MASTER THESIS

REAL-SPACE RENORMALIZATION GROUP STUDY OF THE ANDERSON IMPURITY MODEL

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Abstract

In this thesis, we present a real-space renormalization group study of the Anderson impurity model(AIM) that describes itinerant fermions hybridized with an impurity. We use the quantum renormalization group method to map the exact RG equation to a path integral of the dynamical hopping and hybridization fields. We solve the saddle-point equation of the bulk theory that governs the fluctuating renormalization group flow, which becomes exact in the large N limit. From the solution, we show that the ranges of the renormalized hopping and hybridization are determined by the energy gap of the system. When the system is at the critical point, those ranges increase indefinitely both in the spatial and temporal directions as the RG scale increases.

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List of Symbols

EOM	Equation of Motion
DE	Differential equation
RG	Renormalization Group
AdS	Anti-de Sitter space, space with negative scalar curvature
CFT	Conformal Field theory
D	Dimension of the boundary Spacetime
(D+1)	Dimension of the bulk space (AdS)
z	Radial bulk coordinate/RG time
$x'\,,\tau'$	Renormalized spatial and temporal coordinates
$g_{\mu\nu}$	Metric of the AdS spacetime
$g^{\mu\nu}$	Inverse metric of the AdS spacetime

e^{μ}_{α}	Vierbein of the curved AdS spacetime
T	Size of the temporal lattice before taking the continuum limit
S	Size of the spatial lattice before taking the continuum limit
$\dagger$	Hermitian conjugate
σ	Flavor label
$i, j, k, l \ldots$	Boundary spatial indices with spacing $\frac{\Delta r}{2}$
k_1, k_2	Boundary Fourier indices from spatial indices/ Boundary momentum indices
ω	Boundary Fourier indices from temporal indices/ Boundary frequency indices
$\tau_1, \tau_2, \tau_3 \ldots$	Boundary temporal indices with spacing $\Delta\tau$
$\psi_{i\tau_1\tau_2\sigma}$	Grassmannian fields describing the lattice of itinerant electrons
$f_{\tau_1\tau_2\sigma}$	Grassmannian fields describing the impurity
$t_{ij\tau_1\tau_2}$	Hopping amplitude of the lattice of electrons
$V_{i\tau_1\tau_2}$	Hopping amplitude from the lattice of electrons to the impurity
$W_{i\tau_1\tau_2}$	Hopping amplitude from the impurity to the lattice of electrons
$\mu_{\tau_1\tau_2}$	Hopping amplitude of the impurity
m_{ψ}	Absolute value of the on-site energy of the electrons on the lattice

m_f	Absolute value of the on-site energy of the impurity.
$U(N)$	Group of N×N Unitary matrices

Declaration of Academic Achievement

This work is done in collaboration with Sung-Sik Lee. The augmented matrix form of the RG equation at the saddle point level with the presence of impurity and hybridization is new and obtained by the author. The exact solution to the RG equation at the saddle point with the presence of impurity and hybridization is new and obtained by the author.

Chapter 1

Introduction

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence has revolutionized our understanding of quantum gravity and quantum field theory [1][2][3]. On the one hand, it opened up the possibility to address issues of quantum gravity through the lens of quantum field theory and quantum information theory. On the other hand, one can understand the dynamics of strongly interacting quantum field theories through weakly coupled gravitational duals in the large N limits. However, using the AdS/CFT correspondence to understand condensed matter systems requires a constructive derivation of holographic duals for general quantum field theories beyond what can be realized in string theory.

Quantum Renormalization Group (QRG) is one approach that allows one to construct holographic duals for general quantum field theories[4][5][6] based on the idea that the extra dimension of the bulk spacetime corresponds to the length scale [7][8][9]. QRG differs from the conventional RG in that it keeps only a subset of couplings called single-trace couplings. The RG flow defined in that subspace can still capture the exact RG flow as the single-trace couplings are promoted to dynamical

variables subject to quantum fluctuations. The fluctuations of the single-trace couplings encode the information about multi-trace couplings that are not included in the RG flow [10]. In QRG, the exact beta function for a D-dimensional field theory is replaced with a $(D+1)$-dimensional action that governs the dynamics of the fluctuating RG flow of the single-trace couplings, where the extra dimension corresponds to the RG scale. Since the single-trace coupling that sources the energy-momentum tensor is nothing but the metric, the promotion of that coupling to a dynamical variable results in a theory of dynamical metric in the bulk spacetime that combines the original spacetime and an extra dimension associated with the RG scale. Earlier, QRG has been applied to the critical $U(N)$ vector models. It has been shown that the D-dimensional bosonic model, which describes the $U(N)$ symmetry-breaking critical point, is mapped into a bi-local field theory in the bulk whose saddle-point solution exhibits the $(D+1)$-dimensional AdS geometry[10]. On the other hand, the fermionic model that describes the metal-insulator transition is shown to exhibit a Lifshitz geometry[11].

The goal of this thesis is to study the Anderson impurity model (AIM) [12][13] through QRG[14]. AIM describes the hybridization between a localized impurity state and the conduction electrons of a metallic host. In AIM, the impurity is represented by a single localized energy level that interacts with the sea of itinerant electrons through both hybridization and the on-site interaction. The model is crucial for understanding a wide range of physical systems, including magnetic impurities in metals and quantum dots, making it a central model in studying the interplay between criticality and defects.

The thesis is organized as follows. In Chapter 2, we apply QRG method to AIM

and derive the bulk equation of motion for the single-trace couplings that correspond to hopping amplitudes and hybridization. In Chapter 3, we present the exact solution to the equation of motion in the non-interacting limit.

Chapter 2

Quantum Renormalization Group(QRG) for the AIM

2.1 Partition function as an overlap of states

We consider the (1+1)-dimensional Anderson impurity model (AIM), in which the action in discretized space and continuous time reads

$$S = \int d\tau \left(\bar{\psi}_{i\sigma} \partial_\tau \psi_{i\sigma} + \bar{f}_\sigma \partial_\tau f_\sigma + (-1)^i m_\psi \bar{\psi}_{i\sigma} \psi_{i\sigma} + m_f \bar{f}_\sigma f_\sigma - t_{ij} \bar{\psi}_{i\sigma} \psi_{j\sigma} \right.$$
$$\left. - V_i \bar{\psi}_{i\sigma} f_\sigma - W_i \bar{f}_\sigma \psi_{i\sigma} + U(\psi, f) \right). \tag{2.1.1}$$

Here τ is imaginary time, σ is the $U(N)$ flavor index, i, j are discrete indices for spatial coordinate. $\psi_{i\sigma}$ and f_σ are Grassmannian variable describing the itinerant fermion at site i and impurity, respectively. m_ψ is the staggered chemical potential for ψ while $-m_f$ is the chemical potential for the impurity. The staggered chemical potential is introduced to tune the gap for the itinerant fermions. $t_{ij\tau_1\tau_2}$ is the hopping

amplitude for the itinerant fermion. μ is the hopping amplitude of the impurity. V_i is the hybridization amplitudes between the itinerant fermion at site (i) and impurity, while W_i is the complex conjugate of V_i.

The action could be regularized on a two-dimensional Euclidean spacetime lattice, where the regularized action reads

$$\begin{aligned} S = &(-1)^i m_\psi \bar{\psi}_{i\tau_1\sigma}\psi_{i\tau_1\sigma} + m_f \bar{f}_{\tau_1\sigma} f_{\tau_1\sigma} - t^{[0]}_{ij\tau_1\tau_2}\bar{\psi}_{i\tau_1\sigma}\psi_{j\tau_2\sigma} - V^{[0]}_{i\tau_1\tau_2}\bar{\psi}_{i\tau_1\sigma}f_{\tau_2\sigma} \\ &- W^{[0]}_{i\tau_1\tau_2}\bar{f}_{\tau_1\sigma}\psi_{i\tau_2\sigma} - \mu^{[0]}_{\tau_1\tau_2}\bar{f}_{\tau_1\sigma}f_{\tau_2\sigma} + U(\psi, f), \end{aligned} \tag{2.1.2}$$

with all indices summed over τ_1, τ_2 are indices for temporal coordinates. $\psi_{i\tau\sigma}$ and $f_{\tau\sigma}$ are the discretized $\psi_{i\sigma}$ and f_σ. m_f is the chemical potential for the impurity. $t^{[0]}_{ij\tau_1\tau_2}$, $\mu^{[0]}_{\tau_1\tau_2}$, $V^{[0]}_{i\tau_1\tau_2}$, $W^{[0]}_{i\tau_1\tau_2}$ are the regularized amplitudes of t_{ij}, μ, V_i, W_i on the spacectime lattice respectively. Unless mentioned otherwise, all repeated indices are understood to be summed over throughout the thesis.

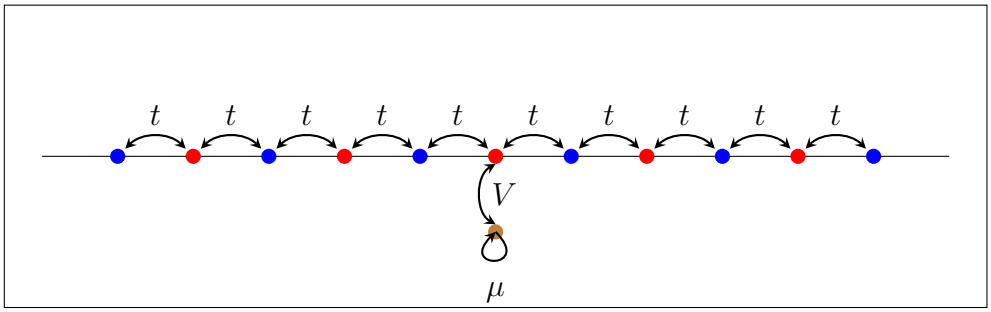

Figure 2.1: An illustration of the Anderson impurity model with a staggered chemical potential and the nearest neighbour hopping for the itinerant fermions. The itinerant fermion can also hop to the impurity site through hybridization.

We define a set of U(N) singlet operators

$$\mathcal{O} = \{\bar{\psi}_{i\tau_1\sigma}\psi_{j\tau_2\sigma}, \bar{\psi}_{i\tau_1\sigma}f_{\tau_2\sigma}, \bar{f}_{\tau_1\sigma}\psi_{i\tau_2\sigma}, \bar{f}_{\tau_1\sigma}f_{\tau_2\sigma}\} \tag{2.1.3}$$

to be single-trace operators. The name "single-trace" is derived from the fact that these operators have one flavour contraction. Other operators with multiple contractions of flavour indices are called multi-trace operators. Single-trace operators are special in that all U(N) invariant multi-trace operators can be written as composites of these single-trace operators.

We now write the action as the sum of two parts: $S = S_0 + S_1$, where

$$S_0 = (-1)^i m_\psi \bar{\psi}_{i\tau_1\sigma}\psi_{i\tau_1\sigma} + m_f \bar{f}_{\tau_1\sigma} f_{\tau_1\sigma}, \tag{2.1.4}$$

$$S_1 = -t_{ij\tau_1\tau_2}\bar{\psi}_{i\tau_1\sigma}\psi_{j\tau_2\sigma} - V_{i\tau_1\tau_2}\bar{\psi}_{i\tau_1\sigma}f_{\tau_2\sigma} - W_{i\tau_1\tau_2}\bar{f}_{\tau_1\sigma}\psi_{i\tau_2\sigma} - \mu_{\tau_1\tau_2}\bar{f}_{\tau_1\sigma}f_{\tau_2\sigma} + U(\psi, f). \tag{2.1.5}$$

It is noted that S_0 is ultra-local in spacetime and represents an insulating fixed point of the renormalization group (RG) flow. With $m_\psi > 0$ and $m_f > 0$, S_0 describes the fixed point where the first (second) sublattice of the lattice for the itinerant fermion is fully empty (occupied) and the impurity site is fully empty. S_1 serves as a deformation added to the IR fixed-point.

To begin with QRG, we note that the partition function can be written as the inner product between two states,

$$Z = \int D\psi Df\, e^{-S} = \int D\psi Df\, e^{-S_0 - S_1} = \langle S_0^* || S_1 \rangle. \tag{2.1.6}$$

Here

$$|S_0\rangle = \int D\psi Df\, e^{-(-1)^i m_\psi \bar{\psi}_{i\tau\sigma}\psi_{i\tau\sigma} - m_f \bar{f}_{\tau\sigma} f_{\tau\sigma}} |\psi, f\rangle, \tag{2.1.7}$$

$$|S_1\rangle = \int D\psi Df\, e^{t^{[0]}_{ij\tau_1\tau_2}\bar{\psi}_{i\tau_1\sigma}\psi_{j\tau_2\sigma}+V^{(0)}_{i\tau_1\tau_2}\bar{\psi}_{i\tau_1\sigma}f_{\tau_2\sigma}+W^{(0)}_{i\tau_1\tau_2}\bar{f}_{\tau_1\sigma}\psi_{i\tau_2\sigma}+\mu^{(0)}_{\tau_1\tau_2}\bar{f}_{\tau_1\sigma}f_{\tau_2\sigma}-U(\psi,f)}|\psi,f\rangle \tag{2.1.8}$$

represent quantum states defined on the two-dimensional spacetime lattice with respective wavefunctions e^{-S_0} and e^{-S_1} in the basis

$$|\psi,f\rangle := \prod_{\tau,\sigma}\left\{\left(1-\bar{f}_{\tau\sigma}c^{(f)\sigma\dagger}_{1\tau}\right)\left(1-f_{\tau\sigma}c^{(f)\sigma\dagger}_{2\tau}\right)\prod_i\left(1-\bar{\psi}_{i\tau\sigma}c^{(\psi)\sigma\dagger}_{1i\tau}\right)\left(1-\psi_{i\tau\sigma}c^{(\psi)\sigma\dagger}_{2i\tau}\right)\right\}|0\rangle, \tag{2.1.9}$$

where $c^{(\psi)\sigma\dagger}_{1i\tau}$ ($c^{(\psi)\sigma}_{1i\tau}$) creates (annihilates) a fermion of type ψ at spacetime lattice (i,τ). Similarly, $c^{(f)\sigma\dagger}_{1\tau}$ ($c^{(f)\sigma}_{1\tau}$) creates (annihilates) a fermion of type f. The same definition for the creation and annihilation operators applies to $\bar{\psi}$ and $\bar{f}$, except the subscript 1 is changed to 2. $|0\rangle$ is the vacuum state with $c|0\rangle = 0$ for all particle types. Here, the N-component vectors represent fields with N possible flavours.

The inner product in the spacetime Fock space is defined to be

$$\langle a||b\rangle := \langle a|\hat{O}|b\rangle, \tag{2.1.10}$$

where we choose

$$\hat{O} := \prod_{\tau,\sigma}\left\{\left(c^{(f)\sigma\dagger}_{1\tau}-c^{(f)\sigma}_{1\tau}\right)\left(c^{(f)\sigma\dagger}_{2\tau}-c^{(f)\sigma}_{2\tau}\right)\prod_i\left[\left(c^{(\psi)\sigma\dagger}_{1i\tau}-c^{(\psi)\sigma}_{1i\tau}\right)\left(c^{(\psi)\sigma\dagger}_{2i\tau}-c^{(\psi)\sigma}_{2i\tau}\right)\right]\right\} \tag{2.1.11}$$

such that

$$\langle \psi', f' || \psi, f \rangle = \prod_{\tau,\sigma} \left\{ \delta\left(\bar{f}'_{\tau\sigma} - \bar{f}_{\tau\sigma}\right) \delta\left(f'_{\tau\sigma} - f_{\tau\sigma}\right) \prod_i \left[\delta\left(\bar{\psi}'_{i\tau\sigma} - \bar{\psi}_{i\tau\sigma}\right) \delta\left(\psi'_{i\tau\sigma} - \psi_{i\tau\sigma}\right) \right] \right\}. \tag{2.1.12}$$

Since the partition function Z is a physical quantity, it remains unchanged under a coarse graining transformation. Therefore, there exists an operator $\hat{H}$ which acts on the spacetime Fock space and generates the RG flow such that

$$Z = \langle S_0^* || S_1 \rangle = \langle S_0^* || e^{-dz\hat{H}} | S_1 \rangle, \tag{2.1.13}$$

where $|S_0\rangle$ satisfies

$$\hat{H}^\dagger \hat{O}^\dagger | S_0^* \rangle = 0. \tag{2.1.14}$$

This represents the fact that S_0 is a fixed point that is invariant under the coarse graining transformation. $\hat{H}$ that satisfies (2.1.14) is

$$\begin{aligned} \hat{H} = \sum_\tau \Bigg(\sum_i \Bigg(& -\frac{2}{(-1)^i m_\psi} c_{1i\tau}^{(\psi)\dagger} \cdot c_{2i\tau}^{(\psi)\dagger} + c_{1i\tau}^{(\psi)\dagger} \cdot c_{1i\tau}^{(\psi)} + c_{2i\tau}^{(\psi)\dagger} \cdot c_{2i\tau}^{(\psi)} \Bigg) \\ & - \frac{2}{m_f} c_{1\tau}^{(f)\dagger} \cdot c_{2\tau}^{(f)\dagger} + c_{1\tau}^{(f)\dagger} \cdot c_{1\tau}^{(f)} + c_{2\tau}^{(f)\dagger} \cdot c_{2\tau}^{(f)} \Bigg). \end{aligned} \tag{2.1.15}$$

In the wavefunction basis, (2.1.15) can be written as

$$\begin{aligned} \tilde{H} = \sum_\tau \Bigg(\sum_i \Bigg(& \frac{2}{(-1)^i m_\psi} \frac{\delta}{\delta \psi_{i\tau}} \cdot \frac{\delta}{\delta \bar{\psi}_{i\tau}} + \psi_{i\tau} \cdot \frac{\delta}{\delta \psi_{i\tau}} + \bar{\psi}_{i\tau} \cdot \frac{\delta}{\delta \bar{\psi}_{i\tau}} \Bigg) \\ & + \frac{2}{m_f} \frac{\delta}{\delta f_\tau} \cdot \frac{\delta}{\delta \bar{f}_\tau} + f_\tau \cdot \frac{\delta}{\delta f_\tau} + \bar{f}_\tau \cdot \frac{\delta}{\delta \bar{f}_\tau} \Bigg). \end{aligned} \tag{2.1.16}$$

One can keep inserting $e^{-\hat{H}dz}$ between the bra and ket without modifying the partition function to generate a non-trivial RG flow of the deformation. This is captured by the evolution $e^{-\hat{H}z}|S_1\rangle$ with increasing z, which represents the flow of the deformation under the coarse-graining transformation that leaves S_0 invariant. We note that the exact Polchinski RG equation can be reproduced in this manner with the choice of the free Gaussian action as S_0 and the coarse-graining operator that leaves $|S_0\rangle$ invariant [15]. Here, we use the ultra-local action as S_0 that gives rise to a real-space coarse-graining scheme. The exact RG flow is now viewed as an evolution of the quantum state defined in spacetime. Formally, the RG flow can be viewed as a gradual projection of $|S_1\rangle$ (UV state) to $|S_0\rangle$ (IR state).

2.2 Derivation of the Bulk theory

In this section, we aim to write the coarse graining 'Hamiltonian' in terms of the dynamical couplings for the single-trace operators. Since RG flow is viewed as an evolution of wavefunction, it is natural to identify a complete basis of wavefunction. It turns out that any spacetime wavefunction can be written as a linear superposition of wavefunctions that only include the single-trace operators. This follows from the fact that multi-trace operators are composites of single-trace ones. To express $|S_1\rangle$ as a linear superposition of the basis wavefunctions, we first introduce a set of Lagrange

multipliers to write (2.1.8) as

$$|S_1\rangle = \int Dj D\psi Df \, \Big\{ e^{-N\left(W_{i\tau_1\tau_2}\left(W^*_{i\tau_1\tau_2} - \frac{1}{N}\bar{f}_{\tau_1\sigma}\psi_{i\tau_2\sigma}\right) + \mu_{\tau_1\tau_2}\left(\mu^*_{\tau_1\tau_2} - \frac{1}{N}\bar{f}_{\tau_1\sigma}f_{\tau_2\sigma}\right)\right)} e^{-N\left(t_{ij\tau_1\tau_2}\left(t^*_{ij\tau_1\tau_2} - \frac{1}{N}\bar{\psi}_{i\tau_1\sigma}\psi_{j\tau_2\sigma}\right) + V_{i\tau_1\tau_2}\left(V^*_{i\tau_1\tau_2} - \frac{1}{N}\bar{\psi}_{i\tau_1\sigma}f_{\tau_2\sigma}\right)\right)} e^{Nt^{(0)}_{ij\tau_1\tau_2}t^*_{ij\tau_1\tau_2} + NV^{(0)}_{i\tau_1\tau_2}V^*_{i\tau_1\tau_2} + NW^{(0)}_{i\tau_1\tau_2}W^*_{i\tau_1\tau_2} + N\mu^{(0)}_{\tau_1\tau_2}\mu^*_{\tau_1\tau_2} - U(\psi,f)}|\psi, f\rangle \Big\}, \tag{2.2.1}$$

where j represents the collection of single-trace couplings and

$$Dj = DtDVDWD\mu Dt^*DV^*DW^*D\mu^*. \tag{2.2.2}$$

$$|S_1\rangle = \int D\psi Df Dj \left(e^{t_{ij\tau_1\tau_2}\bar{\psi}_{i\tau_1\sigma}\psi_{j\tau_2\sigma} + V_{i\tau_1\tau_2}\bar{\psi}_{i\tau_1\sigma}f_{\tau_2\sigma} + W_{i\tau_1\tau_2}\bar{f}_{\tau_1\sigma}\psi_{i\tau_2\sigma} + \mu_{\tau_1\tau_2}\bar{f}_{\tau_1\sigma}f_{\tau_2\sigma} - U(\psi,f)} e^{-N\left(t^*_{ij\tau_1\tau_2}\left(t_{ij\tau_1\tau_2} - t^{(0)}_{ij\tau_1\tau_2}\right) + V^*_{i\tau_1\tau_2}\left(V_{i\tau_1\tau_2} - V^{(0)}_{i\tau_1\tau_2}\right) + W^*_{i\tau_1\tau_2}\left(W_{i\tau_1\tau_2} - W^{(0)}_{i\tau_1\tau_2}\right) + \mu^*_{\tau_1\tau_2}\left(\mu_{\tau_1\tau_2} - \mu^{(0)}_{\tau_1\tau_2}\right)\right)}|\psi, f\rangle \right). \tag{2.2.3}$$

$$|S_1\rangle = \int Dj \, F(j) \, |j\rangle, \tag{2.2.4}$$

where

$$|j\rangle = \int D\psi Df \, \left(e^{t_{ij\tau_1\tau_2}\bar{\psi}_{i\tau_1\sigma}\psi_{j\tau_2\sigma} + V_{i\tau_1\tau_2}\bar{\psi}_{i\tau_1\sigma}f_{\tau_2\sigma} + W_{i\tau_1\tau_2}\bar{f}_{\tau_1\sigma}\psi_{i\tau_2\sigma} + \mu_{\tau_1\tau_2}\bar{f}_{\tau_1\sigma}f_{\tau_2\sigma}}|\psi, f\rangle \right), \tag{2.2.5}$$

and

$$F(j) :=$$

$$e^{-N\left(t^*_{ij\tau_1\tau_2}\left(t_{ij\tau_1\tau_2}-t^{(0)}_{ij\tau_1\tau_2}\right)+V^*_{i\tau_1\tau_2}\left(V_{i\tau_1\tau_2}-V^{(0)}_{i\tau_1\tau_2}\right)+W^*_{i\tau_1\tau_2}\left(W_{i\tau_1\tau_2}-W^{(0)}_{i\tau_1\tau_2}\right)+\mu^*_{\tau_1\tau_2}\left(\mu_{\tau_1\tau_2}-\mu^{(0)}_{\tau_1\tau_2}\right)\right)-U'(j^*)}. \tag{2.2.6}$$

Applying one step of the RG $e^{-dz\hat{H}}$ to (2.2.4), we get:

$$\begin{aligned} &e^{-dz\hat{H}}|S_1\rangle \\ &\quad= \int Dj \left(e^{-N\left(t^*_{ij\tau_1\tau_2}\left(t_{ij\tau_1\tau_2}-t^{(0)}_{ij\tau_1\tau_2}\right)+V^*_{i\tau_1\tau_2}\left(V_{i\tau_1\tau_2}-V^{(0)}_{i\tau_1\tau_2}\right)+W^*_{i\tau_1\tau_2}\left(W_{i\tau_1\tau_2}-W^{(0)}_{i\tau_1\tau_2}\right)+\mu^*_{\tau_1\tau_2}\left(\mu_{\tau_1\tau_2}-\mu^{(0)}_{\tau_1\tau_2}\right)\right)} \right. \\ &\quad \left. e^{-dz\hat{H}}|j\rangle \right), \end{aligned} \tag{2.2.7}$$

which simplifies to:

$$\begin{aligned} &e^{-dz\hat{H}}|S_1\rangle = \\ &\int Dj \left(e^{-Nt^*_{ij\tau_1\tau_2}\left(t_{ij\tau_1\tau_2}-t^{(0)}_{ij\tau_1\tau_2}\right)} e^{-N\mu^*_{\tau_1\tau_2}\left(\mu_{\tau_1\tau_2}-\mu^{(0)}_{\tau_1\tau_2}\right)} e^{-Nv^*_{i\tau_1\tau_2}\left(v_{i\tau_1\tau_2}-v^{(0)}_{i\tau_1\tau_2}\right)} e^{-Nw^*_{i\tau_1\tau_2}\left(w_{i\tau_1\tau_2}-w^{(0)}_{i\tau_1\tau_2}\right)} \right. \\ &\left. e^{g\left(j^{(0)},j^*\right)}|j\rangle \right). \end{aligned} \tag{2.2.8}$$

where

$$\int Dj \left(e^{-Nj^*(j-j^{(0)})} g\left(j^{(0)},\, j^*\right)|j\rangle \right) = -dz \int Dj \left(e^{-Nj^*(j-j^{(0)})} \hat{H}|j\rangle \right). \tag{2.2.9}$$

Taking the continuum limit of RG time and letting

$$\mathcal{H}\left(j^{(0)},\, j^*\right) := -\frac{g\left(j^{(0)},\, j^*\right)}{dzN}, \tag{2.2.10}$$

we obtain

$$Z = \langle S_0^*||e^{-z'\hat{H}}|S_1\rangle$$

$$= \int Dj(z) \left(e^{-N\left(\int_0^{z'} dz\ t^*_{ij\tau_1\tau_2}\partial_z t_{ij\tau_1\tau_2} + \mu^*_{\tau_1\tau_2}\partial_z\mu_{\tau_1\tau_2} + v^*_{i\tau_1\tau_2}\partial_z v_{i\tau_1\tau_2} + w^*_{i\tau_1\tau_2}\partial_z w_{i\tau_1\tau_2} + \mathcal{H}(j(z), j^*(z))\right)} \langle S_0^* || j\rangle \right) \tag{2.2.11}$$

with

$$\begin{aligned}\hat{\mathcal{H}} = & \frac{2}{(-1)^i m_\psi} t_{ii\tau_1\tau_1} - \frac{2}{(-1)^i m_\psi} t^\dagger_{kj\tau_3\tau_2} t_{ki\tau_3\tau_1} t_{ij\tau_1\tau_2} - \frac{2}{(-1)^i m_\psi} v^\dagger_{k\tau_3\tau_2} v_{i\tau_1\tau_2} t_{ki\tau_3\tau_1} \\ & - \frac{2}{(-1)^i m_\psi} w^\dagger_{j\tau_3\tau_2} w_{i\tau_3\tau_1} t_{ij\tau_1\tau_2} - \frac{2}{(-1)^i m_\psi} \mu^\dagger_{\tau_3\tau_2} v_{i\tau_1\tau_2} w_{i\tau_3\tau_1} + 2t^\dagger_{ij\tau_1\tau_2} t_{ij\tau_1\tau_2} + \frac{2}{m_f}\mu_{\tau_1\tau_1} \\ & - \frac{2}{m_f} t^\dagger_{ij\tau_3\tau_2} v_{i\tau_3\tau_1} w_{j\tau_1\tau_2} - \frac{2}{m_f}\mu^\dagger_{\tau_3\tau_2}\mu_{\tau_3\tau_1}\mu_{\tau_1\tau_2} - \frac{2}{m_f} v^\dagger_{i\tau_2\tau_3} v_{i\tau_2\tau_1}\mu_{\tau_1\tau_3} - \frac{2}{m_f} w^\dagger_{i\tau_2\tau_3} w_{i\tau_1\tau_3}\mu_{\tau_2\tau_1} \\ & + 2\mu^\dagger_{\tau_1\tau_2}\mu_{\tau_1\tau_2} + 2v^\dagger_{i\tau_1\tau_2} v_{i\tau_1\tau_2} + 2w^\dagger_{i\tau_1\tau_2} w_{i\tau_1\tau_2}.\end{aligned} \tag{2.2.12}$$

Eq. (2.2.12) represents the coarse-graining Hamiltonian in terms of the single-trace sources and their conjugate variables. The canonical commutation relation between the sources X and their conjugate fields P_X is

$$[X,\, P_X] = \frac{1}{N}. \tag{2.2.13}$$

It is noted that t_{ij} and $t^\dagger_{ij}$ correspond to the annihilation and creation operators of the hopping field between sites i and j. The term $-\frac{2}{(-1)^i m_\psi} t^\dagger_{kj\tau_3\tau_2} t_{ki\tau_3\tau_1} t_{ij\tau_1\tau_2}$ in the Hamiltonian describes the process in which a hopping between sites k, j, which are far from each other, can be generated from shorter-range shopping $t_{ki\tau_3\tau_1}$ and $t_{ij\tau_1\tau_2}$. For the free theory, the Hamiltonian is linear in the conjugate field $t^\dagger$ which becomes Lagrangian multipliers. In general interacting theories, non-linear terms in the conjugate fields appear in the Hamiltonian[10].

2.3 Bulk EOM Derivation

In the large N limit, quantum fluctuations of the RG path are suppressed and hence the saddle point approximation becomes exact. To derive the evolution of the single-trace couplings under the RG flow at the saddle point level and write it in a compact way, we first group the sites into two sub-lattices as

$$\Lambda_+ = \{i \in \Lambda \mid m_i > 0\}, \quad \Lambda_- = \{i \in \Lambda \mid m_i < 0\}, \tag{2.3.1}$$

where Λ be the set of all sites. We then define

$$t_{f(i)f(j)\tau_1\tau_2 ab} = \begin{cases} t_{ij\tau_1\tau_2} & \text{if } i \in \Lambda_a,\ j \in \Lambda_b, \\ 0 & \text{otherwise} \end{cases}, \tag{2.3.2}$$

$$V_{f(i)\tau_1\tau_2 a} = \begin{cases} v_{i\tau_1\tau_2} & \text{if } i \in \Lambda_a \\ 0 & \text{otherwise} \end{cases}, \tag{2.3.3}$$

$$W_{f(i)\tau_1\tau_2 a} = \begin{cases} w_{i\tau_1\tau_2} & \text{if } i \in \Lambda_a \\ 0 & \text{otherwise} \end{cases}, \tag{2.3.4}$$

where $a, b = \pm$, $f(i) = \left[\frac{i}{2}\right]$ with [...] representing the floor-function. $f(i)$ represents the unit-cell index. The same definition is applied to the conjugate fields. From now on, $f(i)$, $f(j)$ are replaced by i, j for simplicity.

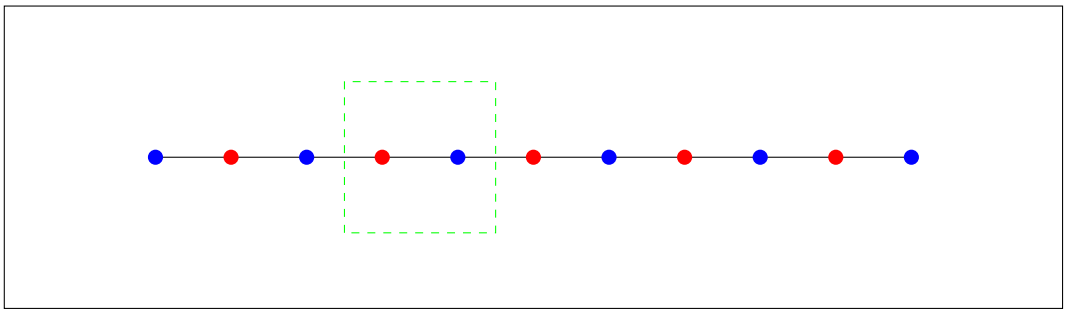

Figure 2.2: A unit cell (green dashed lines) of the lattice is made of two sites with positive and negative chemical potential.

The equation of motion can be obtained from the Heisenberg equation for the dynamical sources and their conjugate momenta,

$$(\partial_z \hat{X})_H = [\hat{\mathcal{H}}, \hat{X}]_H, \tag{2.3.5}$$

$$(\partial_z \hat{P_X})_H = [\hat{\mathcal{H}}, \hat{P_X}]_H, \tag{2.3.6}$$

which becomes the classical Hamilton's equation of motion at the saddle-point level. Denoting $\bar{j}$ to be the saddle-point solution of j, the system of EOMs for $\bar{j}$ reads

$$\partial_z \bar{t}_{ij\tau_1\tau_2 ab} = -2\bar{t}_{ij\tau_1\tau_2 ab} + 2\left(\frac{c}{m_\psi}\bar{t}_{ik\tau_1\tau_3 ac}\bar{t}_{kj\tau_3\tau_2 cb} + \frac{1}{m_f}\bar{V}_{i\tau_1\tau_3 a}\bar{W}_{j\tau_3\tau_2 b}\right), \tag{2.3.7}$$

$$\partial_z \bar{V}_{i\tau_1\tau_2 a} = -2\bar{V}_{i\tau_1\tau_2 a} + 2\left(\frac{c}{m_\psi}\bar{t}_{ik\tau_1\tau_3 ac}\bar{V}_{k\tau_3\tau_2 c} + \frac{1}{m_f}\bar{\mu}_{\tau_3\tau_2}\bar{V}_{i\tau_1\tau_3 a}\right), \tag{2.3.8}$$

$$\partial_z \bar{W}_{i\tau_1\tau_2 a} = -2\bar{W}_{i\tau_1\tau_2 a} + 2\left(\frac{c}{m_\psi}\bar{W}_{k\tau_1\tau_3 c}\bar{t}_{ki\tau_3\tau_2 ca} + \frac{1}{m_f}\bar{\mu}_{\tau_1\tau_3}\bar{W}_{i\tau_3\tau_2 a}\right), \tag{2.3.9}$$

$$\partial_z \bar{\mu}_{\tau_1\tau_2} = -2\bar{\mu}_{\tau_1\tau_2} + \frac{2}{m_f}\bar{\mu}_{\tau_1\tau_3}\bar{\mu}_{\tau_3\tau_2} + \frac{2c}{m_\psi}\bar{V}_{i\tau_1\tau_3 c}\bar{V}_{i\tau_3\tau_2 c}. \tag{2.3.10}$$

It is noted that c is summed over the sub-lattice index ± 1. In the non-interacting case, the equation of motion for the source is decoupled from the conjugate momenta.

One convenient way to write out the EOM for the saddle-point configurations is to use an augmented matrix compatible with the hybridization and the staggered mass effect. For every τ_1, τ_2, we therefore define $(2S+1) \times (2S+1)$ matrices

$$T'(z)_{\tau_1\tau_2} := \begin{pmatrix} t'(z)_{\tau_1\tau_2} & V'(z)_{\tau_1\tau_2} \\ W'(z)_{\tau_1\tau_2} & \frac{\mu(z)_{\tau_1\tau_2}}{m_f} \end{pmatrix}, \tag{2.3.11}$$

where S is the number of unit cells and

$$(t'(z)_{\tau_1\tau_2})_{ij} := \frac{1}{m_\psi} \begin{pmatrix} t_{ij\tau_1\tau_2++} & -it_{ij\tau_1\tau_2+-} \\ -it_{ij\tau_1\tau_2-+} & -t_{ij\tau_1\tau_2--} \end{pmatrix}, \tag{2.3.12}$$

$$(V'(z)_{\tau_1\tau_2})_i := \frac{1}{\sqrt{m_\psi m_f}} \begin{pmatrix} V_{i\tau_1\tau_2+} \\ -iV_{i\tau_1\tau_2-} \end{pmatrix}, \tag{2.3.13}$$

$$(W'(z)_{\tau_1\tau_2})_i := \frac{1}{\sqrt{m_\psi m_f}} \begin{pmatrix} W_{i\tau_1\tau_2+} & -iW_{i\tau_1\tau_2-} \end{pmatrix}. \tag{2.3.14}$$

The definition for the unprimed $t(z)_{\tau_1\tau_2}$, $V(z)_{\tau_1\tau_2}$, $W(z)_{\tau_1\tau_2}$, $\mu(z)_{\tau_1\tau_2}$ follows without the normalization by the chemical potential. The attachment of factors of i and -1 on the components are also not needed for the unprimed matrices. With the above definitions, the system of EOM could be compactly written as

$$\partial_z \bar{T}'(z)_{\tau_1\tau_2} = \left(-2\bar{T}'(z)_{\tau_1\tau_2} + 2\bar{T}'(z)_{\tau_1\tau_3}\bar{T}'(z)_{\tau_3\tau_2}\right), \tag{2.3.15}$$

where $\bar{T}'(z)_{\tau_1\tau_2}$ is the saddle-point solution to $T'(z)_{\tau_1\tau_2}$.

Since the initial condition is translationally invariant in the temporal direction,

that is, it only depends on $\tau_1 - \tau_2$, and saddle-point evolution preserves the symmetry, it is more convenient to look at the bulk solution in the frequency basis. In such basis, only one frequency label ω is required. The formal solution of (2.3.15) therefore reads

$$\bar{T}'(z,\omega) = \left((1-e^{2z})I + e^{2z}\bar{T}'(0,\omega)^{-1}\right)^{-1}, \tag{2.3.16}$$

with the convention of frequency Fourier transform chosen to be

$$\bar{T}'(z,\omega) = \frac{1}{\Phi}\sum_{\tau_1\tau_2} \bar{T}'(z)_{\tau_1\tau_2} e^{-i\omega(\tau_1-\tau_2)}, \tag{2.3.17}$$

where Φ is the temporal size of the system. The derivation from (2.3.15) to (2.3.16) can be found in Appendix A. One can further introduce variables that are conjugate to the unit cell variables. In space, one generally needs two momenta because the translational symmetry is broken by the impurity. In the momentum space, (2.3.16) can be written as

$$\bar{T}'(z,\omega)_{k_1k_2} = \left((1-e^{2z})I + e^{2z}\bar{T}'(0,\omega)^{-1}\right)^{-1}_{k_1k_2}, \tag{2.3.18}$$

where

$$T'(z,\omega)_{k_1k_2} := \begin{pmatrix} t'(z,\omega)_{k_1k_2} & V'(z,\omega)_{k_1} \\ W'(z,\omega)_{k_2} & \frac{\mu(z,\omega)}{m_f} \end{pmatrix} \tag{2.3.19}$$

with

$$t'(z,\omega)_{k_1k_2} = \frac{1}{S}\sum_{r_1r_2} t'(z,\omega)_{r_1r_2} e^{-i(k_1r_1-k_2r_2)}, \tag{2.3.20}$$

$$V'(z,\omega)_{k_1} = \frac{1}{\sqrt{S}}\sum_{r_1} V'(z,\omega)_{r_1} e^{-ik_1r_1}, \tag{2.3.21}$$

$$W'(z,\omega)_{k_1} = \frac{1}{\sqrt{S}} \sum_{r_1} W'(z,\omega)_{r_1} e^{ik_1 r_1}. \tag{2.3.22}$$

Chapter 3

Exact solution to the EOM

In this section, we present the exact solution of the bulk equation of motion in the non-interacting case.

3.1 Without hybridization

We first consider the simple case without the impurity. We choose the UV theory that has only the nearest neighbor hoppings for the itinerant fermion. In real space, the UV boundary condition of the saddle-point hopping field of the itinerant fermion $(\bar{t}(0)_{\tau_1\tau_2})_{ij}$ reads

$$(\bar{t}(0)_{\tau_1\tau_2})_{ij} :=$$

$$\begin{pmatrix} \begin{cases} \lambda\delta_{r_i r_j}, & \text{if } \tau_1 = \tau_2 \\ \frac{1}{2}\delta_{r_i r_j}, & \text{if } \tau_1 - \tau_2 = 1 \\ \frac{-1}{2}\delta_{r_i r_j}, & \text{if } \tau_1 - \tau_2 = -1 \\ 0, & \text{else} \end{cases} & \begin{cases} t\delta_{\tau_1\tau_2}, & \text{if } r_i - r_j = 1 \text{ or } r_i = r_j \\ 0, & \text{else} \end{cases} \\ \begin{cases} t\delta_{\tau_1\tau_2}, & \text{if } r_i - r_j = -1 \text{ or } r_i = r_j \\ 0, & \text{else} \end{cases} & \begin{cases} -\lambda\delta_{r_i r_j}, & \text{if } \tau_1 = \tau_2 \\ \frac{1}{2}\delta_{r_i r_j}, & \text{if } \tau_1 - \tau_2 = 1 \\ \frac{-1}{2}\delta_{r_i r_j}, & \text{if } \tau_1 - \tau_2 = -1 \\ 0, & \text{else} \end{cases} \end{pmatrix}. \tag{3.1.1}$$

In the Fourier space, the hopping amplitude space becomes

$$\bar{t}(0,\omega)_{k_1 k_2} = \begin{pmatrix} \lambda - isin(\omega) & t(1+e^{-ik_1}) \\ t(1+e^{ik_1}) & -\lambda - isin(\omega) \end{pmatrix} \delta_{k_1 k_2}. \tag{3.1.2}$$

To extract the dispersion relation, we define $\epsilon' := t\,(k - \pi)$, where ϵ' represents the normalized deviation from the Dirac point. The dispersion of the itinerant fermion is determined by the total action, which is the sum of the UV and IR actions. The gap of the itinerant fermion is $\delta = m_\psi - \lambda$. We are interested in the case where the itinerant fermion is critical and set $\lambda = m_\psi$ in the rest of the paper. To leading order in ϵ', ω, the hopping field reads

$$\bar{t}(0,\omega)_{\epsilon'\epsilon'_1} = \begin{pmatrix} m_\psi - i\omega & i\epsilon' \\ -i\epsilon' & -m_\psi - i\omega \end{pmatrix} \delta_{\epsilon'\epsilon'_1}. \tag{3.1.3}$$

In this case, the dispersion, which is determined from the sum of the UV and IR actions, is determined from

$$\text{Eigenvalue}\left(m_\psi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \bar{t}(0,\omega)_{\epsilon'}\right) = \pm\epsilon' + i\omega. \tag{3.1.4}$$

As far as the theory respects the particle-hole symmetry and local, the hopping field is analytic in momentum and frequency space, and the inclusion of further short-ranged hoppings does not change the behavior of the hopping in the low ϵ', ω limit.

We change the label from ϵ', ω to $\epsilon'' = (e^{2z}-1)\frac{\epsilon'}{m_\psi}$, $\omega' = (e^{2z}-1)\frac{\omega}{m_\psi}$ and expand in powers of ω', ϵ'',

$$\bar{t'}^{[0]}(z,\omega')_{\epsilon''\epsilon_1''} = \frac{\begin{pmatrix} 1 - i\frac{e^{2z}}{e^{2z}-1}\omega' & \frac{e^{2z}}{e^{2z}-1}\epsilon'' \\ -\frac{e^{2z}}{e^{2z}-1}\epsilon'' & 1 + i\frac{e^{2z}}{e^{2z}-1}\omega' \end{pmatrix}\delta_{\epsilon''\epsilon_1''}}{1+\omega'^2+\epsilon''^2}, \tag{3.1.5}$$

where the label '[0]' stands for the case without hybridization. The hopping field remains linear in ω and ϵ' at low momentum and frequency. However, their coefficients increase with increasing z. This represents the fact that the hoppings spread in the spacetime lattice with the increasing RG scale.

To examine the renormalized hopping field in real space, we take the continuum limit and define

$$\bar{t'}^{[0]}(z,x,\tau) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega d\epsilon'}{(2\pi m_\psi)^2}\bar{t'}^{[0]}(z,\omega')_{\epsilon''\epsilon''}e^{i\epsilon' x + i\omega\tau}. \tag{3.1.6}$$

From

$$\bar{t}'^{[0]}(z,\omega')_{\epsilon''\epsilon_1''} \sim \frac{\begin{pmatrix} 1-i\omega' & \epsilon'' \\ -\epsilon'' & 1+i\omega' \end{pmatrix} \delta_{\epsilon''\epsilon_1''}}{1+\omega'^2+\epsilon''^2}, \tag{3.1.7}$$

we obtain

$$\bar{t}'(z,x,\tau) = \frac{1}{(e^{2z}-1)^2} \begin{pmatrix} 1-\partial_{\tau'} & -i\partial_{x'} \\ i\partial_{x'} & 1+\partial_{\tau'} \end{pmatrix} J(z,x,\tau), \tag{3.1.8}$$

where

$$J(z,x,\tau) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\omega'\, d\epsilon''}{(2\pi)^2} \frac{e^{i\epsilon'' x' + i\omega'\tau'}}{1+\omega'^2+\epsilon''^2} = \frac{K_0(\|\vec{r'}\|)}{2\pi} \tag{3.1.9}$$

and $x' = \frac{m_\psi x}{e^{2z}-1}$ and $\tau' = \frac{m_\psi \tau}{e^{2z}-1}$ are the variables conjugate to the rescaled momentum and frequency ϵ'' and ω'. Using identities of the Bessel function, we can write the renormalized hopping field at scale z as

$$\bar{t}'^{[0]}(z,x,\tau) = \frac{1}{2\pi(e^{2z}-1)^2} \begin{pmatrix} K_0(\|\vec{r'}\|) - \frac{\tau'}{\|\vec{r'}\|}K_1(\|\vec{r'}\|) & -i\frac{x'}{\|\vec{r'}\|}K_1(\|\vec{r'}\|) \\ i\frac{x'}{\|\vec{r'}\|}K_1(\|\vec{r'}\|) & K_0(\|\vec{r'}\|) + \frac{\tau'}{\|\vec{r'}\|}K_1(\|\vec{r'}\|) \end{pmatrix}. \tag{3.1.10}$$

The hopping field decays exponentially in $\|\vec{r'}\| = \sqrt{x'^2+\tau'^2} \gg 1$. When plotted in the original space and time coordinates, the range of hopping increases exponentially with increasing z as is shown in Fig. 3.3.

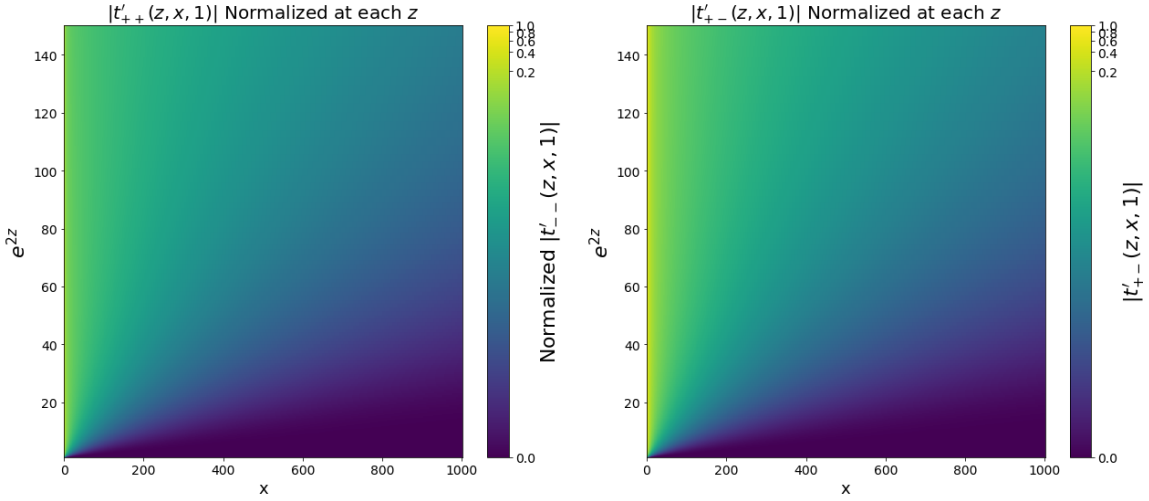

Figure 3.3: Plots of $|t'_{--}(z,x,1)|$ and $|t'_{+-}(z,x,1)|$ normalized by their maximum value at each z. At a fixed z, the hopping fields decay exponentially in x but the range of hopping increases as e^{2z} with increasing RG scale. This is because further neighbour hoppings are generated with an increasing length scale z.

3.2 Evolution of the saddle-point solution with hybridization

Now, we discuss the solution in the presence of impurity. At UV, the hybridization between the impurity and the itinerant fermion is chosen to be

$$\left(\bar{V}(0)_{\tau_1\tau_2}\right)_i := \begin{pmatrix} V\delta_{\tau_1,\tau_2}\delta_{i,0} \\ 0 \end{pmatrix}, \tag{3.2.1}$$

where the hybridization is present only for the first sub-lattice in unit cell 0. Its Fourier component reads

$$\bar{V}(0,\omega)_{\epsilon''} = \begin{pmatrix} \frac{V}{\sqrt{S}} \\ 0 \end{pmatrix}. \tag{3.2.2}$$

At the saddle-point, $\bar{W}(z)_{i\tau_1\tau_2}$ is simply the complex conjugate of $\bar{V}(z)_{i\tau_1\tau_2}$, while the impurity hopping

$$\bar{\mu}(0,\omega) = \bar{\mu}(0,0) - i\omega \tag{3.2.3}$$

. The exact solution for $\bar{t}'(z,\omega')_{\epsilon_1''\epsilon_2''}$ reads

$$\bar{t}'(z,\omega')_{\epsilon_1''\epsilon_2''} = \bar{t}'^{[0]}(z,\omega')_{\epsilon_1''\epsilon_2''} + C(z,\omega')\frac{\begin{pmatrix} -(i+\omega')^2 & \epsilon_2''(1-i\omega') \\ -(1-i\omega')\epsilon_1'' & -\epsilon_1''\epsilon_2'' \end{pmatrix}}{\xi(z,\omega')(1+\epsilon_1''^2+\omega'^2)(1+\epsilon_2''^2+\omega'^2)}, \tag{3.2.4}$$

where

$$C(z,\omega') := -\frac{V^2 e^{2z}\left(\bar{T}'^{-1}_{\mu}(0,\omega')\right)^2}{S m_\psi m_f}\left(\frac{1}{\bar{T}'^{-1}_{\mu}(0,0)} - e^{2z}\left(1-e^{2z}+e^{2z}\bar{T}'^{-1}_{\mu}(0,\omega')\right)^{-1}\right)$$

$$= -\frac{2V^2 e^{2z}\left(e^{2z}-1\right)}{2m_\psi\left(-im_\psi\omega' + \left(\bar{\mu}(0,0)-m_f\right)e^{2z} - \bar{\mu}(0,0)\right) - V^2\left(i\omega' + e^{2z} - 1\right)} \tag{3.2.5}$$

and

$$\xi(z,\omega') := 1+$$

$$Tr\left[e^{2z}\bar{T}'^{-1}_{V}(0,\omega')\left(\frac{1}{\bar{T}'^{-1}_{\mu}(0,\omega')} - e^{2z}\left(1-e^{2z}+e^{2z}\bar{T}'^{-1}_{\mu}(0,\omega')\right)^{-1}\right)\bar{T}'^{-1}_{W}(0,\omega')\bar{t}'^{[0]}(z,\omega')\right] \tag{3.2.6}$$

$$= 1 + \frac{V^2\left(2 - 2i\omega' + e^{2z}\left(i\omega' - 2\right)\right)}{\left(e^{2z}-1\right)\left(V^2\left(i\omega' + e^{2z} - 1\right) - 2m_\psi\left(-im_\psi\omega' + \left(\bar{\mu}(0,0) - m_f\right)e^{2z} - \bar{\mu}(0,0)\right)\right)}, \tag{3.2.7}$$

up to the first order of ω'. Here, $\bar{T'}_t^{-1}(z,\omega)$, $\bar{T'}_V^{-1}(z,\omega)$, $\bar{T'}_W^{-1}(z,\omega)$ and $\bar{T'}_\mu^{-1}(z,\omega)$ are the block matrices within $\bar{T'}^{-1}(z,\omega)$ when written as $\bar{T'}^{-1}(z,\omega) = \begin{pmatrix} \bar{T'}_t^{-1}(z,\omega) & \bar{T'}_V^{-1}(z,\omega) \\ \bar{T'}_W^{-1}(z,\omega) & \bar{T'}_\mu^{-1}(z,\omega) \end{pmatrix}$. To examine the profile of the hybridization in real space, let us consider the hybridization at zero frequency. The exact solution of hybridization is given by

$$\bar{V'}(z,\omega')_{\epsilon''} = -\frac{e^{2z}\int_{-\infty}^{\infty}\frac{d\epsilon_1''}{2\pi(e^{2z}-1)}S\bar{t'}(z,\omega')_{\epsilon''\epsilon_1''}\bar{T'}_V^{-1}(0,\omega')_{\epsilon_1''}}{\bar{T'}_\mu^{-1}(z,\omega')}. \tag{3.2.8}$$

. Therefore,

$$\bar{V}(z,0)_{\epsilon''} = -\frac{V\begin{pmatrix} 1 \\ -i\epsilon'' \end{pmatrix}}{\sqrt{S}\left(1-e^{-2z}\right)\left(\left(e^{2z}-1\right)\left(\frac{\bar{\mu}(0,0)}{m_f} + \frac{V^2}{2m_\psi m_f}\right) - e^{2z}\right)\left(1+\epsilon''^2\right)}. \tag{3.2.9}$$

Its Fourier transformation gives the hybridization in real space,

$$\bar{V}(z,x') := \int_{-\infty}^{\infty}\frac{d\epsilon''}{2\pi(e^{2z}-1)}\sqrt{S}\bar{V}(z,0)_{\epsilon''}e^{i\epsilon''x'} \tag{3.2.10}$$

$$= -\frac{Ve^{-|x'|}\begin{pmatrix} 1 \\ Sgn(x') \end{pmatrix}}{\left(1-e^{-2z}\right)\left(\left(e^{2z}-1\right)\left(\frac{\bar{\mu}(0,0)}{m_f} + \frac{V^2}{2m_\psi m_f}\right) - e^{2z}\right)}. \tag{3.2.11}$$

At a fixed z, the hybridization decays exponentially in $x' = \frac{m_\psi x}{e^{2z}-1}$. Its range is independent of the strength of the hybridization at UV as its decay is controlled by

the range of hopping of the itinerant fermions.

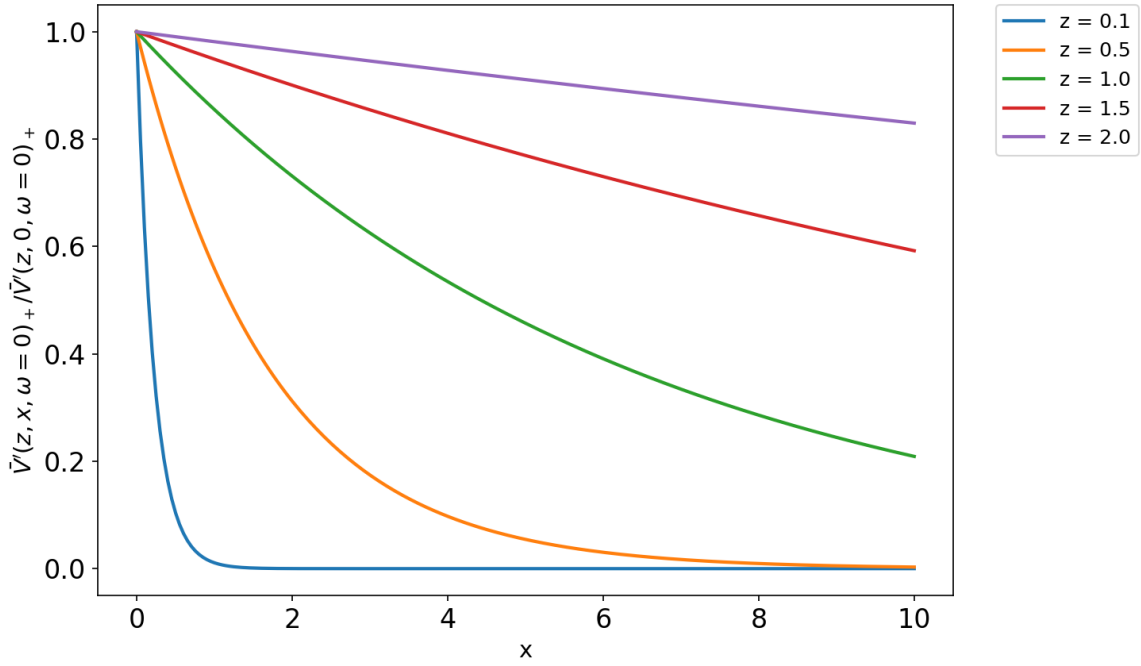

Figure 3.4: Spatial profile of hybridization $\bar{V}'(z, x, \omega = 0)$ plotted as a function of x for different choices z with $m_\psi = m_f = V = 1$.

In Fig. 3.4, the renormalized hybridization is shown as a function of x in real space. With increasing z, the range of the hybridization increases as e^{2z}. This is because the itinerant fermions acquires longer-range hopping at larger z and the impurity also hybridizes with fermions far from the central site 0.

So far, the chemical potential of the impurity is kept general. The spatial profile of the hybridization does not depend on the chemical potential of the impurity. However, the temporal profile is sensitive to the chemical potential. Now, we consider how the temporal profile of the hybridization field is affected by the chemical potential of the impurity. The correction to the renormalized hopping field generated from the

impurity is the second term of (3.2.4). If we take

$$\bar{\mu}(0,\omega) = m_f - i\omega, \tag{3.2.12}$$

(3.2.7) becomes

$$\xi(z,\omega') = 1 + \frac{V^2\left(2 - 2i\omega' + e^{2z}\left(i\omega' - 2\right)\right)}{\left(e^{2z}-1\right)\left(V^2\left(i\omega' + e^{2z} - 1\right) - 2m_\psi\left(-im_\psi\omega' - m_f\right)\right)}. \tag{3.2.13}$$

At z^* that satisfies

$$V^2(e^{2z^*} - 1) = 2m_\psi m_f, \tag{3.2.14}$$

ξ vanishes, which makes the renormalized hopping field singular. This singularity reflects the fact that the full theory is not smoothly connected to the IR fixed point theory[10]. Namely, this singularity along the RG flow arises when we have chosen the IR fixed theory such that the occupation number of the impurity site is zero while the full theory support non-zero occupation number. At the singularity, $\bar{T}'(z,0)$ is divergent and

$$M(z) := m_\psi e^{-2z} det\left[\bar{T}'(z,0)^{-1}\right] = m_\psi e^{-2z} det\left[\left(1 - e^{2z}\right) I + e^{2z} T'(0,0)^{-1}\right] \tag{3.2.15}$$

vanishes.

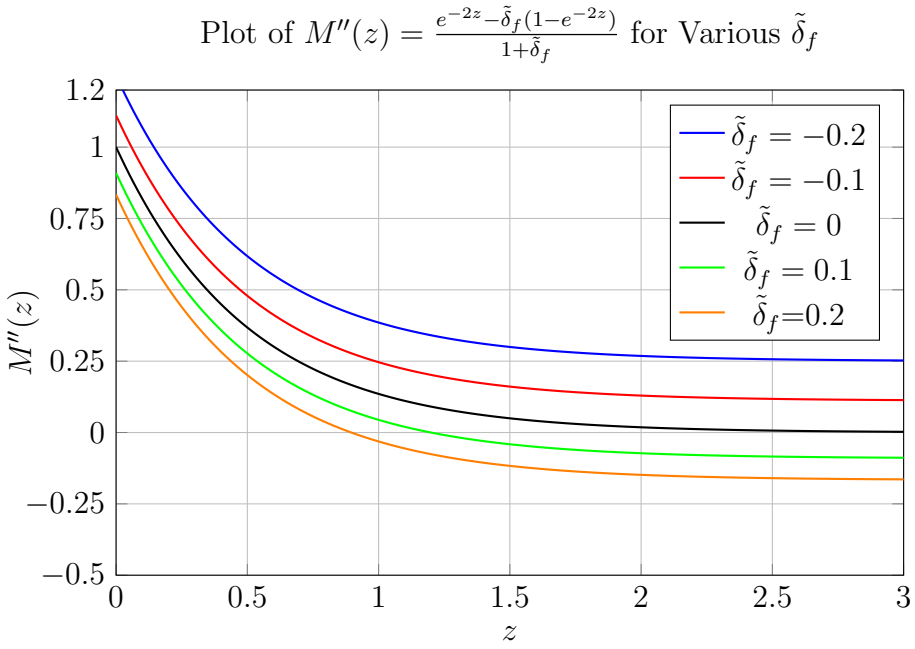

Figure 3.5: $M''(z) = \frac{e^{-2z} - \tilde{\delta}_f(1-e^{-2z})}{1+\tilde{\delta}_f}$ for the impurity plotted as a function for z for various values of the bare chemical potential, where $\tilde{\delta}_f = \delta_f + \frac{V^2}{2m_\psi m_f}$. It is noted that the sign of $M'(z)$ changes at finite z if and only if $\tilde{\delta}_f > 0$. When $\tilde{\delta}_f = 0$, the impurity is critical.

In order to avoid such singularity, we tune the UV chemical potential of the impurity so that $M(z)$ does not change sign under the RG flow. This is done by introducing a new deformation in the UV action,

$$S'_f = -m_f \delta_f \bar{f}_{\tau\sigma} f_{\tau\sigma}, \tag{3.2.16}$$

which tunes the bare chemical potential of the impurity. For sufficiently negative δ_f, the impurity site is fully empty, which guarantees that it is in the same phase described by the IR fixed point action. In this case, the RG flow is smooth at all z. As δ_f is increased, the impurity site undergoes a Lifshitz phase transition as its occupation number jumps. The singularity we encounter earlier implies that without

δ_f the impurity site is in the fully occupied state due to the renormalization of its chemical potential by itinerant fermions. In such cases, we should have chosen a different IR fixed point action with $m_f < 0$ to avoid the singularity in the RG flow. Below, we keep the same IR fixed point action with $m_f > 0$ and tune δ_f to the critical value from below the critical value. This amounts to approaching the critical point within the phase in which the impurity site remains empty. With δ_f, $M(z)$ becomes

$$M(z) \sim M''(z) := \frac{e^{-2z} - \delta_f \left(1 - e^{-2z}\right) - \frac{V^2\left(1-e^{-2z}\right)}{2m_\psi m_f}}{1 + \delta_f - \frac{V^2}{2m_\psi m_f}} \tag{3.2.17}$$

up to a smooth and positive function that becomes constant in the large z limit. The derivation is in Appendix C. At $\delta_f = -\frac{V^2}{2m_\psi m_f}$, which corresponds to the critical point for the impurity, $M''(z)$ approaches 0 asymptotically in the large z limit as is shown in Fig. 3.5.

The energy gap of the impurity mode is determined by $\tilde{\delta_f} := \delta_f + \frac{V^2}{2m_\psi m_f} \leq 0$. For $\tilde{\delta}_f < 0$ ($\tilde{\delta}_f > 0$), the impurity site is fully empty (occupied) and gapped. At $\tilde{\delta}_f = 0$, the impurity is half-filled and gapless as shown in Fig. 3.6.

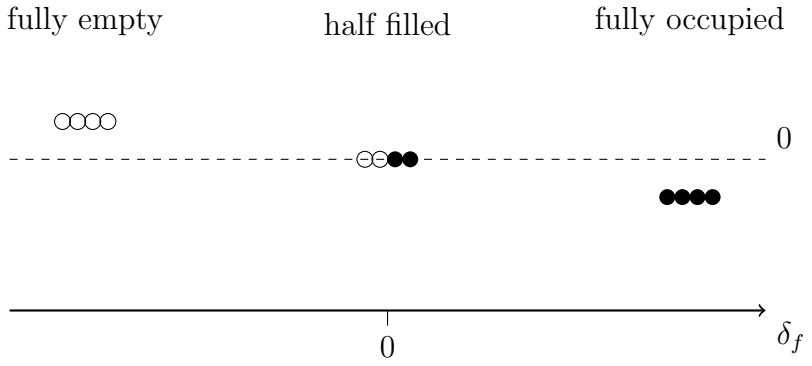

Figure 3.6: Pictorial illustration of the evolution of the occupation number of impurity site as a function of chemical potential.

To see how the gap of the impurity site affects the hybridization field, we consider the hybridization at $x = 0$ as a function of time τ',

$$\bar{V}(z, x = 0, \tau') := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\epsilon'' d\omega'}{\left(2\pi\left(e^{2z}-1\right)\right)^2}\sqrt{S}\bar{V}(z,\omega')_{\epsilon''}e^{i\omega'\tau'} \tag{3.2.18}$$

$$= -\frac{Ve^{2z}\left(a(z)-b(z)\right)e^{\frac{a(z)\tau'}{b(z)}}\theta\left(-\tau'\,\mathrm{sgn}\left(\frac{a(z)}{b(z)}\right)\right)\mathrm{sgn}\left(\frac{a(z)}{b(z)}\right)}{b(z)^2\left(-1+e^{2z}\right)}\begin{pmatrix}1\\0\end{pmatrix}, \tag{3.2.19}$$

where

$$a(z) = \left(e^{2z}-1\right)\tilde{\delta}_f - 1, \tag{3.2.20}$$

$$b(z) = \frac{V^2\left(e^{2z}-1\right)}{2m_\psi m_f} + \frac{m_\psi}{m_f}. \tag{3.2.21}$$

The Fourier transformation gives

$$\bar{V}(z, x = 0, \tau') = \frac{Ve^{2z}\theta\left(\tau'\right)\left(\left(\frac{V^2\left(e^{2z}-1\right)}{2m_\psi m_f}+\frac{m_\psi}{m_f}\right)+1-\left(e^{2z}-1\right)\tilde{\delta}_f\right)e^{\frac{\left(e^{2z}-1\right)\tilde{\delta}_f-1}{\left(\frac{V^2\left(e^{2z}-1\right)}{2m_\psi m_f}+\frac{m_\psi}{m_f}\right)}\tau'}}{\left(e^{2z}-1\right)^2\left(\frac{V^2\left(e^{2z}-1\right)}{2m_\psi m_f}+\frac{m_\psi}{m_f}\right)^2}\begin{pmatrix}1\\0\end{pmatrix}. \tag{3.2.22}$$

It is noted that the original time τ is related to the rescaled time τ', which is the conjugate variable of ω', as $\tau' \sim e^{-2z}\tau$ at large z. For $\tilde{\delta}_f < 0$, the range of the hybridization remains finite in τ' at all z as is shown in Fig. 3.7. In the original time coordinate, the temporal range of the hybridization scales as $\Delta\tau \sim e^{2z}$. As the chemical potential of the impurity is tuned toward the critical value, the hybridization decays more slowly. At the critical point with $\tilde{\delta}_f = 0$, the temporal range of the hybridization diverges as $\Delta\tau \sim e^{4z}$ in the large z limit as is shown in Fig. 3.8. When $\tilde{\delta}_f > 0$, a discontinuity appears in $\bar{V}(z, x = 0, \tau')$ as non-analyticity appears when z

crosses z^*. The temporal hopping therefore, becomes further non-local for $z > z^*$. This non-analyticity arises because of the 'wrong' choice of our IR fixed point action: the full theory supports the occupied impurity site whereas we have chosen the IR fixed point action that describes the empty impurity. The non-analyticity of the RG solution can be viewed as a Lifshitz transition that occurs as a function of the RG scale.

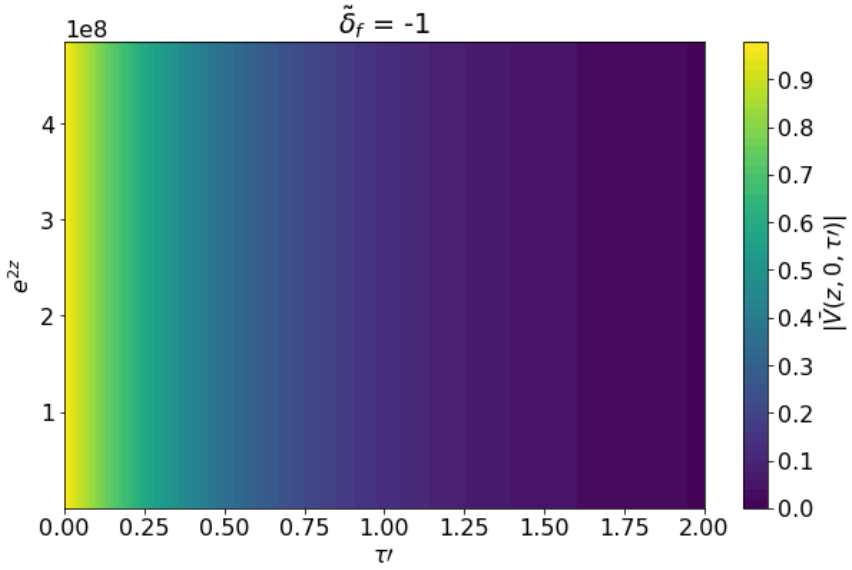

Figure 3.7: Temporal profile of the hybridization $\bar{V}(z, x = 0, \tau')$ normalized by its maximum value at each z for empty impurity with $\tilde{\delta}_f = -1$ and $m_\psi = m_f = V = 1$.

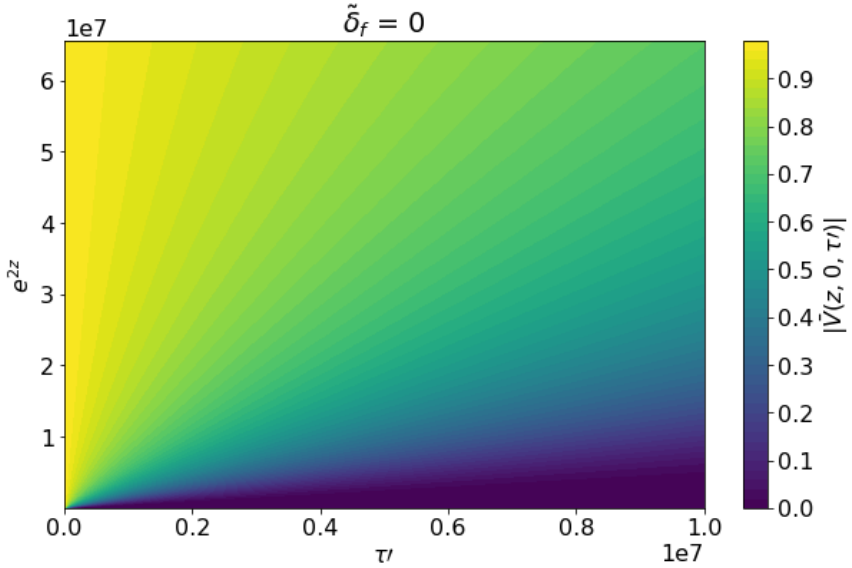

Figure 3.8: Temporal profile of the hybridization $\bar{V}(z, x = 0, \tau')$ normalized by its maximum value at each z for the critical impurity with $\tilde{\delta} = 0$ and $m_\psi = m_f = V = 1$.

Chapter 4

Conclusion

In this work, we applied the quantum renormalization group formalism to construct the holographic dual of the Anderson impurity model. The bulk theory, which describes the fluctuating renormalization group flow for the single-trace couplings, takes the form of the bi-local field theory of the dynamical hopping of itinerant fermions and the hybridization between itinerant fermion and the impurity. In the non-interacting limit, we solved the bulk equation of motion exactly, where the solution represents the real-space renormalization group flow of the hopping and hybridization. At the critical point, the range of hopping and the hybridization increases with increasing renormalization group length scale as longer-range hoppings and hybridizations are generated at low energies.

In the future, we hope to extract the bulk geometry based on the equation of motion for the propagating mode that represents the deformation added to the saddle-point solution. We also plan to extend the formalism developed in this thesis to the interacting theory in the limit that the number of flavours is large.

Appendix A

Derivation from (2.3.15) to (2.3.16)

In this appendix, we derive the formal solution (2.3.16) from (2.3.15). For simplicity, we write (2.3.15) in the Fourier space as

$$\partial_z \bar{T}'(z,\omega') = -2\bar{T}'(z,\omega') + 2\left[\bar{T}'(z,\omega')\right]^2. \tag{A.0.1}$$

We first introduce the substitution

$$G(z,\omega') = e^{2z}\bar{T}'(z,\omega'), \tag{A.0.2}$$

then we differentiate $G(z,\omega')$ with respect to z to obtain

$$\partial_z G(z,\omega') = 2e^{2z}\bar{T}'(z,\omega') + e^{2z}\partial_z \bar{T}'(z,\omega'). \tag{A.0.3}$$

This gives the equation of motion for $G(z,\omega')$,

$$G(z,\omega')^{-1}\partial_z G(z,\omega')G(z,\omega')^{-1} = 2e^{-2z}. \tag{A.0.4}$$

Integrating both sides with respect to z, we obtain

$$\int G(z,\omega')^{-1}\partial_z G(z,\omega')G(z,\omega')^{-1}\,dz = \int 2e^{-2z}\,dz. \tag{A.0.5}$$

From

$$\partial_z\left(G(z,\omega')G(z,\omega')^{-1}\right) = 0, \tag{A.0.6}$$

we obtain

$$G(z,\omega')^{-1}\partial_z G(z,\omega')G(z,\omega')^{-1} = -\partial_z G(z,\omega')^{-1}. \tag{A.0.7}$$

Combining this with (A.0.5), we have

$$-G(z,\omega')^{-1} = (-e^{-2z}+C)I, \tag{A.0.8}$$

where C is determined from the UV boundary condition,

$$C = I - \left[\bar{T}'(0,\omega')\right]^{-1}. \tag{A.0.9}$$

Substituting this back to the expression for $\bar{T}'(z,\omega')$, we obtain the formal solution for the renormalized hopping field,

$$\bar{T}'(z,\omega') = \left((1-e^{2z})I + e^{2z}\left[\bar{T}'(0,\omega')\right]^{-1}\right)^{-1}. \tag{A.0.10}$$

Appendix B

Derivation of Exact Solution of the Saddle Point Equation

In this appendix, we derive the exact solution of the saddle-point equation of motion when the impurity is present. We first need to calculate $\bar{T}'(0,\omega')^{-1}$. Since the $\bar{t}'$ sector is block diagonal in momentum basis, it is more convenient to work in that basis. The block matrix $\bar{T}'(0,\omega')$ takes the form

$$\bar{T}'(0,\omega') = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \tag{B.0.1}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are conformal block matrices.

$$\mathbf{A} = \bar{t}'(0,\omega) = \begin{pmatrix} \ddots & & & 0 \\ & \frac{m_\psi - \delta - i\omega}{m_\psi} & \frac{\epsilon'}{m_\psi} & \\ & \frac{-\epsilon'}{m_\psi} & \frac{m_\psi - \delta + i\omega}{m_\psi} & \\ 0 & & & \ddots \end{pmatrix}, \tag{B.0.2}$$

$$\mathbf{B} = \bar{V}'(0,\omega) = \frac{V}{\sqrt{S m_\psi m_f}} \begin{pmatrix} \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \tag{B.0.3}$$

$$\mathbf{C} = \bar{W}'(0,\omega) = \frac{V}{\sqrt{S m_\psi m_f}} \begin{pmatrix} \cdots & 1, & 0 & \cdots \end{pmatrix}, \tag{B.0.4}$$

$$\mathbf{D} = \frac{\bar{\mu}(0,\omega)}{m_f}. \tag{B.0.5}$$

From the block matrix identity [16], the inverse is written as

$$\bar{T}'(0,\omega')^{-1} = \begin{pmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{pmatrix}. \tag{B.0.6}$$

Each entry of the inverse matrix can be written as

$$\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C} = \begin{pmatrix} \ddots & & & 0 \\ & \frac{m_\psi - \delta - i\omega}{m_\psi} & \frac{\epsilon'}{m_\psi} & \\ & \frac{-\epsilon'}{m_\psi} & \frac{m_\psi - \delta + i\omega}{m_\psi} & \\ 0 & & & \ddots \end{pmatrix} - \frac{V^2}{S m_\psi \bar{\mu}(0,\omega)} \begin{pmatrix} 1 & 0 & \cdots & 1 & 0 \\ 0 & 0 & & 0 & 0 \\ \vdots & & \ddots & \vdots & \\ 1 & 0 & & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{B.0.7}$$

$$(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}) = \frac{\bar{\mu}(0,\omega)}{m_f}$$

$$-\frac{V^2}{Sm_\psi m_f}\begin{pmatrix}\cdots & 1 & , & 0 & \cdots\end{pmatrix}\begin{pmatrix}\ddots & & & 0\\ & \frac{m_\psi-\delta-i\omega}{m_\psi} & \frac{\epsilon'}{m_\psi} & \\ & \frac{-\epsilon'}{m_\psi} & \frac{m_\psi-\delta+i\omega}{m_\psi} & \\ 0 & & & \ddots\end{pmatrix}^{-1}\begin{pmatrix}\vdots\\ 1\\ 0\\ \vdots\end{pmatrix}, \quad \text{(B.0.8)}$$

$$-\mathbf{CA}^{-1}=\frac{-V}{\sqrt{Sm_\psi m_f}}\begin{pmatrix}\cdots & 1, & 0 & \cdots\end{pmatrix}\begin{pmatrix}\ddots & & & 0\\ & \frac{m_\psi-\delta-i\omega}{m_\psi} & \frac{\epsilon'}{m_\psi} & \\ & \frac{-\epsilon'}{m_\psi} & \frac{m_\psi-\delta+i\omega}{m_\psi} & \\ 0 & & & \ddots\end{pmatrix}^{-1}, \quad \text{(B.0.9)}$$

$$-\mathbf{BD}^{-1}=\frac{-V}{\sqrt{Sm_\psi m_f}}\frac{m_f}{\bar{\mu}(0,\omega)}\begin{pmatrix}\vdots\\ 1\\ 0\\ \vdots\end{pmatrix}. \quad \text{(B.0.10)}$$

Since $\mathbf{B}$ and $\mathbf{C}$ are column and row vectors respectively, while $\mathbf{D}$ is a scalar, the matrix $\mathbf{BD}^{-1}\mathbf{C}$ is rank 1. Hence, we use the Sherman-Morrison formula:

$$(\mathbf{A}-\mathbf{BD}^{-1}\mathbf{C})^{-1}=\mathbf{A}^{-1}+\frac{\mathbf{A}^{-1}(\mathbf{BD}^{-1}\mathbf{C})\mathbf{A}^{-1}}{1+g}, \quad \text{(B.0.11)}$$

where $g=\mathrm{Tr}\left\{(-\mathbf{BD}^{-1}\mathbf{C})\mathbf{A}^{-1}\right\}\neq -1$ [17]. Using this, we write the inverse of $\bar{T}'(z,\omega)$ as

$$\bar{T}'(z,\omega)^{-1}=\begin{pmatrix}\bar{T'}_t^{-1}(z,\omega) & \bar{T'}_V^{-1}(z,\omega)\\ \bar{T'}_W^{-1}(z,\omega) & \bar{T'}_\mu^{-1}(z,\omega)\end{pmatrix}, \quad \text{(B.0.12)}$$

where

$$\bar{T'}_V^{-1}(0,\omega)_{\epsilon'_1} = -\frac{V}{\sqrt{Sm_\psi m_f}} \frac{\begin{pmatrix} 1+i\frac{\omega}{m_\psi} \\ \frac{\epsilon'_1}{m_\psi} \end{pmatrix}}{\left(1+\left(\frac{\omega}{m_\psi}\right)^2+\left(\frac{\epsilon'_1}{m_\psi}\right)^2\right)\left(\frac{\bar{\mu}(0,\omega)}{m_f}-\sum_{\epsilon'}\frac{V^2}{Sm_\psi m_f}\frac{1+i\frac{\omega}{m_\psi}}{1+\left(\frac{\omega}{m_\psi}\right)^2+\left(\frac{\epsilon'}{m_\psi}\right)^2}\right)}, \tag{B.0.13}$$

$$\bar{T'}_W^{-1}(0,\omega)_{\epsilon'_2} = -\frac{V}{\sqrt{Sm_\psi m_f}} \frac{\begin{pmatrix} 1+i\frac{\omega}{m_\psi} , & -\frac{\epsilon'_2}{m_\psi} \end{pmatrix}}{\left(1+\left(\frac{\omega}{m_\psi}\right)^2+\left(\frac{\epsilon'_2}{m_\psi}\right)^2\right)\left(\frac{\bar{\mu}(0,\omega)}{m_f}-\sum_{\epsilon'}\frac{V^2}{Sm_\psi m_f}\frac{1+i\frac{\omega}{m_\psi}}{1+\left(\frac{\omega}{m_\psi}\right)^2+\left(\frac{\epsilon'}{m_\psi}\right)^2}\right)}, \tag{B.0.14}$$

$$\bar{T'}_\mu^{-1}(0,\omega) = \left(\frac{\bar{\mu}(0,\omega)}{m_f}-\sum_{\epsilon'}\frac{V^2}{Sm_\psi m_f}\frac{1+i\frac{\omega}{m_\psi}}{\left(1+\left(\frac{\omega}{m_\psi}\right)^2+\left(\frac{\epsilon'}{m_\psi}\right)^2\right)}\right)^{-1}, \tag{B.0.15}$$

$$\bar{T'}_t^{-1}(0,\omega)_{\epsilon'_1\epsilon'_2} = \left(\bar{t}^{[0]}(0,\omega)^{-1}\right)_{\epsilon'_1}\delta_{\epsilon'_1\epsilon'_2} + \bar{T'}_V^{-1}(0,\omega)_{\epsilon'_1}\frac{1}{\bar{T'}_\mu^{-1}(0,\omega)}\bar{T'}_W^{-1}(0,\omega)_{\epsilon'_2}. \tag{B.0.16}$$

$$= \begin{pmatrix} \frac{1+i\frac{\omega}{m_\psi}}{1+\left(\frac{\omega}{m_\psi}\right)^2+\left(\frac{\epsilon'}{m_\psi}\right)^2} & \frac{-\frac{\epsilon'}{m_\psi}}{1+\left(\frac{\omega}{m_\psi}\right)^2+\left(\frac{\epsilon'}{m_\psi}\right)^2} \\ \frac{\frac{\epsilon'}{m_\psi}}{1+\left(\frac{\omega}{m_\psi}\right)^2+\left(\frac{\epsilon'}{m_\psi}\right)^2} & \frac{1-i\frac{\omega}{m_\psi}}{1+\left(\frac{\omega}{m_\psi}\right)^2+\left(\frac{\epsilon'}{m_\psi}\right)^2} \end{pmatrix}\delta_{\epsilon'_1\epsilon'_2} + \frac{\frac{V^2}{Sm_\psi m_f}}{\frac{\bar{\mu}(0,\omega)}{m_f}-\sum_{\epsilon'}\frac{V^2}{Sm_\psi m_f}\frac{1+i\frac{\omega}{m_\psi}}{1+\left(\frac{\omega}{m_\psi}\right)^2+\left(\frac{\epsilon'}{m_\psi}\right)^2}}$$

$$\frac{1}{\left(1+\left(\frac{\omega}{m_\psi}\right)^2+\left(\frac{\epsilon'_1}{m_\psi}\right)^2\right)\left(1+\left(\frac{\omega}{m_\psi}\right)^2+\left(\frac{\epsilon'_2}{m_\psi}\right)^2\right)}\begin{pmatrix} \left(-i+\frac{\omega}{m_\psi}\right)\left(i-\frac{\omega}{m_\psi}\right) & \left(1+i\frac{\omega}{m_\psi}\right)\frac{\epsilon'_2}{m_\psi} \\ \frac{\epsilon'_1}{m_\psi}\left(1+i\frac{\omega}{m_\psi}\right) & -\frac{\epsilon'_2\epsilon'_1}{{m_\psi}^2} \end{pmatrix}. \tag{B.0.17}$$

It is convenient to use rescaled momentum and frequency defined by $\epsilon'' = \epsilon' \frac{e^{2z}-1}{m_\psi}$ and $\omega' = \omega \frac{e^{2z}-1}{m_\psi}$ to write

$$\bar{T}'(z,\omega)_{\epsilon_1'\epsilon_2'} = \bar{T}'(z,\omega')_{\epsilon_1''\epsilon_2''} = \left((1-e^{2z})I + e^{2z}\bar{T}'^{-1}(0,\omega')\right)^{-1}_{\epsilon_1''\epsilon_2''}. \tag{B.0.18}$$

To extract the hopping field for the itinerant fermion, we need to solve $\bar{t}'(z,\omega')_{\epsilon_1'',\epsilon_2''}$ explicitly. The identity (B.0.11) allows us to calculate $\bar{T}'^{-1}_V(0,\omega)_{\epsilon_1'} \frac{1}{\bar{T}'^{-1}_\mu(0,\omega)} \bar{T}'^{-1}_W(0,\omega)_{\epsilon_2'}$. Since $\bar{T}'^{-1}_V(0,\omega')_{\epsilon_1'}$ and $\bar{T}'^{-1}_W(0,\omega')_{\epsilon_2'}$ are column and row vectors respectively, their product must be of rank 1. By the block matrix inverse identities [16], we obtain

$$\bar{t}'(z,\omega')$$

$$= \left\{ e^{2z}\bar{T}'^{-1}_t(0,\omega') + (1-e^{2z})I - e^{4z}\bar{T}'^{-1}_V(0,\omega')\left(1-e^{2z}+e^{2z}\bar{T}'^{-1}_\mu(0,\omega')\right)^{-1}\bar{T}'^{-1}_W(0,\omega')\right\}^{-1}. \tag{B.0.19}$$

Together with (B.0.13), (B.0.14), (B.0.15), and (B.0.16), we reach the expression for the renormalized hopping field of itinerant fermion at scale z,

$$\bar{t}'(z,\omega')_{\epsilon_1''\epsilon_2''} = \bar{t}'^{[0]}(z,\omega')_{\epsilon_1''\epsilon_2''} + C(z,\omega')\frac{\begin{pmatrix} (1-i\omega')^2 & \epsilon_2''(1-i\omega') \\ -(1-i\omega')\epsilon_1'' & -\epsilon_1''\epsilon_2'' \end{pmatrix}}{\xi(z,\omega')(1+\epsilon_1''^2+\omega'^2)(1+\epsilon_2''^2+\omega'^2)}, \tag{B.0.20}$$

where

$$C(z,\omega') := -\frac{V^2e^{2z}\left(\bar{T}'^{-1}_\mu(0,\omega')\right)^2}{Sm_\psi m_f}\left(\frac{1}{\bar{T}'^{-1}_\mu(0,\omega')} - e^{2z}\left(1-e^{2z}+e^{2z}\bar{T}'^{-1}_\mu(0,\omega')\right)^{-1}\right)$$

$$= -\frac{2V^2 e^{2z}\left(e^{2z}-1\right)\sqrt{\frac{\omega'^2}{\left(e^{2z}-1\right)^2}+1}}{2m_\psi\sqrt{\frac{\omega'^2}{\left(e^{2z}-1\right)^2}+1}\left(-im_\psi\omega'+\left(\bar{\mu}(0,0)-m_f\right)e^{2z}-\bar{\mu}(0,0)\right)-V^2\left(i\omega'+e^{2z}-1\right)} \tag{B.0.21}$$

and

$$\xi(z,\omega') := 1+$$

$$\mathrm{Tr}\left[e^{2z}\bar{T'}_V^{-1}(0,\omega')\left(\frac{1}{\bar{T'}_\mu^{-1}(0,\omega')}-e^{2z}\left(1-e^{2z}+e^{2z}\bar{T'}_\mu^{-1}(0,\omega')\right)^{-1}\right)\bar{T'}_W^{-1}(0,\omega')\bar{t'}^{[0]}(z,\omega')\right] \tag{B.0.22}$$

$$=1+\frac{V^2\left(i\omega'+e^{2z}-1\right)\left(-e^{2z}\left(\sqrt{\omega'^2+1}\sqrt{\frac{\omega'^2}{\left(e^{2z}-1\right)^2}+1}-i\omega'+1\right)+\sqrt{\omega'^2+1}\sqrt{\frac{\omega'^2}{\left(e^{2z}-1\right)^2}+1}+\omega'^2+1\right)}{\left(e^{2z}-1\right)\sqrt{\left(\omega'^2+1\right)\left(\frac{\omega'^2}{\left(e^{2z}-1\right)^2}+1\right)}\left(V^2\left(i\omega'+e^{2z}-1\right)-2m_\psi\sqrt{\frac{\omega'^2}{\left(e^{2z}-1\right)^2}+1}\left(-im_\psi\omega'+\left(\bar{\mu}(0,0)-m_f\right)e^{2z}-\bar{\mu}(0,0)\right)\right)}. \tag{B.0.23}$$

Taking leading order dependence of ω' yields (3.2.5) and (3.2.7).

Appendix C

Derivation of the Profile of $M(z)$

In this appendix, we derive (3.2.17) from (3.2.15). The latter reads

$$M(z) := m_\psi e^{-2z} \det\left[\bar{T}'(z,0)^{-1}\right] = m_\psi e^{-2z} \det\left[\left(1-e^{2z}\right) I + e^{2z} T'(0,0)^{-1}\right]. \quad \text{(C.0.1)}$$

By the block matrix determinant identity [18],

$$\det \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det\left[\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right] \det[\mathbf{D}], \quad \text{(C.0.2)}$$

we can write $\det\left[\bar{T}'(z,\omega')^{-1}\right]$ as

$$\det\left[\bar{T}'(z,\omega')^{-1}\right] = \det\left[\bar{T}'_t(z,\omega')^{-1} - \frac{\bar{T}'_V(z,\omega')^{-1}\bar{T}'_W(z,\omega')^{-1}}{\bar{T}'_\mu(z,\omega')^{-1}}\right] \det\left[\bar{T}'_\mu(z,\omega')^{-1}\right]. \quad \text{(C.0.3)}$$

We first note that $\bar{T'}_\mu(z,\omega')^{-1} \neq 0$ for all z, ω'. [1] Hence, we can write

$$\det\left[\bar{T'}_t(z,\omega')^{-1} - \frac{\bar{T'}_V(z,\omega')^{-1}\bar{T'}_W(z,\omega')^{-1}}{\bar{T'}_\mu(z,\omega')^{-1}}\right] = \frac{\det\left[\bar{T'}(z,\omega')^{-1}\right]}{\bar{T'}_\mu(z,\omega')^{-1}}. \tag{C.0.8}$$

We then observe that

$$\det\left[\bar{T'}_t(z,\omega')^{-1} - \frac{\bar{T'}_V(z,\omega')^{-1}\bar{T'}_W(z,\omega')^{-1}}{\bar{T'}_\mu(z,\omega')^{-1}}\right] = \det\left[\left(\bar{t'}^{[0]}(z,\omega')\right)^{-1} + V_2(z,\omega')\right], \tag{C.0.9}$$

where

$$V_2(z,\omega')_{\epsilon_1''\epsilon_2''} := -C(z,\omega')\frac{\begin{pmatrix} (1-i\omega')^2 & (1-i\omega')\epsilon_2'' \\ -(1-i\omega')\epsilon_1'' & -\epsilon_1''\epsilon_2'' \end{pmatrix}}{(1+\epsilon_1''^2+\omega'^2)(1+\epsilon_2''^2+\omega'^2)}. \tag{C.0.10}$$

Using the matrix determinant lemma [18], we rewrite (C.0.9) as

$$\det\left[\bar{t'}^{[0]}(z,\omega') + V_2(z,\omega')\right] = \left(1 + u^T\left(\bar{t'}^{[0]}(z,\omega')\right)^{-1} v\right)\det\left[\left(\bar{t'}^{[0]}(z,\omega')\right)^{-1}\right], \tag{C.0.11}$$

---

[1] If

$$\bar{T'}_\mu(z,\omega')^{-1} = 0 \tag{C.0.4}$$

for a $z > 0$ and ω',

$$\bar{T'}_\mu(z,\omega')^{-1} = 1 - e^{2z} + e^{2z}\bar{T'}_\mu(0,\omega')^{-1} = 0. \tag{C.0.5}$$

This implies

$$\mathrm{Re}\left[\frac{1-e^{2z}}{\bar{T'}_\mu(0,\omega')^{-1}} + e^{2z}\right] = 0 \tag{C.0.6}$$

and

$$1 + \frac{V^2\left(e^{2z}-1\right)}{2m_\psi m_f\sqrt{1+\dfrac{\omega'^2}{\left(e^{2z}-1\right)^2}}} = 0. \tag{C.0.7}$$

This leads to a contradiction because the left-hand side is strictly positive for all $z > 0$, ω'.

where

$$V_2(z,\omega') = vu^T.$$

This decomposition is guaranteed to exist and unique up to orthogonal transformation because $V_2(z,\omega')$ is rank 1. Here, we take

$$u = \begin{pmatrix} \vdots \\ 1 - i\omega' \\ -\epsilon_1'' \\ \vdots \end{pmatrix}, \tag{C.0.12}$$

$$v = \begin{pmatrix} \vdots \\ 1 - i\omega' \\ \epsilon_1'' \\ \vdots \end{pmatrix}. \tag{C.0.13}$$

Since $\bar{t}'^{[0]}(z,\omega')$ is invertible, it suffices to focus on

$$\xi(z,\omega') \equiv \left(1 + u^T\left(\bar{t}'^{[0]}(z,\omega')\right)v\right) = 1 - C(z,\omega')\sum_{\epsilon''} \frac{\frac{\epsilon''^2}{e^{2z}-1} + (i+\omega')\left(-i + \frac{\omega'}{e^{2z}-1}\right)}{(1 + \frac{\omega'^2}{(e^{2z}-1)^2} + \frac{\epsilon''^2}{(e^{2z}-1)^2})(1 + \omega'^2 + \epsilon''^2)} \tag{C.0.14}$$

which has a root if and only if $\omega' = 0$. Therefore, we define $M'(z)$ as

$$M'(z) := \left(1 + u^T\left(\bar{t}'^{[0]}(z,0)\right)v\right) = 1 - C(z,0)e^{-2z}$$

$$= 1 - \frac{2V^2(e^{2z}-1)}{2m_\psi m_f + (e^{2z}-1)V^2}. \tag{C.0.15}$$

This has a root at $z = z^*$.

Now, let us consider the case with the chemical potential of the impurity tuned with δ_f. Since δ_f does not modify the frequency dependence nor the imaginary part of $\bar{\mu}(0,\omega')$, all roots of $\det\left[\bar{T'}_t(z,\omega')^{-1} - \frac{\bar{T'}_V(z,\omega')^{-1}\bar{T'}_W(z,\omega')^{-1}}{\bar{T'}_\mu(z,\omega')^{-1}}\right]$, $\det\left[(\bar{T'}_\mu(z,\omega')^{-1}\right]$ and hence $\det\left[\bar{T'}(z,\omega')^{-1}\right]$ can occur only at $\omega' = 0$. Therefore, it is sufficient to consider $\omega' = 0$. While $\bar{T'}_\mu(z,0)^{-1}$ itself may vanish for a general δ_f, $\det\left[\bar{T'}(z,\omega')^{-1}\right]$ does not vanish when $\bar{T'}_\mu(z,0)^{-1} = 0$.[2] From (C.0.8), (C.0.9), and (C.0.11), we see that $M(z)$ is proportional to $M''(z)$,

$$M''(z) := e^{-2z}\bar{T'}_\mu(z,0)^{-1}M'(z) \tag{C.0.20}$$

$$= e^{-2z}\left(\bar{T'}_\mu(z,0)^{-1} - \frac{V^2\left(e^{2z}-1\right)}{m_\psi m_f\left(1+\delta_f - \frac{V^2}{2m_\psi m_f}\right)}\right)$$

[2] We note that

$$\bar{T'}_\mu(z,0)^{-1} = 1 - e^{2z} + e^{2z}\bar{T'}_\mu^{-1}(0,0) = 0 \tag{C.0.16}$$

for

$$e^{2z} - 1 = \frac{1}{\delta_f - \frac{V^2}{2m_\psi m_f}}. \tag{C.0.17}$$

There exists z that satisfies this equation if

$$\delta_f - \frac{V^2}{2m_\psi m_f} \geq 0. \tag{C.0.18}$$

However, this does not make $M(z)$ to vanish. For a general δ_f, Eq. (C.0.15) becomes

$$M'(z) = 1 - \frac{2V^2(e^{2z}-1)}{V^2(e^{2z}-1) + 2m_\psi m_f(1+\delta_f) - 2e^{2z}m_\psi m_f\delta_f}$$

$$= 1 - \frac{V^2(e^{2z}-1)}{(m_\psi m_f)(1+\delta_f - \frac{V^2}{2m_\psi m_f})\bar{T'}_\mu(z,0)^{-1}}. \tag{C.0.19}$$

Since $\delta_f \geq \frac{V^2}{2m_\psi m_f}$, we find that $\lim_{z\longrightarrow z'}\left(M'(z)\bar{T'}_\mu(z,0)^{-1}\right) \neq 0$ as long as $V \neq 0$, where z' is defined by $\bar{T'}_\mu(z',0)^{-1} = 0$. Therefore, the root of $M(z)$ is the same as that for $M'(z)$.

$$= \frac{e^{-2z} - \delta_f \left(1 - e^{-2z}\right) - \frac{V^2\left(1-e^{-2z}\right)}{2m_\psi m_f}}{1 + \delta_f - \frac{V^2}{2m_\psi m_f}}. \tag{C.0.21}$$

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