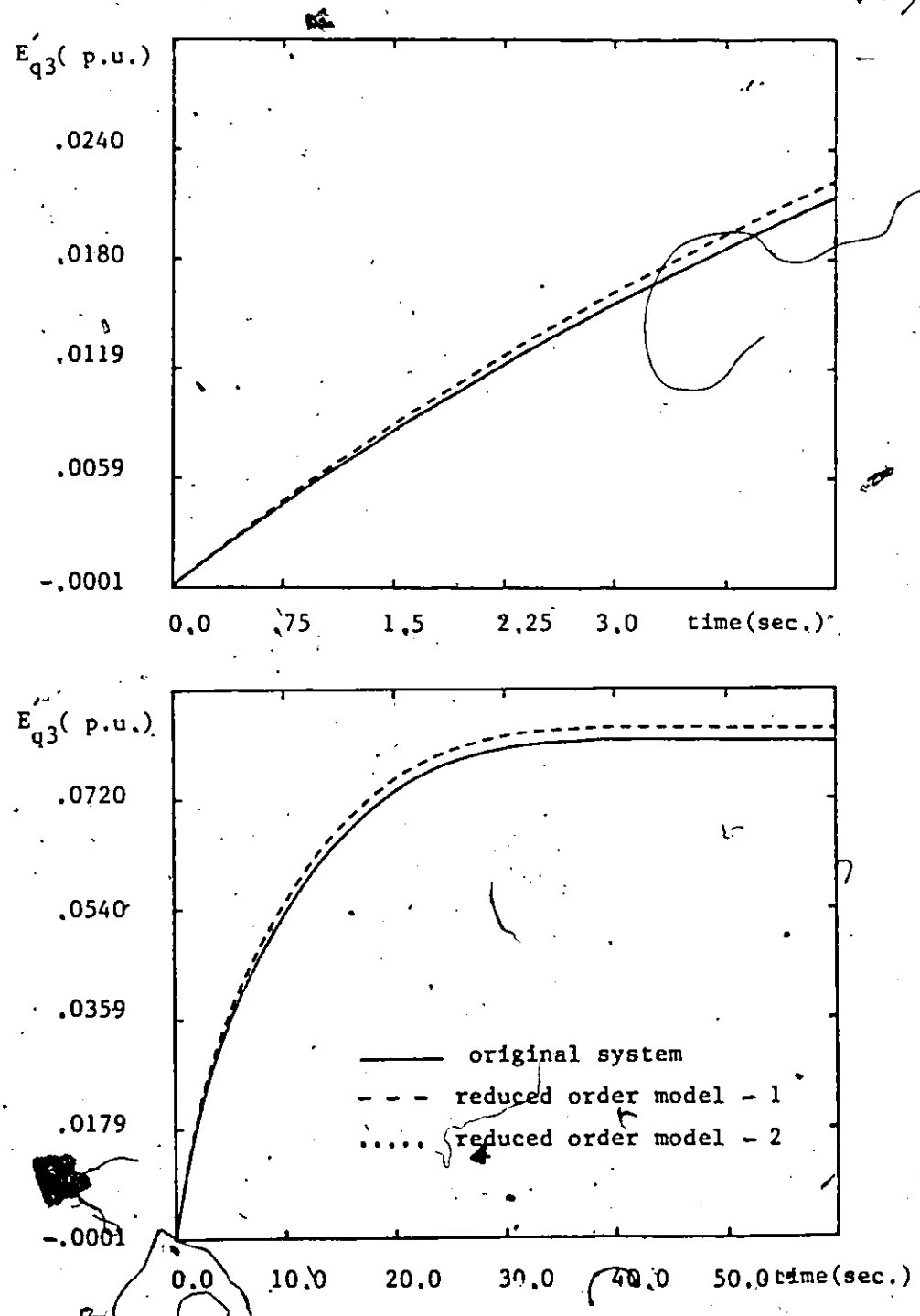


Figure 6.8 E'_{q3} response following a 10% step change in E_{fd3}.

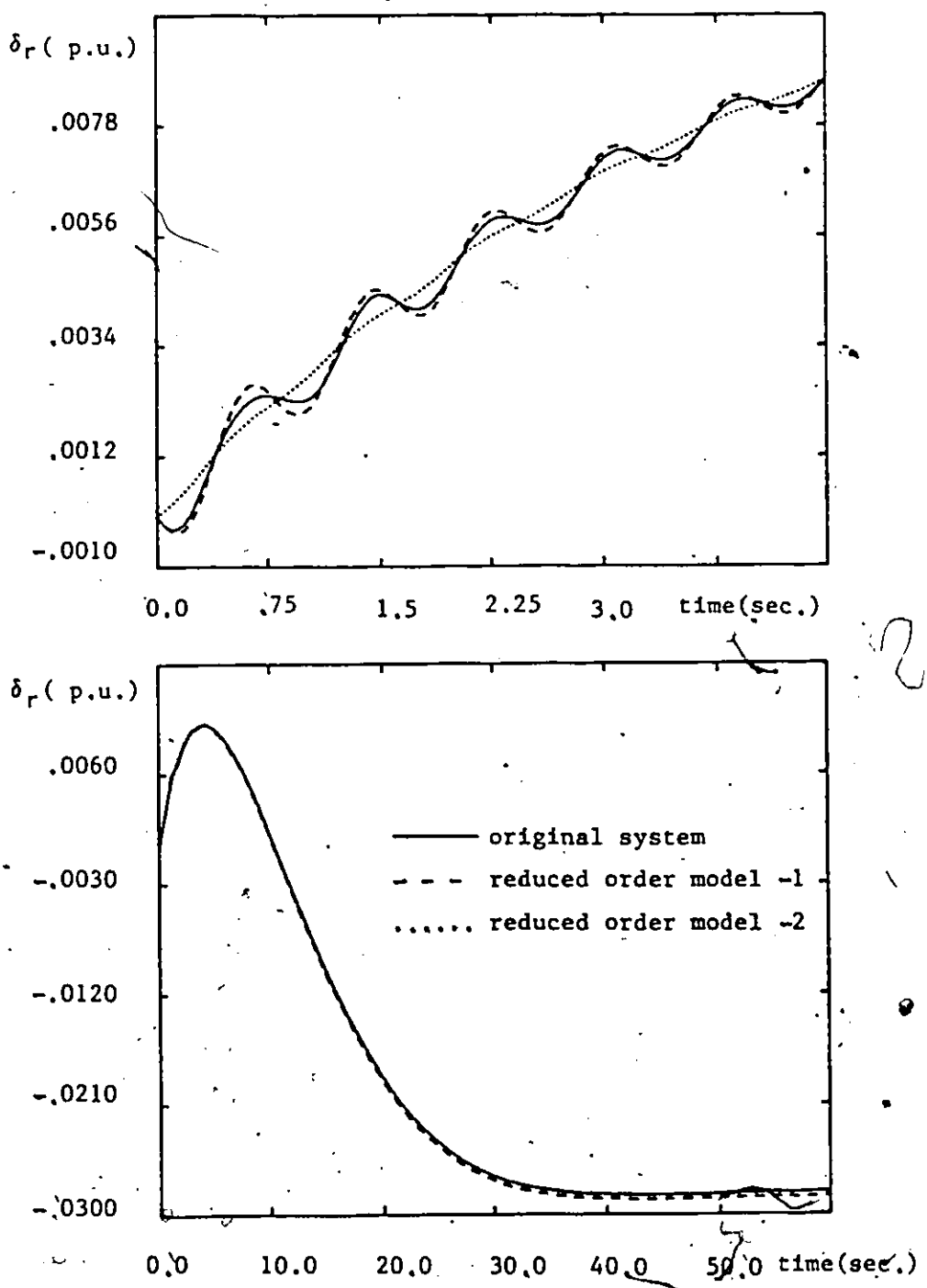


Figure 6.9 δ_r (p u.) response following a 10% step change in E_{fd3}.

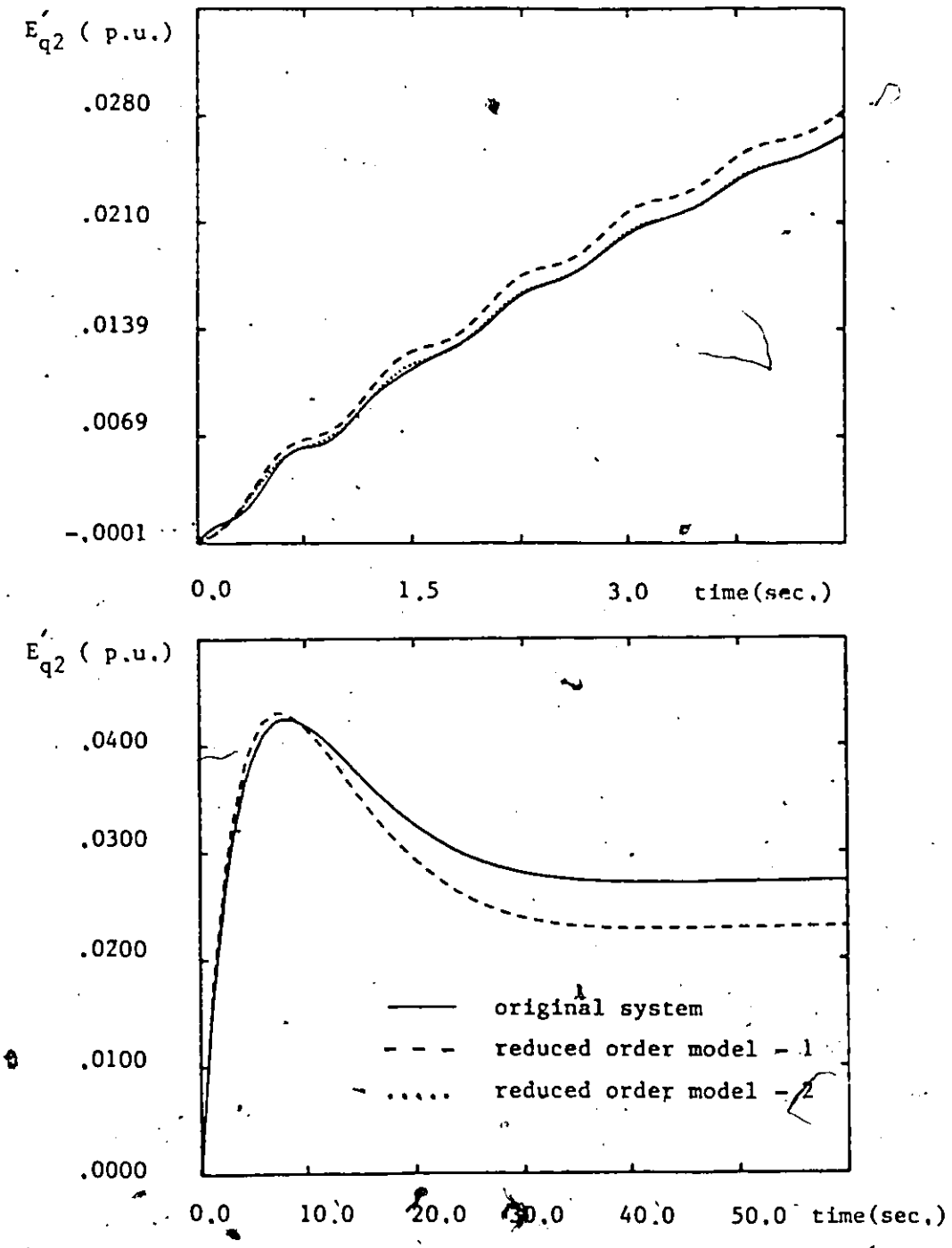


Figure 6.10 E'_{q2} response following a 10% step change in T_{m3}

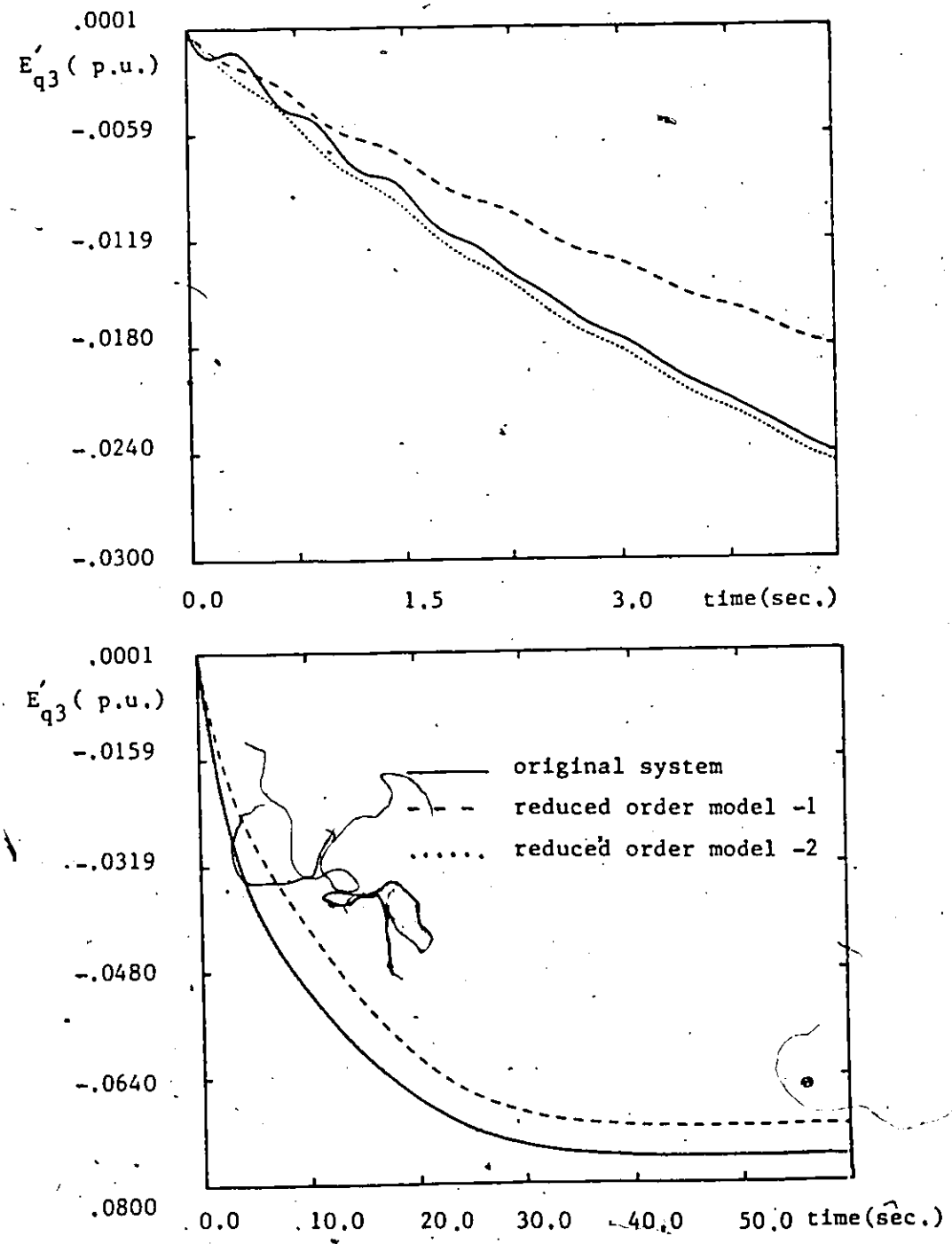
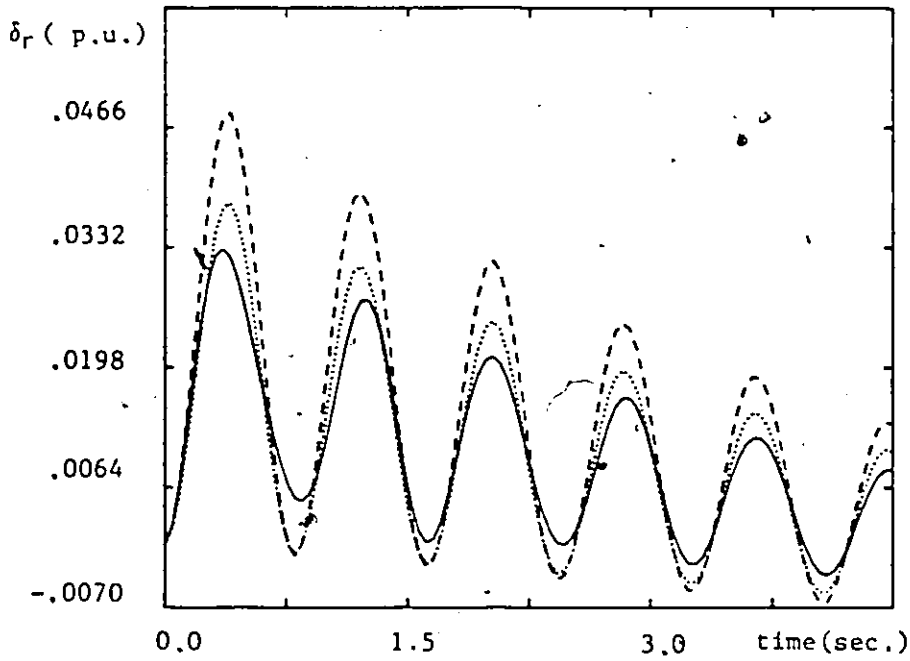


Figure 6.11 E'_{q3} response following a 10% step change in T_{m3}.



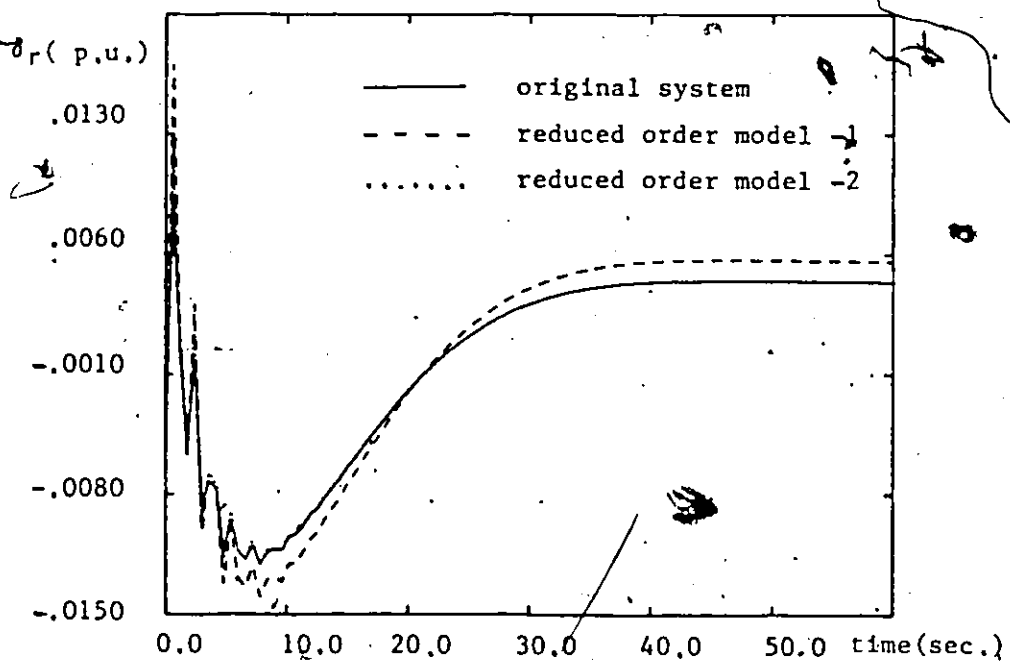


Figure 6.12 δ_r (p.u.) response following a 10% step in T_{m3}.

model the field current has an input from the mechanical power, and the rotor speed has an input from the field voltage.

In conclusion, we may say that the proposed model reduction technique has provided us with two different reduced order models. Both models have indicated satisfactory transient responses compared with those of the original system when they are excited by the same disturbance. Although the first reduced order model ($\xi = 0$) has less accurate transient response than the second one ($\xi = 0$), it preserves the physical structure of the system (the interaction between the state variables and the forcing variables). The choice between these two reduced models depends upon the purposes of the study (simulation or controller design).

6.3 An Algorithm for Identifying Coherent Generators of Multimachine Systems

6.3.1 Concepts of Cost Decomposition

Let a linear time invariant dynamic system having n components with state variables $x_i \in R^{ni}$ be described by

$$\dot{\underline{x}}_i = \sum_{j=1}^{n} A_{ij}\,\underline{x}_j + B_i\,\underline{u}, \qquad \underline{y} = \sum_{j=1}^{n} C_j\,\underline{x}_j \tag{6.10}$$

where $y \in R^{\ell}$, $u \in R^{m}$ and $u(t)$ can be modeled as zero mean white noise with unit intensity I.

The total system has an associated norm of y which is called the system value, or "cost function", V defined by

$$V = \lim_{t\to\infty} E||y||^2, \qquad ||y||^2 = y^T y \tag{6.11}$$

Component cost analysis consists of the decomposition of V into the sum of contributions V_i associated with each component state x_i, where the V_i satisfy the cost decomposition property

$$V = \sum_{i=1}^{n} V_i \tag{6.12}$$

In order to compute the expected value of the norm of y in equation (6.11) the random excitation of equation (6.10) is described. The random excitation is used because, as shown in [34], the optimum disturbance for detecting coherent groups that depend on power system structure is not deterministic.

6.3.2 Disturbance Models

The task in this subsection is to define the form of the probabilistic input $\underline{u}(t)$. This $\underline{u}(t)$ will then be used to derive the linear power system model of Chapter 4, "equation (4.10)", for determining coherent groups.

First, decompose $\underline{u}(t)$ into two functions $\underline{u}_1(t)$ and $\underline{u}_2(t)$, i.e., $\underline{u}(t) = \underline{u}_1(t) + \underline{u}_2(t)$.

The function $\underline{u}_1(t)$ is defined as

$$\underline{u}_1(t) = \begin{cases} \underline{u}_1 & \text{for } t > 0 \\ \underline{0} & \text{for } t < 0 \end{cases} \tag{6.13}$$

That is, $\underline{u}_1(t)$ is a vector step function, initiated at time $t = 0$.

In the linearlized model, $\underline{u}(t)$ represents deviations in mechanical power to the generators and deviations in electric power at load buses.

Since $\underline{u}_1(t)$ is a step function, non zero entries in $\underline{u}_1(t)$ will model

(1) Loss of generation.

(2) Loss of load due to load schedding.

(3) Line switching.

If $\underline{u}_1(t)$ is to represent the random occurrence of such events, then

$$E\{\underline{u}_1(t)\} = \begin{bmatrix} \underline{m}_{11} \\ \underline{m}_{12} \end{bmatrix} = \underline{m}_1 \tag{6.14}$$

and

$$E\{[\underline{u}_1(t) - \underline{m}_1][\underline{u}_1(t) - \underline{m}_1]^T\} = \left[\begin{array}{c:c} R_{11} & 0 \\ \hdashline 0 & R_{12} \end{array}\right] = [R_1] \tag{6.15}$$

where $\underline{m}_{11}$ and $[R_{11}]$ describe the uncertainty in the location and magnitude of generation changes $\Delta\underline{P}_m$. Also, $\underline{m}_{12}$ and $[R_{22}]$ describes the uncertainty int he location and magnitude of power injections at buses due to either loads being shed or lines being switched.

To model a fault, define

$$\underline{u}_2(t) = \begin{cases} 0 & t > T_1 \\ \underline{u}_2(t), & 0 \leq t \leq T_1 \\ 0 & t < 0 \end{cases} \tag{6.16}$$

That is $\underline{u}_2(t)$ represents a pulse of duration T_1, occurring at time $t = 0$. The faults are represented by changing the mechanical power to a genrator. Thus

$$\underline{u}_2(t) = \begin{bmatrix} \Delta\underline{P}_m \\ \hline \underline{0} \end{bmatrix} \tag{6.18}$$

If $\underline{u}_2(t)$ is to be probabilistic, then define

$$E\{\underline{u}_2(t)\} = \begin{bmatrix} m_{21} \\ \hline \underline{0} \end{bmatrix} = m_2$$

and

$$E\{[\underline{u}_2(t) - \underline{m}_2][\underline{u}_2(t) - \underline{m}_2]^T\} = \left[\begin{array}{c|c} R_{21} & 0 \\ \hline 0 & R_{12} \end{array}\right] = [R_2] \tag{6.19}$$

The initial conditions are also assumed to be random

$$E\{\underline{x}(0)\} = 0$$
$$E\{\underline{x}(o)\,\underline{x}^T(0)\} = P_x(0) \tag{6.20}$$

The expected deviation from any operating state is zero but the variance of such deviation is non zero.

This assumption reflects the idea that for a given steady-state operating point, the power system is expected, on the average, to be at the operating point, although instantaneously it may be subject to transient fluctuations.

The initial conditions are assumed to be uncorrelated with $\underline{u}_1(t)$ and $\underline{u}_2(t)$, i.e.,

$$E\{\underline{x}(0)\,\underline{u}_1^T(t)\} = 0$$
$$E\{\underline{x}(0)\,\underline{u}_2^T(t)\} = 0 \tag{6.21}$$

Finally, it is assumed that $\underline{u}_1(t)$ and $\underline{u}_2(t)$ are uncorrelated with respect to one another. This assumption is based on the fact that the model is only used to represent one type of contingency at a time.

6.3.3 Linear Differential Systems Driven by White Noise

In this section we obtain some of the statistical properties of the state differential system with a white noise process as input. In particular, we compute the mean, and the covariance matrix of the state x.

Suppose that x(t) is the solution of

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t), \qquad \underline{x}(0) = \underline{x}_o \tag{6.22}$$

where u(t) is white noise with intensity R and x_o is a stochastic variable independent of u(t), with mean m_o and

$$P_o = E\{ (x_o - m_o)(x_o - m_o)^T \}$$

as its covariance matrix. Also, consider without loss of generality that A, B and R are constant matrices (time-invariant case). Then x(t) has mean

$$m_x(t) = \exp(A(t-t_o)) \cdot m_o \tag{6.23}$$

and the covariance matrix of x(t) is the solution of the following matrix differential equation

$$\dot{P}_x(t) = A\,P_x(t) + P_x(t)\,A^T + B\,R\,B^T \tag{6.24}$$

Also, the covariance matrix can be written as

$$P_x(t) = \exp[A(t-t_0)]\, P_o \exp[A^T(t-t_o)] + \int_o^t \exp[A(t-\tau)]\, B\, R\, B^T \exp[A^T(t-\tau)]\, d\tau \tag{6.25}$$

It is not difficult to see that if, and only if, A is asymptotically stable, $P_x(t)$ has the following limit for arbitrary P_o:

$$\lim_{t\to\infty} P_x(t) = P_x = \int_o^\infty \exp(A\tau)\, B\, R\, B^T \exp[A^T\tau)\, d\tau \tag{6.26}$$

where P_x is the solution of

$$AP + PA^T + B\, R\, B^T = 0 \tag{6.26}$$

If we define the matrix of the expected value of the energy in the process as

$$S(t) = \frac{1}{T} \int_o^T P_x(t)\, dt \tag{6.28}$$

The criterion for identifying coherency suggested in this algorithm is the "component cost", V_i, which is defined as

$$V_i = \text{trace}\,[S(t)\, C^T C] \tag{6.29}$$

where C is given by

$$\underline{y} = C\underline{x} \tag{6.30}$$

and y are the process outputs.

Let the components λ_i be arranged in order of their component costs

$$\underbrace{V_1 \geq V_2 \geq \cdots\cdots \geq V_r}_{\text{Retain (non-coherent)}} > \underbrace{V_{r+1} \geq \cdots\cdots \geq V_{N-1}}_{\text{Truncate (coherent)}} \tag{6.31}$$

6.3.4 A Reduction Algorithm for Obtaining Dynamic Equivalents

The overall procedure for obtaining coherency-based dynamic equivalents can be divided into four basic steps:

(i) A study system and external system are defined.

(ii) Identification of groups of coherent generators which are valid for faults in the study system.

(iii) Reduction of generator buses.

(iv) Dynamic aggregation of generating unit models.

In the following subsections, steps (ii) through (iv) of the procedure are described.

6.3.4.1 Identification of Coherent Generators

The determination of the coherent groups is done using the linearized model of the power system described in Chapter 4.

Rewriting equation (4.10)

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \qquad (6.32)$$

where

$$\underline{x} = \begin{bmatrix} \Delta\underline{\hat{\delta}} \\ \Delta\underline{\hat{w}} \end{bmatrix}, \qquad \underline{u} = \begin{bmatrix} \Delta\underline{P}_m \\ \Delta\underline{P}_\ell \end{bmatrix}$$

$$A = \left[\begin{array}{c|c} 0 & 2\pi f_o I_{N-1,N-1} \\ \hline -\hat{M}T & -\beta I_{N-1,N-1} \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{c|c} 0 & 0 \\ \hline \hat{M}^{-1} & \hat{M}L \end{array}\right]$$

Note, we choose the reference angle as the torque angle of one of the machines in the external system. As the statistic of the disturbances mentioned in subsection (6.3.2) is similar to the statistics of a random constant model, generated as the output of an integrator with no input, but with an initial condition with specified ~~mean~~ m and variance R. Thus the covariance matrix of equation (6.22) is the solution of the following differential equation

$$\begin{aligned} \dot{P}_x(t) &= A\,P_x(t) + P_x(t)\,A^T + A^{-1}\,B\,Q\,B^T + B\,Q\,B^T\,A^{-T} \\ &\quad - A^{-1}\exp\,[A(t-t_o)]\,B\,Q\,B^T - B\,Q\,B^T\exp[A^T(t-t_o)]\,A^{-T} \end{aligned} \qquad (6.33)$$

where $Q = R + mm^T$

As $t \to \infty$ P_x will be the unique solution of the following matrix equation

$$AP_x + P_x A^T + A^{-1} B Q B^T + B Q B^T A^{-T} = 0 \qquad (6.34)$$

Rewriting the matrix expression $A^{-1} B Q B^T + B Q B^T A^{-T}$ in terms of the statistics of the disturbance Q and the power system structure (A and B), we get

$$A^{-1} B Q B^T + B Q B^T A^{-T} = \begin{bmatrix} 0 & Z_{12} \\ \hline Z_{12}^T & 0 \end{bmatrix} = Z \qquad (6.35)$$

where the matrix Z_{12} is equal to

$$Z_{12} = -(\hat{M}T)^{-1} [\hat{M} (R_{11} + m_{11}m_{11}^T) \hat{M}^T + (\hat{M}L) m_{12}m_{11}^T \hat{M}^T$$

$$+ \hat{M} m_{11}m_{12}^T (\hat{M}L)^T + (\hat{M}L) (R_{12} + m_{12}m_{12}^T) (\hat{M}L)^T]$$

The simplification in the structure of the matrix was possible by selecting the states in a certain order.

Now, rewriting P_x in equation (6.34) as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ \hline P_{12}^T & P_{22} \end{bmatrix} \qquad (6.36)$$

Then substituting A and Z matrices in equation (6.34) we obtain the following four algebraic equations:

$$P_{12}^T + P_{12} = 0 \qquad (6.37\text{-a})$$

$$2\pi f_o P_{22} - P_{11}(\hat{M}T)^T - \beta P_{12} = Z_{12} \qquad (6.36\text{-b})$$

$$2\pi f_o P_{22} - (\hat{M}T)\, P_{11} - \beta P_{12}^T = Z_{12} \qquad (6.37\text{-c})$$

$$-(\hat{M}T)\, P_{12} - 2\beta P_{22} - P_{12}^T\, (\hat{M}T)^T = 0 \qquad (6.37\text{-d})$$

Note that equation (6.37-a) has infinite number of solutions and equation (6.37-b) is the transpose of equation (6.37-c) setting the entries of the matrix P_{12} equal to zero as a solution of (6.37-a) and substituting in equation (6.37-d), we obtain a solution for the matrix P_{22} as $P_{22} = [0]$. Then substituting the matrix P_{12} and P_{22} in equation (6.37-b) or (6.37-c), we get

$$P_{11} = Z_{12}\, (\hat{M}T)^{-T} \qquad (6.38)$$

From equation (6.29), we obtain the cost value of the states

$$\hat{\delta}_1,\ \hat{\delta}_2,\ \ldots\ldots\ \hat{\delta}_{N-1},$$

which is simply equal to the corresponding diagonal element of the matrix P_{11}. According to the criterion given in equation (6.31) we

are able to identify generators that are coherent with the one chosen as a reference. Suppose that not all the generators in the external system are coherent. We repeat the whole procedure again with another reference machine chosen from those in the external system that are not coherent with the ones chosen before. We then identify another group of coherent machines. Once the coherent groups are determined the next step is the reduction of the coherent generator buses.

6.3.4.2 Reduction of Generator Buses

Let m machines swing together. These m coherent machines can be replced by an equivalent machine such that

(i) The power output of the equivalent machine is equal to the power output of all m machines.

(ii) The power received at each tie bus from the equivalent machine is the same as the total power received from those m machines.

(iii) The inertia of the equivalent machine is the sum of the inertias of the m machines.

If the subsystem of m machines contains n_t tie buses, the admittance matrix of that subsystem can be written as

$$[Y] = \begin{matrix} \uparrow \\ m \\ \downarrow \\ \uparrow \\ n_t \\ \downarrow \end{matrix} \begin{bmatrix} Y_{11} & \vdots & Y_{12} \\ - - - & \vdots & - - - \\ Y_{21} & \vdots & Y_{22} \end{bmatrix} \qquad (6.39)$$

which is obtained after eliminating all the buses other than the tie buses and the internal buses of th m generators. Now let

$$S_i = P_i + jQ_i \tag{6.40}$$

be the power output of the i^{th} machine at its terminal. Then the current output of the i^{th} machine at its internal bus will be

$$S_i = E_i I_i^* \tag{6.41}$$

where

$$E_i = V_i + I_i Z_i \tag{6.42}$$

The total power output of all these m machines is given by

$$S_T = \sum_{i=1}^{m} E_i I_i^* \tag{6.43}$$

This should be equal to the power output of the equivalent machine at its internal bus. That is

$$E_e I_T^* = \sum_{i=1}^{m} E_i I_i^* \tag{6.44}$$

where,

$$I_T = \sum_{i=1}^{m} I_i \tag{6.45}$$

is the current output of the equivalent machine. Hence, we have,

$$E_e = |E_e| \angle \delta_e$$
$$= \sum_{i=1}^{m} E_i I_i^* \Big/ \sum_{i=1}^{m} I_i^* \tag{6.46}$$

from which the internal voltage of the equivalent machine and the load angle are determined. The power equation at the k^{th} tie bus is

$$V_k I_k^* = V_k \left(\sum_{\ell=1}^{m} \hat{y}_{k\ell} E_\ell^* + \sum_{\ell=m+1}^{n_t} \hat{y}_{k\ell} V_\ell^* \right)$$
$$V_k I_k^* = V_k \left(\hat{y}_{ke} E_e^* + \sum_{\ell=m+1}^{n_t} \hat{y}_{k\ell} V_\ell^* \right) \tag{6.47}$$

for which we can write

$$\hat{y}_{ek} = \sum_{\ell=1}^{m} \hat{y}_{k\ell} E_\ell / E_e \tag{6.48}$$

Since the admittance matrix is to be symmetric we take

$$\hat{y}_{ek} = \hat{y}_{ke} \tag{6.49}$$

From the above equations, the power output equation of the equivalent machine is

$$E_e^* I_T = E_e^* (\hat{y}_{ee} E_e + \sum_{\ell=m+1}^{n_t} \hat{y}_{e\ell} V_\ell)/E_e \tag{6.50}$$

and hence

$$\hat{y}_{ee} = (\sum_{I=1}^{m} I_I - \sum_{\ell=m+1}^{m} \hat{y}_{e\ell} V_\ell)/E_e \tag{6.51}$$

Now the admittance matrix of the equivalent system is

$$[Y] = \begin{bmatrix} \hat{y}_{ee} & \hat{y}_{el} & \hat{y}_{en_t} \\ \hat{y}_{1e} & & \\ & Y_{22} & \\ \hat{y}_{n_t e} & & \end{bmatrix} \tag{6.52}$$

6.3.4.3 Dynamic Aggregation of Generating Unit Models

With the coherent groups determined and the generator buses reduced, the next step is the modeling of the equivalent generators that represent each coherent group.

The criteria for an acceptable equivalent model of generating units, from the dynamic viewpoint, is that its electric power output

response matches the total electric power output of the units it replaces, and that the voltage response at its terminal bus matches the voltage response of all the individual units models.

The aggregation method relies upon the consideration that the units to be aggregated, being attached to the same bus, have the same terminal voltage, and the assumption that these units, being coherent, have the same speed. Thus, every excitation system among a group of coherent units measures the same input voltage signal. Also every governor-turbine system among a group of coherent units measures the same input speed signal. In the present research only a very simple classical model of the generator is used. This involves only the rotor dynamics of the generator. The rotor dynamics for the equivalent generator are easily derived, and are shown below.

The basic differential equation representing the rotor dynamics used is:

$$2M_i \frac{d\Delta\omega_i}{dt} = \Delta P_{mi} - \Delta P_{gi} - D_i \Delta\omega_i \qquad (6.53)$$

with
- $\Delta\omega$ p.u. speed deviation from synchronous speed
- M inertia constant in MWS/MVA
- P_m mechanical power in pu
- P_g electromagnetic power in pu
- D damping constant in pu
- i machine subscript

Because of the coherency assumption, all the machines of a group have the same speed deviation. Thus, summing over the machine equations (6.60) of the group results in

$$2\ \frac{d\Delta\omega}{dt}\sum_i M_i = \sum_i \Delta P_{mi} - \sum_i \Delta P_{gi} - \Delta\omega \sum_i D_i \qquad (6.54)$$

Thus for the equivalent machine:

(a) The inertia constant is the sum of the inertia constants of the machines of the coherent group.

(b) The damping factor is the sum of the damping factors of the machines of the coherent group.

(c) The mechanical power is the sum of the mechanical powers of the machines of the coherent group.

(d) The electrical power is the sum of the electrical powers of the machines of the coherent group.

Results (c) and (d) are in agreement with the power conservation assumption already made for the coherent bus reduction. The details of the other components (exciter and governor-turbine) are not included here because our purpose is mainly to identify the coherent generators. The procedure discussed in this section is illustrated by considering the 39 bus network of the New England area.

6.3.5 Testing the Reduction Algorithm on the 39 Bus New England System

To test the algorithm formalized in the previous subsection, simulations were made using a 39 bus model of the New England system.

A schematic of this system is shown in Figure (6.13). The transmission line data, bus data and generator data are given in Tables (6.8), (6.9) and (6.10), respectively. The large system is divided into an internal system containing generators 1, 8, 9 and 10 and an external system containing the other generators. Generator 10 is used throughout as the reference.

As a first step to test the reduction algorithm, a zero mean, independent, inertially weighted (ZMIIW) disturbance was used to identify the coherent generators.

$$R_{11} = \text{diag}\{M_1^2, M_8^2, M_9^2, M_{10}^2, 0, 0, 0, 0, 0, 0\}, \quad R_{12} = [0]$$

$$m_{11} = [0], \quad m_{12} = [0]$$

Table (6.11) illustrates the coherency measures for ZMIIW disturbance of these four generators. A careful look at Table (6.11) recommends the following aggregation levels:

Aggregation Levels	Generators to be Aggregated
1	4-7
2	4-7 and 2-3
3	4-6-7 and 2-3
4	4-6-7-2-3
5	4-6-7-2-3-5

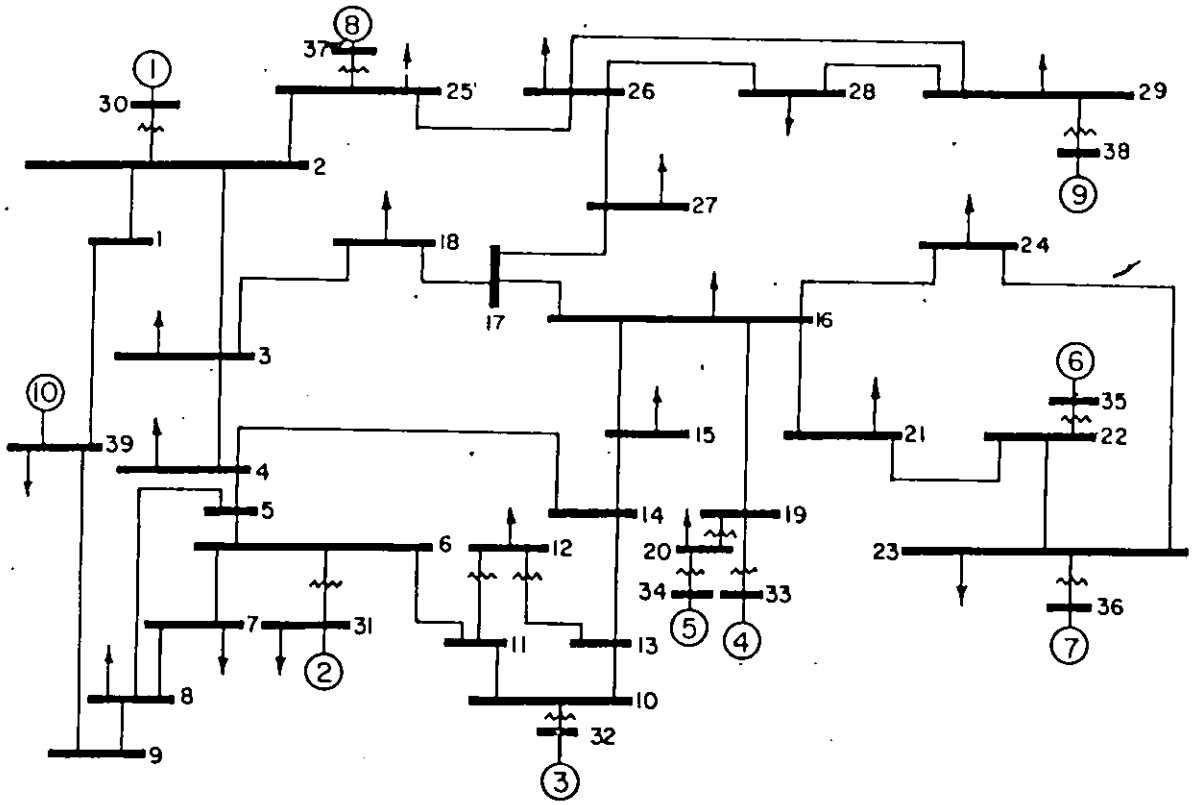


Figure 6.13 The standard New England test system*.

* Reference [29] .

Table 6.8 New England Test System Transmission Line Data in pu on 100 MVA Base

Lines		Characteristics				
From	To	R	X	wC[1]	TAP	RATIO
1	2	0.0035	0.0411	0.6987		0.0
1	39	0.0010	0.0250	0.7500		0.0
2	3	0.0013	0.0151	0.2572		0.0
2	30	0.0070	0.0086	0.1460		0.0
3	4	0.0013	0.0213	0.2214		0.0
3	18	0.0011	0.0133	0.2138		0.0
4	5	0.0008	0.0128	0.1342		0.0
4	14	0.0008	0.0129	0.1382		0.0
5	6	0.0002	0.0025	0.0434		0.0
5	8	0.0008	0.0112	0.1476		0.0
6	7	0.0006	0.0092	0.1130		0.0
6	11	0.0007	0.0082	0.1389		0.0
7	8	0.0004	0.0046	0.0780		0.0
8	9	0.0023	0.0363	0.3804		0.0
9	39	0.0010	0.0250	1.2000		0.0
10	11	0.0004	0.0043	0.07290		0.0
10	13	0.0004	0.0043	0.0729		0.0
13	14	0.0009	0.0101	0.1723		0.0
14	15	0.0018	0.0217	0.3660		0.0
15	16	0.0009	0.0094	0.1710		0.0
16	17	0.0007	0.0089	0.1342		0.0
16	19	0.0016	0.0195	0.3040		0.0
16	21	0.0008	0.0135	0.2548		0.0
16	24	0.0003	0.0059	0.0680		0.0
17	18	0.0007	0.0082	0.1319		0.0
17	27	0.0013	0.0173	0.3216		0.0
21	22	0.0008	0.0140	0.2565		0.0
22	23	0.0006	0.0096	0.1846		0.0
23	24	0.0022	0.0350	0.3610		0.0
25	26	0.0032	0.0323	0.513		0.0
26	27	0.0014	0.0147	0.2396		0.0
26	28	0.0043	0.0474	0.7802		0.0
26	29	0.0057	0.0625	1.0290		0.0
28	29	0.0014	0.0151	0.2490		0.0
12	11	0.0016	0.0435	0.0		1.006
12	13	0.0016	0.0435	0.0		1.006
6	31	0.0000	0.0250	0.0		1.070
10	32	0.0000	0.0200	0.0		1.070
19	33	0.0007	0.0142	0.0		1.070
29	34	0.0009	0.0180	0.0		1.009
22	35	0.0000	0.0143	0.0		1.025
23	36	0.0005	0.0272	0.0		1.00
25	37	0.0006	0.0232	0.0		1.025
2	30	0.0000	0.0181	0.0		1.025
29	38	0.0008	0.0156	0.0		1.025
19	20	0.0007	0.0138	0.0		1.060

Table 6.9 New England Test System Bus Data

Bus	Volts	Angle	Load (MW)	Load (MVAR)	Gen (MW)	Gen (MVAR)
1	1.048	-9.574	0.0	0.0	0.0	0.0
2	1.049	-7.015	0.0	0.0	0.0	0.0
3	1.030	-9.862	322.00	2.4	0.0	0.0
4	1.004	-10.658	500.00	184.0	0.0	0.0
5	1.005	-9.471	0.0	0.0	0.0	0.0
6	1.007	-8.769	0.0	0.0	0.0	0.0
7	0.997	-10.973	233.80	84.0	0.0	0.0
8	0.996	-11.479	522.00	176.60	0.0	0.0
9	1.028	-11.303	0.0	0.0	0.0	0.0
10	1.017	-6.384	0.0	0.0	0.0	0.0
11	1.012	-7.198	0.0	0.0	0.0	0.0
12	1.000	-7.213	8.50	88.00	0.0	0.0
13	1.014	-7.099	0.0	0.0	0.0	0.0
14	1.012	-8.768	0.0	0.0	0.0	0.0
15	1.016	-9.186	320.00	153.0	0.0	0.0
16	1.032	-8.783	329.40	32.30	0.0	0.0
17	1.034	-8.780	0.0	0.0	0.0	0.0
18	1.031	-9.621	158.88	30.00	0.0	0.0
19	1.050	-3.158	0.0	0.0	0.0	0.0
20	0.991	-4.570	680.00	103.00	0.0	0.0
21	1.032	-5.377	274.00	115.00	0.0	0.0
22	1.050	-0.930	0.0	0.0	0.0	0.0
23	1.045	-1.128	247.50	84.60	0.0	0.0
24	1.038	-7.663	308.60	-92.20	0.0	0.0
25	1.058	-5.653	224.00	47.20	0.0	0.0
26	1.052	-6.910	139.00	17.00	0.0	0.0
27	1.038	-8.922	281.00	75.50	0.0	0.0
28	1.050	-3.399	206.00	27.60	0.0	0.0
29	1.050	-0.639	283.50	26.90	0.0	0.0
30	0.048	-4.596	0.0	0.0	250.0	145.10
31	0.982	0.0	9.20	4.60	572.9	207.3
32	1.983	1.613	0.0	0.0	650.0	206.70
33	1.997	2.059	0.0	0.0	632.0	109.10
34	1.012	0.620	0.0	0.0	608.0	167.00
35	1.049	4.031	0.0	0.0	650.0	211.30
36	1.063	6.724	0.0	0.0	560.0	100.50
37	1.028	1.131	0.0	0.0	540.0	0.70
38	1.026	6.424	0.0	0.0	830.0	22.80
39	1.030	-11.112	1104.00	250.0	1000.0	88.88

Table 6.10 New England Test System Generator Data

Unit Number	H	R_a	x'_d	x'_q	x_d	x_q	T'_{do}	T'_{qo}	x_ℓ
1	500	0	0.006	0.006	0.02	0.019	7.0	0.7	0.003
2	30.3	0	0.0647	0.0697	0.295	0.282	6.56	1.5	0.035
3	35.8	0	0.0531	0.0531	0.2495	0.237	5.7	1.5	0.0304
4	28.6	0	0.0436	0.0436	0.262	0.258	5.69	1.5	0.0295
5	26.0	0	0.132	0.132	0.67	0.62	5.4	0.44	0.054
6	34.8	0	0.05	0.05	0.254	0.241	7.3	0.4	0.0224
7	26.4	0	0.049	0.049	0.295	0.292	5.66	1.5	0.0322
8	24.3	0	0.057	0.057	0.290	0.280	7.7	0.41	0.028
9	34.5	0	0.057	0.057	0.2106	0.205	4.79	1.96	0.0298
10	42	0	0.031	0.031	0.1	0.069	10.2	0.0	0.0125

Table 6.11 Coherency Measures for ZMIIW Disturbance of Generators 1, 8, 9, 10

Generator Pair	Coherency Measure	Generator Pair	Coherency Measure
1-2	0.1315	5-6	0.0872
1-3	0.1326	5-7	0.0616
1-4	0.1566	5-8	0.1956
1-5	0.2083	5-9	0.2545
1-6	0.1458	6-7	0.026
1-7	0.1593	6-8	0.1356
1-8	0.1473	6-9	0.2529
1-9	0.2987	7-8	0.148
2-3	0.0155	7-9	0.2467
2-4	0.0811	10-2	0.2849
2-5	0.1449	10-1	0.4536
2-6	0.0588	10-2	0.4444
2-7	0.0851	10-3	0.4598
2-8	0.1246	10-4	0.525
2-9	0.279	10-5	0.589
3-4	0.0656	10-6	0.502
3-5	0.1295	10-7	0.529
3-6	0.0433	10-8	0.459
3-7	0.0696	10-9	0.617
3-8	0.1248		
3-9	0.2715		
4-5	0.0657		
4-6	0.022		
4-7	0.0004		
4-8	0.1459		
4-9	0.2469		

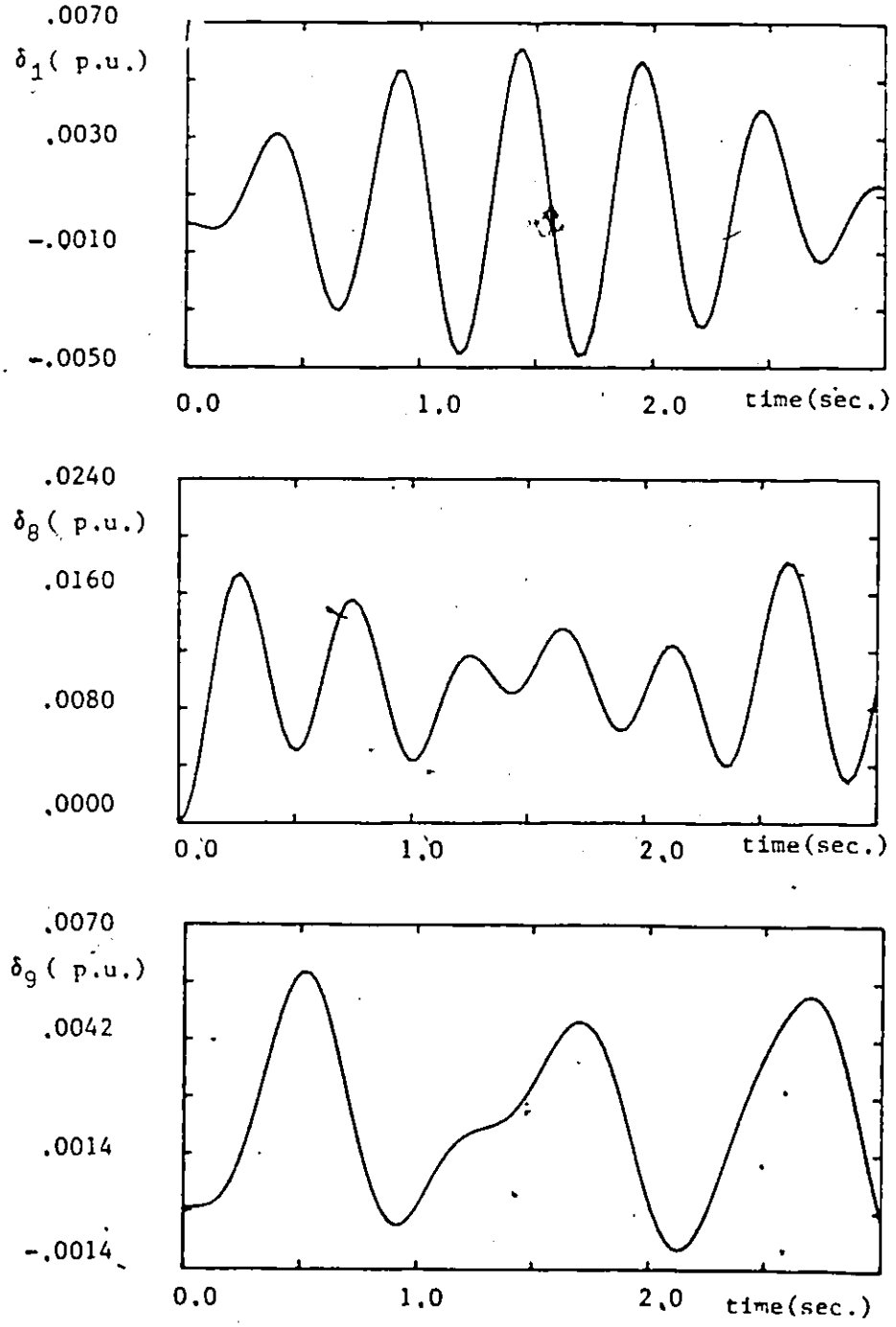


Figure 6.14 Torque angle responses at buses 1,8 and 9 following a 10% step change in mechanical power at bus-9; aggregation level- 1.

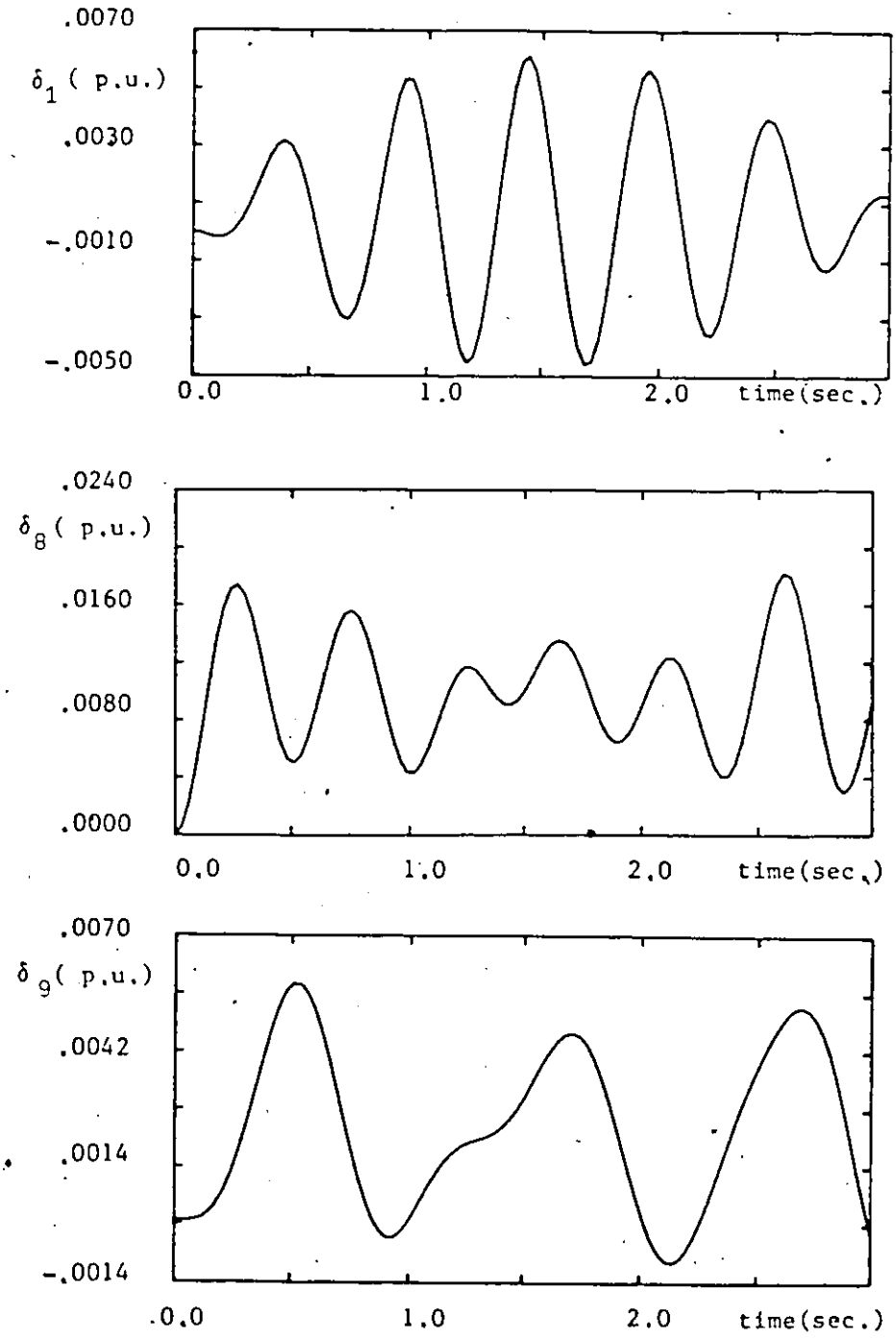


Figure 6.15 Torque angle responses at buses 1,8 and 9 following a 10% step change in mechanical power at bus-9;aggregation level - 2.

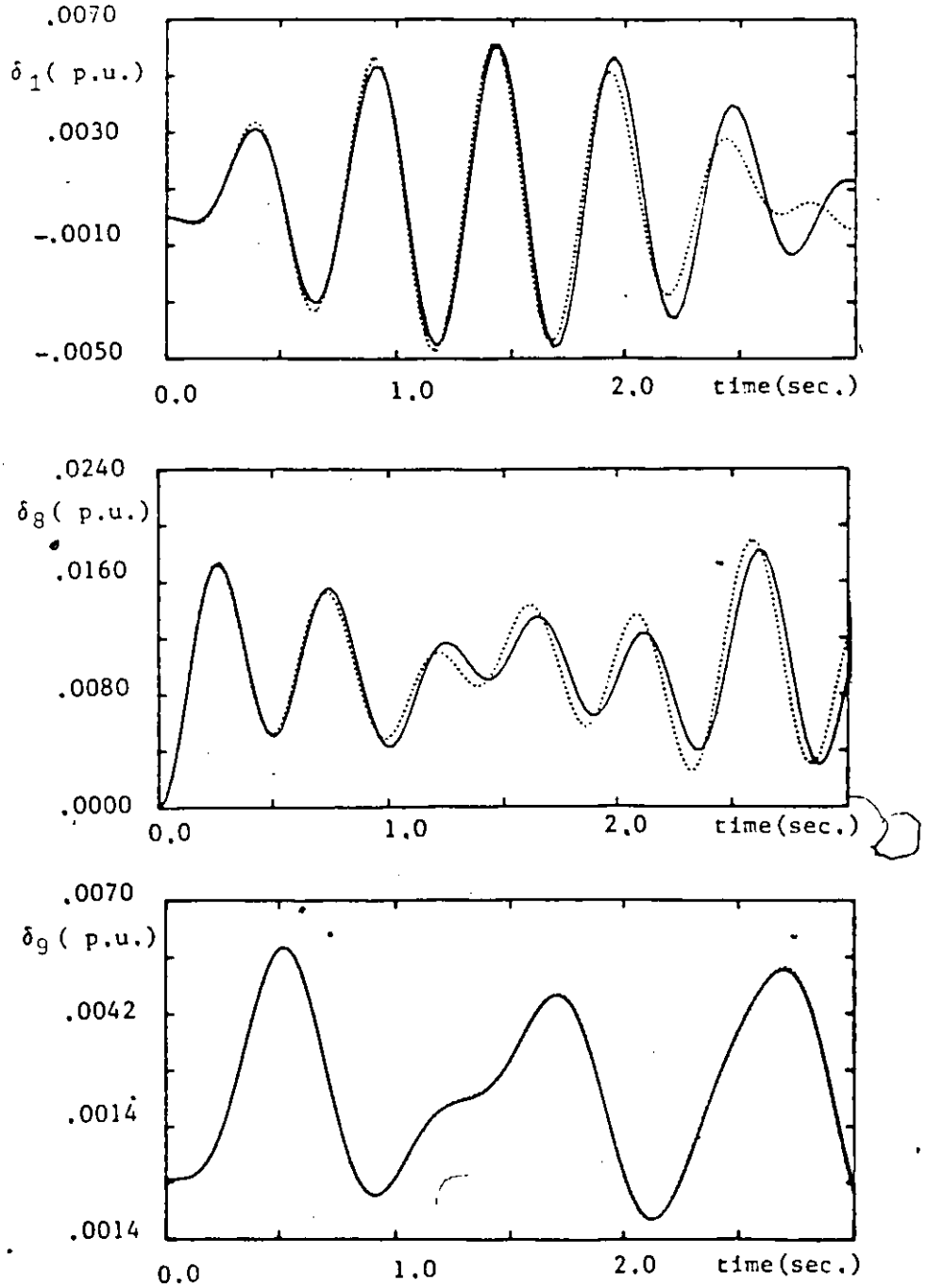


Figure 6.16 Torque angle responses at buses 1,8and 9 following a 10% step change in mechanical power at bus 9; aggregation level - 3.

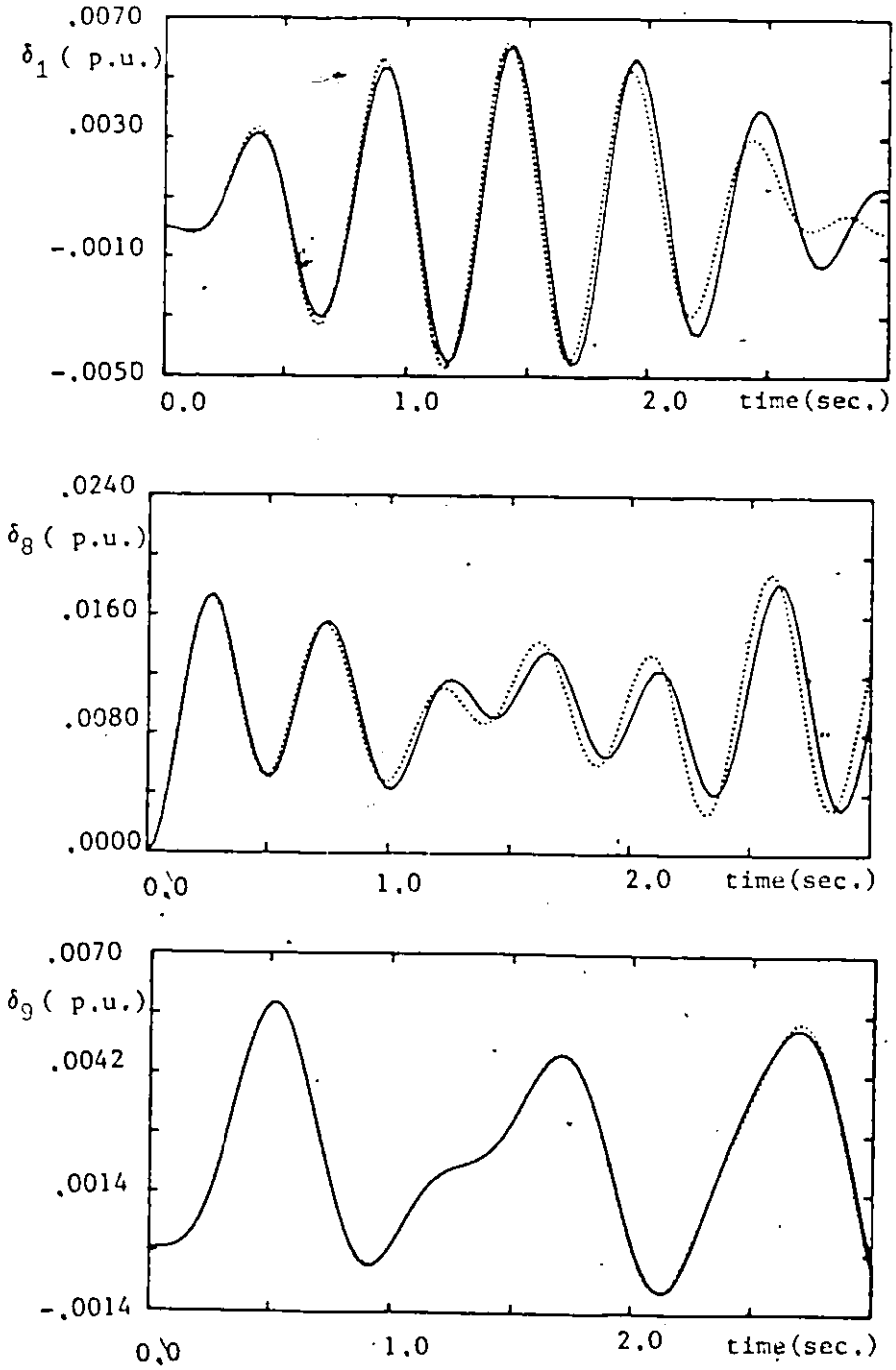


Figure 6.17 Torque angle responses at buses 1,8 and 9 following a 10% step change in mechanical power at bus-9; aggregation level -4.

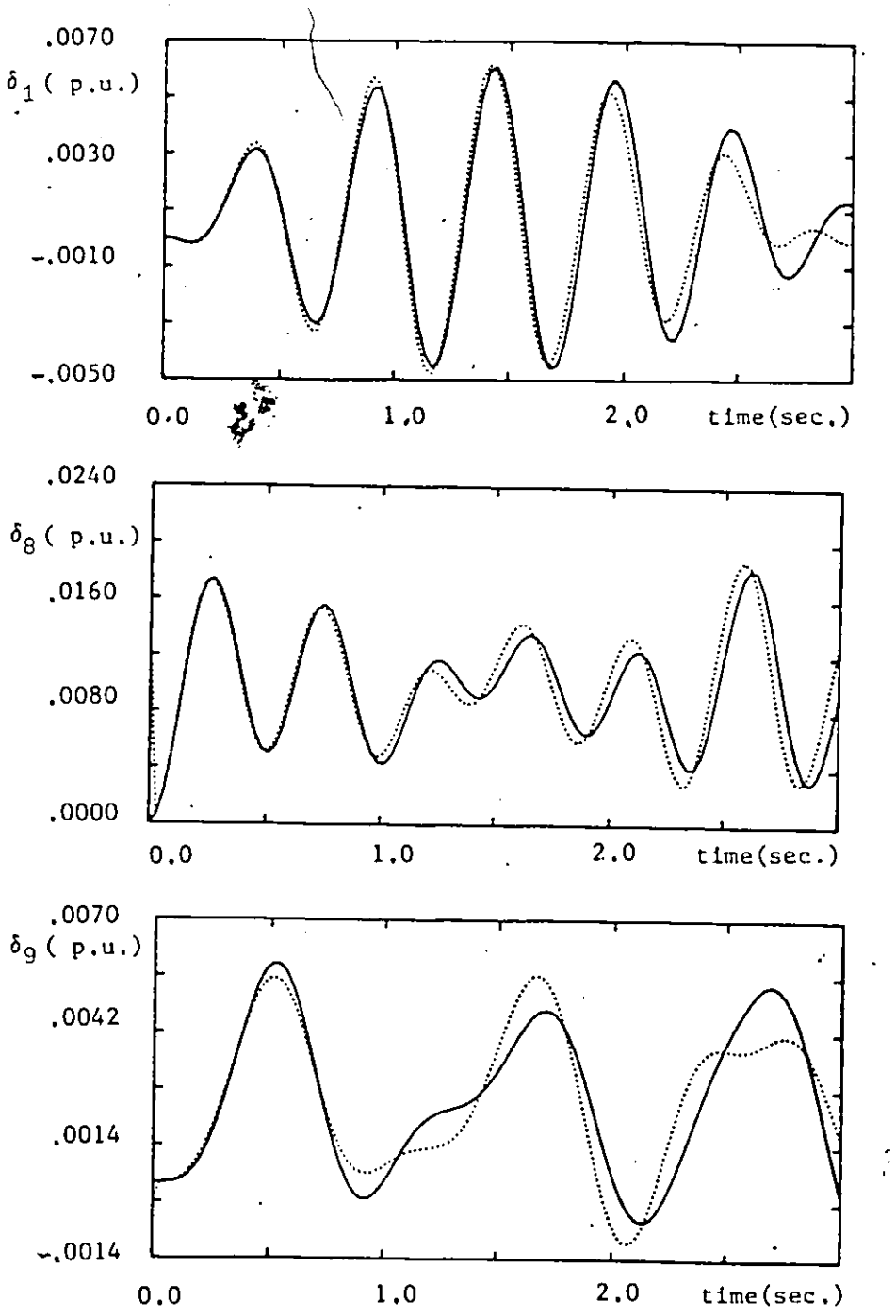


Figure 6.18 Torque angle responses at buse 1,8 and 9 following a 10% step change in mechanical power at bus-9; aggregation level - 5 .

Figures (6.14) through (6.18) are simulations comparing the response of the full 39 bus system to the reduced models at the five levels of aggregation. The disturbance used in the simulation is 10% step change in the mechanical power at generator 8. These simulations indicate that the reduced-order model responses start to deviate at aggregation level 4. At level 5, the response quality has decayed badly on generator 8. Note that the main degradation of the reduced model occurs when a generator is aggregated into a group with which it is not very coherent.

6.4 Conclusions

This chaper is divided into two main sections. In the first section, a criterion is proposed for selecting the most important state variables of the original system to be retained in the reduced-order model. Those state variables which are strongly influenced by the eigenvalues are chosen to be state vector components of the reduced-order model. Thus the proposed method of reduction retains the physical significance of the state variables, while retaining the dominant eigenvalues of the original system. This approach can be regarded as the combination of two well known methods, singular perturbation and aggregation. The application of this approach to the power system examples studied in this chapter indicates that although the reduced-order models obtained by setting the derivative of the state variables (the unretained state variables) equal to zero have dynamic responses close to that of the original system they may

introduce some interactions between the state variables and the forcing variables which cannot be interpreted easily in terms of the actual physical system.

In the second section, a procedure for obtaining dynamic equivalents of multimachine systems is proposed. This procedure utilizes the idea of component cost analysis for identifying the coherent generators. It was shown that by using the classical model for power system generating units (only for identifying the coherent generators) we are able to obtain the coherency measures without solving Lyapunov equation.

The procedure has been applied to the New England system model. The results obtained verify that the reduction algorithm for producing dynamic equivalents is a valid approach for producing dynamic equivalents.

CHAPTER 7

A COMPARISON OF MODEL REDUCTION METHODS FOR COMPENSATOR DESIGN

7.1 Introduction

Several existing model reduction techniques have been reviewed in Chapter 3 through Chapter 4. Moreover, we have proposed two procedures for obtaining reduced-order models in Chapter 5 and Chapter 6. A close examination of the majority of these methods has revealed that they fall into two categories those which simplify the high order system by deriving a low-order transfer function or state space model retaining the dominant modes or preserving part of the Taylor (or Laurent) series expansion of the system transfer function (i.e., some time moments or Markov parameters of the reduced-order model are matched to those of the original system), and those which involve fitting a low-order model such that either its time response to a specified input, or its frequency response is an optimum approximation to the corresponding response of the original system. All of these methods attempt to obtain a reduced-order model which approximates, as far as possible, certain characteristics of the original system.

It is obviously desirable that the reduction technique should be computationally simple and applicable to a wide class of systems.

Furthermore, it should yield a stable reduced-order model for a given stable system. As a reduced-order model cannot characterize a given system as completely as the large-order model, it would be desirable to determine how closely and in what sense a particular aspect of the behaviour of a given large system has been approximated by a particular reduced-order model. This aspect has been discussed in Chapter 2. However, obtaining a good approximation of the system response to specified inputs should not be the only goal of a model reduction technique, more important is the application of model reduction to the design of a feedback control system. The object of this chapter is to examine the aspect for several different methods of model reduction when the reduced-order model is utilized for designing, either compensators through the classical frequency response approach, or designing PID controllers so as to minimize the integral square error as a performance index. The comparison is made for an 11^{th} order system representing a synchronous machine connected to an infinite bus through a transmission line (Ex. 5.4.2). Four methods are utilized for obtaining 5^{th} order models. These are the methods of (i) continued fraction expansion, (ii) frequency domain approximation (Elliott and Wolovich [17]), (iii) the method developed in Chapter 5 with the characteristic equation of the reduced model obtained using either the Routh table or retaining the dominant eigenvalues, and (iv) the aggregation procedure developed in Chapter 6 for both cases ($\xi = 0$ and $\dot{\xi} = 0$). These methods have been chosen as they are representative of the different reduction approaches. A

compensator and PID controller are designed for each case, and the performance with the original system is examined. This provides a very interesting comparison and gives good insight into the application of these methods for such design.

7.2 System Description

The system to be controlled is that of one-machine connected to an infinite bus through a transmission line. This system comprises a single synchronous generator and excitation system. The models used for the synchronous generator and the exciter are those given in Appendix I. We would like to remind the reader that the data is previously given in example 5.4.2 in Chapter 5. As our goal is to compare between the different model reduction methods for compensator design, we study the case of a single-input - single-output system. Thus the system is of 11^{th} order and has the following state, input, and output, respectively:

$$\underline{x}^T = [\underline{x}_g^T, \underline{x}_e^T] \quad , \quad V_{ref} \text{ and } v_t$$

where

$$\underline{x}_g^T = [i_d, i_f, i_{kd}, i_q, i_{kq}, \omega, \delta]^T, \text{ and } \underline{x}_e^T = [V_1, V_3 V_R, E_{fd}]^T$$

$\underline{x}_g$ denotes the state variables of the synchronous machine.

$\underline{x}_e$ denotes the state variables of the excitation system.

Fifth-order models of the original system were obtained using the various methods mentioned in the previous section. Again, these methods are:

(1) Continued fraction expansion.

(2) Frequency domain approximation (Elliott and Wolovich).

(3) The aggregation technique described in Chapter 5,

a - the characteristic equation of the reduced model is obtained using the Routh table

b - the characteristic equation of the reduced model is obtained by retaining the dominant modes.

(4) The aggregation technique developed in Chapter 6,

a - the state variables ξ are set equal zero.

b - the derivative of the state variables ξ are set equal zero.

The eigenvalues (poles) and the zeros of the original system as well as the reduced-order models are given in Table 7.1 and Table 7.2, respectively. Also, the transient response to a unit step input in V_{ref} and the frequency response of the original system and each of the six reduced-order models are shown in Figures (7.1) through (7.2).

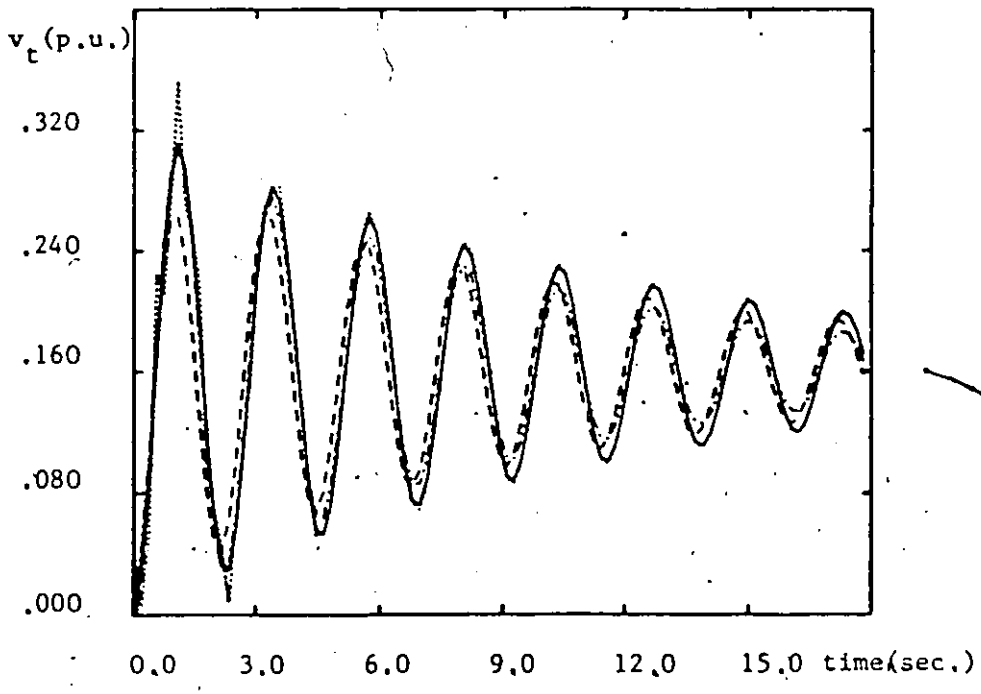


Figure 7.1-a Original system and reduced order model responses following a step disturbance.

——— original system

-.-.-.- reduced order model - 1

- - - - reduced order model - 3-a

. . . . . reduced order model - 3-b

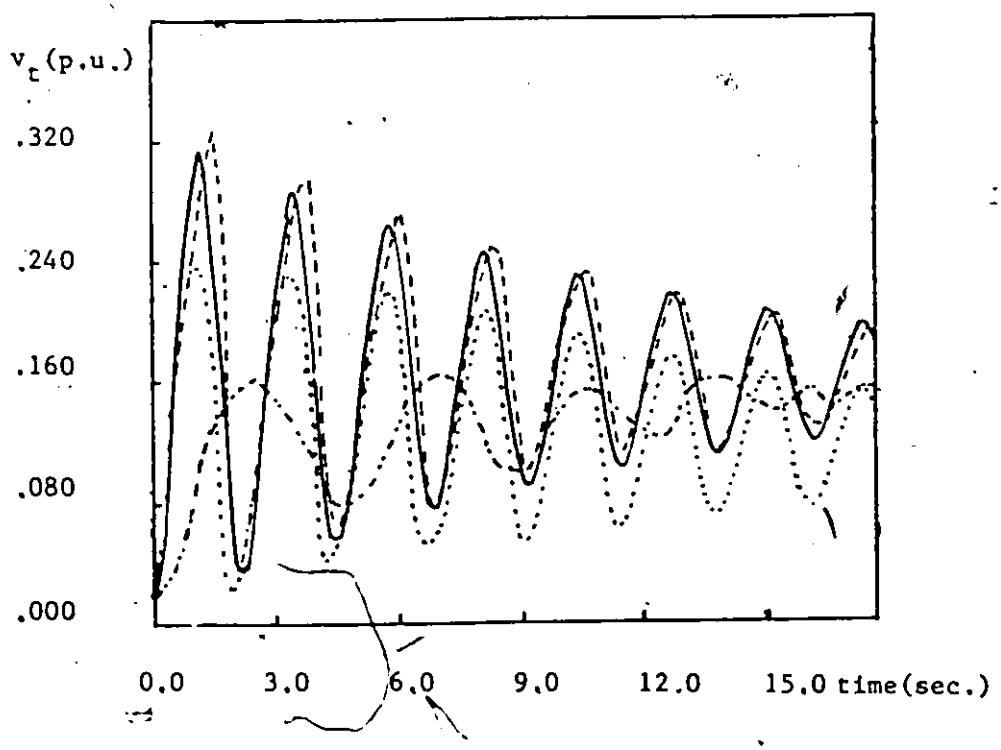


Figure 7.1-b Original system and reduced order model responses following a step disturbance.

——— original system

-.-.-.- reduced order model - 2

- - - - reduced order model - 4-b

..... reduced order model - 4-a

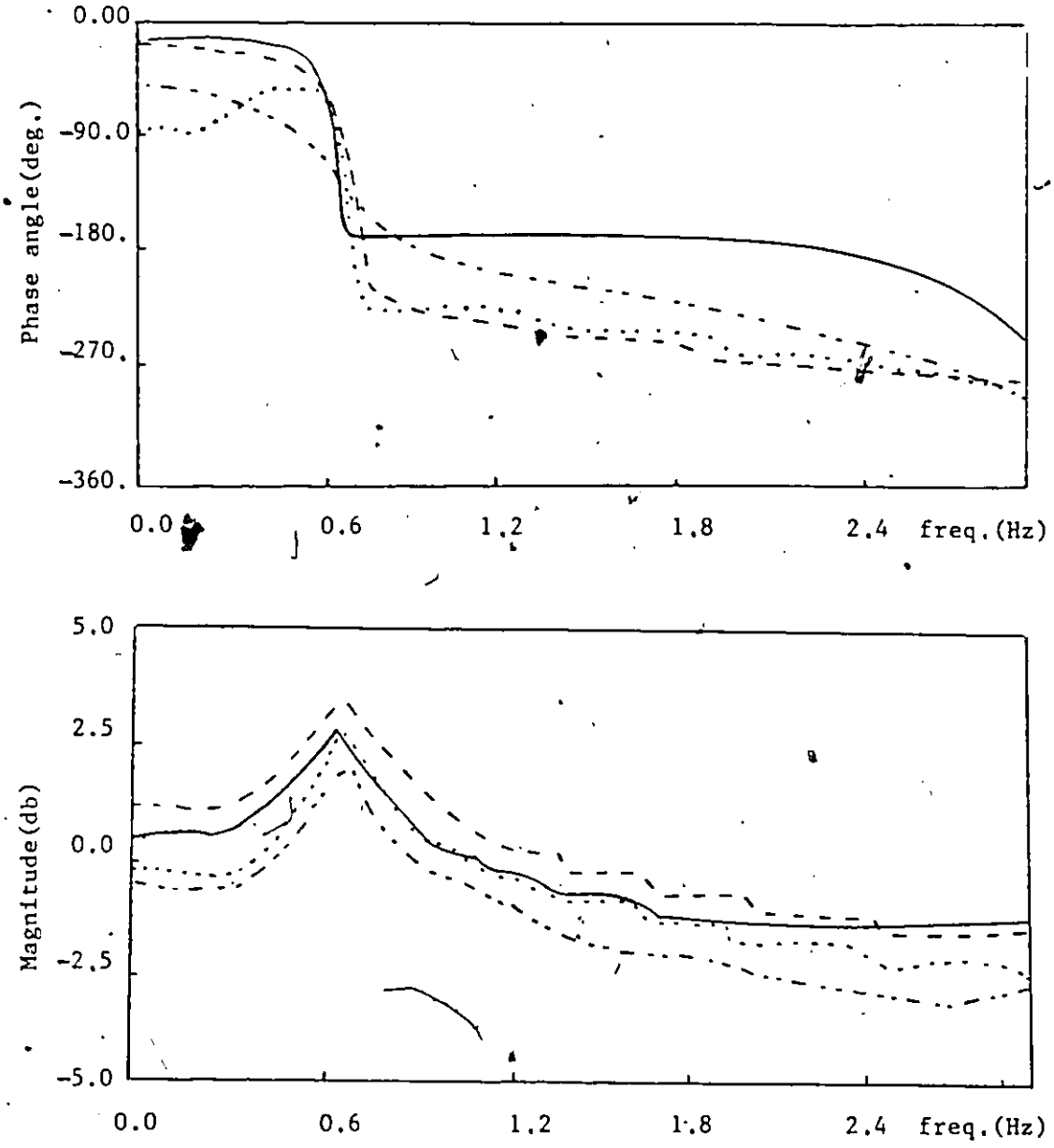


Figure 7.2-a Frequency response of the original system and reduced order models.

—— original system

-.-.- reduced order model -1

---- reduced order model -3-a

.... reduced order model -3-b

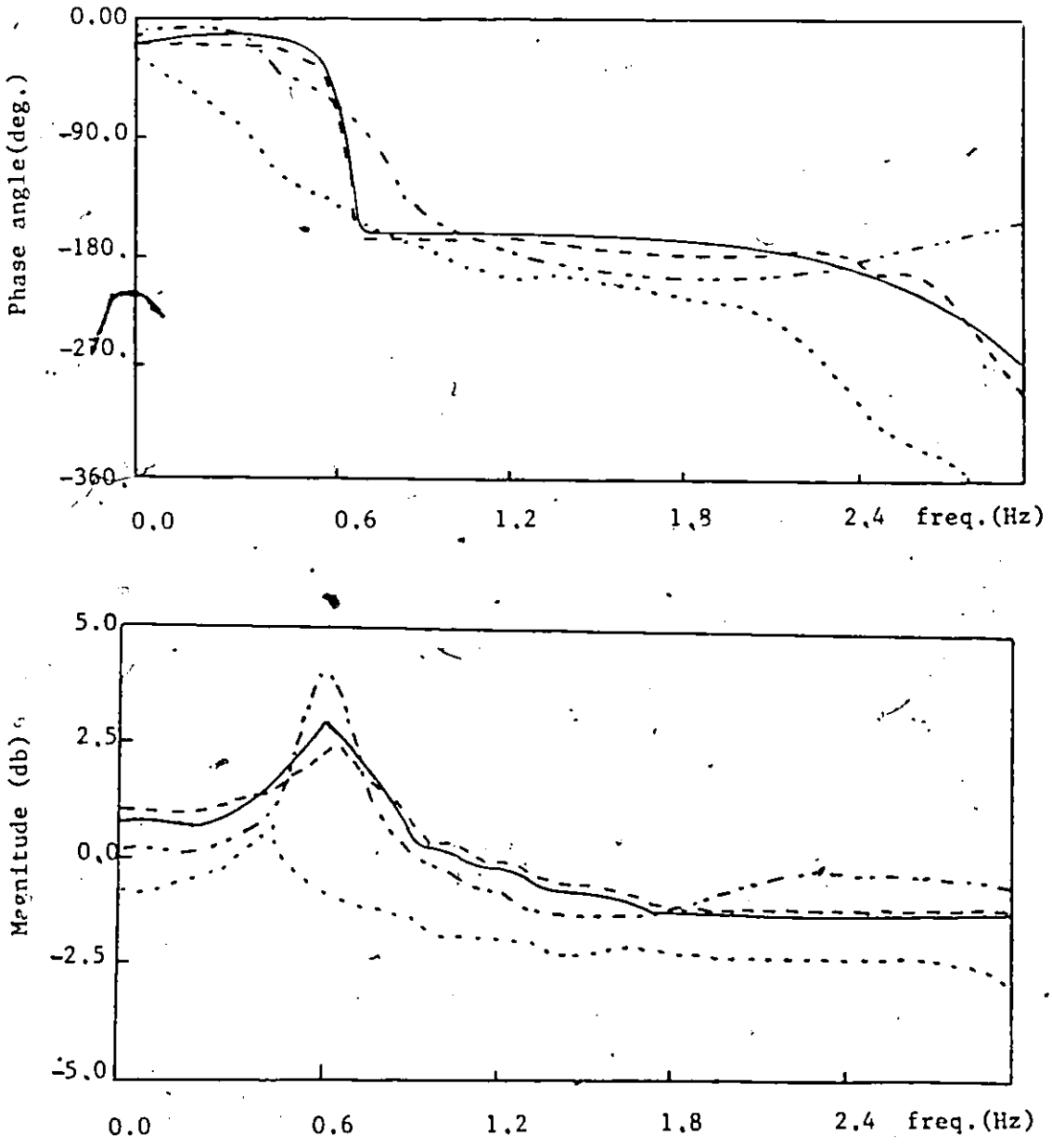


Figure 7.2-b Frequency response of the original system and the reduced order models.

——— original system

-.-.-. reduced order model - 2

- - - reduced order model - 4-a

..... reduced order model - 4-b

Table 7.1 Eigenvalues of the Original System as Well as the Six Reduced-Order Models

Original System	Reduced-Order Models					
	1	2	3-a	3-b	4-a	4-b
-0.265	-0.0265	-0.00015	-0.00369	-0.0548	-0.0548	-0.0548
-0.1217	-0.0237	-0.00058	-0.0329 ±j0.0203	-0.0015 ±j0.029	-0.0015 ±j0.029	-0.0015 ±j0.029
-0.0986	-0.00037	-0.0039	-0.0002 ±j0.0064	-0.0002 ±j0.0064	-0.0002 ±j0.0064	-0.0002 ±j0.0064
-0.0548	-0.0026 ±j0.0066	-0.006 ±j0.002				
-0.0037						
-0.0035 ±j0.998						
-0.0015 ±j0.029						
-0.0002 ±j0.0064						

Table 7.2 Zeros of the Original System as Well as the Six Reduced-Order Models

Original System	Reduced-Order Models					
	1	2	3-a	3-b	4-a	4-b
-0.00313 ±j0.035	-0.0037	-0.0029	-0.0025 ±j0.0032	-0.0096 ±j0.0119	-0.004 ±j0.036	-0.01 ±j0.02
-0.0274 ±j0.994	-0.0109	-0.006 ±j0.005	-0.1715	0.0116 ±j0.01216	-0.578	-0.006
-0.0037	-0.0359		-0.1425			-0.04
-0.118						
-0.265						
-0.236						
-1.0709						

The examination of Table (7.1) indicates that the first three reduced-order models (continued fraction expansion, frequency domain approximation by Elliott and Wolovich, and the aggregation method using the Routh table) have some eigenvalues which are different from those of the original system.

It is important to mention that third-order models as well as fourth-order models obtained using the continued fraction approach and the frequency domain approach, respectively, are unstable. This illustrates that these approaches do not always guarantee the stability of the reduced-order models if the original system is stable. Also, the examination of Table (7.2) illustrates that most of the reduced-order models obtained based on matching some time moments

or steady-state responses of the original system have zeros which are different from those of the original system. Column 6 (model 4-a) in Table 7.2 shows that the reduced-order models obtained by setting selected state variables (ξ) equal to zero are similar to the reduced order models obtained by disregarding the less important eigenvalues and zeros of the original system. The examination of Figures (7.1) through (7.2) shows that the steady-state responses of all reduced-order models matches that of the original system, except that one obtained by aggregation (4-a) which gives a noticeable error at steady-state. The phase characteristic of the reduced-order models obtained by the frequency domain approach (2) and the aggregation (4-a) are closest to that of the original system in the range of interest (0.2 Hz to 2 Hz). The investigation of the original system responses shows that the dynamic system performance is dominated by two pairs of complex eigenvalues ($-0.0015 \pm j0.029$ and $-0.0002 \pm j0.0064$) and one real eigenvalue -0.0548. Also, it indicates that the system damping is inherently low. Accordingly, compensation may be used to improve the system performance. In the following section, two procedures for designing compensators will be utilized. The first procedure involves designing PID controllers so as to minimize the integral square error as a performance index, while the second procedure utilizes lead or lead-lag compensators so that the frequency response of the compensated system will satisfy the system specifications.

7.3 Compensator Design Using Simplified Models

We now present the main results of using simplified models in compensator design of linear dynamical systems. These results are divided into two groups:

1) Design of PID controllers so as to minimize the integral square error.

2) Design of a lead or lead-lag compensators so that the frequency rsponse of the compensated system satisfies the system specifi[illegible]ons.

7.3.1 The Design of a PID Controller

The problem is to design a PID controller acting on the error $e(t) = V_{ref}(t) - V_t(t)$ so that the integral square error is minimized, where $V_{ref}(t)$ is the reference voltage and $V_t(t)$ is the actual terminal voltage. The PID controller has the following form

$$U(t) = K_p[e(t) + \frac{1}{T_i}\int e(t)dt + T_d \frac{de(t)}{dt}] \qquad (7.1)$$

To insure a unique value for K_p, T_i and T_d, other constraints can be placed on the response. These constraints are: (1) a decay ratio of ¼, (2) a value 0.5 for the dimensionless group K_pT_d/τ, where τ is the time necessary to reach 63.2 percent of the final value.

The design procedure involves four steps. The first step is to express the error as a function of the complex frequency s. This function will involve the parameters of the controllers as unknown coefficients. The second step is to express the integral-square error I in terms of the error transform E(s) by Parseval's theorem; i.e.,

$$I = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} E(s)\, E(-s)\, ds \tag{7.2}$$

At this stage, provided E(s) is a rational function, the integral square error will appear in the form

$$I = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{b(s)\, b(-s)}{a(s)\, a(-s)}\, ds \tag{7.3}$$

where b(s) and a(s) are polynomials in (s). The third step is to evaluate the integral. Fortunately, definite integrals of this form have been evaluated in terms of the coefficients appearing in the polynomials [59]. The fourth step is to adjust the values of the controller parameters in such a way as to minimize the integral square error.

Now, for the original system and each of the reduced models, a PID controller is designed so as to satisfy the above constraints.

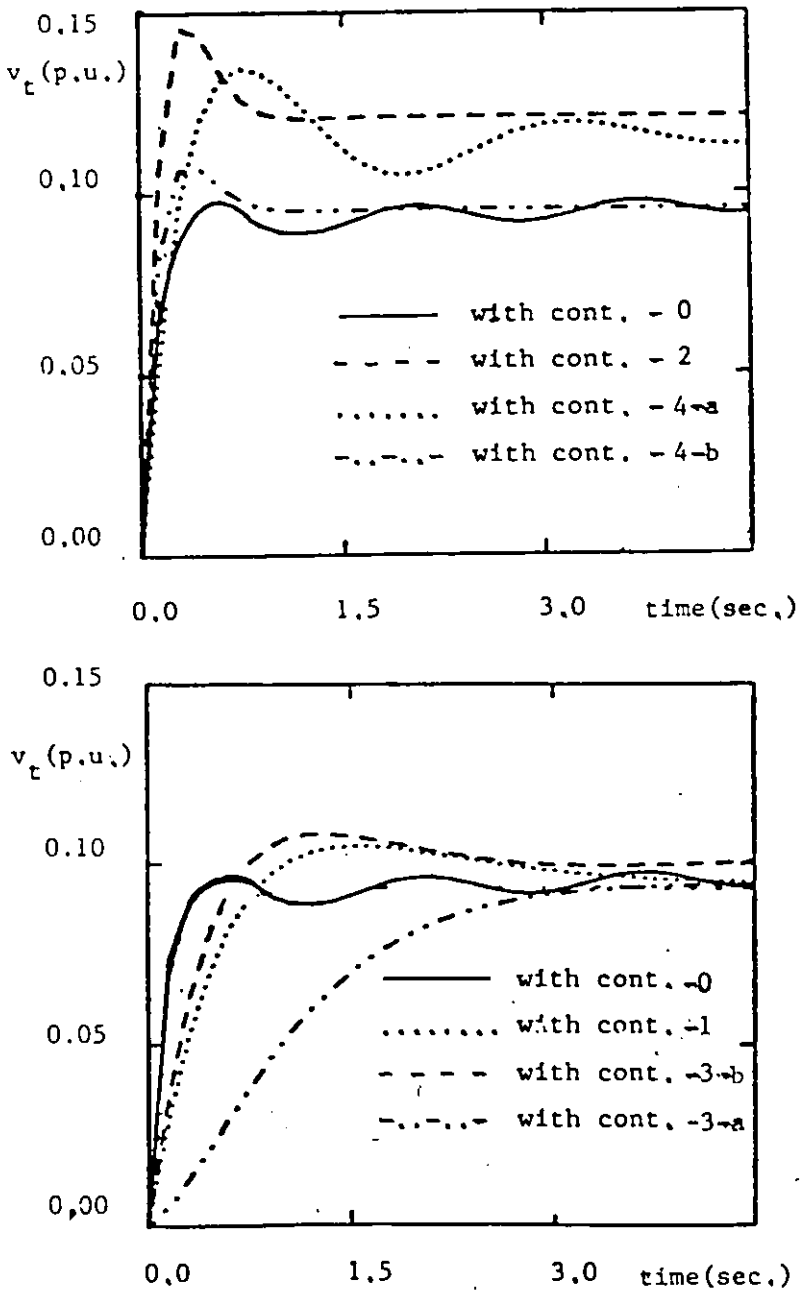


Figure 7.3 Original system equipped with various PID controllers responses following a step disturbance.

The values of the controller coefficients as well as the integral square error are illustrated in the following table.

Controller	K_p	T_i	T_d	I
0	0.669	0.9655	0.0132	0.0661
1	0.623	1.061	0.0101	0.056
2	0.324	2.495	0.017	0.067
3-a	0.526	0.9056	0.022	0.078
3-b	0.594	1.287	0.0165	0.084
4-a	0.841	2.626	0.011	0.07
4-b	0.4912	1.446	0.019	0.088

In the above table 0 is used to denote the controller designed with the original system, while 1 to 4-b are used to denote use of the corresponding reduced-order models. It should be noted that the controllers are quite different. The closed loop response of the original system with each controller to a unit step input is shown in Figure (7.3). Examination of these responses indicates that the responses using controllers 1, 3-b and 4-b are reasonably close to the response using the controller designed with the original system. Controller 3-a makes the system relatively over damped while controllers 2 and 4-a introduce relatively high overshoot and a steady-state offset. Also, it is seen from the calculation of the integral square error that controllers 1, 3-a, 3-b and 4-b give a relatively close

value to that of controller 0, while controllers 2 and 4-a give different values. These integrals are 0.085, 2.206, 0.096, 0.091, 1.97 and 0.11, respectively.

7.3.2 The Design of a Lead or Lead-Lag Compensator

In this section, it is desired to design a compensator based on the original system and each of the reduced models. This compensator must compensate for the phase lag shown in Figure (7.2) over the desired frequency range (0.2 to 2 Hz). Besides providing the required damping by adding phase lead, the control loop itself should be stable with the chosen compensator gain. This is achieved by making the control loop gain margin equal to 10 db and the phase margin 45°. The transfer functions of the compensators are, respectively;

$$g_0 = \frac{10.33\ (s + 0.0031)}{s + 0.0342}$$

$$g_1 = \frac{13.26\ (s + 0.00146)}{s + 0.009}$$

$$g_2 = \frac{9.63\ (s + 0.00336)}{s + 0.0342}$$

$$g_{3-a} = \frac{7.96\ (s + 0.0029)}{s + 0.0192}$$

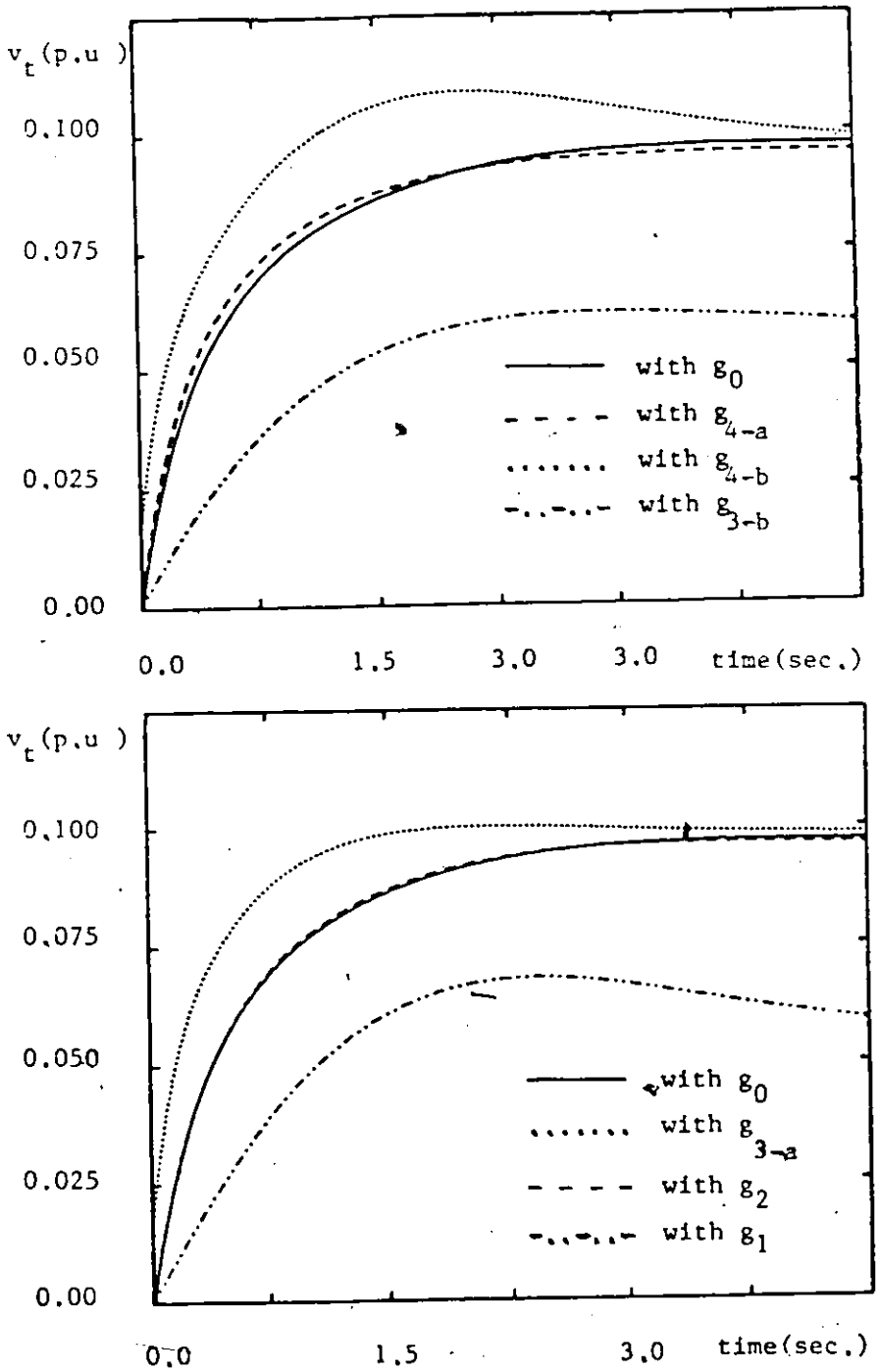


Figure 7.4 Original system equipped with various compensators responses following a step disturbance.

$$g_{3-b} = \frac{5.4\ (s + 0.002)}{s + 0.023}$$

$$g_{4-a} = \frac{11.3\ (s + 0.0034)}{s + 0.038}$$

$$g_{4-b} = \frac{7.4\ (s + 0.0015)}{s + 0.025}$$

Again, the subscript 0 is used for the compensator based on the original system, while subscripts 1 to 4-b are used to denote use of the corresponding reduced-order models. It should be noted that the compensators designed using the reduced-order model obtained by the frequency domain approximation is quite close to that one designed using the original system.

The closed loop responses of the original system with each compensator to unit a step input are shown in Figure (7.4). It is seen that each of the g_2 and the g_{3-b} responses is quite close to that of the g_0 response while the rest of the compensators produce quite different transient and steady-state responses.

7.4 Conclusions

Several model reduction techniques have been compared for designing PID controllers so as to minimize the integral square error as well as designing compensators satisfying frequency domain

specifications. Four methods are utilized for obtaining 5^{th}-order models from an 11^{th}-order model of a synchronous machine connected to an infinite bus with excitation system included. These methods are: (i) continued fraction expansion, (ii) frequency domain approximation, (iii) the aggregation technique developed in Chapter 5, and (iv) the aggregation technique developed in Chapter 6). These methods have been chosen as they are representative of the different reduction approaches. It was found that most of the responses of the reduced-order model to a unit step input are quite close to that of the original system except those obtained by the aggregation ($\xi = 0$) and the frequency domain approximation.

Comparing the responses of the original model equipped with the various PID controllers with that of the original system with its own controller indicate that some of these responses are quite close to that of the original system with its own controller and the others are quite different. Among the various controllers designed with the reduced-order models, the best results were obtained with the methods using the continued fraction expansion, the procedure developed in Chapter 5 and the procedure developed in Chapter 6 when the derivatives of the unretained state variables are set equal to zero ($\dot{\xi} = 0$).

In another comparison, the controller was designed to meet frequency domain specifications of gain margin and phase margin. As expected, the reduced-order models which provide a good match to the frequency response of the original system gave good compensators. The

best compensator was obtained with the reduced-order model obtained by the method of frequency domain approximation.

It would appear from the above comparison that the usefulness of the reduced model for the design of suboptimal controller for the original system depends very much upon the criteria used for designing the controller.

CHAPTER 8

CONCLUSIONS

This research was initiated to continue the process of rationalizing, unifying, and improving existing model reduction techniques. A close examination of many existing reduction methods has revealed that they may be classified in two main groups, those which preserve eigenvalues (aggregated models), and those which preserve part of the Taylor (or Laurent) series of the system transfer function (partial Padé approximation). A method of reduction has been developed in Chapter 5 which benecifially combines the method of aggregation and partial Padé approximation.

A procedure for selecting state variables for retention in the reduced-order model has been developed in Chapter 6 and applied to the singular perturbation method for general model reduction and also for the identification of coherent groups of generating units.

In Chapter 7, several of the described model reduction techniques have been examined and compared for application to PID controller and compensator design. This comparison has indicated that the choice of the model reduction scheme depends very much upon the criteria used for designing the controller. Among the various PID controllers from the reduced-order models, the best results have been

obtained with the methods which provide a good match to the time response. However, the best compensators have been obtained with the methods that provide a good match to the frequency response of the original system.

6.1 Aggregation with Moment Matching

The method of aggregation suffers from the drawback that an eigenvalue/eigenvector calculation must be accomplished for the entire state [A] matrix, which will generally be assymmetric. After transformation to Hessenberg form, several iterations of the QR algorithm, requiring $4\ n^2$ operation per iteration, are needed to determine each eigenvalue. Calculation of the eiggnvectors will take at least as many operations and thus the computer time involved may become prohibitive for a large matrix ($n>50$). It is worth noting that the procedure presented in Chapter 5 does not require the calculation of the system eigenvectors. However, it requires that the state equations of the system be transformed to either the controllable or observable companion form. This transformation is quite straightforward and can be performed in a routine manner. Moreover, with this transformation we may decompose the original (multivariable) system into a number of blocks less than or equal to the number of inputs, and obtain the system eigenvalues by solving for the eigenvalues of subsystems of smaller order with the advantage of reducing the computation effort.

On the other hand, traditional moment matching methods require the least amount of computation but suffer from the serious drawback

that the stability of the reduced model cannot be guaranteed. Also, there have traditionally been difficulties in extending the method to multivariable systems. It is worth mentioning that the new method proposed in Chapter 5 maintains the separate advantages of the aggregation and partial Padé approximation methods as described in the following paragraph.

The retention of the invariance property under linear state variable feedback as well as the invariance property in the matched time moments makes it possible to solve the suboptimal regulator and servomechanism problem. Also, the use of canonical realization allows the aggregation procedure to be accomplished without eigenvector calculation. Moreover, the stability of the reduced-order models is guaranteed and the relationship between the state variables of the reduced-order models and the original system is maintained.

8.2 Aggregation Combined with Singular Perturbation

Although the previous method has these desirable features, it suffers from the drawback that the physical significance of the states is lost. This creates a proplem when the original system is a part of a large interconnected power system, as transient stability programs are written to accept dynamic equivalents expressed in terms of equivalent lines and generators.

The singular perturbation method is an attractive approach for model reduction since the physical nature of the model is preserved. The main difficulty with the method is the problem of deciding the

proper partitioning of the state vector into slow and fast parts. The problem is complicated by the fact that, in general, the states are not decoupled. To overcome this difficulty a criterion has been proposed in Chapter 6 for selecting the most important state variables of the original system to be retained in the reduced-order model.

The choice of the state variables is based on the participation of the dominant eigenvalues in the step response of each state. Based on the previous criterion, a reduction technique has been proposed. Ths original system is transformed into two subsystems. The first subsystem is described by the subset of the actual state variables chosen to be retained, while the other is described by canonical differential equations representing the less-dominant eigenvalues in such a way that the second subsystem acts on the first one while there is no coupling path from the first to the second subsystem.

The reduced-order model is obtained, either by setting the value of the state variables of the second subsystem equal to zero or by considering their derivatives equal to zero. Thus, the reduced-order models obtained retain the physical significance of the state variables of the original system while retaining the dominant eigenvalues. This approach can be regarded as the combination of two well known methods, singular perturbation and aggregation. This method requires knowledge of the eigenvalues and eigenvectors of the original system which is computationally expensive. Fortunately, in dynamic stability studies, we are often interested in those lightly damped, oscillatory modes in the frequency range of 0.1 to 2 Hz, that are

associated with power oscillations, thus a partial eigensolution for the original system is sufficient. This will provide us with the set of eigenvalues of largest absolute magnitude together with the corresponding eigenvectors.

Also, in Chapter 6, a procedure for obtaining dynamic equivalents of multimachine systems has been proposed. This procedure utilizes the idea of component cost analysis for identifying the coherent generators. The original system is divided into an internal subsystem where the disturbance is assumed to occur and whose detailed behavior is of interest, and an external subsystem whose effect on the internal system must be accounted for. Assuming that the disturbances are applied to the generators of the internal subsystem, and that the reference generator is selected from within the external subsystem, a generator is identified to be coherent with the reference generator if the value of its component cost is very small (ideal coherency will exist when the component cost is zero). A zero component cost for such a generator means that either this generator and the reference generator respond to the disturbance in the same way, or they are both uncontrollable. It has been shown that by using the classical model for each generating unit (for identifying the coherent generator only) we are able to obtain the coherency measures without solving a Lyapunov equation. This procedure has been applied to the New England system model. Although the results obtained verify that this reduction algorithm for producing dynamic equivalents is feasible, it would be

desirable to test the algorithm on a much larger system, e.g., one with at least fifty generators.

8.3 Contributions of the Thesis

The specific contributions of the overall study are considered to be as follows:

(1) The development of a model reduction technique which combines the excellent qualities of aggregation and moment matching. The important aspects are; guaranteeing the stability of the reduced-order model, saving computation time, and retaining the invariance property under state variable feedback conditions.

(2) The development of a criterion for selecting the state variables of the original system to be retained in the reduced-order model. This criterion is based on the participation of the system eigenvalues in the step response of each state.

(3) The development of a reduction technique which can be regarded as the combination of the methods of aggregation and singular perturbation. The reduced-order model obtained has the advantage of retaining the physical significance of the state variables and the dominant eigenvalues of the original system.

(4) The development of a procedure for identifying coherent groups of generators in an interconnected power system, utilizing the concept of component cost analysis.

(5) A comparison of various reduction techniques in order to illustrate their usefulness in designing suboptimal controllers to obtain suboptimal control performance.

(6) The verification of the methods developed in this thesis by implementation using a variety of realistic power system models including a single synchronous machine connected to an infinite bus (with and without excitation system), a three machine system with six buses and nine lines, and a 10 machine system with 39 buses and 46 lines representing the New-England area. These applications include simulation, analysis and simple controller design.

8.4 Suggestions for Future Work

Specific topics which seem worthy of future study are:

(1) In this thesis we have proposed two techniques for model order reduction. Although the implementation using a variety of realistic power system models verify quite conclusively that these methods are reasonable and viable approaches to producing dynamic equivalents, some important work remains to be done. It would be of interest if these methods were to be adopted by a utility where they can be applied to the analysis of existing systems of larger sizes.

(2) The procedure developed in Chapter 6 for identifying the coherent generators, together with the aggregation technique developed in

the beginning of the same chapter, permits the obtaining of dynamic equivalents for multimachine systems which preserve the physical nature of the system as well as retain the dominant eigenvalues and the associated eigenvectors. Similar application by utlities would provide valuable operational experience.

(3) Much of the concern of model simplification has been about linear systems. Extension to more realistic non-linear models would seem appropriate for the attention of researchers; for example, the extension of aggregation concept to a certain class of non-linear systems. Specifically, such developments would permit analysis of "large" disturbances.

REFERENCES

[1] Anderson, L., "Decoupling of two-time scale systems", Proc. of JACC, pp. 153-163, 1978.

[2] Altalib, H.T. and Krause, P.C., "Dynamic equivalents by combination of system components", IEEE Trans. Power Apparatus and Systems, Vol. PAS-95, pp. 1535-1543, 1976.

[3] Aoki, M., "Control of large-scale dynamic systems by aggregation", IEEE Trans. Automatic Control, Vol. AC-13, pp. 246-253, 1968.

[4] Athans, M., "The matrix minimum principle", Information and Control, Vol. 11, pp. 592-606, 1968.

[5] Chang, A. and Adibi, M., "Power system dynamic equivalents", IEEE Trans Power Apparatus and Systems, Vol. PAS-89, pp. 1737-1744, 1970.

[6] Chen, C.F. and Shieh, L.S., "A novel approach to linear model simplification", Int. J. Control, Vol. 8, No. 6, pp. 561-570, 1968.

[7] Chidamabara, R.M., "On a method for simplifying linear dynamic systems", IEEE Trans. Automatic Control, Vol. AC-12, pp. 119-121, 1967.

[8] Chidamabara, R.M., "Two simple techniques for the simplification of large dynamic systems", Proc. JACC, pp. 669-674, 1969.

[9] Chow, J.H. and Kokotovic, P.V., "Eigenvalue placement in two-time scale system", Proc. IFAC Symp. Large-Scale Systems, pp. 321-326, 1976.

[10] Chow, J.H. and Kokotovic, P.V., "A decomposition of near-optimum regulators for systems with slow and fast modes", IEEE Trans. Automatic Control, Vol. AC-21, pp. 701-705, 1976.

[11] Chow, J.H., Allemong, J.J. and Kokotovic, P.V., "Singular perturbation analysis of systems with sustained high frequency oscillations", Automatica, Vol. 14, pp. 271-279, 1978.

[12] Davison, E.J., "A method for simplifying linear dynamic systems", IEEE Trans. Automatic Control, Vol. AC-11, pp. 93-101, 1966.

[13] Davison, E.J., "A new method for simplifying linear dynamic systems", IEEE Trans. Automatic Control, Vol. AC-13, pp. 214-215, 1968.

[14] Di Caprio, U., "Theoretical and practical dynamic equivalents in multimachine power systems, Part 1: Construction of coherency-based theoretical equivalent", Int. J. Elect. Power & Energy Syst. Vol. 4, pp. 224-232, 1982.

[15] Di Caprio, U., "Theoretical and practical dynamic equivalents in multimachine power systems Part 2: Construction of practical equivalent", Int. J. Elect. Power & Energy Syst., Vol. 5, pp. 40-54, 1983.

[16] Dorsey, J. and Schlueter, R.A., "Global and local dynamic equivalents based on structural archetypes for coherency", IEEE Trans. Power Apparatus and Systems, Vol. PAS-102, pp. 1793-1801, 1983.

[17] Elliott, H. and Wolovich, W.A., "A frequency domain model reduction procedure", Automatica, Vol. 16, pp. 167-178, 1980.

[18] El-Nahas, I., Sinha, N.K. and Alden, R.T.H., "Reduction of high order systems with application to controller design", Proc. 19th Annual Alerton Conf. on Communication, Control and Computing, pp. 163-173, 1981.

[19] El-Nahas, I., Sinha, N.K. and Alden, R.T.H., "A computer program for obtaining aggregated models of a multivariable linear system by matching specified time moments and Markov parameters", Internal Reports in Simulation, Optimization and Control, No. SOC-299, Faculty of Eng.

[20] El-Nahas, I., Sinha, N.K., and Alden, R.T.H., "Padé and Routh approximation in the time domain", To be published in Proceeding of the 22nd IEEE Conf. on Decision and Control, Dec. 1983.

[21] Hickin, J. and Sinha, N.K., "Aggregation matrices for a class of low-order models for large-scale systems", Electronic Letters, Vol. 11, pp. 186, 1975.

[22] Hickin, J. and Sinha, N.K., "A new method for reducing multivariable systems", Proc. 7th Annual Pittsburgh Conf. on Modeling and Simulation, pp. 259-268, 1976.

[23] Hickin, J. and Sinha, N.K., "Model Reduction for linear multivariable systems", IEEE Trans. Automatic Control, Vol. AC-25, pp. 1121-1127, 1980.

[24] Hutton, M.F., "Routh approximation method for high order linear systems", Ph.D. Dissertation, Polytech. Inst. of New York, 1974.

[25] Hutton, M.F. and Friedland, B., "Routh approximation for reducing order of linear time-invariant systems", IEEE Trans. Automatic Control, Vol. AC-20, pp. 329-337, 1975.

[26] Jennings, A. and Stewart, W.J., "Simultaneous iteration for partial eigensolution of real matrices", J. Inst. Math. Appl., Vol. 15, pp. 351-361, 1975.

[27] Kokotovic, P.V., "A Riccati equation for block diagonalization of ill conditioned systems", IEEE Trans. Automatic Control, Vol. AC-20, pp. 812-814, 1975.

[28] Lal, M., and Mitra, R., "A comparison of transfer function simplification methods", IEEE Trans. A Control, Vol. AC-19, pp. 602-603, 1974.

[29] Meisel, J., Sen. A., Leguarre, J. and Gille, M., "Emergency operating state control of bulk interconnected power systems", System Engineering for Power: Emergency Operating State Control, Vol. 2, pp. 112-116, Sept. 30 - Oct. 5, 1979.

[30] Moore, B., "Principal component analysis in linear systems: controllability, observability and model reduction", IEEE Trans. on Automatic Control, Vol. AC-26, pp. 17-32, 1981.

[31] Padé, H., "Sur la representation approchee d'une fonction par des fractions rationnelles", Thesis, Ann: de l'Ec. Nor., pp. 1-93, 1892.

[32] Podmore, R. and Germond, A., "Development of dynamic equivalents for transient stability studies", Report on EPRI project 763, 1977.

[33] Rao, C.R. and Mitra, S.K., "Generalized inverse of matrices and its applications", John Wiley & Sons, Inc., 1971.

[34] Rozsa, P., Sinha, N.K., and Lastman, G.J., "On estimating state variable participation for model reduction", Proc. 13th Annual Conf. on Modeling and Simulation, Pittsburgh, PA., 1982.

[35] Sandell, N.R., Jr. Variaru, P., Athans, M. and Safanov, M.G., "Survey of decentralized control methods for large-scale systems", IEEE Trans. Automatic Control, Vol. AC-23, pp. 108-128, 1978.

[36] Schlueter, R.A., Akhtar, H. and Modir, H., "An RMS Coherency measure: A basis for unification of coherency and modal analysis, model aggregtion techniques", IEEE P.E.S. Summer Meeting, 1978.

[37] Shamash, Y., "Stable reduced-order models using Padé type approximations", IEEE Trans. Automatic Control, Vol. AC-19, pp. 615-616, 1974.

[38] Shamash, Y., "Linear system reduction using Padé approximation to allow retention of dominant modes", Int. J. Control, Vol. 21, No. 2, pp. 257-272, 1975.

[39] Sinha, N.K., and Rozsa, P., "Some canonical forms for linear multivariable systems", Int. J. Control, Vol. 22, pp. 865-883, 1976.

[40] Sinha, N.K., El-Nahas, I., and Alden, R.T.H., "Reduction of high-order systems with application to compensator design", Proc. IFAC Symp. on Theory and Application of Digital Control, New Delhi, India, pp. 4.20-4.25, 1982.

[41] Sinha, N.K., El-Nahas, I. and Alden, R.T.H., "Routh approximation for multivariable systems", Problems of Control and Information Theory, Vol. 11, pp. 195-204, 1982.

[42] Sinha, N.K., "Reduction of high-order systems: A critical survey", Proceedings of the 1982 Canadian Conf. on Industrial Computer Systems, McMaster University, pp. 39.1-39.7, 1982.

[43] Siret, J.M., Michailesco, G. and Bertrand, P., "Representation of linear dynamical systems by aggregated models", Int. J. Control, Vol. 26, No. 1, pp. 121-128, 1977.

[44] Skelton, K.E., "Cost decomposition of linear systems with application to model reduction", Int. J. Control, Vol. 32, No. 6, pp. 1031-1055, 1980.

[45] Skelton, K.E. and Yousuff, A., "Component cost analysis of large scale systems", Int. J. Control, Vol. 37, No. 2, pp. 285-304, 1983.

[46] Stanton, K.N., "Dynamic energy balance studies for simulation of power frequency transients", Proc. PICA Conf., Boston, MA, 1971.

[47] Stewart, W.J. and Jennings, A., "A simultaneous iteration algorithm for real matrices", ACM Trans. Math. Softw., Vol. 7, No. 2, pp. 184-198, 1981.

[48] Stewart, W.J. and Jennings, A., "Algorithm 570-LOPSI: A simultaneous iteration algorithm for real matrices", ACM Trans. Math. Softw., Vol. 7, No. 2, pp. 230-232, 1981.

[49] Syrcos, G.P. and Sannuti, P., "Transformation of physical models into singularly perturbed form", Proceedings of IFAC Symposium on Theory and Appl. of Digital Control, India, January 1982.

[50] Syrcos, G.P. and Sannuti, P., "Singular perturbation modelling of continuous and discrete physical systems", Int. J. Control, Vol. 37, No. 5, pp. 1007-1022, 1983.

[51] Tinney, W.E., Powell, W.L. and Peterson, N.M., "Sparsity-Oriented network reduction", Proc. 8th PICA Conf., Minneapolis, MN, pp. 384-390, 1973.

[52] Undrill, J.M. and Turner, A.E., "Construction of Power systems electromechanical equivalents by modal analysis", IEEE Trans. Power Apparatus and Systems, Vol. PAS-90, pp. 2049-2059, 1971.

[53] VanNess, J.E., Zimmer, H. and Cultu, M., "Reduction of dynamic models of power systems", Proc. PICA Conf., pp. 105-112, 1973.

[54] Verghese, G.C., Pérez-Arriagu, I.J. and Schweppe, F.C., "Selective modal analysis with applications to electric power system, Part 1: Heuristic introduction", IEEE Trans. Power Apparatus and Systems, Vol. PAS-101, pp. 3117-3125, 1982.

[55] Wilson, D.A., "Optimum solution of model reduction problem", Proc. IEE, Vol. 117, pp. 1161-1165, 1970.

[56] Zakian, "Simplification of linear-time invariant systems by moment approximants", Int. J. Control, Vol 18, pp. 455-460, 1973.

GENERAL REFERENCES

[57] Anderson, P., "Analysis of faulted power systems", Iowa State University Press, Ames, IA, 1973.

[58] Anderson, P., and Fouad, A., "Power systems control and stability", Iowa State University Press, Ames, IA, 1977.

[59] Fossard, A., "Multivariable system control", North-Holland Pub. Co., 1977.

[60] Kwakernaak, H. and Sivan, R., "Linear optimal control systems", Wiley-Interscience, a Division of John Wiley & Sons, Inc., 1972.

[61] Newton, G., Gould, L. and Kaiser, J., "Analytical design of linear feedback controls", John Wiley & Sons, Inc., 1957.

APPENDIX 1

SUBSYSTEM MODELS

In this Appendix the equations describing the performance of each subsystem will be presented. The models are taken directly from the text by Anderson [58] but the equations representing them are rearranged in a matrix form.

1.1 Synchronous Machines

The modeling of a synchronous machine in state space form has been considered by many authors. Two different approaches have been used by Anderson in choosing the states of the model. The first approach uses the stator current and rotor currents (referred to the machine rotor frame) as states, while the second approach uses the stator and rotor fluxes (referred to the machine rotor frame) as states. In this Appendix, the first approach will be used in describing the synchronous machine model. The equations of a model based on linear approximation around an appropriate operating condition, for a synchronous machine, are taken directly from reference [58]. These equations are presented in matrix form as follows.

The linearized differential equations for a synchronous machine together with the mechanical equations representing the motion of the

rotor (not including the load equations) are with the Δ prefix omitted for convenience.

$$
\begin{bmatrix}
x_d & x_{af} & x_{ad} & & & & & \\
x_{af} & x_f & x_{fd} & & 0 & & & \\
x_{ad} & x_{fd} & x_{kdf} & & & & & \\
 & & & x_q & x_{aq1} & x_{aq2} & & \\
 & 0 & & x_{aq1} & x_{kq1} & x_{kq12} & & \\
 & & & x_{aq2} & x_{kq12} & x_{kq2} & & \\
 & & & & & & -2H\omega_o & \\
 & & & & & & 1 &
\end{bmatrix}
\begin{bmatrix} \dot{i}_d \\ \dot{i}_f \\ \dot{i}_{kd1} \\ \dot{i}_q \\ \dot{i}_{kq1} \\ \dot{i}_{kq2} \\ \dot{\omega} \\ \dot{\delta} \end{bmatrix} =
$$

$$
\begin{bmatrix}
R_a & 0 & 0 & \omega_o x_q & \omega_o x_{aq1} & \omega_o x_{aq2} & \psi_{qo} & 0 \\
0 & k_f & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & R_{kd1} & 0 & 0 & 0 & 0 & 0 \\
-\omega_o x_d & -\omega_o x_{af} & -\omega_o x_{ad} & R_a & 0 & 0 & -\psi_{do} & 0 \\
0 & 0 & 0 & 0 & R_{kq1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & R_{kq2} & 0 & 0 \\
\frac{\psi_{qo}-x_d\psi_{qo}}{3} & -\frac{x_{af}i_{qo}}{3} & -\frac{x_{ad}i_{qo}}{3} & -\frac{\psi_{do}+x_q i_{do}}{3} & \frac{x_{aq1}i_{do}}{3} & -\frac{x_{aq2}i_{do}}{3} & D & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix} i_d \\ i_f \\ i_{kd1} \\ i_q \\ i_{kq1} \\ i_{kq2} \\ \omega \\ \delta \end{bmatrix} -
\begin{bmatrix} v_d \\ -E_{fd} \\ 0 \\ v_q \\ 0 \\ 0 \\ T_m \\ 0 \end{bmatrix} \qquad (1.1)
$$

The flux linkage equations are:

$$\begin{bmatrix} \psi_d \\ \psi_f \\ \psi_{kd1} \\ \psi_q \\ \psi_{kq1} \\ \psi_{kq2} \end{bmatrix} = \begin{bmatrix} x_d & x_{af} & x_{ad} & & & \\ x_{af} & x_f & x_{fd} & 0 & & \\ x_{ad} & x_{fd} & x_{kd1} & & & \\ & & & x_q & x_{aq1} & x_{aq2} \\ & 0 & & x_{aq1} & x_{kq1} & x_{kq12} \\ & & & x_{aq2} & x_{kq2} & x_{kq2} \end{bmatrix} \begin{bmatrix} i_f \\ i_f \\ i_{kd1} \\ i_q \\ i_{kq1} \\ i_{kq2} \end{bmatrix} \quad (1.2)$$

The terminal voltage equation is:

$$v_t = \left[\frac{v_{do}}{v_{to}} \; ; \; \frac{v_{qo}}{v_{to}}\right] \begin{bmatrix} v_d \\ - \\ v_q \end{bmatrix} \quad (1.3)$$

The electrical torque equation is:

$$T_e = [i_{qo} \; ; \; -i_{do}] \begin{bmatrix} \psi_d \\ - \\ \psi_q \end{bmatrix} + [-\psi_{qo} \; ; \; \psi_{do}] \begin{bmatrix} i_d \\ - \\ i_q \end{bmatrix} \quad (1.4)$$

1.2 Excitation Systems

Throughout this thesis one type of exciter is used. This is a type 1 rotating exciter. The block diagram representing a type 1

rotating exciter is taken directly from reference [58]. This is depicted in Fig. (A.1). the first summing point compares the regulator reference with the output of the voltage sensor to determine the voltage error input to the regulator amplifier. The second summing point combines voltage error with the excitation major damping loop signal. The next summing point subtracts a signal which represents the saturation function of the exciter. The equations describing the performance of a type 1 exciter are:

$$\begin{bmatrix} \dot{V}_R \\ \dot{V}_1 \\ \dot{V}_3 \\ \dot{E}_{fd} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau_R} & 0 & 0 & 0 \\ -\frac{K_A}{\tau_A} & -\frac{1}{\tau_A} & -\frac{K_A}{\tau_A} & 0 \\ 0 & \frac{K_f}{\tau_E\tau_F} & -\frac{1}{\tau_F} & -\frac{-K_R(K_E+S_E)}{\tau_F\tau_E} \\ 0 & \frac{1}{\tau_E} & 0 & \frac{-(K_E+S_E)}{\tau_E} \end{bmatrix} \begin{bmatrix} V_R \\ V_1 \\ V_3 \\ E_{fd} \end{bmatrix} + \begin{bmatrix} \frac{1}{\tau_R} & 0 \\ 0 & \frac{K_A}{\tau_A} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_t \\ V_{ref} \end{bmatrix} \tag{1.5}$$

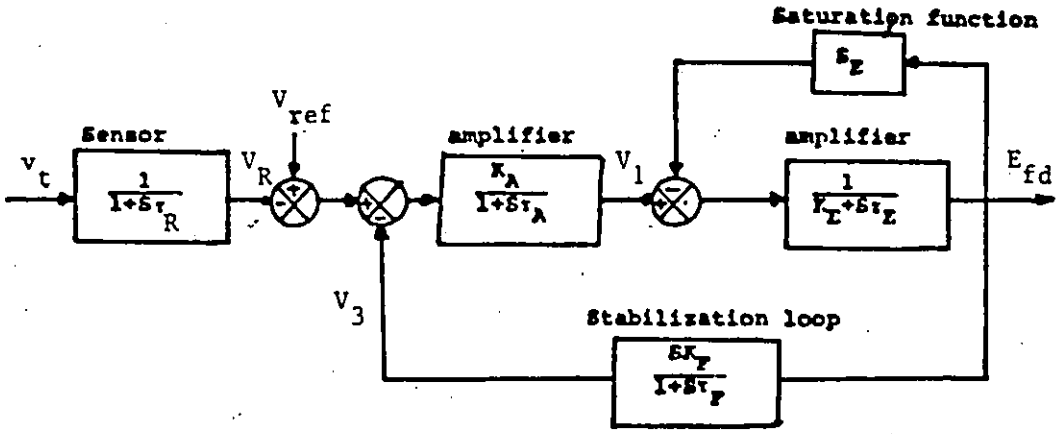


Figure A.1 Type-1 exciter block diagram

APPENDIX 2

FORMULATION OF THE NETWORK ADMITTANCE MATRIX

The bus admittance matrix $[Y_B]$ relates the network bus currents to bus voltages, including non-generator load buses. This matrix can, generally, be arranged in the following partitioned form:

$$\begin{bmatrix} \underline{I}_N \\ \hdashline \underline{I}_L \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ \hdashline Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} \underline{V}_N \\ \hdashline \underline{V}_L \end{bmatrix} \qquad (2.1)$$

where $\underline{I}_N$ and $\underline{V}_N$ are the currents and voltages of all generator buses. $\underline{I}_L$ and $\underline{V}_L$ are the currents and voltages of all non-generator (load) buses.

The network admittance matrix, $[Y_N]$, relates the generator bus currents to the generator bus voltages. The construction of this matrix from the bus admittance matrix can be achieved by eliminating all the non-generator (load) buses. This can, generally, be performed by two methods.

2.1 Partitioning Method

Expanding equation (2.1) yields:

$$\begin{aligned} \underline{I}_N &= [Y_{11}]\ \underline{V}_N + [Y_{12}]\ \underline{V}_L \\ \underline{I}_L &= [Y_{21}]\ \underline{V}_N + [Y_{22}]\ \underline{V}_L \end{aligned} \tag{2.2}$$

If all system loads are represented by constant admittance the current-voltage relationship at all non-generator load buses can be described as:

$$\underline{I}_L = -[Y_{LL}]\ \underline{V}_L \tag{2.3}$$

where $[Y_{LL}]$ is a diagonal complex matrix with each element representing the load admittance at the corresponding load bus. Combining equations (2.2) and (2.3) and upon reduction one obtains:

$$I_N = [Y_N]\ \underline{V}_N \tag{2.4}$$

where

$$[Y_N] = \{[Y_{11}] - [Y_{22}]\ ([Y_{LL}] + [Y_{22}])^{-1}\ [Y_{21}]\} \tag{2.5}$$

Thus, the construction of the $[Y_N]$ matrix from the $[Y_B]$ matrix, using this method, requires a complex matrix inversion of an order equal to the number of all non-generator load buses in the system.

2.2 Elimination Method

This method is a modification of the partitioning method in order to avoid the inversion of a complex matrix which can be of high order. In this method the load buses are eliminated one by one. Consequently, if we start with the last load bus arranged in equation (2.1), $[Y_{12}]$ will be a column vector, $[Y_{21}]$ a row vector, and hence, Y_{22} and Y_{LL} will become complex elements. Repeating the process in equation (2.3) we finally obtain the $[Y_N]$ matrix.