AN EXACT THEORY OF STRAIN IN RODS OF FINITE TRANSVERSE DIMENSIONS

AN EXACT THEORY OF STRAIN IN RODS OF FINITE TRANSVERSE DIMENSIONS

BY

MICHAEL R. TROTH, B.Sc. (Eng.)

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS

FOR THE DEGREE

MASTER OF ENGINEERING

McMASTER UNIVERSITY

FEBRUARY 1971

MASTER OF ENGINEERING (1971)
(Civil Engineering and
Engineering Mechanics)

McMASTER UNIVERSITY
Hamilton, Ontario

TITLE: AN EXACT THEORY OF STRAIN IN RODS OF FINITE TRANSVERSE DIMENSIONS

AUTHOR: MICHAEL RICHARD TROTH, B.Sc. (Eng.),
Queen Mary College, University of London

SUPERVISOR: Professor G. AE. Oravas

NUMBER OF PAGES: xii, 114, A.10,B.18,C.6,D.11

SCOPE AND CONTENTS:

An exact analysis for the state of strain in a three dimensional rod continuum is presented. The exact geometrical description of the rod involves the evaluation of a power series expansion of the radius vector. It is shown however, that by a suitable choice of coördinates in the reference configuration and an interpretation of the deformation gradient as a material transformation, the strain tensor may be evaluated to the degree of accuracy inherent in using the full power series expansion of the radius vector without necessitating the explicit evaluation of the power series. Some concepts from the theory of multipolar media are used in order to make this three dimensional analysis compatible with the exact analysis of one dimensional rods.

ACKNOWLEDGEMENTS

I would like to thank my Research Supervisor, Professor G. AE. Oravas, under whose guidance and inspiration this work became a privilege and a pleasure.

During the preparation of this work I was financially supported by the National Research Council of Canada whose grant, number A-0904, was appreciated.

My acknowledgements would not be complete without an expression of thanks to my parents for their inestimable devotion, generosity and encouragement.

Michael Troth

Michael Troth

TABLE OF CONTENTS

NOTATION

$\bar{\bar{T}}_C$	The subscript, C, for any tensor indicates the conjugate of the tensor.
$(\ )_C$	Refers to quantities defined on the axis in the reference configuration, C.
$(\ )_c$	Refers to quantities defined on the axis in the current configuration, c.
	Repeated indices imply Einsteinian summation convention unless indices are underlined in which case summation is suspended.
	Greek indices take the values 1, 2.
	Latin indices take the values 1, 2, 3.
$\bar{a}_k$	Axial directors in current configuration = $(\bar{d}_k)_c$
$\bar{A}_K$	Axial directors in reference configuration = $(\bar{D}_K)_C$
c	Rod axis in current configuration
C	Rod axis in reference configuration
$\bar{\bar{C}}$	The Green deformation tensor $(= \bar{\bar{F}} \cdot \bar{\bar{F}}_C)$
$\bar{d}_k$	Directors in current configuration
$\bar{D}_K$	Directors in reference configuration
δ^i_j	Kroenecker delta
$d\bar{r}$	Material element in current configuration
$d\bar{R}$	Material element in reference configuration
e_{ijk}	Permutation symbol
$\bar{\bar{\varepsilon}}$	Euler strain tensor

$\bar{\bar{E}}$	Euler-Lagrange strain tensor
E_3	Affine Proper Euclidean three space
$\bar{\bar{F}}$, $\bar{F}$	Deformation gradient
$\bar{g}_k$	Base vectors in current configuration
$\bar{G}_K$	Base vectors in reference configuration
$\bar{\bar{g}}$	Metric tensor in current configuration
$\bar{\bar{G}}$	Metric tensor in reference configuration
$\bar{\bar{I}}$, $\bar{\bar{I}}$	Identity tensor
$\bar{\bar{\mu}}$	Transformation tensor in current configuration
$\bar{\bar{M}}$	Transformation tensor in reference configuration
$\bar{\bar{Q}}$	An Orthogonal tensor
$\bar{r}$	Position vector in current configuration
$\bar{R}$	Position vector in reference configuration
$\bar{r}(x^3)$, $\bar{r}_c$	Position vector of rod axis in current configuration
$\bar{R}(X^3)$	Position vector of rod axis in reference configuration
$\bar{u}$, $\bar{U}$	Displacement vector
x^k	Curvilinear coördinates in current configuration
X^K	Curvilinear coördinates in current configuration
ξ^k	Generalised curvilinear coördinates in current configuration
Ξ^K	Generalised curvilinear coördinates in reference configuration
$\binom{n}{r}$	Binomial coefficient $= \frac{n!}{r!(n-r)!}$
$\left\{ {k \atop \ell \ m} \right\}$	Christoffel symbol of second kind

EXPLANATION OF TERMS

A physical rod is modeled by a mathematical continuum consisting of a space manifold of points. When discussing the physical rod the adjectives "undeformed" and "deformed" refer to the states of the physical rod before and after deformation. When discussing the mathematical model the adjectives "reference" and "current" refer to the configuration of the space manifold occupied by the rod before and after a deformation is applied to the physical rod. The term "mapping" refers to process undergone by the space manifold corresponding to the deformation of the physical rod. The term "configuration" is applied to both the physical rod and the space manifold.

CHAPTER 1

MULTIPOLAR MEDIA THEORY IN CONTINUUM MECHANICS

Definition of a Body in Continuum Mechanics[1]*

In the classical continuum mechanics a body, B, consists of a set of particles sometimes called a manifold of material points. Each particle of the body is given a label or name, R. In the mechanics of discrete mass points labeling is usually done by means of a subscript, e.g., we may talk of the k^{th} particle having mass m_k and being at the position $\bar{R}_k$. The particles of the body may occupy various positions of a three dimensional Euclidean point space, E_3. (See Appendix A)

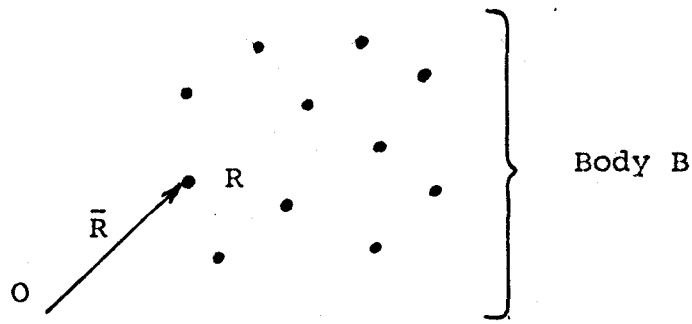


Figure 1

The complete specification of the positions of the particles of a body is called a *configuration* of the body. Such a configuration can be represented by a one-one mapping $\bar{\chi}$ or its inverse χ^{-1}. (See Appendix A)

---

* Superscripts in parentheses refer to references on page 114

$$\bar{R} = \bar{\chi}\ (R) \qquad R = \chi^{-1}\ (\bar{R}) \qquad \text{(I. 1)}$$

where $\bar{R}$, or R^i is a point in E_3 and is shown in Figure 1. Consider another configuration,

$$\bar{r} = \bar{\theta}\ (R) \qquad R = \theta^{-1}\ (\bar{r}) \qquad \text{(I. 2)}$$

The two configurations $\bar{\chi}$ and $\bar{\theta}$ are related by,

$$\bar{r} = \bar{\theta}\ (R) = \bar{\theta}(\chi^{-1}(\bar{R})) = \bar{\kappa}(\bar{R}) \qquad \text{(I. 3 i)}$$

$$\bar{R} = \bar{\kappa}^{-1}\ (\bar{r}) \qquad \text{(I. 3 ii)}$$

Thus far the body B is simply a set of points. We complete the definition of the body B by specifying that the transformation $\bar{R} = \bar{\kappa}^{-1}(\bar{r})$ be piecewise continuous*. Continuity means that two points arbitrarily close in the configuration $\bar{\chi}$ are arbitrarily close in $\bar{\theta}$. We shall also require that $\bar{R} = \bar{\kappa}^{-1}(\bar{r})$ be piecewise continuously differentiable to whatever order may be required.

Multipolar Media[3]

Multipolar media theory introduces a new concept into the classical model of a manifold of material points by postulating a set of vectors associated with each point. These vectors are called *directors*, the number of vectors may in general be arbitrary but normally the number is the same as the dimension of the space. These directors define *directions associated with the point* and are susceptible of rotations and stretches independant of the deformation of the material elements.

---

* See Appendix A. § "Limits and Continuity"

Such a model of a *directed medium* should include, amongst other possibilities representation of a molecular concept in which the molecules have an internal structure. These ideas will now be expressed in mathematical form.

In addition to the classical deformation characterised by a mapping,

$$\bar{r} = \bar{r}\ (\bar{R}) \tag{I. 4}$$

We have a transformation,

$$\bar{d}_b = \bar{d}_b(\bar{D}_a) = \bar{d}_b\ (\bar{R}) \tag{I. 5}$$

which may be defined independently.

Where $\bar{D}_a$ are the *directors in the reference configuration* and $\bar{d}_b$ are the *directors in the current configuration.*

The directors may be defined with respect to a local curvilinear basis at the point in the following way.

We have, $\bar{D}_a = \bar{D}_a \cdot \bar{\bar{I}}$

$$= \bar{D}_a \cdot \bar{G}^K \bar{G}_k$$

$$\therefore \bar{D}_a = D_a^K\ \bar{G}_K \tag{I. 6}$$

where $D_a^K \equiv \bar{D}_a \cdot \bar{G}^K$

We define the reciprocal directors, $\bar{D}^b$, such that,

$$\delta_a^b = \bar{D}_a \cdot \bar{D}^b = D_a^K\ \bar{G}_K \cdot D_M^b\ \bar{G}^M \qquad \text{by (I. 6)}$$

$$= D_a^K\ D_M^b\ \delta_K^M$$

$$\therefore \delta_a^b = D_a^K\ D_K^b \tag{I. 7}$$

and so,

$$1 = \bar{D}_a \cdot \bar{D}^a = D_a^K \, \bar{G}_K \cdot D_M^a \, \bar{G}^M = D_a^K \, D_M^a \, \delta_K^M$$

$$\therefore D_a^K \, D_M^a = \delta_M^K \qquad \text{(I. 8)}$$

Having now defined a director triad at every point in the continuum together with the reciprocal director triad, we may use this new basis at any point as a new frame of reference in the local Euclidean point space.

The metric tensor components for this director basis may be calculated as,

$$D_{ab} = \bar{D}_a \cdot \bar{D}_b = D_a^K \, D_b^M \, G_{KM} \qquad \text{(I. 9.i)}$$

and

$$D^{ab} = \bar{D}^a \cdot \bar{D}^b = D_K^a \, D_M^b \, G^{KM} \qquad \text{(I. 9.ii)}$$

We define a tensor $\bar{\bar{W}}$ by,

$$\bar{\bar{W}} = W^K_{MP} \, \bar{G}_K \, \bar{G}^M \equiv \frac{\partial}{\partial X^P} [\bar{D}_a] \quad \bar{D}^a$$

or,

$$\bar{\bar{W}} = W^K_{MP} \, \bar{G}_K \, \bar{G}^M \equiv D^K_{a,P} \, D^a_M \, \bar{G}_K \quad \bar{G}^M \qquad \text{(I.10.i)}$$

so that,

$$W^K_{MP} \equiv D^K_{a,P} \, D^a_M \qquad \text{(I.10.ii)}$$

where $D^{K}_{a,P}$ is the partial covariant derivative (See Appendix B). This tensor, $\bar{\bar{W}}$, is called the *wryness of the director frame* in the undeformed material. It may be observed that for the special case in which the magnitude of the directors and the angles between them are constant, as is the case for example if the directors are chosen as an orthogonal unit triad, and if the point $\bar{R}$ is made to traverse the X^{P} coördinate line at unit speed, the quantities $W^{K}{}_{MP}$ are the components of the angular velocity of the director frame, $\bar{D}_{a}$, carried by $\bar{R}$; and we have,

$$\frac{\partial}{\partial X^{P}} [\bar{D}_{a}] = \bar{\bar{W}} \cdot \bar{D}_{a} \qquad \text{(I.11)}$$

We could similarly define the wryness of the director frame in the deformed configuration but a wryness tensor so defined would not afford a comparison between the deformed and undeformed configurations since it refers only to the relative configurations of the director frames at different points in the deformed material. In order to compare reference and current configurations we use the generalisation of the angular velocities of the director frame at $\bar{r}$ relative to those of the director frame at $\bar{R}$ when $\bar{r}$ traverses the curve into which the path of $\bar{R}$ is deformed. Introducing the *relative wryness tensor*, $\bar{\bar{\Psi}}$, where,

$$\bar{\bar{\Psi}} = \Psi^{k}{}_{mP}\, \bar{g}_{k}\, \bar{g}^{m} \equiv \frac{\partial}{\partial X^{P}} [\bar{d}_{a}]\, \bar{d}^{a}$$

$$\therefore \quad \bar{\bar{\Psi}} = \Psi^{k}_{\ mP}\ \bar{g}_{k}\ \bar{g}^{m} \equiv d^{k}_{a;P}\ \ d^{a}_{m}\ \bar{g}_{k}\ \bar{g}^{m} \qquad \text{(I.12.i)}$$

so that,

$$\Psi^{k}_{\ mP} \equiv d^{k}_{a;P}\ \ d^{a}_{m} \qquad \text{(I.12.ii)}$$

where

$$d^{k}_{a;P} \equiv d^{k}_{a,\,p}\ \frac{\partial x^{p}}{\partial X^{P}}$$

is the total covariant derivative. (See Appendix B)

Special Cases of the Directors

We may summarise by stating, that in a directed medium a deformation consists of a transformation carrying $\bar{R}$ into $\bar{r}$ and the directors $\bar{D}_{b}$ at $\bar{R}$ into directors $\bar{d}_{a}$ at $\bar{r}$.

i.e. $\bar{r} = \bar{r}\ (\bar{R}) \qquad \bar{d}_{a} = \bar{d}_{a}\ (\bar{D}_{b})$ (I.13)

Conceptually, *a deformation consists of a displacement of the point and independent stretches and rotations of the directors*. Within this general concept we may define the directors to suit our convenience. They may be purely geometrical frames of reference transforming completely independently of the material continuum, or on the other hand they may be defined to represent some aspect of the material, possibly it's microstructure, transforming accordingly. For example in the

special case when,

$$\bar{d}_a = \bar{D}_a \cdot \bar{\bar{F}} \tag{I.14}$$

where $\bar{\bar{F}} \equiv \frac{\partial \bar{r}}{\partial \bar{R}}$, the material deformation gradient, the directors are material elements and their presence adds nothing to the classical description of strain. In the special case when,

$$\bar{d}_a = \bar{D}_a \cdot \bar{\bar{I}} \tag{I.15}$$

i.e.

$$\bar{d}_a = \bar{D}_a$$

the directors are invariable elements and again add nothing new.

CHAPTER 2

DEVELOPMENT OF THE ANALYSIS OF RODS OF FINITE DIMENSIONS

Rod Analysis

Rod analysis as a branch of continuum mechanics seeks to analyse the behaviour of continua having geometrical properties such that the space occupied by the rod comprises the neighbourhood of a mathematical curve given by the parametric equations $X^K = X^K(S)$, bounded by a surface $f(X^1, X^2, X^3) = 0$. The curve, C, is called the axis of the rod and the axial dimension of the rod is assumed to be of a higher order than either of the other two rod dimensions, which are called cross-sectional dimensions. The limiting case of a rod is a mathematical line in which the cross-sectional dimensions are reduced to zero.

In tracing the modern development of the analysis of rods of finite dimensions we shall examine the work of Hay, Ericksen and Truesdell, Suhubi, Antman and Green, Laws and Naghdi.

The Analysis of Hay[2]

The first work on the modern approach to rod analysis was done by G.E. Hay in 1942. Hay's analysis is a rigorous mathematical treatment of a rod of uniform cross section with

the external forces acting only at the ends. The scope of the paper is however limited to rods whose cross-sectional dimensions are of infinitesimal order.

Hay's Description of Rod Geometry

In Hay's description of the rod geometry in the reference configuration, the curvilinear coördinates are defined as the two principal axes of inertia of the cross section and the axis of the rod. The cross section of the unstrained rod is therefore defined plane and the coördinates identified with principal axes of inertia are defined straight. This assumption of straight coördinates is justified within the limits of the cross sectional dimensions since any curvilinear coördinates may be regarded as straight over an infinitesimal length. The unstrained cross sections are defined normal to the rod axis. In the description of the strained rod, it is acknowledged that the unstrained plane cross section will have deformed into a general surface. The lengths of the coördinates at a point in the unstrained rod cross section are constrained to remain constant throughout deformation, so that within an infinitesimal distance from the rod axis the cross-section in-plane deformation is accounted for in the transformation of the base vectors defined at every point on the rod axis.

In Hay's analysis the configuration of the unstrained rod is completely described by the rod axis and the cross section at each point on the axis, where the axis is the locus

of the cross-sectional centroids. Consequently the description of the rod axis, which is an arbitrary space curve, is of fundamental importance to the analysis. In order to describe the reference configuration of the rod axis Hay introduces a rotation vector instead of the classical Frenêt-Serrêt description of a curve. The base vector triads defined at each point on the rod axis are all orthogonal in the reference configuration as a consequence of defining the curvilinear coördinates as the two principal axes of inertia of the normal cross section and the rod axis, since each base vector is tangential to a curvilinear coördinate. In the reference configuration Hay considers two adjacent cross sections separated by an elemental increment of axial arc length. He then defines a vector at each point on the axis which describes the rotation of the base vector triad at that point which would be required for it to align itself with the adjacent triad, in the direction of increasing arc length. In the strained configuration such a vector may not be so simply defined since the base vector triads along the axis are no longer geometrically similar. A rotation vector, in this case, is defined again by considering two adjacent cross sections. In order for each cross section to align itself with the adjacent cross section, a certain displacement of the cross section would be necessary and a hypothetical strain is introduced in order to describe this displacement. A vector is in this case introduced at each point on the rod axis which

describes the rotation of the principal axes of this hypothetical strain between adjacent cross sections which would be necessary in order to align each cross section with its adjacent cross section, in the direction of increasing arc length.

Hay's Analysis of Strain

In his analysis of strain Hay's approach is to treat the rod as a three dimensional continuum, defining the strain tensor components e_{ij} by,

$$2e_{ij} = g_{ij} - \bar{g}_{ij} \qquad \text{(II. 1)}$$

where g_{ij} and $\bar{g}_{ij}$ are the metric tensor components in the unstrained and strained configurations respectively at an arbitrary point within the rod continuum.

Hay's Method of Approximation

In the strained rod configuration Hay expresses various parameters of the problem such as the metric tensor, Christoffel symbol, stress tensor and strain tensor at any point within the rod continuum in Taylor power series expansions about the corresponding parameters defined on the strained rod axis. The coefficients of the power series are found for the first few terms by use of constitutive equations and compatibility relations.

The Analysis of Ericksen and Truesdell[3]

In their paper Ericksen and Truesdell develop an exact analysis of the stress and strain of a unidimensional continuum which they call a rod but which is actually a mathematical space curve composed of material elements. Their geometrical description is thus more fundamental than that of Hay since their rod has zero cross sectional dimensions instead of first order infinitesimal cross-sectional dimensions. The rod is defined as a unidimensional Cosserat continuum with directors defined at each point of the curve subject to the restriction that the directors are defined as material elements. Thus the directors, which are vectors, transform as material elements. In a three dimensional continuum this transformation would take the form,

$$d_a^k = x^k_{;K} \; D_a^K \qquad \text{(II. 2)}$$

where $x^k_{;K}$ is the deformation gradient. However a three dimensional deformation gradient cannot be defined in a one dimensional continuum so that a tensor $A^k{}_K$ is defined by the directors of the one dimensional continuum such that,

$$A^k{}_K = d_a^k \; D_K^a$$

then,

$$d_a^k = A^k{}_K \; D_a^K \qquad \text{(II. 3)}$$

where $A^k{}_K \equiv x^k_{;K}$ along the axis of the rod.

This type of transformation is only admissible for

vectors of infinitesimal magnitude which is satisfied in this case since the directors have been defined to be material elements.

The orientation of the director triads in the reference configuration is described by a wryness tensor and the relative orientation of the director triads in the current configuration is described by the relative wryness tensor. In Hay's analysis the directors are collinear with the base vectors whereas in Ericksen and Truesdell's more general approach this would be a special case. It is demonstrated that the relative wryness includes and generalises Hay's rotation vector.

Ericksen and Truesdell's Analysis of Strain

Since only a unidimensional curve is considered, the strain in the rod is completely described by the stretch, λ, which is defined by,

$$\lambda \equiv \frac{ds}{dS} \qquad \text{(II. 4)}$$

where s and S represent arc lengths in the deformed and undeformed rod respectively.

Ericksen and Truesdell's Analysis of Stress

The analysis of stress by Ericksen and Truesdell is based on the stress principal which they define as:-"At each point on a rod, the action of the material to one side upon the

material to the other is equipollent to that of a stress resultant vector, $\bar{S}$, and a couple resultant vector, $\bar{M}$." An elemental length of the rod is considered and the conditions of its equilibrium give two equations relating;

i) the stress resultant vector and the external force

and

ii) the couple resultant vector, external force and external couple.

The Analysis of Suhubi[6]

Suhubi's paper is an attempt to adapt the methods of rod analysis to a physical rod of finite cross sectional dimensions in order to develop a method of analysis to meet practical engineering needs.

Suhubi's Description of Rod Geometry

In order to describe any point within the rod, Suhubi introduces normal coördinate systems associated with the axis of the rod. That is, in both the reference and current configurations he introduces at each point on the rod axis an orthonormal vector triad, one vector of which is tangential to the rod axis. The rod axis is described by a curvilinear coördinate X^3. The coördinate system is such that the transverse coördinates are rectilinear and define a plane cross section which is normal to the rod axis. On the curvilinear coördinate X^3 which is associated with the rod axis the

transverse coördinates have a zero value, $X^1 = 0 = X^2$. The transverse base vectors defined on the axis are collinear with the transverse coördinates. See Figure 2.

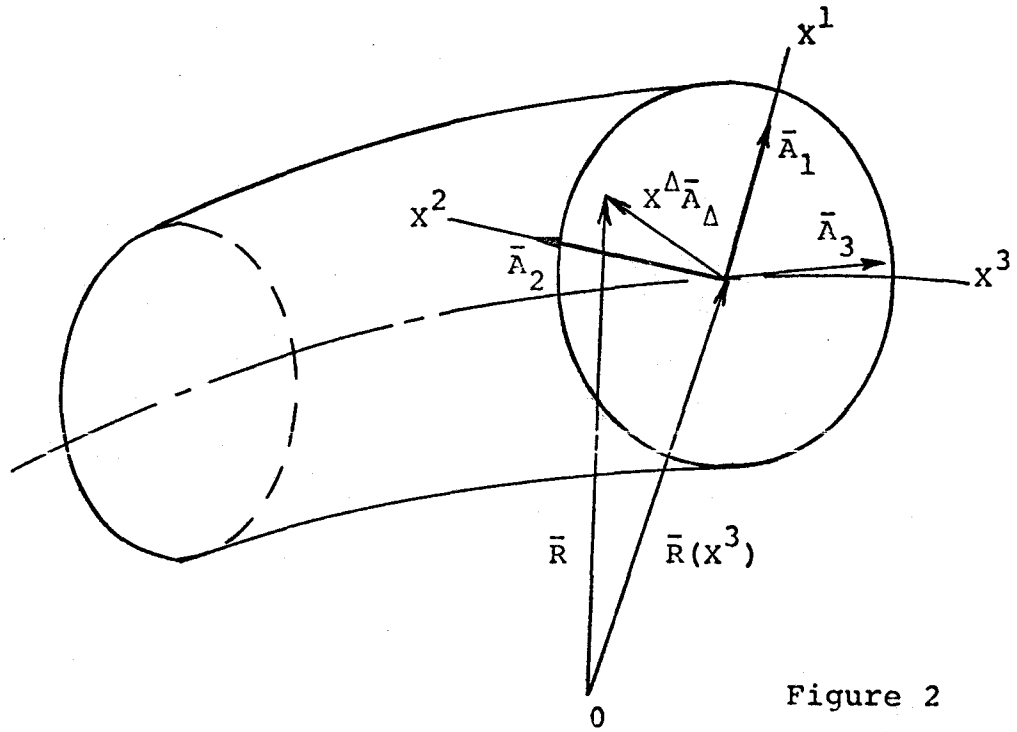


Figure 2

The position vector in the reference configuration to an arbitrary point in the rod cross section is then,

$$\bar{R} = \bar{R}(X^3) + X^\Delta \bar{A}_\Delta \qquad \text{(II. 5)}$$

where $\bar{R}(X^3)$ is the position vector to the rod axis and $\bar{A}_\Delta$ are the base vectors defined in the plane, normal cross-section. Similarly in the current configuration,

$$\bar{r} = \bar{r}(x^3) + x^\alpha \bar{a}_\alpha \qquad \text{(II. 6)}$$

the geometry of the current configuration being defined in the same way as that of the reference configuration.

Between the reference configuration and the current configuration there is the mapping,

$$\bar{r} = \bar{r}(\bar{R}) \qquad \text{(II. 7.i)}$$

which is considered to be invertible such that,

$$\bar{R} = \bar{R}(\bar{r}) \qquad \text{(II. 7.ii)}$$

Since in the reference configuration the rod axis is defined to lie along the X^3 curvilinear coördinate for which $X^1 = 0 = X^2$ and in the current configuration the rod axis is defined to lie along the x^3 curvilinear coördinate for which $x^1 = 0 = x^2$, the transformation imposes the restriction that the mapping between X^3 and x^3 must be isomorphic. (See Appendix A.)

i.e.

$$x^3 = x^3(X^3) \qquad \text{(II. 8.i)}$$

also,

$$X^3 = X^3(x^3) \qquad \text{(II. 8.ii)}$$

The transformation of the X^1 and X^2 coördinates however must be of a form such that rectilinear coördinates X^1 and X^2 are mapped into rectilinear coördinates x^1 and x^2 during an arbitrarily imposed deformation. Therefore the mapping between X^Δ and x^Δ cannot be isomorphic, and is defined

to be of the form,

$$x^{\alpha} = x^{\alpha}(X^{K}) \qquad (\alpha = 1,2;\ K = 1,2,3) \qquad \text{(II. 9.i)}$$

whose inverse is,

$$X^{\Delta} = X^{\Delta}(x^{k}) \qquad (\Delta = 1,2;\ k = 1,2,3) \qquad \text{(II. 9.ii)}$$

Suhubi then calls the quantities $\frac{\partial x^{k}}{\partial X^{K}}$ and $\frac{\partial X^{K}}{\partial x^{k}}$ the deformation gradients.

The Transformation Tensor

A useful geometrical development of Suhubi is the evaluation of the form of the transformation tensor, $\bar{\bar{\mu}}$, between the base vectors, $\bar{g}_i$, at an arbitrary point within the rod and the base vectors, $\bar{a}_j$, defined on the rod axis in the same cross section, such that,

$$\bar{g}_i = \bar{\bar{\mu}} \cdot \bar{a}_i \qquad \text{(II.10)}$$

or,

$$= \mu^{j}{}_{m}\, \bar{a}_j\, \bar{a}^{m} \cdot \bar{a}_i$$

so that in Suhubi's notation,

$$\bar{g}_i = \mu^{j}{}_{i}\, \bar{a}_j \qquad \text{(II.11)}$$

Using this transformation, all tensor quantities defined at any point within the rod with respect to the local base vector system may be referred to the base vector system defined on the axis in the same cross section.

Suhubi's Analysis of Stress and Strain

Suhubi deals with stress and strain as for a three dimensional body. His developments are only unique to rod analysis in that the vector and tensor quantities are referred to the axial base vector systems by use of the transformation tensor, $\bar{\bar{\mu}}$.

Suhubi's Analysis of Deformation

For thin rods Suhubi proposes the functional form for the transformation,

$$x^{\alpha} = x^{\alpha} \ (X^{K})$$

to be a power series in X^{Δ} with coefficients which are functions of X^{3}, truncated to the linear terms. Similarly he proposes a power series for

$$X^{\Delta} = X^{\Delta} \ (x^{k})$$

Criticism of Suhubi's Geometrical Description of a Rod and his Definition of Deformation

The criticism of Suhubi's geometrical description is that his definition of each cross section as being plane and normal to the axis in both the reference and current configurations will give, for an arbitrary space curve axis, a multi-valued position vector for each point within the rod for any cross sectional dimension greater than first order infinitesimal.

His geometrical description of the cross-section in the

reference and current configurations.

viz $$\bar{R} = \bar{R}(X^3) + X^\Delta \bar{A}_\Delta$$

and $$\bar{r} = \bar{r}(x^3) + x^\alpha \bar{a}_\alpha$$

while being mathematically correct has the disadvantage that the two cross sections so described are, in general, not the same material surface in the reference and current configurations respectively. This is an immediate consequence of the anisomorphic mapping of the two transverse coördinates, i.e.

$$x^\alpha = x^\alpha (X^K)$$

The deformation gradient used by Suhubi does not therefore describe the material deformation but has only a purely geometrical interpretation.

The Form of the Radius Vector Suggested by Antman[4] and Green, Laws and Naghdi[5]

A more general approach to the geometrical description of a rod which takes into account rod cross sections which are not plane was suggested by Antman. His radius vector takes the form of a function which is analytic with respect to x^α such that,

$$\bar{r} = \sum_{m+n=0}^{\infty} (x^1)^m (x^2)^n \frac{(m,n)}{r} (x^3)$$

where

$$\underset{r}{(m,n)} = \frac{1}{m!n!}\left[\frac{\partial^{m+n}\ \bar{r}}{\partial(x^1)^m\ \partial(x^2)^n}\right]$$

This is a generalisation of Suhubi's form of the radius vector and it may be observed that by truncating this series to the linear terms we may reproduce Suhubi's form of the radius vector. Antman's proposal of the form for the radius vector has the structure of a Taylor power series expansion about $\bar{r}\ (x^3)$. Antman describes the base vectors, the directors and the displacement vector in terms of this symbolic representation of the radius vector. The summation is not explicitly evaluated.

The power series expansion form for the radius vector is also mentioned by Green, Laws and Naghdi in the form,

$$\bar{r} = \bar{r}(\theta^3) + \sum_n \theta^{\alpha_1}\ \theta^{\alpha_2} \ \ldots\ \theta^{\alpha_n}\ \bar{d}_{\alpha_1\alpha_2\ \ldots\ \alpha_n}$$

where it is noted that the $\bar{d}_{\alpha_1\alpha_2\ \ldots\ \alpha_n}$ are completely symmetric in α_1, α_2, ..., α_n. Symbolic forms are given for the differential coefficient of $\bar{r}$, but the power series expansion is not explicitly evaluated. This is also of the form of a Taylor power series expansion about $\bar{r}\ (\theta^3)$ and the symmetric property of $\bar{d}_{\alpha_1\alpha_2\ldots\alpha_n}$ in α_1, α_2, ..., α_n may be demonstrated for a Taylor power series expansion.

This work as a Natural Consequence of the Previous Analyses

While it has been recognised in previous works that the form of the radius vector to an arbitrary point within the rod is a power series expansion, this power series has only been expressed in symbolic form and development of the expressions for the displacement vector and the strain tensor have utilised the truncated series only. It is therefore a natural step to investigate the consequences of the full power series expansion for the radius vector to a point within the rod.

The Thesis

The geometry of the rod and its form in previous works have been described by the radius vector and its functional form respectively. This approach restricts the form of the radius vector expression to the linear terms of the power series expansion due to the complex nature of any calculations based on the full power series expansion for the radius vector. In this work it is shown how, by judicious choice of the coördinate system in the reference configuration and an interpretation of the deformation gradient as a material transformation, the strain tensor may be evaluated with the degree of accuracy inherent in using the full power series expansion, without recourse to evaluating the power series and its derivatives.

An investigation was also made to determine the

applicability of Ericksen and Truesdell's analysis of a one-dimensional rod, viewed as a multipolar medium, to the problem of the three dimensional rod.

CHAPTER 3

GEOMETRY OF RODS OF FINITE CROSS SECTIONAL DIMENSIONS

Curvilinear Coördinates and the Rod Axis, c

In the current configuration we define, without loss of generality, the affine space occupied by the rod to be spanned by material parametric coördinates, x^k, such that the curve c lies along that x^3 coördinate associated with $x^1 = 0 = x^2$. A cross section of the rod at any point on c is that surface defined by x^3 = constant. We shall not define the cross sections to be planes normal to c in any configuration since this would entail defining the x^α coördinates perpendicular to c which would imply, for an arbitrary space curve axis, that at distances from c greater than infinitesimals of the first order the cross sections along c would intersect.

We define, in general, the cross sections along c to be arbitrary non-intersecting surfaces. The position vector, $\bar{r}$, of any point on c is given by,

$$(\bar{r})_c = \bar{r}(0,0,x^3) \equiv \bar{r}(x^3) \qquad \text{(III. 1)}$$

Since we have specified that the line c lies along an x^3 coördinate we may reduce the equation of the bounding surface for a prescribed cross section to,

$$f(x^1,x^2) = 0 \qquad \text{(III. 2)}$$

After Green[5], we define the relationship of the curve c to the boundary $f(x^1, x^2) = 0$ by the condition,

$$\iint \rho x^\alpha \, da = 0 \qquad \text{(III. 3)}$$

where ρ is the mass density and the integration is taken over any section, x^3 = constant, which is bounded by $f(x^1, x^2) = 0$.

Geometry of the Rod

We have the structure of the rod space in the current configuration,

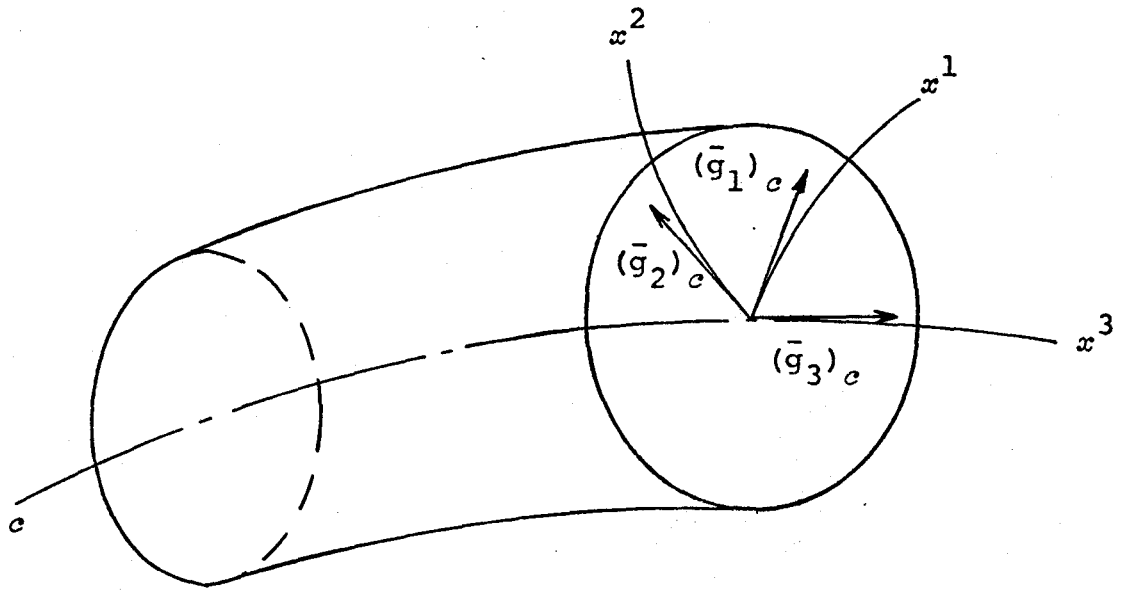


Figure 3

where $(\bar{g}_k)_c$ indicates the base vector $\bar{g}_k$ defined on the axis c. See Figure 3.

Let us consider a cross section, x^3 = constant as shown in Figure 4.

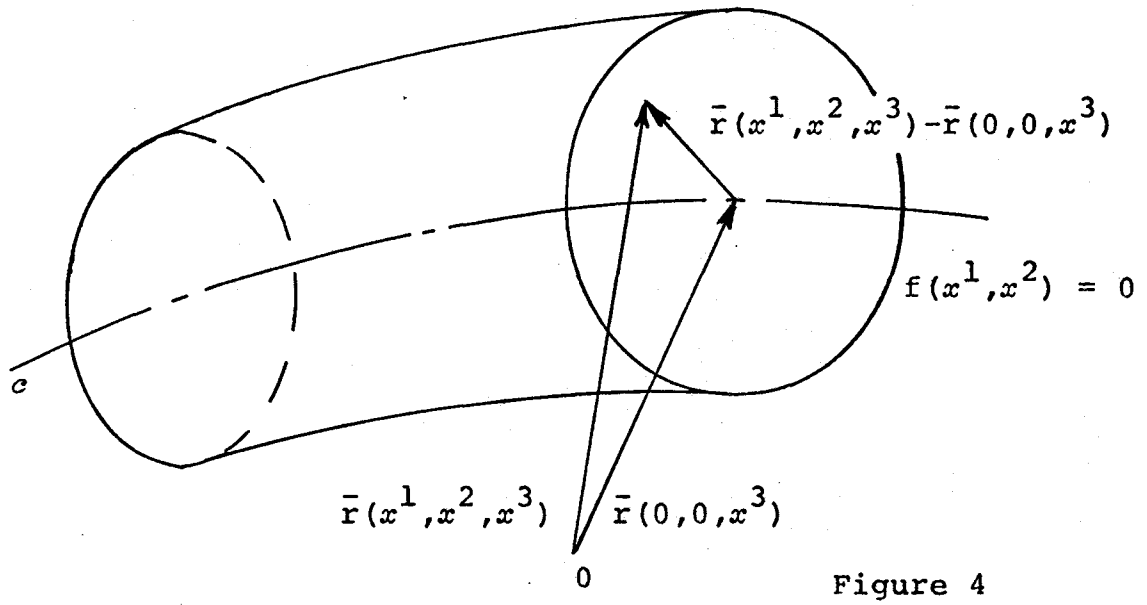


Figure 4

The position vector of any point on the cross section, x^3 = constant, is given by,

$$\bar{r}(x^1,x^2,x^3) = \bar{r}(0,0,x^3) + \left[\bar{r}(x^1,x^2,x^3) - \bar{r}(0,0,x^3)\right] \qquad \text{(III. 4)}$$

The expression,

$$\bar{r}(x^1,x^2,x^3) - \bar{r}(0,0,x^3)$$

may be expanded into a Taylor power series in the following way;

$$\bar{r}(x^1,x^2,x^3) - \bar{r}(0,0,x^3)$$

$$= x^1 \frac{\partial}{\partial x^1} \bar{r}(0,0,x^3) + x^2 \frac{\partial}{\partial x^2} \bar{r}(0,0,x^3) +$$

$$+ \frac{1}{2!}\left[x^1 x^1 \frac{\partial^2 \bar{r}(0,0,x^3)}{\partial (x^1)^2} + 2x^1 x^2 \frac{\partial^2 \bar{r}(0,0,x^3)}{\partial x^1 \partial x^2} + \right.$$

$$\left. + x^2 x^2 \frac{\partial^2 \bar{r}(0,0,x^3)}{\partial (x^2)^2} \right] + \ldots\ldots +$$

$$+ \frac{1}{n!}\left[(x^1)^n \frac{\partial^n \bar{r}(0,0,x^3)}{\partial (x^1)^n} + \binom{n}{1} (x^1)^{n-1} x^2 \frac{\partial^n \bar{r}(0,0,x^3)}{\partial (x^1)^{n-1} \partial x^2} + \ldots + \right.$$

$$\left. + (x^2)^n \frac{\partial^n \bar{r}(0,0,x^3)}{\partial (x^2)^n} \right] + \ldots\ldots \qquad \text{(III. 5)}$$

where

$$\frac{\partial^n \bar{r}(0,0,x^3)}{\partial (x^1)^{n-r} \partial (x^2)^r} \equiv \left[\frac{\partial^n \bar{r}(x^1,x^2,x^3)}{\partial (x^1)^{n-r} \partial (x^2)^r} \right] x^\alpha = 0 \qquad \text{(III. 6)}$$

Alternatively we may write,

$$\bar{r}(x^1,x^2,x^3) - \bar{r}(0,0,x^3) =$$

$$\sum_n \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} (x^1)^{n-r} (x^2)^r \frac{\partial^n \bar{r}(0,0,x^3)}{\partial (x^1)^{n-r} \partial (x^2)^r} \qquad n=1,2,3,\ldots$$

which may be expressed as,

$$\bar{r}(x^1,x^2,x^3) - \bar{r}(0,0,x^3)$$

$$= \sum_n \frac{1}{n!} \left[(x^1)^n \frac{\partial^n \bar{r}(0,0,x^3)}{\partial (x^1)^n} + (x^2)^n \frac{\partial^n \bar{r}(0,0,x^3)}{\partial (x^2)^n} + \right.$$

$$\left. + \sum_{r=1}^{n-1} \binom{n}{r} (x^1)^{n-r} (x^2)^r \frac{\partial^n \bar{r}(0,0,x^3)}{\partial (x^1)^{n-r} \partial (x^2)^r} \right] \qquad \text{(III. 7)}$$

$$n = 1,2,3,\ldots$$

Finally,

$$\bar{r}(x^1,x^2,x^3) - \bar{r}(0,0,x^3) = x^1(\bar{g}_1)_c + x^2(\bar{g}_2)_c +$$

$$+ \sum_n \frac{1}{n!} \left[(x^1)^n \frac{\partial^n \bar{r}(0,0,x^3)}{\partial (x^1)^n} + (x^2)^n \frac{\partial^n \bar{r}(0,0,x^3)}{\partial (x^2)^n} + \right.$$

$$\left. + \sum_{r=1}^{n-1} \binom{n}{r} (x^1)^{n-r} (x^2)^r \frac{\partial^n \bar{r}(0,0,x^3)}{\partial (x^1)^{n-r} \partial (x^2)^r} \right] \qquad \text{(III. 8)}$$

$$n \geqslant 2$$

where,

$$\frac{\partial \bar{r}(0,0,x^3)}{\partial x^\alpha} = (\bar{g}_\alpha)_c \qquad \text{(III. 9)}$$

Consequently we have,

$$\frac{\partial^2 \bar{r}(0,0,x^3)}{\partial(x^1)\,\partial(x^2)} = \left(\frac{\partial}{\partial x^2}\,\bar{g}_1\right)_c = \left\{\begin{matrix} k \\ 1\ 2 \end{matrix}\right\}_c (\bar{g}_k)_c \qquad \text{(III.10)}$$

where $\left\{\begin{matrix} k \\ 1\ 2 \end{matrix}\right\}$ is the Christoffel symbol of the second kind which we assume is analytic in x^k where,

$$\left\{\begin{matrix} m \\ k\ p \end{matrix}\right\} = \frac{1}{2}\, g^{m\ell}\left[\frac{\partial g_{\ell p}}{\partial x^k} + \frac{\partial g_{k\ell}}{\partial x^p} - \frac{\partial g_{kp}}{\partial x^\ell}\right] \qquad \text{(III.11)}$$

and it is easily demonstrated, that for symmetric rod space,

$$\left\{\begin{matrix} m \\ k\ p \end{matrix}\right\} = \left\{\begin{matrix} m \\ p\ k \end{matrix}\right\} \qquad \text{(III.12)}$$

We shall calculate the repeated differentials for an arbitrary radius vector, $\bar{r}(x^1,x^2,x^3)$, and then evaluate the result at $(0,0,x^3)$.

Consequently,

$$\bar{r}(x^1,x^2,x^3) - \bar{r}(0,0,x^3) = x^\alpha(\bar{g}_\alpha)_c +$$

$$+ \sum_n \frac{1}{n!}\left[(x^1)^n\left(\frac{\partial^{n-2}}{\partial(x^1)^{n-2}}\left\{\begin{matrix} k \\ 1\ 1 \end{matrix}\right\}\bar{g}_k\right)_c + (x^2)^n\left(\frac{\partial^{n-2}}{\partial(x^2)^{n-2}}\left\{\begin{matrix} k \\ 2\ 2 \end{matrix}\right\}\bar{g}_k\right)_c +\right.$$

$$\left. + \sum_{r=1}^{n-1}\binom{n}{r}(x^1)^{n-r}(x^2)^r\left(\frac{\partial^{n-2}}{\partial(x^1)^{n-r-1}\partial(x^2)^{r-1}}\left\{\begin{matrix} k \\ 1\ 2 \end{matrix}\right\}\bar{g}_k\right)_c\right] \qquad \text{(III.13)}$$

or,

$$\bar{r}(x^1,x^2,x^3) - \bar{r}(0,0,x^3) = x^\alpha(\bar{g}_\alpha)_c +$$

$$+ \sum_n \frac{1}{n!}(x^\alpha)^n\left(\frac{\partial^{n-2}}{\partial(x^\alpha)^{n-2}}\begin{Bmatrix} k \\ \alpha\ \alpha \end{Bmatrix}\bar{g}_k\right)_c +$$

$$+ \sum_n \frac{1}{n!}\sum_{r=1}^{n-1}\binom{n}{r}(x^1)^{n-r}(x^2)^r\left(\frac{\partial^{n-2}}{\partial(x^1)^{n-r-1}\partial(x^2)^{r-1}}\begin{Bmatrix} k \\ 1\ 2 \end{Bmatrix}\bar{g}_k\right)_c$$

(III.14)

In order to evaluate the higher order derivatives we use the Leibniz theorem for repeated differentiation which states: "If u and v are functions of x, then,

$$D^n(uv) = u_0v_n + \binom{n}{1}u_1v_{n-1} + \binom{n}{2}u_2v_{n-2} + \ldots +$$

$$+ \binom{n}{r}u_rv_{n-r} + \ldots + \binom{n}{n-1}u_{n-1}v_1 + u_nv_0 \qquad \text{(III.15)}$$

where suffices denote differentiation with respect to x; e.g., $u_0 \equiv u$; $u_rv_{n-r} \equiv D^r u\, D^{n-r}v$."

We may rewrite as a summation,

$$D^n(uv) = \sum_{r=0}^{n}\binom{n}{r}u_rv_{n-r} \qquad \text{(III.16)}$$

We now consider $\dfrac{\partial^{n-2}}{\partial (x^{\underline{\alpha}})^{n-2}} \left(\left\{ \begin{matrix} k \\ \underline{\alpha}\ \underline{\alpha} \end{matrix} \right\} \bar{g}_k \right)$.

By (III.16)

$$\frac{\partial^{n-2}}{\partial (x^{\underline{\alpha}})^{n-2}} \left(\left\{ \begin{matrix} k \\ \underline{\alpha}\ \underline{\alpha} \end{matrix} \right\} \bar{g}_k \right) = \sum_{r=0}^{n-2} \binom{n-2}{r} \frac{\partial^{n-2-r}}{\partial (x^{\underline{\alpha}})^{n-2-r}} \left\{ \begin{matrix} k \\ \underline{\alpha}\ \underline{\alpha} \end{matrix} \right\} \frac{\partial^{r}}{\partial (x^{\underline{\alpha}})^{r}} \bar{g}_k \qquad \text{(III.17)}$$

Where $\underline{\alpha}$ implies no summation with respect to α.

And

$$\frac{\partial^{r}}{\partial (x^{\alpha})^{r}} \bar{g}_k = \frac{\partial^{r-1}}{\partial (x^{\underline{\alpha}})^{r-1}} \left(\left\{ \begin{matrix} k_1 \\ k\ \ \underline{\alpha} \end{matrix} \right\} \bar{g}_{k_1} \right)$$

$$= \sum_{r_1=0}^{r-1} \binom{r-1}{r_1} \frac{\partial^{r-1-r_1}}{\partial (x^{\underline{\alpha}})^{r-1-r_1}} \left\{ \begin{matrix} k_1 \\ k\ \ \underline{\alpha} \end{matrix} \right\} \frac{\partial^{r_1}}{\partial (x^{\underline{\alpha}})^{r_1}} \bar{g}_{k_1} \quad \text{by (III.16)}$$

$$\therefore \quad \frac{\partial^{r}}{\partial (x^{\alpha})^{r}} \bar{g}_k =$$

$$\sum_{r_1=0}^{r-1} \binom{r-1}{r_1} \frac{\partial^{r-1-r_1}}{\partial (x^{\underline{\alpha}})^{r-1-r_1}} \left\{ \begin{matrix} k_1 \\ k\ \ \underline{\alpha} \end{matrix} \right\} \left[\sum_{r_2=0}^{r_1-1} \binom{r_1-1}{r_2} \frac{\partial^{r_1-1-r_2}}{\partial (x^{\underline{\alpha}})^{r_1-1-r_2}} \left\{ \begin{matrix} k_2 \\ k_1\ \ \underline{\alpha} \end{matrix} \right\} \left[\sum_{r_3=0}^{r_2-1} \cdots \right. \right.$$

$$\cdots\left[\sum_{r_p=0}^{r_{p-1}-1}\binom{r_{p-1}-1}{r_p}\frac{\partial^{r_{p-1}-1-r_p}}{\partial(x^{\underline{\alpha}})^{r_{p-1}-1-r_p}}\begin{Bmatrix}k_p\\ k_{p-1}\underline{\alpha}\end{Bmatrix}\frac{\partial^{r_p}}{\partial(x^{\underline{\alpha}})^{r_p}}\bar{g}_{k_p}\right]\cdots\Bigg]\Bigg] \tag{III.18}$$

to as many terms as are applicable for each value of r_1, r_2, . . ., in the summation until

$$r_{p-1}-1 = 0$$

when r_p is only summed over $r_p = 0$.

and then,

$$\sum_{r_p=0}^{r_{p-1}-1}\binom{r_{p-1}-1}{r_p}\frac{\partial^{r_{p-1}-1-r_p}}{\partial(x^{\underline{\alpha}})^{r_{p-1}-1-r_p}}\begin{Bmatrix}k_p\\ k_{p-1}\underline{\alpha}\end{Bmatrix}\frac{\partial^{r_p}}{\partial(x^{\underline{\alpha}})^{r_p}}\bar{g}_{k_p} = \begin{Bmatrix}k_p\\ k_{p-1}\alpha\end{Bmatrix}\bar{g}_{k_p}$$

Terms involving derivatives of the form,

$\frac{\partial^{r_{m-1}-1-r_m}}{\partial(x^{\underline{\alpha}})^{r_{m-1}-1-r_m}}\begin{Bmatrix}k_m\\ k_{m-1}\underline{\alpha}\end{Bmatrix}$ will disappear if the order of $\begin{Bmatrix}k_m\\ k_{m-1}\alpha\end{Bmatrix}$

in x^{α} is less than $(r_{m-1}-1-r_m)$.

Hence (III.17) becomes

$$\frac{\partial^{n-2}}{\partial (x^{\underline{\alpha}})^{n-2}}\left(\left\{ \begin{matrix} k \\ \underline{\alpha}\ \underline{\alpha} \end{matrix} \right\} \bar{g}_k\right)$$

$$= \sum_{r=0}^{n-2} \binom{n-2}{r} \frac{\partial^{n-2-r}}{\partial (x^{\underline{\alpha}})^{n-2-r}} \left\{ \begin{matrix} k \\ \underline{\alpha}\ \underline{\alpha} \end{matrix} \right\} \left[\sum_{r_1=0}^{r-1} \binom{r-1}{r_1} \frac{\partial^{r-1-r_1}}{\partial (x^{\underline{\alpha}})^{r-1-r_1}} \left\{ \begin{matrix} k_1 \\ k\ \underline{\alpha} \end{matrix} \right\} \right.$$

$$\left. \sum_{r_2=0}^{r_1-1} \binom{r_1-1}{r_2} \frac{\partial^{r_1-1-r_2}}{\partial (x^{\underline{\alpha}})^{r_1-1-r_2}} \left\{ \begin{matrix} k_2 \\ k_1\underline{\alpha} \end{matrix} \right\} \left[\sum_{r_3=0}^{r_2-1} \cdots \left[\left\{ \begin{matrix} k_p \\ k_{p-1}\underline{\alpha} \end{matrix} \right\} \bar{g}_{k_p} \right] \cdots \right] \right] \tag{III.19}$$

where it should be understood that the term

$$\left\{ \begin{matrix} k_p \\ k_{p-1}\underline{\alpha} \end{matrix} \right\} \bar{g}_{k_p}$$

is not constant over the summation but rather represents the terminal term of each expression constituting the summation. The value of p will be different for each combination of r_1, r_2, ..., r_{p-1}.

Equation (III.19 represents a series which may be finite or infinite depending on the functional form of the

Christoffel symbols. If it is infinite it may be evaluated to any degree of accuracy and if it is finite it may be evaluated exactly for any given n and a coördinate system of prescribed geometrical properties.

We now consider the term, $\frac{\partial^{n-2}}{\partial (x^1)^{n-r-1} \partial (x^2)^{r-1}} \left(\left\{ \begin{matrix} k \\ 1 \ 2 \end{matrix} \right\} \bar{g}_k \right)$

$$\frac{\partial^{n-2}}{\partial (x^1)^{n-r-1} \partial (x^2)^{r-1}} \left(\left\{ \begin{matrix} k \\ 1 \ 2 \end{matrix} \right\} \bar{g}_k \right) = \frac{\partial^{n-r-1}}{\partial (x^1)^{n-r-1}} \frac{\partial^{r-1}}{\partial (x^2)^{r-1}} \left(\left\{ \begin{matrix} k \\ 1 \ 2 \end{matrix} \right\} \bar{g}_k \right) \tag{III.20}$$

Now by (III.16),

$$\frac{\partial^{r-1}}{\partial (x^2)^{r-1}} \left(\left\{ \begin{matrix} k \\ 1 \ 2 \end{matrix} \right\} \bar{g}_k \right) = \sum_{s=0}^{r-1} \binom{r-1}{s} \frac{\partial^{r-1-s}}{\partial (x^2)^{r-1-s}} \left\{ \begin{matrix} k \\ 1 \ 2 \end{matrix} \right\} \frac{\partial^{s}}{\partial (x^2)^{s}} \bar{g}_k$$

and by (III.18)

$$\therefore \quad \frac{\partial^{r-1}}{\partial (x^2)^{r-1}} \left(\left\{ \begin{matrix} k \\ 1 \ 2 \end{matrix} \right\} \bar{g}_k \right)$$

$$= \sum_{s=0}^{r-1} \binom{r-1}{s} \frac{\partial^{r-1-s}}{\partial (x^2)^{r-1-s}} \left\{ \begin{matrix} k \\ 1 \ 2 \end{matrix} \right\} \left[\sum_{s_1=0}^{s-1} \binom{s-1}{s_1} \frac{\partial^{s-1-s_1}}{\partial (x^2)^{s-1-s_1}} \left\{ \begin{matrix} k_1 \\ k \ 2 \end{matrix} \right\} \left[\right.$$

$$\left. \sum_{s_2=0}^{s_1-1} \binom{s_1-1}{s_2} \frac{\partial^{s_1-1-s_2}}{\partial (x^2)^{s_1-1-s_2}} \left\{ \begin{matrix} k_2 \\ k_1 \ 2 \end{matrix} \right\} \left[\sum_{s_3=0}^{s_2-1} \cdots \left[\left\{ \begin{matrix} k_p \\ k_{p-1} \ 2 \end{matrix} \right\} \bar{g}_{k_p} \right] \cdots \right] \right] \right] \tag{III.21}$$

where the condition for $\left\{ {k_p \atop k_{p-1}\,2} \right\} \bar{g}_{k_p}$ stated on page 32 is understood.

To evaluate $\frac{\partial^{n-r-1}}{\partial (x^1)^{n-r-1}} \frac{\partial^{r-1}}{\partial (x^2)^{r-1}} \left(\left\{ {k \atop 1\ 2} \right\} \bar{g}_k \right)$ we must calculate $\frac{\partial^{n-r-1}}{\partial (x^1)^{n-r-1}}$ [equation (III.21)]. Since each term in the final expression for equation (III.21) will have one element from each of the individual summations, we may differentiate equation (III.21) term by term as for the differential of an ordinary product.

Consequently,

$$\frac{\partial^{n-r-1}}{\partial (x^1)^{n-r-1}} \frac{\partial^{r-1}}{\partial (x^2)^{r-1}} \left(\left\{ {k \atop 1\ 2} \right\} \bar{g}_k \right)$$

$$= \sum_{s=0}^{r-1} \binom{r-1}{s} \frac{\partial^{n-2-s}}{\partial (x^1)^{n-r-1} \partial (x^2)^{r-1-s}} \left\{ {k \atop 1\ 2} \right\} \left[\sum_{s_1=0}^{s-1} \binom{s-1}{s_1} \frac{\partial^{s-1-s_1}}{\partial (x^2)^{s-1-s_1}} \left\{ {k_1 \atop k\ 2} \right\} \left[\right. \right.$$

$$\left. \left. \sum_{s_2=0}^{s_1-1} \binom{s_1-1}{s_2} \frac{\partial^{s_1-1-s_2}}{\partial (x^2)^{s_1-1-s_2}} \left\{ {k_2 \atop k_1\ 2} \right\} \left[\sum_{s_3=0}^{s_2-1} \cdots \left[\left\{ {\ell \atop \ell_0\ 2} \right\} \bar{g}_\ell \cdots \right] \right] \right] \right] +$$

$$+ \sum_{s=0}^{r-1} \binom{r-1}{s} \frac{\partial^{r-1-s}}{\partial (x^2)^{r-1-s}} \left\{ {k \atop 1\ 2} \right\} \left[\sum_{s_1=0}^{s-1} \binom{s-1}{s_1} \frac{\partial^{n+s-r-s_1-2}}{\partial (x^1)^{n-r-1} \partial (x^2)^{s_1-1-s_1}} \left\{ {k_1 \atop k\ 2} \right\} \left[\right. \right.$$

$$\sum_{s_2=0}^{s_1-1}\binom{s_1-1}{s_2}\frac{\partial^{s_1-1-s_2}}{\partial(x^2)^{s_1-1-s_2}}\left\{{k_2 \atop k_1\ 2}\,{}_2\right\}\left[\sum_{s_3=0}^{s_2-1}\cdots\left[\left\{{\ell \atop \ell_0\ 2}\right\}\bar{g}_\ell\right]\cdots\right]\Bigg]\Bigg] +$$

$$+\sum_{s=0}^{r-1}\binom{r-1}{s}\frac{\partial^{r-1-s}}{\partial(x^2)^{r-1-s}}\left\{{k \atop 1\ 2}\right\}\left[\sum_{s_1=0}^{s-1}\binom{s-1}{s_1}\frac{\partial^{s-1-s_1}}{\partial(x^2)^{s-1-s_1}}\left\{{k_1 \atop k\ \ 2}\right\}\right[$$

$$\sum_{s_2=0}^{s_1-1}\binom{s_1-1}{s_2}\frac{\partial^{n+s_1-r-s_2-2}}{\partial(x^1)^{n-r-1}\partial(x^2)^{s_1-1-s_2}}\left\{{k_2 \atop k_1\ 2}\,{}_2\right\}\left[\sum_{s_3=0}^{s_2-1}\cdots\left[\left\{{\ell \atop \ell_0\ 2}\right\}\bar{g}_\ell\right]\cdots\right]\Bigg]\Bigg] +$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots +$$

$$+\sum_{s=0}^{r-1}\binom{r-1}{s}\frac{\partial^{r-1-s}}{\partial(x^2)^{r-1-s}}\left\{{k \atop 1\ 2}\right\}\left[\sum_{s_1=0}^{s-1}\binom{s-1}{s_1}\frac{\partial^{s-1-s_1}}{\partial(x^2)^{s-1-s_1}}\left\{{k_1 \atop k\ \ 2}\right\}\right[$$

$$\sum_{s_2=0}^{s_1-1}\binom{s_1-1}{s_2}\frac{\partial^{s_1-1-s_2}}{\partial(x^2)^{s_1-1-s_2}}\left\{{k_2 \atop k_1\ 2}\,{}_2\right\}\left[\sum_{s_3=0}^{s_2-1}\cdots\right.$$

$$\cdots\left[\sum_{t=0}^{n-r-1}\binom{n-r-1}{t}\frac{\partial^{n-r-1-t}}{\partial(x^1)^{n-r-1-t}}\left\{{\ell \atop \ell_0\ 2}\right\}\left(\sum_{t_1=0}^{t-1}\binom{t-1}{t_1}\frac{\partial^{t-1-t_1}}{\partial(x^1)^{t-1-t_1}}\left\{{\ell_1 \atop \ell\ \ 1}\right\}\right(\right.$$

$$\sum_{t_2=0}^{t_1-1}\binom{t_1-1}{t_2}\frac{\partial^{t_1-1-t_2}}{\partial(x^1)^{t_1-1-t_2}}\left\{\begin{matrix}\ell_2\\ \ell_1\ 1\end{matrix}\right\}\left(\sum_{t_3=0}^{t_2-1}\cdots\left(\left\{\begin{matrix}\ell_q\\ \ell_{q-1}\ 1\end{matrix}\right\}\bar{g}_{\ell_q}\right)\cdots\right)\cdots\Bigg]\Bigg]\Bigg] \qquad \text{(III.22)}$$

the last two lines being the expansion of,

$$\frac{\partial^{n-r-1}}{\partial(x^1)^{n-r-1}}\left[\left\{\begin{matrix}\ell\\ \ell_o\ 2\end{matrix}\right\}\bar{g}_\ell\right] = \sum_{t=0}^{n-r-1}\binom{n-r-1}{t}\frac{\partial^{n-r-1-t}}{\partial(x^1)^{n-r-1-t}}\left\{\begin{matrix}\ell\\ \ell_o\ 2\end{matrix}\right\}\frac{\partial^t}{\partial(x^1)^t}\,\bar{g}_\ell$$

by (III.16) and (III.18)

where $\ell_o \equiv k_{p-1}$; $\quad \ell = k_p$: and the condition for terms $\left\{\begin{matrix}\ell\\ \ell_o\ 2\end{matrix}\right\}\bar{g}_\ell$ and $\left\{\begin{matrix}\ell_q\\ \ell_{q-1}\ 1\end{matrix}\right\}\bar{g}_{\ell_q}$ stated on page 32 is understood.

Having expanded the derivatives of the terms in equation (III.14), we now evaluate the expressions at $x^\alpha = 0$ so that equation (III. 8) becomes,

$$\bar{r}(x^1,x^2,x^3) - \bar{r}(0,0,x^3) = x^\alpha(\bar{g}_\alpha)_c +$$

$$+\sum_n \frac{1}{n!}(x^\alpha)^n\left[\sum_{r=0}^{n-2}\binom{n-2}{r}\frac{\partial^{n-2-r}}{\partial(x^\alpha)^{n-2-r}}\left\{\begin{matrix}k\\ \alpha\ \alpha\end{matrix}\right\}\left[\sum_{r_1=0}^{r-1}\binom{r-1}{r_1}\frac{\partial^{r-1-r_1}}{\partial(x^\alpha)^{r-1-r_1}}\left\{\begin{matrix}k_1\\ k\ \alpha\end{matrix}\right\}\right.\right.$$

$$\sum_{r_2=0}^{r_1-1}\binom{r_1-1}{r_2}\frac{\partial^{r_1-1-r_2}}{\partial(x^\alpha)^{r_1-1-r_2}}\left\{{k_2 \atop k_1\ \alpha}\right\}\left[\sum_{r_3=0}^{r_2-1}\cdots\left[\left\{{k_p \atop k_{p-1}\alpha}\right\}\bar{g}_{k_p}\right]\cdots\right]\Bigg]\Bigg]_{x^\alpha=0} +$$

$$+\sum_n \frac{1}{n!}\sum_{r-1}^{n-1}\binom{n}{r}(x^1)^{n-r}(x^2)^r\left\{\sum_{s=0}^{r-1}\binom{r-1}{s}\frac{\partial^{n-2-s}}{\partial(x^1)^{n-r-1}\partial(x^2)^{r-1-s}}\left\{{k \atop 1\ 2}\right\}\right[$$

$$\sum_{s_1=0}^{s-1}\binom{s-1}{s_1}\frac{\partial^{s-1-s_1}}{\partial(x^2)^{s-1-s_1}}\left\{{k_1 \atop k\ \ 2}\right\}\left[\sum_{s_2=0}^{s_1-1}\cdots\left[\left\{{\ell \atop \ell_o\ 2}\right\}\bar{g}_\ell\right]\cdots\right]\Bigg] +$$

$$+\quad \sum_{s=0}^{r-1}\binom{r-1}{s}\frac{\partial^{r-1-s}}{\partial(x^2)^{r-1-s}}\left\{{k \atop 1\ 2}\right\}\Bigg[$$

$$\sum_{s_1=0}^{s-1}\binom{s-1}{s_1}\frac{\partial^{n+s-r-s_1-2}}{\partial(x^1)^{n-r-1}\partial(x^2)^{s-1-s_1}}\left\{{k_1 \atop k\ \ 2}\right\}\left[\sum_{s_2=0}^{s_1-1}\cdots\left[\left\{{\ell \atop \ell_o 2}\right\}\bar{g}_\ell\right]\cdots\right]\Bigg] +$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots + $$

$$+\sum_{s=0}^{r-1}\binom{r-1}{s}\frac{\partial^{r-1-s}}{\partial(x^2)^{r-1-s}}\left\{\begin{matrix}k\\1\ 2\end{matrix}\right\}\Bigg[$$

$$\sum_{s_1=0}^{s-1}\binom{s-1}{s_1}\frac{\partial^{s-1-s_1}}{\partial(x^2)^{s-1-s_1}}\left\{\begin{matrix}k_1\\k\ 2\end{matrix}\right\}\Bigg[\sum_{s_2=0}^{s_1-1}\cdots$$

$$\cdots\Bigg[\sum_{t=0}^{n-r-1}\binom{n-r-1}{t}\frac{\partial^{n-r-1-t}}{\partial(x^1)^{n-r-1-t}}\left\{\begin{matrix}\ell\\\ell_o\ 2\end{matrix}\right\}\Bigg(\sum_{t_1=0}^{t-1}\binom{t-1}{t_1}\frac{\partial^{t-1-t_1}}{\partial(x^1)^{t-1-t_1}}\left\{\begin{matrix}\ell_1\\\ell_1\ 1\end{matrix}\right\}\Bigg($$

$$\sum_{t_2=0}^{t_1-1}\binom{t_1-1}{t_2}\frac{\partial^{t_1-1-t_2}}{\partial(x^1)^{t_1-1-t_2}}\left\{\begin{matrix}\ell_2\\\ell_1\ 1\end{matrix}\right\}\Bigg(\sum_{t_3=0}^{t_2-1}\cdots$$

$$\cdots\left(\left\{\begin{matrix}\ell_q\\\ell_{q-1}\ 1\end{matrix}\right\}\bar{g}_{\ell_q}\right)\cdots\Big)\Big)\Big)\Big]\cdots\Big]\Big]\Big]\Big\}_{x^\alpha=0}$$

(III.23)

Giving the general expression for the radius vector to an arbitrary point within a rod continuum. Equation (III.23) is a series which may or may not be infinite, depending on the analytic form of the Christoffel symbols. If the Christoffel

symbols are expressed in such an analytic form that they are only differentiable a finite number of times with respect to the curvilinear parameters x^k, e.g. if the Christoffel symbols are expressed as a finite power series in x^k, then equation (III.23) is a finite series. If however the Christoffel symbols are expressed in such an analytic form that there is no maximum number of differentiations, e.g. if the Christoffel symbols are expressed in terms of trigononometric ratios of x^k, then equation (III.23) is an infinite power series and may be evaluated to any required degree of accuracy by prescribing a limit for n.

The Reference Configuration

Equation (III.23) gives the most general form for the radius vector to an arbitrary point within a rod in an arbitrary configuration occupying a space of arbitrary geometry.

We could, therefore, obtain the corresponding radius vector for a reference configuration of arbitrary geometry by substituting reference configuration parameters for current configuration parameters directly into equation (III.23).

Since we shall study the interrelationships between the reference and current configurations, we are at liberty to choose arbitrarily the coördinates which span one of these configurations within the limitations discussed previously. We are not at liberty to also choose the coördinates spanning the second configuration, since these are completely determined by the choice of one set and the deformation mapping.

Consequently, without loss of generality, we may

specify coördinates spanning the reference configuration. Since we have previously defined the axis of the rod to lie along an X^3 coördinate, no restriction can be placed on that X^3 coördinate. All the X^Δ coördinates, however, may be defined arbitrarily as long as they give a single valued representation to any point within the rod space. Consider the field of the X^1 coördinates to be a field of parallel straight lines. Consider the field of X^2 coördinates to be another field of parallel straight lines which are orthogonal to the X^1 coördinates. The rod axis is then an arbitrary space curve passing through these coördinate fields. The surfaces defined by X^3 = constant, on which lie the X^Δ coördinates for that value of X^3, will be plane but will not in general be normal to the rod axis.

Finally, in order to complete the description of the coördinates spanning the reference configuration, we must define the field of the X^3 coördinates with the exception of that coördinate for which $X^\Delta = 0$, which is the prescribed rod axis. The only condition for defining the rest of the field is that the X^3 coördinates must not intersect each other. We define the field of the X^3 coördinates to be such that for a particular cross section, the angles that any X^3 coördinate in the field subtends with the X^1 and X^2 coördinates of the cross section are the same as the angles which the X^3 coördinate defined as the axis, where $X^\Delta = 0$, subtends

with the X^1 and X^2 coördinates respectively in the same cross section. Thus for a particular cross section the base vector triad at an arbitrary point in the cross section is given by,

$$\bar{G}_K = (\bar{G}_K)_C \qquad \text{(III.24)}$$

CHAPTER 4

DEFORMATION

The Deformation Mapping

We describe the deformation of the rod by the mapping,

$$\bar{r} = \bar{r}(\bar{R}) \qquad \text{(IV. 1)}$$

characterised by the deformation gradient,

$$\bar{\bar{F}} \equiv \frac{\partial \bar{r}}{\partial \bar{R}} \qquad \text{(IV. 2)}$$

Since the axis of the rod in the reference configuration must map into the axis of the rod in the current configuration we have the condition on the mapping,

$$x^3 = x^3(X^3) \qquad \text{(IV. 3)}$$

We shall differ from Suhubi's definition of the mapping of the X^Δ curvilinear coördinates and define the mapping for each of the X^Δ coördinates to be of the same form as the mapping of the X^3 coördinate; i.e. we define the mapping for each of the curvilinear coördinates to be isomorphic such that,

$$x^k = x^k(X^k) \qquad \text{(IV. 4.i)}$$

$$\text{or} \qquad x^1 = x^1(X^1) \qquad \text{(IV. 4.ii)}$$

$$x^2 = x^2(X^2) \qquad \text{(IV. 4.iii)}$$

and

$$x^3 = x^3(X^3) \qquad \text{(IV. 4.iv)}$$

The reason for using this deformation mapping will become apparent later.

Then,

$$\frac{\partial x^i}{\partial X^j} = \delta^{\underline{i}}_{\underline{j}} \frac{\partial x^{\underline{i}}}{\partial X^{\underline{j}}} \qquad \text{(IV. 5)}$$

Therefore the Jacobian of the transformation (IV. 1) exists, i.e.

$$\left| \frac{\partial x^k}{\partial X^k} \right| \neq 0$$

and so we may define the inverse transformation,

$$\bar{R} = \bar{R}(\bar{r}) \qquad \text{(IV. 6)}$$

or

$$X^K = X^{\underline{K}}(x^{\underline{K}}) \qquad \text{(IV. 7)}$$

Transformation of Base Vectors

The base vectors in the reference configuration of a general curvilinear E_3 space are defined by,

$$\bar{G}_K \equiv \frac{\partial \bar{R}}{\partial X^K} \qquad \text{(IV. 8)}$$

Similarly, the base vectors for the current configuration,

$$\bar{g}_k \equiv \frac{\partial \bar{r}}{\partial x^k} \tag{IV. 9}$$

Now consider

$$\bar{g}_k \equiv \frac{\partial \bar{r}}{\partial x^k}$$

$$= \frac{\partial \bar{R}}{\partial x^k} \cdot \frac{\partial \bar{r}}{\partial \bar{R}}$$

$$= \frac{\partial X^K}{\partial x^k} \frac{\partial \bar{R}}{\partial X^K} \cdot \frac{\partial \bar{r}}{\partial \bar{R}}$$

where $\frac{\partial}{\partial \bar{R}} \equiv \bar{G}^K \frac{\partial}{\partial X^K}$

$\therefore$ $$\bar{g}_k = \frac{\partial X^K}{\partial x^k} \bar{G}_K \cdot \bar{\bar{F}}$$

or by (IV. 5)

$$\bar{g}_k = \frac{\partial X^{\underline{k}}}{\partial x^{\underline{k}}} \bar{G}_{\underline{k}} \cdot \bar{\bar{F}} \tag{IV.10}$$

where the deformation gradient $\bar{\bar{F}}$ is the transformation tensor for material elements, $d\bar{r}$, of the continuum; viz.

$$d\bar{r} = d\bar{R} \cdot \frac{\partial \bar{r}}{\partial \bar{R}} = d\bar{R} \cdot \bar{\bar{F}}$$

Thus the base vectors of a general continuum do not

transform according to the law governing the transformation of material elements.

The Directors

From equation (IV.10), we have for the base vectors,

$$\bar{g}_k = \frac{\partial X^{\underline{k}}}{\partial x^{\underline{k}}} \bar{G}_{\underline{k}} \cdot \bar{\bar{F}}$$

or,

$$\frac{\partial x^{\underline{k}}}{\partial X^{\underline{k}}} \bar{g}_{\underline{k}} = \bar{G}_k \cdot \bar{\bar{F}} \qquad \text{(IV.11)}$$

Now we define vectors $\bar{d}_k$ in the current configuration such that,

$$\bar{d}_k = \frac{\partial x^{\underline{k}}}{\partial X^{\underline{k}}} \bar{g}_{\underline{k}} \qquad \text{(IV.12)}$$

Each vector $\bar{d}_k$ will be collinear with the corresponding base vector $\bar{g}_k$ but their magnitudes will differ by the scalar multiple $\frac{\partial x^{\underline{k}}}{\partial X^{\underline{k}}}$.

In the reference configuration the corresponding vector, $\bar{D}_K$ will be,

$$\bar{D}_K = \frac{\partial X^{\underline{K}}}{\partial X^{\underline{K}}} \bar{G}_{\underline{K}}$$

or

$$\bar{D}_K = \bar{G}_K \qquad \text{(IV.13)}$$

Then equation (IV.11) becomes,

$$\bar{d}_k = \bar{D}_k \cdot \bar{\bar{F}} \tag{IV.14}$$

We notice that equation (IV.14) expresses a "material" transformation law between vectors $\bar{d}_k$ and $\bar{D}_k$ defined in the current and reference configurations respectively. This transformation law is identical to that proposed by Ericksen and Truesdell for the transformation of their directors of a line. For this reason we shall call the vectors $\bar{d}_k$ and $\bar{D}_k$ defined by equations (IV.12) and (IV.13) the directors of the three-dimensional rod.

It will be noticed that although the directors we have defined have the same transformation law as the directors of a line defined by Ericksen and Truesdell, they differ from the directors of Ericksen and Truesdell in one important respect, their order of magnitude. Ericksen and Truesdell's directors represent material elements and are therefore of infinitesimal magnitude. Our rod directors are, as may be observed from equations (IV.12) and (IV.13), of finite magnitude. The analogy between the directors of a line and the directors of a rod holds therefore only in the mode of transformation of the directors and not in the geometrical description of the directors.

It may be seen that director triads defined at each point within the space occupied by the rod will constitute

unique non-coplanar base vector systems and therefore represent a linearly independant set of vectors. The reciprocal directors $\bar{d}^k$ therefore exist and the director triads may be used as alternative bases for the local Euclidean point spaces instead of the corresponding curvilinear base vector triads.

In order to calculate the relationship between the reciprocal directors and the reciprocal curvilinear base vectors we use the identity,

$$e_{ijk}\,\bar{g}^i = \frac{\bar{g}_j \times \bar{g}_k}{\bar{g}_1 \cdot \bar{g}_2 \times \bar{g}_3}$$

where e_{ijk} is the permutation symbol, whose indices never take part in a summation.

Then,

$$e_{ijk}\bar{g}^i = \frac{\frac{\partial X^j}{\partial x^j}\,\bar{d}_j \times \frac{\partial X^k}{\partial x^k}\,\bar{d}_k}{\frac{\partial X^1}{\partial x^1}\,\bar{d}_1 \cdot \frac{\partial X^2}{\partial x^2}\,\bar{d}_2 \times \frac{\partial X^3}{\partial x^3}\,\bar{d}_3}$$

$$= \frac{\bar{d}_j \times \bar{d}_k}{\frac{\partial X^i}{\partial x^i}\,(\bar{d}_1 \cdot \bar{d}_2 \times \bar{d}_3)}$$

$$e_{ijk}\bar{g}^{i} = \frac{e_{ijk}\,\bar{d}^{i}}{\frac{\partial X^{\underline{i}}}{\partial x^{\underline{i}}}}$$

$$\therefore \qquad \bar{g}^{i} = \frac{\partial x^{\underline{i}}}{\partial X^{\underline{i}}}\,\bar{d}^{\underline{i}} \qquad \text{(IV.15.i)}$$

$$\text{or} \qquad \bar{d}^{i} = \frac{\partial X^{\underline{i}}}{\partial x^{\underline{i}}}\,\bar{g}^{\underline{i}} \qquad \text{(IV.15.ii)}$$

giving the relationship between the reciprocal curvilinear base vectors and the reciprocal basis $\bar{d}^{k}$.

Later on, for reasons which will become apparent, we shall replace the geometrical description of the rod in terms of the axial curvilinear base vectors with a geometrical description in terms of the axial directors.

The Generalised Curvilinear Coördinates

Consider again the expression for the position vector in the current configuration, equation (III.23). The right hand side of the equation is a vector summation, the result of which we may refer to its components with respect to the base vector system in the cross section on the axis, c.

i.e., we may write,

$$\bar{r}(x^1,x^2,x^3) - \bar{r}(0,0,x^3) = \xi^{-k}(\bar{g}_k)_c \qquad \text{(IV.16)}$$

where $\xi^{-k}(\bar{g}_k)_c$ is identical to the right hand side of equation (III.23). Using equation (IV.12) we may express $(\bar{g}_k)_c$, and hence the vector summation $\xi^{-k}(\bar{g}_k)_c$ in terms of the director base system at the same point on the rod axis. ∴ we may write,

$$\bar{r} = \bar{r}(x^3) + \xi^k(\bar{d}_k)_c$$

where,

$$\xi^k = \xi^{-k}\left(\frac{\partial X^k}{\partial x^k}\right)_c$$

or,

$$\bar{r} = \bar{r}(x^3) + \xi^k\bar{a}_k \qquad \text{(IV.17)}$$

where

$$\bar{a}_k = (\bar{d}_k)_c \qquad \text{(IV.18)}$$

and where henceforth,

$$\bar{r}(x^1,x^2,x^3) \equiv \bar{r}$$

and $$\bar{r}(0,0,x^3) \equiv \bar{r}(x^3)$$

We shall designate the ξ^k as the *Generalised Curvilinear Coördinates* of a point in a cross section of the deformed rod configuration.

Similarly by substituting reference configuration parameters for current configuration parameters in equation (III.23) we obtain,

$$\bar{R} = \bar{R}(X^3) + \Xi^K \, (\bar{G}_K)_C$$

where $\Xi^K \, (\bar{G}_K)_C$ is the vector summation in the reference configuration corresponding to $\xi^k (\bar{g}_k)_c$ in the current configuration. Since we have defined,

$$\bar{G}_K \equiv \bar{D}_K$$

Then let,

$$(\bar{G}_K)_C \equiv \bar{A}_K$$

and we may write immediately,

$$\bar{R} = \bar{R}(X^3) + \Xi^K \, \bar{A}_K \qquad \text{(IV.19)}$$

We shall refer to the Ξ^K as the Generalised Curvilinear Coördinates of a point in a cross section of the undeformed rod. After the simplification of the reference configuration geometry we have,

$$\left\{ \begin{matrix} K \\ 1 \; 2 \end{matrix} \right\} = 0 = \left\{ \begin{matrix} K \\ 2 \; 1 \end{matrix} \right\}$$

and

$$\left\{ \begin{matrix} K \\ \underline{\Delta}\ \underline{\Delta} \end{matrix} \right\} = 0 \qquad \Delta = 1 \text{ or } 2.$$

Therefore from equation (III.23) we have,

$$\bar{R} = \bar{R}(X^3) + X^{\Delta}(\bar{G}_{\Delta})_C$$

$$\therefore \quad \Xi^K = \delta^K_{\Delta} X^{\Delta} \tag{IV.20}$$

where δ^K_{Δ} is the Kroenecker delta.

Since the current configuration is completely prescribed by the reference configuration and the deformation gradient, all properties of the current configuration geometry may be derived from them. We have seen how the base vectors and directors in the current configuration may be calculated by equations (IV.10) and (IV.14). We shall now demonstrate how the metric tensor components, reciprocal base vectors and Christoffel symbols of the current configuration may be calculated.

Calculation of the Metric Tensor Components

The metric tensor in the reference configuration for the curvilinear base vector system is,

$$\bar{\bar{G}} = \bar{\bar{I}} = \bar{G}_i \, \bar{G}^i$$

$$= \bar{G}_i \, \bar{G}^i \cdot \bar{\bar{I}} \qquad \text{(See Appendix B)}$$

$$= \bar{G}_i \, \bar{G}^i \cdot \bar{G}^j \, \bar{G}_j$$

$$\therefore \bar{\bar{G}} = G^{ij} \, \bar{G}_i \, \bar{G}_j$$

where

$$G^{ij} = \bar{G}^i \cdot \bar{G}^j$$

Similarly,

$$\bar{\bar{G}} = G_{ij} \, G^i \, \bar{G}^j$$

where,

$$G_{ij} = \bar{G}_i \cdot \bar{G}_j$$

Now since

$$\bar{D}_K \equiv \bar{G}_K \, ,$$

the metric tensor components of the director basis in the reference configuration are,

$$D_{ij} = G_{ij} \qquad \text{(IV.21.i)}$$

or after the simplification of the reference configuration geometry,

$$D_{ij} = (G_{ij})_C = G_{ij} \qquad \text{(IV.21.ii)}$$

Similarly,

$$D_{ij} = (G^{ij})_C = G^{ij} \qquad \text{(IV.21.iii)}$$

The components of the metric tensor in the current configuration,

$$g_{ij} = \bar{g}_i \cdot \bar{g}_j$$

$$= \frac{\partial X^i}{\partial x^i} (\bar{G}_i \cdot \bar{\bar{F}}) \cdot \frac{\partial X^j}{\partial x^j} (\bar{G}_j \cdot \bar{\bar{F}}) \qquad \text{by (IV.10)}$$

$$= \frac{\partial X^i}{\partial x^i} \frac{\partial X^j}{\partial x^j} (\bar{G}_i \cdot \bar{\bar{F}}) \cdot (\bar{\bar{F}}_C \cdot \bar{G}_j)$$

$$= \frac{\partial X^i}{\partial x^i} \frac{\partial X^j}{\partial x^j} \bar{G}_i \cdot (\bar{\bar{F}} \cdot \bar{\bar{F}}_C) \cdot \bar{G}_j$$

$$= \frac{\partial X^i}{\partial x^i} \frac{\partial X^j}{\partial x^j} \bar{G}_i \bar{G}_j : (\bar{\bar{F}} \cdot \bar{\bar{F}}_C)$$

$$\therefore \quad g_{ij} = \frac{\partial X^i}{\partial x^i} \frac{\partial X^j}{\partial x^j} \bar{G}_i \bar{G}_j : \bar{\bar{C}}$$

where $\bar{\bar{C}} = C_{ij} \bar{G}^i \bar{G}^j \equiv \bar{\bar{F}} \cdot \bar{\bar{F}}_C$, is the *Green deformation tensor* and the double dot product is defined by

$$\bar{G}_i \cdot (\bar{\bar{F}} \cdot \bar{\bar{F}}_C) \cdot \bar{G}_j \equiv \bar{G}_i \bar{G}_j : (\bar{\bar{F}} \cdot \bar{\bar{F}}_C)$$

so that,

$$g_{ij} = \frac{\partial X^{\underline{i}}}{\partial x^{\underline{i}}} \frac{\partial X^{\underline{j}}}{\partial x^{\underline{j}}} C_{\underline{ij}} \tag{IV.22}$$

Similarly,

$$d_{ij} = \bar{D}_i \bar{D}_j : \bar{\bar{C}} \tag{IV.23}$$

To calculate g^{ij} we use the identity,

$$e_{i\ell k} \bar{g}^i \equiv \frac{\bar{g}_\ell \times \bar{g}_k}{\bar{g}_1 \cdot \bar{g}_2 \times \bar{g}_3}$$

where $e_{i\ell k}$ is the permutation symbol defined by;

$e_{123} = e_{231} = e_{312} = +1$; $e_{213} = e_{132} = e_{321} = -1$

Similarly,

$$e_{jmp}\bar{g}^j = \frac{\bar{g}_m \times \bar{g}_p}{\sqrt{|\bar{\bar{g}}|}}$$

where $\sqrt{|\bar{\bar{g}}|} \equiv \bar{g}_1 \cdot \bar{g}_2 \times \bar{g}_3$

$\therefore |\bar{\bar{g}}| \; e_{i\ell k} \bar{g}^i \cdot e_{jmp}\bar{g}^j = (\bar{g}_\ell \times \bar{g}_k) \cdot (\bar{g}_m \times \bar{g}_p)$

$$\therefore\ |\bar{\bar{g}}|e_{i\ell k}e_{jmp}\bar{g}^{ij} = \left[\frac{\partial X^{\underline{\ell}}}{\partial x^{\underline{\ell}}}(\bar{G}_{\underline{\ell}}\cdot\bar{\bar{F}})\times\frac{\partial X^{\underline{k}}}{\partial x^{\underline{k}}}(G_{\underline{k}}\cdot\bar{\bar{F}})\right]\cdot\left[\frac{\partial X^{\underline{m}}}{\partial x^{\underline{m}}}(\bar{G}_{\underline{m}}\cdot\bar{\bar{F}})\times\frac{\partial X^{\underline{p}}}{\partial x^{\underline{p}}}(\bar{G}_{\underline{p}}\cdot\bar{\bar{F}})\right]$$

or

$$|\bar{\bar{g}}|e_{i\ell k}e_{jmp}g^{ij}$$

$$= \frac{\partial X^{\underline{\ell}}}{\partial x^{\underline{\ell}}}\frac{\partial X^{\underline{k}}}{\partial x^{\underline{k}}}\frac{\partial X^{\underline{m}}}{\partial x^{\underline{m}}}\frac{\partial X^{\underline{p}}}{\partial x^{\underline{p}}}\left[(\bar{G}_{\underline{\ell}}\cdot\bar{\bar{F}})\times(\bar{G}_{\underline{k}}\cdot\bar{\bar{F}})\right]\cdot\left[(\bar{G}_{\underline{m}}\cdot\bar{\bar{F}})\times(\bar{G}_{\underline{p}}\cdot\bar{\bar{F}})\right]$$

$$= \frac{\partial X^{\underline{\ell}}}{\partial x^{\underline{\ell}}}\frac{\partial X^{\underline{k}}}{\partial x^{\underline{k}}}\frac{\partial X^{\underline{m}}}{\partial x^{\underline{m}}}\frac{\partial X^{\underline{p}}}{\partial x^{\underline{p}}}\left[\left\{(\bar{G}_{\underline{\ell}}\cdot\bar{\bar{F}})\times(\bar{G}_{\underline{k}}\cdot\bar{\bar{F}})\right\}\times(\bar{G}_{\underline{m}}\cdot\bar{\bar{F}})\right]\cdot(\bar{G}_{\underline{p}}\cdot\bar{\bar{F}})$$

$$= \frac{\partial X^{\underline{\ell}}}{\partial x^{\underline{\ell}}}\frac{\partial X^{\underline{k}}}{\partial x^{\underline{k}}}\frac{\partial X^{\underline{m}}}{\partial x^{\underline{m}}}\frac{\partial X^{\underline{p}}}{\partial x^{\underline{p}}}\Big[(\bar{G}_{\underline{\ell}}\cdot\bar{\bar{F}})\cdot(\bar{G}_{\underline{m}}\cdot\bar{\bar{F}})\,(\bar{G}_{\underline{k}}\cdot\bar{\bar{F}})$$

$$-(\bar{G}_{\underline{k}}\cdot\bar{\bar{F}})\cdot(\bar{G}_{\underline{m}}\cdot\bar{\bar{F}})\,(\bar{G}_{\underline{\ell}}\cdot\bar{\bar{F}})\Big]\cdot(\bar{G}_{\underline{p}}\cdot\bar{\bar{F}})$$

and writing,

$$\bar{G}_m\cdot\bar{\bar{F}} = \bar{\bar{F}}_C\cdot\bar{G}_m \qquad \text{etc.}$$

$$= \frac{\partial X^{\underline{\ell}}}{\partial x^{\underline{\ell}}} \frac{\partial X^{\underline{k}}}{\partial x^{\underline{k}}} \frac{\partial X^{\underline{m}}}{\partial x^{\underline{m}}} \frac{\partial X^{\underline{p}}}{\partial x^{\underline{p}}} \Big[\bar{G}_{\underline{\ell}}\ \bar{G}_{\underline{m}} : (\bar{\bar{F}} \cdot \bar{\bar{F}}_C)\ (\bar{G}_{\underline{k}} \cdot \bar{\bar{F}})$$

$$- \bar{G}_{\underline{k}}\ \bar{G}_{\underline{m}} : (\bar{\bar{F}} \cdot \bar{\bar{F}}_C)\ (\bar{G}_{\underline{\ell}} \cdot \bar{\bar{F}}) \Big] \cdot (\bar{G}_{\underline{p}} \cdot \bar{\bar{F}})$$

$$= \frac{\partial X^{\underline{\ell}}}{\partial x^{\underline{\ell}}} \frac{\partial X^{\underline{k}}}{\partial x^{\underline{k}}} \frac{\partial X^{\underline{m}}}{\partial x^{\underline{m}}} \frac{\partial X^{\underline{p}}}{\partial x^{\underline{p}}} \Big[(\bar{G}_{\underline{\ell}}\ \bar{G}_{\underline{m}} : \bar{\bar{C}})\ (\bar{G}_{\underline{k}} \cdot \bar{\bar{F}}) \cdot (\bar{G}_{\underline{p}} \cdot \bar{\bar{F}})$$

$$- (\bar{G}_{\underline{k}}\ \bar{G}_{\underline{m}} : \bar{\bar{C}})\ (\bar{G}_{\underline{\ell}} \cdot \bar{\bar{F}}) \cdot (\bar{G}_{\underline{p}} \cdot \bar{\bar{F}}) \Big]$$

$$= \frac{\partial X^{\underline{\ell}}}{\partial x^{\underline{\ell}}} \frac{\partial X^{\underline{k}}}{\partial x^{\underline{k}}} \frac{\partial X^{\underline{m}}}{\partial x^{\underline{m}}} \frac{\partial X^{\underline{p}}}{\partial x^{\underline{p}}} \Big[(\bar{G}_{\underline{\ell}}\ \bar{G}_{\underline{m}} : \bar{\bar{C}})\ (\bar{G}_{\underline{k}}\ \bar{G}_{\underline{p}} : \bar{\bar{C}}) - (\bar{G}_{\underline{k}} \bar{G}_{\underline{m}} : \bar{\bar{C}})\ (\bar{G}_{\underline{\ell}}\ \bar{G}_{\underline{p}} : \bar{\bar{C}}) \Big]$$

$$\therefore\ |\bar{\bar{g}}|\, e_{i\ell k} e_{jmp} g^{ij} = \frac{\partial X^{\underline{\ell}}}{\partial x^{\underline{\ell}}} \frac{\partial X^{\underline{k}}}{\partial x^{\underline{k}}} \frac{\partial X^{\underline{m}}}{\partial x^{\underline{m}}} \frac{\partial X^{\underline{p}}}{\partial x^{\underline{p}}} \Big[C_{\underline{\ell m}}\ C_{\underline{kp}} - C_{\underline{km}}\ C_{\underline{\ell p}} \Big] \qquad \text{(IV.24)}$$

Similarly,

$$|\bar{\bar{d}}|\, e_{i\ell k} e_{jmp} d^{ij}$$

$$= (\bar{D}_{\ell}\ \bar{D}_{m} : \bar{\bar{C}})\ (\bar{D}_{k}\ D_{p} : \bar{\bar{C}}) - (\bar{D}_{k}\ \bar{D}_{m} : \bar{\bar{C}})\ (\bar{D}_{\ell}\ \bar{D}_{p} : \bar{\bar{C}}) \qquad \text{(IV.25)}$$

Calculation of Reciprocal (or Dual) Base Vectors

We have,

$$\bar{g}^i = \bar{g}^i \cdot \bar{\bar{I}}$$

$$= \bar{g}^i \cdot \bar{g}^j \, \bar{g}_j$$

$$\therefore \quad \bar{g}^i = g^{ij} \, \bar{g}_j \tag{IV.26}$$

$\bar{g}_j$ is given by equation (IV.10) and g^{ij} is given by equation (IV.23). Hence the reciprocal base vector, $\bar{g}^i$, may be calculated.

Similarly,

$$\bar{d}^i = d^{ij} \, \bar{d}_j \tag{IV.27}$$

where $\bar{d}_j$ is given by equation (IV.12) and d^{ij} is given by equation (IV.24).

Calculation of Christoffel Symbols

We have,

$$\left\{ \begin{matrix} k \\ \ell \; m \end{matrix} \right\} \equiv \frac{\partial \bar{g}_\ell}{\partial x^m} \cdot \bar{g}^k$$

$$= \frac{\partial X^M}{\partial x^m} \, \frac{\partial \bar{g}_\ell}{\partial X^M} \cdot \bar{g}^k$$

$$\therefore \left\{ \begin{matrix} k \\ \ell \; m \end{matrix} \right\} = \frac{\partial X^{\underline{m}}}{\partial x^{\underline{m}}} \frac{\partial \bar{g}_{\ell}}{\partial X^{\underline{m}}} \cdot \bar{g}^{k} \qquad \text{(IV.28)}$$

Now $\frac{\partial X^{\underline{m}}}{\partial x^{\underline{m}}}$ is a component of the prescribed deformation gradient, $\bar{g}_{\ell}$ is given by equation (IV.10) and $\bar{g}^{k}$ is given by equation (IV.25). Hence the Christoffel symbols in the current configuration may be calculated. Therefore the connection of the rod space as a mathematical structure is fully established.

CHAPTER 5

THE TRANSFORMATION TENSOR

Derivation of the Curvilinear Base Vectors

In a current configuration the radius vector of an arbitrary point within the rod is,

$$\bar{r} = \bar{r}(x^3) + \xi^k \bar{a}_k$$

where the vector $\xi^k \bar{a}_k$ is given by the right hand side of equation (III.23).

The curvilinear base vectors at any point are defined by,

$$\bar{g}_i \equiv \frac{\partial \bar{r}}{\partial x^i}$$

$$\therefore \bar{g}_i = \frac{\partial \bar{r}}{\partial x^i} = \frac{\partial \bar{r}}{\partial x^i}(x^3) + \frac{\partial}{\partial x^i}(\xi^k \bar{a}_k)$$

Now by equation (III.23) we have the functional form of $\xi^k \bar{a}_k$ as,

$$\xi^k \bar{a}_k = x^\alpha (\bar{g}_\alpha)_c + \sum_n \frac{1}{n!} (x^\alpha)^n \left[A_\alpha(x^3) \right] +$$

$$+ \sum_n \frac{1}{n!} \sum_{r=1}^{n} \binom{n}{r} (x^1)^{n-r} (x^2)^r \left[B(x^3) \right] \qquad \text{(V. 2)}$$

where $A_\alpha(x^3)$ and $B(x^3)$ may be obtained by comparison of equation (V. 2) with equation (III.23).

Therefore,

$$\frac{\partial}{\partial (x^\alpha)} (\xi^k \bar{a}_k) = (\bar{g}_\alpha)_c + \sum_n \frac{1}{(n-1)!} (x^\alpha)^{n-1} \left[\underset{\alpha}{A}(x^3) \right] +$$

$$+ \sum_n \frac{1}{n!} \sum_{r=1}^{n} \binom{n}{r} \left[\delta^1{}_\alpha (n-r) (x^1)^{n-r-1} (x^2)^r + \delta^2{}_\alpha (x^1)^{n-r} r (x^2)^{r-1} \right] B(x^3)$$

(V. 3.i)

and

$$\frac{\partial}{\partial x^3} (\xi^k \bar{a}_k) = \sum_n \frac{1}{n!} (x^\alpha)^n \frac{\partial}{\partial x^3} \left[A_\alpha(x^3) \right] +$$

$$+ \sum_n \frac{1}{n!} \sum_{r=1}^{n} \binom{n}{r} (x^1)^{n-r} (x^2)^r \frac{\partial}{\partial x^3} \left[B(x^3) \right] \qquad \text{(V. 3.ii)}$$

$A_\alpha(x^3)$ and $B(x^3)$ are products each term of which is a summation. The derivatives $\frac{\partial}{\partial x^3}\left[A_\alpha(x^3)\right]$ and $\frac{\partial}{\partial x^3}\left[B(x^3)\right]$ are obtained from the rule for differentiation of a product, each term being differentiated one at a time. The differentiation is an elementary operation and the full expression is not included here because of its length. Each derivative $\frac{\partial}{\partial x^\alpha}(\xi^k \bar{a}_k)$ and $\frac{\partial}{\partial x^3}(\xi^k \bar{a}_k)$ will be itself a vector summation and we may express the result of $\frac{\partial}{\partial x^i}(\xi^k \bar{a}_k)$ in terms of its components with respect to the director base system.

We define the quantity $\frac{D\xi^k}{Dx^i}$ such that,

$$\frac{\partial}{\partial x^i}(\xi^k \bar{a}_k) \equiv \frac{D\xi^k}{Dx^i}\bar{a}_k \qquad \text{(V. 4.i)}$$

Similarly we define $\frac{D\Xi^K}{DX^i}$ for the reference configuration such that,

$$\frac{\partial}{\partial X^i}(\Xi^K \bar{A}_K) \equiv \frac{D\Xi^K}{DX^i}\bar{A}_K \qquad \text{(V. 4.ii)}$$

After simplification of the reference configuration geometry,

$$\frac{D\Xi^{\Delta}}{DX^{i}} = \delta^{\Delta}_{i} \qquad \text{(V. 5.i)}$$

and

$$\frac{D\Xi^{3}}{DX^{i}} = 0 \qquad \text{(V. 5.ii)}$$

by (IV.20).

The Transformation Tensor

Consider equation (V. 1)

i.e.

$$\bar{g}^{i} = \frac{\partial \bar{r}(x^{3})}{\partial x^{i}} + \frac{\partial}{\partial x^{i}} (\xi^{k}\, \bar{a}_{k})$$

or,

$$\bar{g}_{i} = \delta^{3}_{i}\, \frac{\partial \bar{r}(x^{3})}{\partial x^{3}} + \frac{D\xi^{k}}{Dx^{i}}\, \bar{a}_{k}$$

then by equation (IV.12) we have,

$$\bar{d}_{i} = \frac{\partial x^{\underline{i}}}{\partial X^{\underline{i}}} \left[\delta^{3}_{i}\, \frac{\partial \bar{r}(x^{3})}{\partial x^{3}} + \frac{D\xi^{k}}{Dx^{\underline{i}}}\, \bar{a}_{k} \right] \qquad \text{(V. 6)}$$

$$\therefore \qquad \bar{d}_{\alpha} = \frac{\partial x^{\underline{\alpha}}}{\partial X^{\underline{\alpha}}}\, \frac{D\xi^{k}}{Dx^{\underline{\alpha}}}\, \bar{a}_{k} \qquad \text{(V. 7.i)}$$

and, $$\bar{d}_3 = \frac{\partial x^3}{\partial X^3}\left[(\bar{g}_3)_c + \frac{D\xi^k}{Dx^3}\,\bar{a}_k\right]$$

or, $$= \frac{\partial x^3}{\partial X^3}\left[\left(\frac{\partial X^3}{\partial x^3}\right)_c \bar{a}_3 + \frac{D\xi^k}{Dx^3}\,\bar{a}_k\right]$$

by equations (IV.12) and (IV.18)

$$\therefore\quad \bar{d}_3 = \frac{\partial x^3}{\partial X^3}\left[\frac{D\xi^\beta}{Dx^3}\,\bar{a}_\beta + \left\{\left(\frac{\partial X^3}{\partial x^3}\right)_c + \frac{D\xi^3}{Dx^3}\right\}\bar{a}_3\right] \qquad \text{(V. 7.ii)}$$

where,

$$\frac{D\xi^\beta}{Dx^i} = \left[\frac{\partial}{\partial x^i}\,(\xi^k\,\bar{a}_k)\right]\cdot\,\bar{a}^\beta \qquad \text{(V. 8.i)}$$

and

$$\frac{D\xi^3}{Dx^i} = \left[\frac{\partial}{\partial x^i}\,(\xi^k\,\bar{a}_k)\right]\cdot\,\bar{a}^3 \qquad \text{(V. 8.ii)}$$

Now we have,

$$\bar{d}_i = \delta^k_i\,\bar{d}_k$$

$$= (\bar{a}_i \cdot \bar{a}^k)\bar{d}_k$$

$$\therefore \bar{d}_i = \bar{a}_i \cdot [\bar{a}^k \bar{d}_k] = \bar{a}_i \cdot \bar{\bar{\mu}} \qquad \text{(V. 9)}$$

where the product $\bar{a}^k \bar{d}_k$ defines a second order tensor[8]

$$\bar{a}^k \bar{d}_k \equiv \bar{\bar{\mu}} \qquad \text{(V.10)}$$

We may express this tensor completely in terms of the axial director basis in the following way;
We have,

$$\bar{d}_i = \bar{a}_i \cdot [\bar{a}^k \bar{d}_k] \qquad \text{by (V. 9)}$$

$$= \bar{a}_i \cdot [\bar{a}^k (\bar{d}_k \cdot \bar{\bar{I}})]$$

$$= \bar{a}_i \cdot [\bar{a}^k \bar{d}_k \cdot \bar{a}^j \bar{a}_j]$$

$$\therefore \bar{d}_i = \bar{a}_i \cdot [(\bar{d}_k \cdot \bar{a}^j) \bar{a}^k \bar{a}_j] \qquad \text{(V.11)}$$

comparing with equation (V. 9)

$$\therefore \bar{\bar{\mu}} = (\bar{d}_k \cdot \bar{a}^j) \bar{a}^k \bar{a}_j \qquad \text{(V.12)}$$

or

$$= \mu_k{}^j \, \bar{a}^k \, \bar{a}_j$$

where

$$\mu_k{}^j \equiv \bar{d}_k \cdot \bar{a}^j \qquad \text{(V.13)}$$

Calculation of the Transformation Tensor

Consider, $\bar{d}_i = \bar{d}_i \cdot \bar{\bar{I}}$

$$= \bar{d}_i \cdot \bar{a}^j \, \bar{a}_j$$

$$= (\bar{d}_i \cdot \bar{a}^j) \, \bar{a}_j$$

$$\therefore \bar{d}_i = \mu_i{}^j \, \bar{a}_j \qquad \text{by (V.13)} \qquad \text{(V.14)}$$

Therefore by (V. 7.i) and (V. 7.ii), $\bar{\bar{\mu}}$ must have the form,

$$\bar{\bar{\mu}} = \begin{bmatrix} \frac{\partial x^1}{\partial X^1} \frac{D\xi^1}{Dx^1} \bar{a}^1 \bar{a}_1 + \frac{\partial x^1}{\partial X^1} \frac{D\xi^2}{Dx^1} \bar{a}^1 \bar{a}_2 + \frac{\partial x^1}{\partial X^1} \frac{D\xi^3}{Dx^1} \bar{a}^1 \bar{a}_3 \\ + \frac{\partial x^2}{\partial X^2} \frac{D\xi^1}{Dx^2} \bar{a}^2 \bar{a}_1 + \frac{\partial x^2}{\partial X^2} \frac{D\xi^2}{Dx^2} \bar{a}^2 \bar{a}_2 + \frac{\partial x^2}{\partial X^2} \frac{D\xi^3}{Dx^2} \bar{a}^2 \bar{a}_3 \\ + \frac{\partial x^3}{\partial X^3} \frac{D\xi^1}{Dx^3} \bar{a}^3 \bar{a}_1 + \frac{\partial x^3}{\partial X^3} \frac{D\xi^2}{Dx^3} \bar{a}^3 \bar{a}_2 + \frac{\partial x^3}{\partial X^3} \left[\left(\frac{\partial X^3}{\partial x^3} \right)_c + \frac{D\xi^3}{Dx^3} \right] \bar{a}^3 \bar{a}_3 \end{bmatrix} \qquad \text{(V.15)}$$

The Inverse Transformation Tensor

Consider equation (V.10). By an elementary rule of tensor algebra,

$$\bar{\bar{\mu}}^{-1} = (\bar{a}^k \bar{d}_k)^{-1}$$

$$= (\bar{d}_k)^{-1} (\bar{a}^k)^{-1}$$

$$\therefore \quad \bar{\bar{\mu}}^{-1} = \bar{d}^k \bar{a}_k \qquad \text{(V.16)}$$

Now we have,

$$\bar{a}_i = \hat{o}_i^k \bar{a}_k$$

$$= \bar{d}_i \cdot \bar{d}^k \bar{a}_k$$

$$\therefore \quad \bar{a}_i = \bar{d}_i \cdot \bar{\bar{\mu}}^{-1} \qquad \text{by (V.16)} \qquad \text{(V.17)}$$

Consider,

$$\bar{a}_i = \bar{d}_i \cdot \bar{d}^k \bar{a}_k$$

$$= \bar{d}_i \cdot [\bar{d}^k (\bar{a}_k \cdot \bar{\bar{I}})]$$

$$= \bar{d}_i \cdot [\bar{d}^k \bar{a}_k \cdot \bar{d}^j \bar{d}_j]$$

or $$\bar{a}_i = \bar{d}_i \cdot [(\bar{a}_k \cdot \bar{d}^j)\ \bar{d}^k\ \bar{d}_j] \qquad (V.18)$$

Comparing with equation (V.17)

$$\therefore\ \bar{\bar{\mu}}^{-1} = (\bar{a}_k \cdot \bar{d}^j)\ \bar{d}^k\ \bar{d}_j$$

$$= (\mu^{-1})_k{}^j\ \bar{d}^k\ \bar{d}_j \qquad (V.19)$$

where

$$(\mu^{-1})_k{}^j \equiv \bar{a}_k \cdot \bar{d}^j \qquad (V.20)$$

Now we have,

$$\bar{a}_k = \bar{a}_k \cdot \bar{\bar{I}}$$

$$= \bar{a}_k \cdot \bar{d}^i\ \bar{d}_i$$

or $$\bar{a}_k = (\mu^{-1})_k{}^i\ \bar{d}_i \qquad (V.21)$$

Substituting equation (V.21) into equation (V.14) we have,

$$\bar{d}_i = \mu_i{}^j (\mu^{-1})_j{}^k \bar{d}_k$$

from which we deduce

$$\mu_i{}^j \ (\mu^{-1})_j{}^k = \delta_i^k \qquad \text{(V.22)}$$

Similarly, substituting equation (V.14) into equation (V.21) we have,

$$\bar{a}_i = (\mu^{-1})_i{}^j \ \mu_j{}^k \ \bar{a}_k$$

from which we deduce,

$$(\mu^{-1})_i{}^j \ \mu_j{}^k = \delta_i^k \qquad \text{(V.23)}$$

<u>Director Metric Tensor Components in Terms of the Axial Director Metric Tensor Components</u>

We have,

$$d_{ij} = \bar{d}_i \cdot \bar{d}_j$$

$$= \mu_i{}^s \, \bar{a}_s \cdot \mu_j{}^t \, \bar{a}_t \qquad \text{by (V.14)}$$

$$\therefore d_{ij} = \mu_i{}^s \, a_{st} \, \mu_j{}^t \qquad \text{(V.24)}$$

Also,

$$a_{ij} = \bar{a}_i \cdot \bar{a}_j$$

$$= (\mu^{-1})_i{}^s \, \bar{d}_s \cdot (\mu^{-1})_j{}^t \, \bar{d}_t \qquad \text{by (V.21)}$$

$$\therefore a_{ij} = (\mu^{-1})_i{}^s \, d_{st} (\mu^{-1})_j{}^t \qquad \text{(V.25)}$$

Consider,

$$\bar{\bar{I}} = d^{ij} \, \bar{d}_i \, \bar{d}_j = a^{mn} \, \bar{a}_m \, \bar{a}_n$$

$$\therefore \; d^{ij} \, \mu_i{}^s \, \bar{a}_s \, \mu_j{}^t \, \bar{a}_t = a^{mn} \, \bar{a}_m \, \bar{a}_n \qquad \text{(by (V.14)}$$

Taking scalar products with $\bar{a}^m$ as prefactor and $\bar{a}^n$ as post-

factor,

$$\delta^m_s \, d^{ij} \, \mu_i{}^s \, \mu_j{}^t \, \delta^n_t = a^{mn}$$

$$\therefore \quad \mu_i{}^m \, d^{ij} \, \mu_j{}^n = a^{mn} \tag{V.26}$$

Also, from,

$$\bar{\bar{I}} = d^{ij} \, \bar{d}_i \, \bar{d}_j = a^{mn} \, \bar{a}_m \, \bar{a}_n$$

we have,

$$d^{ij} \, \bar{d}_i \, \bar{d}_j = a^{mn} \, (\mu^{-1})_m{}^s \, \bar{d}_s \, (\mu^{-1})_n{}^t \, \bar{d}_t \quad \text{by (V.21)}$$

Taking scalar products with $\bar{d}^i$ as prefactor and $\bar{d}^j$ as post-factor,

$$\therefore \; d^{ij} = a^{mn} \, \delta^i_s \, (\mu^{-1})_m{}^s \, \delta^j_t \, (\mu^{-1})_n{}^t$$

$$\therefore \quad d^{ij} = (\mu^{-1})_m{}^i \, a^{mn} \, (\mu^{-1})_n{}^j \qquad \text{(V.27)}$$

Relationship of Reciprocal Directors to Reciprocal Axial Directors,

We have,

$$\bar{d}^i = \bar{\bar{I}} \cdot \bar{d}^i$$

$$= (\bar{d}_j \, \bar{d}^j) \cdot \bar{d}^i$$

$$= \bar{d}_j \, d^{ji}$$

$$= \mu_j{}^k \, \bar{a}_k \, d^{ji} \qquad \text{by (V.14)}$$

$$= \mu_j{}^k \, \bar{a}_k \, (\mu^{-1})_m{}^i \, a^{mn} \, (\mu^{-1})_n{}^j \qquad \text{by (V.27)}$$

$$= \bar{a}_k \, (\mu^{-1})_m{}^i \, a^{mn} \, \delta_n^k \qquad \text{by (V.23)}$$

$$= \bar{a}_k \; a^{mk} \; (\mu^{-1})_m{}^i$$

$$= \bar{a}_k \; \bar{a}^k \cdot \bar{a}^m \; (\mu^{-1})_m{}^i$$

$$= \bar{\bar{I}} \cdot \bar{a}^m \; (\mu^{-1})_m{}^i$$

$$\therefore \qquad \bar{d}^i = \bar{a}^m \; (\mu^{-1})_m{}^i \qquad \text{(V.28)}$$

Similarly,

$$\bar{a}^i = \bar{d}^m \; \mu_m{}^i \qquad \text{(V.29)}$$

Calculation of the Inverse Transformation Tensor

We may write equation (V.15) in the trinomial form,

$$\bar{\bar{\mu}} = \bar{h}_1 \; \bar{a}_1 + \bar{h}_2 \; \bar{a}_2 + \bar{h}_3 \; \bar{a}_3$$

where,

$$\bar{h}_1 = \frac{\partial x^1}{\partial X^1} \frac{D\xi^1}{Dx^1} \bar{a}^1 + \frac{\partial x^2}{\partial X^2} \frac{D\xi^1}{Dx^2} \bar{a}^2 + \frac{\partial x^3}{\partial X^3} \frac{D\xi^1}{Dx^3} \bar{a}^3$$

$$\bar{h}_2 = \frac{\partial x^1}{\partial X^1}\frac{D\xi^2}{Dx^1}\bar{a}^1 + \frac{\partial x^2}{\partial X^2}\frac{D\xi^2}{Dx^2}\bar{a}^2 + \frac{\partial x^3}{\partial X^3}\frac{D\xi^2}{Dx^3}\bar{a}^3$$

$$\bar{h}_3 = \frac{\partial x^1}{\partial X^1}\frac{D\xi^3}{Dx^1}\bar{a}^1 + \frac{\partial x^2}{\partial X^2}\frac{D\xi^3}{Dx^2}\bar{a}^2 + \frac{\partial x^3}{\partial X^3}\left[\left(\frac{\partial X^3}{\partial x^3}\right)_c + \frac{D\xi^3}{Dx^3}\right]\bar{a}^3$$

Now by an elementary rule of tensor algebra, the reciprocal tensor,

$$\bar{\bar{\mu}}^{-1} = \bar{a}^1\bar{h}^1 + \bar{a}^2\bar{h}^2 + \bar{a}^3\bar{h}^3$$

where,

$$\bar{h}^1 = \frac{\bar{h}_2 \times \bar{h}_3}{\bar{h}_1 \cdot \bar{h}_2 \times \bar{h}_3} \quad \text{etc.}$$

Therefore we may calculate the reciprocal transformation tensor with respect to the axial director base system as long as $\bar{h}^i$ represents a linearly independant vector set,

i.e. $$\bar{h}_1 \cdot \bar{h}_2 \times \bar{h}_3 \neq 0$$

Now we have already defined

$$\bar{\bar{\mu}}^{-1} = (\mu^{-1})_i{}^j \; \bar{d}^i \; \bar{d}_j$$

but

$$= (\mu^{-1})_i{}^j \; \bar{a}^m (\mu^{-1})_m{}^i \; \mu_j{}^k \; \bar{a}_k$$

by (V.28) and (V.14)

$$= \delta_i^k \; \bar{a}^m \; (\mu^{-1})_m{}^i \; \bar{a}_k \qquad \text{by (V.23)}$$

$$\therefore \bar{\bar{\mu}}^{-1} = (\mu^{-1})_m{}^k \; \bar{a}^m \; \bar{a}_k$$

Thus,

$$(\mu^{-1})_i{}^j \; \bar{d}^i \; \bar{d}_j = (\mu^{-1})_m{}^k \; \bar{a}^m \; \bar{a}_k \qquad \text{(V.30)}$$

From which we observe that the coefficients of the inverse transformation tensor are the same whether the inverse tensor is referred to the $\bar{a}_i$ base system or the $\bar{d}_i$ base system.

Consequently we finally obtain the inverse transformation tensor in terms of the curvilinear director base system

as,

$$\bar{\bar{\mu}}^{-1} = \begin{bmatrix} (\mu^{-1})_1{}^1\, \bar{d}^1\bar{d}_1 + (\mu^{-1})_1{}^2\, \bar{d}^1\bar{d}_2 + (\mu^{-1})_1{}^3\, \bar{d}^1\bar{d}_3 \\ + (\mu^{-1})_2{}^1\, \bar{d}^2\bar{d}_1 + (\mu^{-1})_2{}^2\, \bar{d}^2\bar{d}_2 + (\mu^{-1})_2{}^3\, \bar{d}^2\bar{d}_3 \\ + (\mu^{-1})_3{}^1\, \bar{d}^3\bar{d}_1 + (\mu^{-1})_3{}^2\, \bar{d}^3\bar{d}_2 + (\mu^{-1})_3{}^3\, \bar{d}^3\bar{d}_3 \end{bmatrix}$$

where (V.31

$$(\mu^{-1})_1{}^1 = \frac{1}{|\bar{\bar{\mu}}|}\frac{\partial x^2}{\partial X^2}\frac{\partial x^3}{\partial X^3}\left[\frac{D\xi^2}{Dx^2}\left\{\left(\frac{\partial X^3}{\partial x^3}\right)_c + \frac{D\xi^3}{Dx^3}\right\} - \frac{D\xi^2}{Dx^3}\frac{D\xi^3}{Dx^2}\right]$$

$$(\mu^{-1})_1{}^2 = \frac{1}{|\bar{\bar{\mu}}|}\frac{\partial x^2}{\partial X^2}\frac{\partial x^3}{\partial X^3}\left[\frac{D\xi^1}{Dx^3}\frac{D\xi^3}{Dx^2} - \frac{D\xi^1}{Dx^2}\left\{\left(\frac{\partial X^3}{\partial x^3}\right)_c + \frac{D\xi^3}{Dx^3}\right\}\right]$$

$$(\mu^{-1})_1{}^3 = \frac{1}{|\bar{\bar{\mu}}|}\frac{\partial x^2}{\partial X^2}\frac{\partial x^3}{\partial X^3}\left[\frac{D\xi^1}{Dx^2}\frac{D\xi^2}{Dx^3} - \frac{D\xi^1}{Dx^3}\frac{D\xi^2}{Dx^2}\right]$$

$$(\mu^{-1})_2{}^1 = \frac{1}{|\bar{\bar{\mu}}|}\frac{\partial x^1}{\partial X^1}\frac{\partial x^3}{\partial X^3}\left[\frac{D\xi^2}{Dx^3}\frac{D\xi^3}{Dx^1} - \frac{D\xi^2}{Dx^1}\left\{\left(\frac{\partial X^3}{\partial x^3}\right)_c + \frac{D\xi^3}{Dx^3}\right\}\right]$$

$$(\mu^{-1})_2{}^2 = \frac{1}{|\bar{\bar{\mu}}|}\frac{\partial x^1}{\partial X^1}\frac{\partial x^3}{\partial X^3}\left[\frac{D\xi^1}{Dx^1}\left\{\left(\frac{\partial X^3}{\partial x^3}\right)_c + \frac{D\xi^3}{Dx^3}\right\} - \frac{D\xi^1}{Dx^3}\frac{D\xi^3}{Dx^1}\right]$$

$$(\mu^{-1})_2{}^3 = \frac{1}{|\bar{\bar{\mu}}|}\frac{\partial x^1}{\partial X^1}\frac{\partial x^3}{\partial X^3}\left[\frac{D\xi^1}{Dx^3}\frac{D\xi^2}{Dx^1} - \frac{D\xi^1}{Dx^1}\frac{D\xi^2}{Dx^3}\right]$$

$$(\mu^{-1})_3{}^1 = \frac{1}{|\bar{\bar{\mu}}|}\frac{\partial x^1}{\partial X^1}\frac{\partial x^2}{\partial X^2}\left[\frac{D\xi^2}{Dx^1}\frac{D\xi^3}{Dx^2} - \frac{D\xi^2}{Dx^2}\frac{D\xi^3}{Dx^1}\right]$$

$$(\mu^{-1})_3{}^2 = \frac{1}{|\bar{\bar{\mu}}|}\frac{\partial x^1}{\partial X^1}\frac{\partial x^2}{\partial X^2}\left[\frac{D\xi^1}{Dx^2}\frac{D\xi^3}{Dx^1} - \frac{D\xi^1}{Dx^1}\frac{D\xi^3}{Dx^2}\right]$$

$$(\mu^{-1})_3{}^3 = \frac{1}{|\bar{\bar{\mu}}|}\frac{\partial x^1}{\partial X^1}\frac{\partial x^2}{\partial X^2}\left[\frac{D\xi^1}{Dx^1}\frac{D\xi^2}{Dx^2} - \frac{D\xi^1}{Dx^2}\frac{D\xi^2}{Dx^1}\right]$$

and $|\bar{\bar{\mu}}|$ or det $(\bar{\bar{\mu}})$ is the determinant of the scalar coefficients of the transformation tensor, $\bar{\bar{\mu}}$, which is obtained from the scalar triple products thus,

$$|\bar{\bar{\mu}}| = (\bar{a}_1 \cdot \bar{a}_2 \times \bar{a}_3)(\bar{h}_1 \cdot \bar{h}_2 \times \bar{h}_3) = \det\ (\bar{\bar{\mu}})$$

The Transformation Tensor in the Reference Configuration

We may obtain the corresponding transformation tensor

in the reference configuration, $\bar{\bar{M}}$, by substitution of reference configuration parameters into equation (V.15) such that,

$$\bar{\bar{M}} = \begin{bmatrix} \frac{D\Xi^1}{DX^1}\bar{A}^1\bar{A}_1 + \frac{D\Xi^2}{DX^1}\bar{A}^1\bar{A}_2 + \frac{D\Xi^3}{DX^1}\bar{A}^1\bar{A}_3 \\ + \frac{D\Xi^1}{DX^2}\bar{A}^2\bar{A}_1 + \frac{D\Xi^2}{DX^2}\bar{A}^2\bar{A}_2 + \frac{D\Xi^3}{DX^2}\bar{A}^2\bar{A}_3 \\ + \frac{D\Xi^1}{DX^3}\bar{A}^3\bar{A}_1 + \frac{D\Xi^2}{DX^3}\bar{A}^3\bar{A}_2 + \left[1 + \frac{D\Xi^3}{DX^3}\right]\bar{A}^3\bar{A}_3 \end{bmatrix} \qquad (V.32)$$

where $\frac{D\Xi^K}{DX^1}$ are given by equations (V. 4.ii) and $\bar{\bar{M}}$ is defined by equation (V.10), as,

$$\bar{\bar{M}} \equiv \bar{A}^K \bar{D}_K \qquad (V.33)$$

In the simplification of the reference configuration geometry we observe from equations (V. 5) that $\bar{\bar{M}}$ reduces to the identity tensor, $\bar{\bar{I}}$, such that,

$$\bar{\bar{M}} = \bar{A}^K \bar{A}_K = \bar{\bar{I}} \qquad (V.34)$$

See Appendix B.

Alternatively it is immediately apparent from equations (V.33), (IV.13) and (III.24) that $\bar{\bar{M}}$ reduces to the identity tensor, $\bar{\bar{I}}$.

Derivation of Transformation Tensor, $\bar{\bar{\mu}}$, in the Current Configuration from the Corresponding Transformation Tensor, $\bar{\bar{M}}$, in the Reference Configuration

We have,

$$\bar{\bar{M}} \equiv M_i^{\ j}\ \bar{A}^i\ \bar{A}_j \qquad \text{(V.35)}$$

Now the transformation of the directors has been defined such that,

$$\bar{a}_K = \bar{A}_K \cdot \bar{\bar{F}} \qquad \text{by (IV.14) and (IV.17)}$$

We may write (V.35) as,

$$\bar{\bar{M}}(\bar{A}_p) = M^{ij}\ \bar{A}_i\ \bar{A}_j \qquad \text{(V.36)}$$

Consequently,

$$\bar{\bar{\mu}}(\bar{a}_p) = M^{ij}(\bar{A}_i \cdot \bar{\bar{F}})(\bar{A}_j \cdot \bar{\bar{F}})$$

$$= M^{ij}\ \bar{\bar{F}}_C \cdot \bar{A}_i\ \bar{A}_j \cdot \bar{\bar{F}}$$

$$= \bar{\bar{F}}_C \cdot (M^{ij}\ \bar{A}_i\ \bar{A}_j) \cdot \bar{\bar{F}}$$

$$= \bar{\bar{F}}_C \cdot \bar{\bar{M}}(\bar{A}_p) \cdot \bar{\bar{F}}$$

$$\therefore \quad \bar{\bar{\mu}} = \bar{\bar{F}}_C \cdot \bar{\bar{M}} \cdot \bar{\bar{F}} \qquad \text{(V.37)}$$

Therefore, according to this transformation law, tensor $\bar{\bar{\mu}}$ in the current configuration may be obtained from $\bar{\bar{M}}$, the transformation tensor in the reference configuration and $\bar{\bar{F}}$, the deformation gradient.

In the simplification of the reference configuration geometry when the transformation tensor, $\bar{\bar{M}}$, reduces to the identity tensor, $\bar{\bar{I}}$, the transformation law expressed by equation (V.37) becomes,

$$\bar{\bar{\mu}} = \bar{\bar{F}}_C \cdot \bar{\bar{I}} \cdot \bar{\bar{F}}$$

or,

$$\bar{\bar{\mu}} = \bar{\bar{F}}_C \cdot \bar{\bar{F}} \qquad \text{(V.38)}$$

The Frame-Indifference Property of the Transformation Tensor

Finally in the investigation of the properties of the transformation tensor, we demonstrate the frame-indifference of $\bar{\bar{\mu}}$. Consider equation (V.15). We may rewrite the equation as,

$$\bar{\bar{\mu}} = \frac{\partial x^i}{\partial X^i}\,\bar{a}^i\,\frac{D\xi^n}{Dx^i}\,\bar{a}_n + \frac{\partial x^3}{\partial X^3}\left(\frac{\partial X^3}{\partial x^3}\right)_c \bar{a}^3\,\bar{a}_3 \qquad (V.39)$$

$$= \frac{\partial x^i}{\partial X^i}\,\bar{a}^i\,\frac{\partial}{\partial x^i}\,(\xi^n\,\bar{a}_n) + \frac{\partial x^3}{\partial X^3}\left(\frac{\partial X^3}{\partial x^3}\right)_c \bar{a}^3\,\bar{a}_3$$

$$\therefore\ \bar{\bar{\mu}} = \frac{\partial x^i}{\partial X^i}\,\bar{a}^i\,\frac{\partial \xi^n}{\partial x^i}\,\bar{a}_n + \frac{\partial x^3}{\partial X^3}\,\bar{a}^3\,\xi^n\,\frac{\partial \bar{a}_n}{\partial x^3} + \frac{\partial x^3}{\partial X^3}\left(\frac{\partial X^3}{\partial x^3}\right)_c \bar{a}^3\,\bar{a}_3 \qquad (V.40)$$

Since $\bar{a}_k = \bar{a}_k\,(x^3)$.

Consider a transformation of frame of the form,

$$\bar{r}^* = \bar{c}(t) + \bar{r}\cdot\bar{\bar{Q}}(t)$$

where $\bar{\bar{Q}}\cdot\bar{\bar{Q}}_C = \bar{\bar{I}} = \bar{\bar{Q}}_C\cdot\bar{\bar{Q}}$

and $|\bar{\bar{Q}}| = \pm 1$

In the $\{O^*, \bar{r}^*\}$ frame of reference,

$$\bar{\bar{\mu}}^* = \frac{\partial x^{i*}}{\partial X^{i*}} \bar{a}^{i*} \frac{\partial \xi^{n*}}{\partial x^{i*}} \bar{a}^*_n + \frac{\partial x^{3*}}{\partial X^{3*}} \bar{a}^{3*} \xi^{n*} \frac{\partial \bar{a}^*_n}{\partial x^{3*}} +$$

$$+ \frac{\partial x^{3*}}{\partial X^{3*}} \left(\frac{\partial X^{3*}}{\partial x^{3*}}\right)_c \bar{a}^{3*} \bar{a}^*_3 \qquad \text{(V.41)}$$

Under the change of frame (see Appendix D), the ratios $\frac{\partial x^j}{\partial X^i}$ and the scalar operators $\frac{\partial}{\partial x^i}$ transform as,

$$\frac{\partial x^{j*}}{\partial X^{i*}} = \frac{\partial x^j}{\partial X^i} \qquad \text{by (D.14)}$$

$$\frac{\partial}{\partial x^{i*}} = \frac{\partial}{\partial x^i} \qquad \text{by (D.11)}$$

Consider again,

$$\bar{r}^* = \bar{c} + \bar{r} \cdot \bar{\bar{Q}}$$

Substituting,

$$\bar{r}^* = \bar{r}^*_c + \xi^{k*}\,\bar{a}^*_k$$

and

$$\bar{r} = \bar{r}_c + \xi^n\,\bar{a}_n$$

we have,

$$\bar{r}^*_c + \xi^{k*}\,\bar{a}^*_k = \bar{c} + \left[\bar{r}_c + \xi^n\,\bar{a}_n\right]\cdot\bar{\bar{Q}}$$

or

$$\xi^{k*}\,\bar{a}^*_k = \bar{c} + \bar{r}_c\cdot\bar{\bar{Q}} - \bar{r}^*_c + \xi^n\,\bar{a}_n\cdot\bar{\bar{Q}}$$

But

$$\bar{r}^*_c = \bar{c} + \bar{r}_c\cdot\bar{\bar{Q}}$$

$$\therefore \quad \xi^{k*}\,\bar{a}^*_k = \xi^n\,\bar{a}_n\cdot\bar{\bar{Q}}$$

$$\xi^{k*}\left(\frac{\partial x^{k*}}{\partial X^{k*}}\right)_c(\bar{g}^*_k)_c = \xi^n\left(\frac{\partial x^n}{\partial X^n}\right)_c(\bar{g}_n)_c\cdot\bar{\bar{Q}} \quad \text{by (IV.12)}$$

$$\xi^{k^*}\left(\frac{\partial x^{k^*}}{\partial X^{k^*}}\right)_c(\bar{g}^*_k)_c \;=\; \xi^{n}\left(\frac{\partial x^{n^*}}{\partial X^{n^*}}\right)_c(\bar{g}^*_n)_c$$

by (D.14) and (D. 7)

$$\therefore \qquad \underline{\xi^{k^*}}\left(\frac{\partial x^{\underline{k}^*}}{\partial X^{\underline{k}^*}}\right)_c \;=\; \underline{\xi^{n}}\left(\frac{\partial x^{\underline{n}^*}}{\partial X^{\underline{n}^*}}\right)_c \;\delta^{k}_{\underline{n}}$$

$$\underline{\xi^{k^*}}\left(\frac{\partial x^{\underline{k}^*}}{\partial X^{\underline{k}^*}}\right)_c \;=\; \underline{\xi^{k}}\left(\frac{\partial x^{\underline{k}^*}}{\partial X^{\underline{k}^*}}\right)_c$$

$$\therefore \quad \xi^{\dot{k}^*} \;=\; \xi^{\dot{k}} \qquad \text{(V.42)}$$

and vectors $\bar{a}^k$ and $\bar{a}_k$ transform as,

$$\bar{a}^k \;=\; \bar{a}^k\cdot\bar{\bar{Q}} \qquad \text{(V.43.i)}$$

$$\bar{a}_k \;=\; \bar{a}_k\cdot\bar{\bar{Q}} \qquad \text{(V.43.ii)}$$

$$\therefore \quad \bar{\bar{\mu}}^* = \frac{\partial x^i}{\partial X^i}\,\bar{a}^i\cdot\bar{\bar{Q}}\,\frac{\partial \xi^n}{\partial x^i}\,\bar{a}_n\cdot\bar{\bar{Q}} + \frac{\partial x^3}{\partial X^3}\,\bar{a}^3\cdot\bar{\bar{Q}}\,\xi^n\,\frac{\partial}{\partial x^3}\,(\bar{a}_n\cdot\bar{\bar{Q}})$$

$$+ \frac{\partial x^3}{\partial X^3}\left(\frac{\partial X^3}{\partial x^3}\right)_c \bar{a}^3\cdot\bar{\bar{Q}}\,\bar{a}_3\cdot\bar{\bar{Q}}$$

Now as $\bar{\bar{Q}} = \bar{\bar{Q}}(t)$

$$\frac{\partial \bar{\bar{Q}}}{\partial x^3} = 0$$

$$\therefore \frac{\partial}{\partial x^3}\,(\bar{a}_n\cdot\bar{\bar{Q}}) = \frac{\partial \bar{a}_n}{\partial x^3}\cdot\bar{\bar{Q}}$$

and so,

$$\bar{\bar{\mu}}^* = \bar{\bar{Q}}_C\cdot\left[\frac{\partial x^i}{\partial X^i}\,\bar{a}^i\,\frac{\partial \xi^n}{\partial x^i}\,\bar{a}_n + \frac{\partial x^3}{\partial X^3}\,\bar{a}^3\,\xi^n\,\frac{\partial \bar{a}_n}{\partial x^3} + \right.$$

$$\left. + \frac{\partial x^3}{\partial X^3}\left(\frac{\partial X^3}{\partial x^3}\right)_c \bar{a}^3\,\bar{a}_3\right]\cdot\bar{\bar{Q}}$$

$$= \bar{\bar{Q}}_C\cdot\left[\frac{\partial x^i}{\partial X^i}\,\bar{a}^i\,\frac{\partial}{\partial x^i}\,(\xi^n\,\bar{a}_n) + \frac{\partial x^3}{\partial X^3}\left(\frac{\partial X^3}{\partial x^3}\right)_c \bar{a}^3\,\bar{a}_3\right]\cdot\bar{\bar{Q}}$$

$$= \bar{\bar{Q}}_C \cdot \left[\frac{\partial x^i}{\partial X^i} \frac{D\xi^n}{Dx^i} \bar{a}^i \bar{a}_n + \frac{\partial x^3}{\partial X^3} \left(\frac{\partial X^3}{\partial x^3} \right)_c \bar{a}^3 \bar{a}_3 \right] \cdot \bar{\bar{Q}}$$

$$\therefore \bar{\bar{\mu}}^* = \bar{\bar{Q}}_C \cdot \bar{\bar{\mu}} \cdot \bar{\bar{Q}} \qquad \text{(V.45)}$$

Demonstrating the frame-indifference property of the transformation tensor $\bar{\bar{\mu}}$.

CHAPTER 6

DISPLACEMENT VECTOR AND STRAIN TENSOR

Displacement Vector

The displacement vector is defined as the difference between the position vectors of the same material point in the current and reference configurations respectively,

$$\bar{u} = \bar{r} - \bar{R} \qquad \text{(VI. 1)}$$

or

$$\bar{u} = \bar{r}(x^3) + \xi^k \bar{a}_k - \bar{R}(X^3) - \Xi^K \bar{A}_K$$

$$\therefore \qquad \bar{u} = \bar{r}(x^3) - \bar{R}(X^3) + \xi^k \bar{a}_k - \Xi^K \bar{A}_K \qquad \text{(VI. 2)}$$

Strain Tensor

The Euler and Euler-Lagrange strain tensors are respectively,

$$\bar{\bar{\epsilon}} = \frac{1}{2}\left[\frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\bar{u}\partial}{\partial \bar{r}} - \frac{\partial \bar{u}}{\partial \bar{r}} \cdot \frac{\bar{u}\partial}{\partial \bar{r}}\right] \qquad \text{(VI. 3.i)}$$

$$\bar{\bar{E}} = \frac{1}{2}\left[\frac{\partial \bar{u}}{\partial \bar{R}} + \frac{\bar{u}\partial}{\partial \bar{R}} + \frac{\partial \bar{u}}{\partial \bar{R}} \cdot \frac{\bar{u}\partial}{\partial \bar{R}}\right] \qquad \text{(IV. 3.ii)}$$

(See Appendix C)

Consider the Euler strain tensor. In order to evaluate the expression for the strain tensor we must calculate the quantities

$$\frac{\partial \bar{u}}{\partial \bar{r}} \ , \ \frac{\bar{u}\partial}{\partial \bar{r}} \text{ and } \frac{\partial \bar{u}}{\partial \bar{r}} \cdot \frac{\bar{u}\partial}{\partial \bar{r}}.$$

We have,

$$\frac{\partial \bar{u}}{\partial \bar{r}} \equiv \bar{g}^{i} \frac{\partial \bar{u}}{\partial x^{i}}$$

$$= \bar{g}^{i} \frac{\partial}{\partial x^{i}} \left[\bar{r}(x^{3}) - \bar{R}(X^{3}) + \xi^{k}\bar{a}_{k} - \Xi^{K}\bar{A}_{K} \right]$$

$$= \bar{g}^{i} \left[\frac{\partial \bar{r}}{\partial x^{i}}(x^{3}) - \frac{\partial \bar{R}}{\partial X^{j}}(X^{3}) \frac{\partial X^{j}}{\partial x^{i}} + \frac{\partial}{\partial x^{i}} (\xi^{k}\bar{a}_{k}) \right.$$

$$\left. - \frac{\partial}{\partial X^{j}} (\Xi^{K}\bar{A}_{K}) \frac{\partial X^{j}}{\partial x^{i}} \right]$$

But $\frac{\partial X^{j}}{\partial x^{i}} = \delta^{j}_{\bar{i}} \frac{\partial X^{\bar{j}}}{\partial x^{\bar{i}}}$ by (IV. 5)

and $\frac{\partial \bar{r}(x^{3})}{\partial x^{i}} = \delta^{3}_{i} \frac{\partial \bar{r}(x^{3})}{\partial x^{3}}$

so that,

$$\frac{\partial \bar{u}}{\partial \bar{r}} = \bar{g}^3 \frac{\partial \bar{r}}{\partial x^3}(x^3) - \bar{g}^3 \frac{\partial X^3}{\partial x^3} \frac{\partial \bar{R}}{\partial X^3}(X^3) + \bar{g}^i \frac{D\xi^k}{Dx^i} \bar{a}_k - \bar{g}^i \frac{\partial X^i}{\partial x^i} \frac{D\Xi^K}{DX^i} \bar{A}_K$$

or

$$\frac{\partial \bar{u}}{\partial \bar{r}} = \bar{g}^3 (\bar{g}_3)_c - \frac{\partial X^3}{\partial x^3} \bar{g}^3 (\bar{G}_3)_C + \frac{D\xi^k}{Dx^i} \bar{g}^i \bar{a}_k - \frac{\partial X^i}{\partial x^i} \frac{D\Xi^K}{DX^i} \bar{g}^i \bar{A}_K \qquad \text{(VI. 4.i)}$$

Consequently the conjugate tensor,

$$\frac{\bar{u}\partial}{\partial \bar{r}} = (\bar{g}_3)_c \bar{g}^3 - \frac{\partial X^3}{\partial x^3} (\bar{G}_3)_C \bar{g}^3 + \frac{D\xi^n}{Dx^j} \bar{a}_n \bar{g}^j - \frac{\partial X^j}{\partial x^j} \frac{D\Xi^N}{DX^j} \bar{A}_N \bar{g}^j \qquad \text{(VI. 4.ii)}$$

Using the relations (IV.12) and (IV.15) we may express $\frac{\partial \bar{u}}{\partial \bar{r}}$ and $\frac{\bar{u}\partial}{\partial \bar{r}}$ exclusively in terms of the director base vector system.

$$\therefore \quad \frac{\partial \bar{u}}{\partial \bar{r}} = \frac{\partial x^3}{\partial X^3} \bar{d}^3 \left(\frac{\partial X^3}{\partial x^3}\right)_c \bar{a}_3 - \bar{d}^3 \bar{A}_3 + \frac{D\xi^k}{Dx^i} \frac{\partial x^i}{\partial X^i} \bar{d}^i \bar{a}_k$$

$$- \bar{d}^i \frac{D\Xi^K}{DX^i} \bar{A}_K \qquad \text{(VI. 5.i)}$$

and,

$$\frac{\bar{u}\partial}{\partial\bar{r}} = \frac{\partial x^3}{\partial X^3}\left(\frac{\partial X^3}{\partial x^3}\right)_c \bar{a}_3\bar{d}^3 - \bar{A}_3\bar{d}^3 + \frac{D\xi^n}{Dx^j}\frac{\partial x^j}{\partial X^j}\bar{a}_n\bar{d}^j - \frac{D\Xi^N}{DX^j}\bar{A}_N\bar{d}^j$$

(VI. 5.ii)

And using relationships (V.28) we may express $\frac{\partial\bar{u}}{\partial\bar{r}}$ and $\frac{\bar{u}\partial}{\partial\bar{r}}$ exclusively in terms of the axial director base vector systems.

$$\therefore \quad \frac{\partial\bar{u}}{\partial\bar{r}} = \frac{\partial x^3}{\partial X^3}\left(\frac{\partial X^3}{\partial x^3}\right)_c (\mu^{-1})_u{}^3\bar{a}\bar{u}\bar{a}_3 - (\mu^{-1})_u{}^3\bar{a}\bar{u}\bar{A}_3 +$$

$$+ \frac{D\xi^k}{Dx^i}\frac{\partial x^i}{\partial X^i}(\mu^{-1})_u{}^i\bar{a}\bar{u}\bar{a}_k - \frac{D\Xi^K}{DX^i}(\mu^{-1})_u{}^i\bar{a}\bar{u}\bar{A}_K \qquad \text{(VI. 6.i)}$$

and,

$$\frac{\bar{u}\partial}{\partial\bar{r}} = \frac{\partial x^3}{\partial X^3}\left(\frac{\partial X^3}{\partial x^3}\right)_c (\mu^{-1})_v{}^3\bar{a}_3\bar{a}^v - (\mu^{-1})_v{}^3\bar{A}_3\bar{a}^v +$$

$$+ \frac{D\xi^n}{Dx^j}\frac{\partial x^j}{\partial X^j}(\mu^{-1})_v{}^j\bar{a}_n\bar{a}^v - \frac{D\Xi^N}{DX^j}(\mu^{-1})_v{}^j\bar{A}_N\bar{a}^v \qquad \text{(VI. 6.ii)}$$

and finally we have the product,

$$\frac{\partial \bar{u}}{\partial \bar{r}} \cdot \frac{\bar{u}\partial}{\partial \bar{r}}$$

$$= \left[\frac{\partial x^3}{\partial X^3} \left(\frac{\partial X^3}{\partial x^3} \right)_c (\mu^{-1})_u{}^3 \left\{ \frac{\partial x^3}{\partial X^3} \left(\frac{\partial X^3}{\partial x^3} \right)_c (\mu^{-1})_v{}^3 a_{33} - (\mu^{-1})_v{}^3 (\bar{a}_3 \cdot \bar{A}_3) + \right. \right.$$

$$\left. + \frac{D\xi^n}{Dx^j} \frac{\partial x^j}{\partial X^j} (\mu^{-1})_v{}^j a_{3n} - \frac{D\Xi^N}{DX^j} (\mu^{-1})_v{}^j (\bar{a}_3 \cdot \bar{A}_N) \right\}$$

$$- (\mu^{-1})_u{}^3 \left\{ \frac{\partial x^3}{\partial X^3} \left(\frac{\partial X^3}{\partial x^3} \right)_c (\mu^{-1})_v{}^3 (\bar{A}_3 \cdot \bar{a}_3) - (\mu^{-1})_v{}^3 A_{33} + \right.$$

$$\left. + \frac{D\xi^n}{Dx^j} \frac{\partial x^j}{\partial X^j} (\mu^{-1})_v{}^j (\bar{A}_3 \cdot \bar{a}_n) - \frac{D\Xi^N}{DX^j} (\mu^{-1})_v{}^j A_{3N} \right\}$$

$$+ \frac{D\xi^k}{Dx^i} \frac{\partial x^i}{\partial X^i} (\mu^{-1})_u{}^i \left\{ \frac{\partial x^3}{\partial X^3} \left(\frac{\partial X^3}{\partial x^3} \right)_c (\mu^{-1})_v{}^3 a_{k3} - (\mu^{-1})_v{}^3 (\bar{a}_k \cdot \bar{A}_3) + \right.$$

$$\left. + \frac{D\xi^n}{Dx^j} \frac{\partial x^j}{\partial X^j} (\mu^{-1})_v{}^j a_{kn} - \frac{D\Xi^N}{DX^j} (\mu^{-1})_v{}^j \bar{a}_k \cdot \bar{A}_N \right\}$$

$$- \frac{D\Xi^K}{DX^i} (\mu^{-1})_u{}^i \left\{ \frac{\partial x^3}{\partial X^3} \left(\frac{\partial X^3}{\partial x^3} \right)_c (\mu^{-1})_v{}^3 (\bar{A}_K \cdot \bar{a}_3) - (\mu^{-1})_v{}^3 A_{K3} + \right.$$

$$+ \frac{D\xi^n}{Dx^j} \frac{\partial x^j}{\partial X^j} (\mu^{-1})_v{}^j (\bar{A}_K \cdot \bar{a}_n) - \frac{D\Xi^N}{DX^j} (\mu^{-1})_v{}^j A_{KN} \Bigg\} \Bigg] \bar{a}^u \bar{a}^v$$

(VI. 6.iii)

Equations (VI. 6) may now be evaluated for prescribed reference and current configurations and a prescribed deformation mapping of the coördinates. In principle only a prescribed reference configuration and a prescribed deformation mapping are required, since the current configuration is implicitly prescribed by the other two as we have seen in Chapter IV.

Hence the Euler strain tensor,

$$\bar{\bar{\epsilon}} = \frac{1}{2} \left[\frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\bar{u}\partial}{\partial \bar{r}} + \frac{\partial \bar{u}}{\partial \bar{r}} \cdot \frac{\bar{u}\partial}{\partial \bar{r}} \right]$$

may be evaluated.

The Euler-Lagrange strain tensor may be similarly evaluated using most of the results obtained above.

CHAPTER 7

THE DEFORMATION GRADIENT

The Deformation Gradient as a Material Transformation

Since we have defined the deformation mapping of the curvilinear coördinates as in equations (IV. 4), we may define the coördinates X^K to be material lines embedded in the undeformed rod whose deformed configurations are x^k. The points (X^1, X^2, X^3) and (x^1, x^2, x^3) related by equations (IV. 4) are then the same material point in the reference and current configurations respectively. A deformation gradient defined in conjunction with these isomorphic mappings will therefore describe a material transformation as opposed to the purely geometrical transformation defined by Suhubi. Consequently the material cross section described by,

$$\bar{r} = \bar{r}(x^3) + \xi^k \bar{a}_k$$

in the current configuration will represent the deformed configuration of the material cross section described by,

$$\bar{R} = \bar{R}(X^3) + \Xi^K \bar{A}_K$$

in the reference configuration.

In this chapter we shall investigate the form of the deformation gradient for a rod and the consequences on the deformation gradient of defining the deformation mapping for each curvilinear coördinate as isomorphic.

Calculation of the Deformation Gradient as a Function of the Parametric Rod Space

We have,

$$\bar{\bar{F}} = \frac{\partial \bar{r}}{\partial \bar{R}} = \bar{G}^i \frac{\partial \bar{r}}{\partial X^i}$$

$$= \bar{G}^i \frac{\partial}{\partial X^i} \left[\bar{r}(x^3) + \xi^k \bar{a}_k \right]$$

$$= \bar{G}^i \frac{\partial x^i}{\partial X^i} \frac{\partial}{\partial x^i} \left[\bar{r}(x^3) + \xi^k \bar{a}_k \right]$$

$$= \bar{G}^3 \frac{\partial x^3}{\partial X^3} \frac{\partial \bar{r}(x^3)}{\partial x^3} + \bar{G}^i \frac{\partial x^i}{\partial X^i} \frac{\partial}{\partial x^i} (\xi^k \bar{a}_k)$$

$$= \bar{G}^3 \frac{\partial x^3}{\partial X^3} (\bar{g}_3)_c + \bar{G}^i \frac{\partial x^i}{\partial X^i} \frac{D\xi^k}{Dx^i} \bar{a}_k$$

$$= \bar{G}^3 \frac{\partial x^3}{\partial X^3} \left(\frac{\partial X^3}{\partial x^3} \right)_c \bar{a}_3 + \bar{G}^i \frac{\partial x^i}{\partial X^i} \frac{D\xi^k}{Dx^i} \bar{a}_k$$

by (IV.12)

$$= \bar{G}^3 \frac{\partial x^3}{\partial X^3} \left(\frac{\partial X^3}{\partial x^3}\right)_c (\mu^{-1})_3{}^i \bar{d}_i + \bar{G}^i \frac{\partial x^i}{\partial X^i} \frac{D\xi^k}{Dx^i} (\mu^{-1})_k{}^j \bar{d}_j$$

by (V.21)

$$\therefore \quad \bar{\bar{F}} = \bar{G}^3 \frac{\partial x^3}{\partial X^3} \left(\frac{\partial X^3}{\partial x^3}\right)_c (\mu^{-1})_3{}^i \frac{\partial x^i}{\partial X^i} \bar{g}_i + \bar{G}^i \frac{\partial x^i}{\partial X^i} \frac{D\xi^k}{Dx^i} (\mu^{-1})_k{}^j \frac{\partial x^j}{\partial X^j} \bar{g}_j$$

by (IV.12) (VII. 1)

which may be written

$$\bar{\bar{F}} = \begin{bmatrix} F_1{}^1 \bar{G}^1 \bar{g}_1 + F_1{}^2 \bar{G}^1 \bar{g}_2 + F_1{}^3 \bar{G}^1 \bar{g}_3 \\ + F_2{}^1 \bar{G}^2 \bar{g}_1 + F_2{}^2 \bar{G}^2 \bar{g}_2 + F_2{}^3 \bar{G}^2 \bar{g}_3 \\ + F_3{}^1 \bar{G}^3 \bar{g}_1 + F_3{}^2 \bar{G}^3 \bar{g}_2 + F_3{}^3 \bar{G}^3 \bar{g}_3 \end{bmatrix}$$

(VII. 2)

where

$$F_1{}^1 = \left(\frac{\partial x^1}{\partial X^1}\right)^2 \frac{D\xi^k}{Dx^1} (\mu^{-1})_k{}^1$$

$$F_1{}^2 = \frac{\partial x^1}{\partial X^1} \frac{\partial x^2}{\partial X^2} \frac{D\xi^k}{Dx^1} (\mu^{-1})_k{}^2$$

$$F_1{}^3 = \frac{\partial x^1}{\partial X^1}\frac{\partial x^3}{\partial X^3}\frac{D\xi^k}{Dx^1}(\mu^{-1})_k{}^3$$

$$F_2{}^1 = \frac{\partial x^2}{\partial X^2}\frac{\partial x^1}{\partial X^1}\frac{D\xi^k}{Dx^2}(\mu^{-1})_k{}^1$$

$$F_2{}^2 = \left(\frac{\partial x^2}{\partial X^2}\right)^2\frac{D\xi^k}{Dx^2}(\mu^{-1})_k{}^2$$

$$F_2{}^3 = \frac{\partial x^2}{\partial X^2}\frac{\partial x^3}{\partial X^3}\frac{D\xi^k}{Dx^2}(\mu^{-1})_k{}^3$$

$$F_3{}^1 = \frac{\partial x^1}{\partial X^1}\frac{\partial x^3}{\partial X^3}\left[\left(\frac{\partial X^3}{\partial x^3}\right)_c(\mu^{-1})_3{}^1 + \frac{D\xi^k}{Dx^3}(\mu^{-1})_k{}^1\right]$$

$$F_3{}^2 = \frac{\partial x^2}{\partial X^2}\frac{\partial x^3}{\partial X^3}\left[\left(\frac{\partial X^3}{\partial x^3}\right)_c(\mu^{-1})_3{}^2 + \frac{D\xi^k}{Dx^3}(\mu^{-1})_k{}^2\right]$$

$$F_3{}^3 = \left(\frac{\partial x^3}{\partial X^3}\right)^2\left[\left(\frac{\partial X^3}{\partial x^3}\right)_c(\mu^{-1})_3{}^3 + \frac{D\xi^k}{Dx^3}(\mu^{-1})_k{}^3\right]$$

Now we have,

$$\bar{\bar{F}} \equiv \frac{\partial \bar{r}}{\partial \bar{R}} = \bar{G}^K\frac{\partial \bar{r}}{\partial X^K}$$

$$= \bar{G}^K \frac{\partial x^k}{\partial X^K} \frac{\partial \bar{r}}{\partial x^k}$$

$$= \bar{G}^K \frac{\partial x^K}{\partial X^K} \frac{\partial \bar{r}}{\partial x^K} \qquad \text{by (IV. 5)}$$

$$\therefore \qquad \bar{\bar{F}} = \bar{G}^K \frac{\partial x^K}{\partial X^K} \bar{g}_K$$

Consequently $\bar{\bar{F}}$ must be of the form,

$$\bar{\bar{F}} = \left[\frac{\partial x^1}{\partial X^1} \bar{G}^1 \bar{g}_1 + \frac{\partial x^2}{\partial X^2} \bar{G}^2 \bar{g}_2 + \frac{\partial x^3}{\partial X^3} \bar{G}^3 \bar{g}_3 \right] \qquad \text{(VII. 3)}$$

Since the tensors are mutually independant we may compare coefficients of (VII. 2) with coefficients of (VII. 3) giving,

$$\frac{\partial x^1}{\partial X^1} \frac{D\xi^k}{Dx^1} (\mu^{-1})_k{}^1 = 1 \qquad \therefore \frac{D\xi^k}{Dx^1} (\mu^{-1})_k{}^1 = \frac{\partial X^1}{\partial x^1} \qquad \text{(VII. 4.i)}$$

$$\frac{\partial x^1}{\partial X^1} \frac{\partial x^2}{\partial X^2} \frac{D\xi^k}{Dx^1} (\mu^{-1})_k{}^2 = 0 \qquad \therefore \frac{D\xi^k}{Dx^1} (\mu^{-1})_k{}^2 = 0 \qquad \text{(VII. 4.ii)}$$

since $\frac{\partial x^{\underline{i}}}{\partial X^{\underline{i}}} \neq 0$ in general.

$$\frac{\partial x^1}{\partial X^1}\frac{\partial x^3}{\partial X^3}\frac{D\xi^k}{Dx^1}(\mu^{-1})_k{}^3 = 0 \qquad \therefore \frac{D\xi^k}{Dx^1}(\mu^{-1})_k{}^3 = 0 \qquad \text{(VII. 4.iii)}$$

$$\frac{\partial x^2}{\partial X^2}\frac{\partial x^1}{\partial X^1}\frac{D\xi^k}{Dx^2}(\mu^{-1})_k{}^1 = 0 \qquad \therefore \frac{D\xi^k}{Dx^2}(\mu^{-1})_k{}^1 = 0 \qquad \text{(VII. 4.iv)}$$

$$\frac{\partial x^2}{\partial X^2}\frac{D\xi^k}{Dx^2}(\mu^{-1})_k{}^2 = 1 \qquad \therefore \frac{D\xi^k}{Dx^2}(\mu^{-1})_k{}^2 = \frac{\partial X^2}{\partial x^2} \qquad \text{(VII. 4.v)}$$

$$\frac{\partial x^2}{\partial X^2}\frac{\partial x^3}{\partial X^3}\frac{D\xi^k}{Dx^2}(\mu^{-1})_k{}^3 = 0 \qquad \therefore \frac{D\xi^k}{Dx^2}(\mu^{-1})_k{}^3 = 0 \qquad \text{(VII. 4.vi)}$$

$$\frac{\partial x^1}{\partial X^1}\frac{\partial x^3}{\partial X^3}\left[\left(\frac{\partial X^3}{\partial x^3}\right)_c(\mu^{-1})_3{}^1 + \frac{D\xi^k}{Dx^3}(\mu^{-1})_k{}^1\right] = 0$$

$$\therefore \frac{D\xi^k}{Dx^3}(\mu^{-1})_k{}^1 = -\left(\frac{\partial X^3}{\partial x^3}\right)_c(\mu^{-1})_3{}^1 \qquad \text{(VII. 4.vii)}$$

$$\frac{\partial x^2}{\partial X^2}\frac{\partial x^3}{\partial X^3}\left[\left(\frac{\partial X^3}{\partial x^3}\right)_c(\mu^{-1})_3{}^2 + \frac{D\xi^k}{Dx^3}(\mu^{-1})_k{}^2\right] = 0$$

$$\therefore \frac{D\xi^k}{Dx^3}(\mu^{-1})_k{}^2 = -\left(\frac{\partial X^3}{\partial x^3}\right)_c(\mu^{-1})_3{}^2 \qquad \text{(VII. 4.viii)}$$

$$\frac{\partial x^3}{\partial X^3}\left[\left(\frac{\partial X^3}{\partial x^3}\right)_c (\mu^{-1})_3{}^3 + \frac{D\xi^k}{Dx^3}(\mu^{-1})_k{}^3\right] = 1$$

$$\therefore \frac{D\xi^k}{Dx^3}(\mu^{-1})_k{}^3 = \frac{\partial X^3}{\partial x^3} - \left(\frac{\partial X^3}{\partial x^3}\right)_c (\mu^{-1})_3{}^3 \qquad \text{(VII. 4.ix)}$$

Giving the interrelationships between the components of the inverse transformation tensor and the quantities $\frac{D\xi^k}{Dx^i}$ defined at any point within the rod.

Consider equation (VII. 4.i)

$$\frac{\partial x^1}{\partial X^1}\frac{D\xi^k}{Dx^1}(\mu^{-1})_k{}^i = 1$$

Multiplying both sides by $\mu_1{}^i$

$$\therefore \frac{\partial x^1}{\partial X^1}\frac{D\xi^k}{Dx^1}(\mu^{-1})_k{}^1\,\mu_1{}^i = \mu_1{}^i$$

or $$\frac{\partial x^1}{\partial X^1}\frac{D\xi^k}{Dx^1}\delta_k^i = \mu_1{}^i \qquad \text{by (V.23)}$$

$$\therefore \frac{\partial x^1}{\partial X^1}\frac{D\xi^i}{Dx^1} = \mu_1{}^i$$

Similarly from equation (VII. 4.v)

$$\frac{\partial x^2}{\partial X^2}\frac{D\xi^i}{Dx^2} = \mu_2{}^i$$

It may be observed that these two expressions agree with the component form of the transformation tensor expressed in equation (V.15)

Material Transformation of the Directors

Let us define a second order tensor, $\bar{\bar{F}}$, such that,

$$\bar{\bar{F}} \equiv \bar{D}^a\bar{d}_a \qquad \text{(VII. 5)}$$

over the field of the rod.

Taking a scalar product with $\bar{D}_b$ as prefactor, we have

$$\bar{D}_b\cdot\bar{\bar{F}} = \bar{D}_b\cdot\bar{D}^a\,\bar{d}_a$$

or $$\bar{D}_b\cdot\bar{\bar{F}} = \delta_b^a\,\bar{d}_a$$

$$\therefore\quad \bar{d}_b = \bar{D}_b\cdot\bar{\bar{F}} \qquad \text{(VII. 6)}$$

Comparing equation (VII. 5) with equation (IV.14), we must have,

$\bar{\bar{F}} \equiv \bar{\bar{F}}$, the deformation gradient. Consequently

we observe that equation (VII. 5) may be taken as the definition of the deformation gradient, $\bar{\bar{F}}$.

Alternatively we may consider the deformation gradient, $\bar{\bar{F}}$, to be defined by the equation,

$$d\bar{r} = d\bar{R}\cdot\bar{\bar{F}} \qquad \text{(VII. 7)}$$

representing the transformation between material elements $d\bar{r}$ and $d\bar{R}$ in the current and reference configuration respectively since we have previously specified $\bar{\bar{F}}$ to represent a material transformation.

Comparing equations (VII. 6) and (VII. 7) we observe that due to the way the directors have been defined, the directors transform in deformation in the same way as the vectors representing material elements. Now during deformation these vectors representing material elements will undergo a rotation and a stretching. Consequently we observe that the vectors defined as directors will, during deformation, rotate through the same angle and undergo a proportionate change in length with the vectors representing material elements at the same point in the rod.

Now in order to fully define a deformation for a continuum, we must prescribe the deformation of each material element of the continuum, i.e., we must prescribe the rotation and stretch of each material element. Having prescribed the deformation of each material element in the

material curvilinear coördinate mapping, $x^{\underline{k}} = x^{\underline{k}}(X^{\underline{k}})$, we see that we may immediately deduce the deformed configuration of our directors since, in order to obtain the deformed director from the undeformed director, we rotate the undeformed director through the same angle as the corresponding material element at the same point and apply the same stretch, i.e. change in length per unit length, as the corresponding material element. The deformation gradient may then be obtained from equation (VII. 5).

Thus we see that defining directors in our continuum in the manner we have described enables us to obtain the deformed configuration of the directors and consequently a unique set of base vectors at each point in the continuum immediately from the concept of material transformation, without a knowledge of the differential geometry of the deformed configuration.

If required the curvilinear base vectors may then be obtained from equations (IV.12) where the quantities $\frac{\partial x^{\underline{i}}}{\partial X^{\underline{i}}}$ may be calculated from the deformation mappings given by equations (IV. 4).

CHAPTER 8

ALTERNATIVE METHOD OF CALCULATION OF THE STRAIN TENSOR

In the preceeding chapters a method of calculation of the strain tensor to any required degree of accuracy which is exact within the limits of continuum mechanics and the Taylor power series expansion, has been illustrated. First an exact representation of the radius vector is postulated in the form,

$$\bar{r}(x^1,x^2,x^3) = \bar{r}(0,0,x^3) + \left[\bar{r}(x^1,x^2,x^3) - \bar{r}(0,0,x^3)\right]$$

for a rod of arbitrary geometry defined within an arbitrary system of curvilinear coördinates. The term

$$\left[\bar{r}(x^1,x^2,x^3) - \bar{r}(0,0,x^3)\right]$$

is then expanded in terms of Taylor power series of the geometrical parameters of the rod space about the term $\bar{r}(0,0,x^3)$. By observing the nature of this expansion it is seen that the expansion is a vector summation which may then be expressed in terms of its components with respect to the curvilinear base vector system, $(\bar{g}_i)_c$, defined on the rod axis or, in terms of the axial director base system, $\bar{a}_i$, whose

vectors are collinear with the curvilinear base vectors defined on the rod axis, such that,

$$\bar{r}(x^1,x^2,x^3) = \bar{r}(0,0,x^3) + \xi^k \bar{a}_k$$

The displacement vector, $\bar{u}$, of a point within the rod, which is defined as the difference between the radius vector of a point in the current, or deformed configuration and the radius vector of the point in the reference, or undeformed configuration,

$$\bar{u} = \bar{r} - \bar{R}$$

may then be calculated.

The Euler strain tensor,

$$\overline{\overline{\varepsilon}} = \frac{1}{2}\left[\frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\bar{u}\partial}{\partial \bar{r}} - \frac{\partial \bar{u}}{\partial \bar{r}} \cdot \frac{\bar{u}\partial}{\partial \bar{r}}\right]$$

which requires first calculating the derivative, $\frac{\partial \bar{u}}{\partial \bar{r}}$, of the power series expansion of $\bar{u}$ may then be calculated. All the terms in the resulting expression for the strain tensor which are functions of the Taylor power series representation of the radius vector, appear in the form,

$$\frac{\partial}{\partial x^i}\ (\xi^k \bar{a}_k)$$

When the form of these derivatives of the power series $\xi^k \bar{d}_k$ is

examined it is also seen to be a vector summation which may be expressed in terms of its components with respect to the director base system which we have expressed by,

$$\frac{\partial}{\partial x^i}\,(\xi^k \bar{a}_k) \;\equiv\; \frac{D\xi^k}{Dx^i}\,\bar{a}_k$$

Now if the form of the transformation tensor, $\bar{\bar{\mu}}$, between the curvilinear base vector system at any point in the rod space, and the director base system, is investigated, it is found that $\bar{\bar{\mu}}$ is a function of the terms $\frac{D\xi^k}{Dx^i}$,

where $$\frac{D\xi^k}{Dx^i} \equiv \left[\frac{\partial}{\partial x^i}\,(\xi^k \bar{a}_k)\right] \cdot \bar{a}^i$$

This functional relationship of $\bar{\bar{\mu}}$ to $\frac{D\xi^k}{Dx^i}$ is also illustrated in the investigation of the nature of the deformation gradient, $\bar{\bar{F}}$, in equations (VII. 4).

We now propose to invert this functional relationship and instead of regarding $\bar{\bar{\mu}}$ as a function of $\frac{D\xi^k}{Dx^i}$, regard $\frac{D\xi^k}{Dx^i}$ as a function of $\bar{\bar{\mu}}$. Equation (V.15) indicates that a complete knowledge of the form of tensor $\bar{\bar{\mu}}$, after a prescribed deformation, yields a complete description of the terms $\frac{D\xi^k}{Dx^i}$, such that

$$\mu_\alpha{}^i = \frac{\partial x^{\underline{\alpha}}}{\partial X^{\underline{\alpha}}}\;\frac{D\xi^i}{Dx^{\underline{\alpha}}} \qquad \text{(VIII. 1.i)}$$

$$\mu_3{}^{\alpha} = \frac{\partial x^3}{\partial x^3} \frac{D\xi^{\alpha}}{Dx^3} \qquad \text{(VIII. 1.ii)}$$

$$\mu_3{}^{3} = \frac{\partial x^3}{\partial X^3} \left[\frac{D\xi^3}{Dx^3} + \left(\frac{\partial X^3}{\partial x^3} \right)_c \right] \qquad \text{(VIII. 1.iii)}$$

The functional relationships (VII. 8) could also be used to calculate $\frac{D\xi^k}{Dx^i}$ from a known tensor $\bar{\bar{\mu}}$ since we then have nine equations (VII. 8) for nine unknowns $\frac{D\xi^k}{Dx^i}$. This however involves first the calculation of the inverse tensor $\bar{\bar{\mu}}^{-1}$.

We also note that the transformation tensor, $\bar{\bar{\mu}}$, for an arbitrary configuration of the rod is calculable from a knowledge of the corresponding transformation tensor, $\bar{\bar{M}}$, in the reference configuration and the deformation gradient, $\bar{\bar{F}}$, by equation (V. 37);

viz

$$\bar{\bar{\mu}} = \bar{\bar{F}}_C \cdot \bar{\bar{M}} \cdot \bar{\bar{F}}$$

As outlined in Chapter VII the deformation gradient may be defined as a function of the directors in the reference and current configuration. After the simplification of the reference configuration geometry the directors in this configuration may be calculated without evaluating a power series expansion for the radius vector and the current configuration

directors are obtained by subjecting the reference configuration directors to the same rotation and stretch as vectors representing material elements. The deformation of material elements of the continuum is implicitly prescribed in the mapping of the material curvilinear coördinates,

$$x^k = \underline{x^k}\ (\underline{X^k})$$

In this way the evaluation of the power series expansion is unnecessary in order to evaluate the deformation gradient. If the deformation gradient were considered in purely geometrical terms,

i.e.

$$\bar{\bar{F}} \equiv \frac{\partial \bar{r}}{\partial \bar{R}} = \bar{G}^K \frac{\partial x^K}{\partial X^K} \bar{g}_K$$

its calculation would involve the power series expansion of the radius vector in the current configuration at least.

The transformation tensor in the reference configuration, $\bar{\bar{M}}$, is a function only of the reference configuration geometry and is therefore implicitly prescribed by that geometry. This tensor is not however sufficient by itself to fully describe that geometry since it only describes the vectors of one local point space in terms of the vectors of another local point space. The spatial arrangement of the two points is not included in the tensor. In order to calculate $\bar{\bar{M}}$ for a reference

configuration of arbitrary geometry it would be necessary to first calculate the full power series for $\bar{R}$. However it has been demonstrated that, without any loss in generality, the reference configuration geometry may be defined such that $\bar{\bar{M}}$ reduces to the identity tensor, $\bar{\bar{I}}$. The calculation of the power series for $\bar{R}$ is then unnecessary in order to obtain $\bar{\bar{M}}$.

Consequently it is suggested that the calculation of the strain tensor be approached in the following way. The deformation gradient may be calculated from a consideration of material transformation and the mapping of the material curvilinear coördinates. The transformation tensor, $\bar{\bar{\mu}}$, in the current configuration may then be calculated from the deformation gradient as in equation (V.38). The components of $\bar{\bar{\mu}}$ and the differential coefficients $\frac{\partial x^{\underline{i}}}{\partial X^{\underline{i}}}$ obtained from the mapping functions yield $\frac{D\xi^k}{Dx^i}$ directly according to equation (VIII. 1). Alternatively equations (VII. 4) may be solved for $\frac{D\xi^k}{Dx^i}$. In this way, $\frac{D\xi^k}{Dx^i}$ are obtained without recourse to calculating and differentiating the series expansion which would otherwise have been necessary. The $\frac{D\xi^k}{Dx^i}$ may then be used in the expression for the strain tensor. If we examine all of the other terms involved in the expression for the strain tensor, i.e. equations (VI. 6) we observe that they are all either functions of the differential coefficients $\frac{\partial x^{\underline{i}}}{\partial X^{\underline{i}}}$ obtained from the mapping functions (IV. 4) or functions of the current configuration geometry of the rod axis. The geometry of the rod axis in the

current configuration is not a function of the power series expansion of the radius vector and is prescribed by the axis geometry in the reference configuration and the deformation gradient.

Therefore by this method the strain tensor may be calculated to the degree of accuracy given by the power series expansion of the radius vector without recourse to explicitly evaluating the power series.

CHAPTER 9

CONCLUSIONS

In this thesis, the attempt to reduce the three dimensional analysis of a rod continuum to an analysis in one spatial variable has been made in the following way.

It was first demonstrated how the geometrical description of an arbitrary point in the rod continuum could be uniquely specified by a radius vector whose form was a power series expansion of the one spatial variable, $\bar{r}(x^3)$, which may or may not be infinite according to the functional form of the parameters characterising the geometry of the rod space. It was then demonstrated, by using the fact that the coördinate system of one of the configurations, reference or current, may be prescribed without any loss of generality, that the geometrical parameters of the prescribed space may be considerably simplified. In our case the reference space was chosen to be prescribed.

The strain tensor, which is a function of the displacement vector, which is in turn a function of the radius vectors to a chosen material point in the reference and current configurations respectively, may then, in principle, be calculated. This analysis is termed exact since there is no approximation

in the form of neglecting any terms in view of their relative orders of magnitude as opposed to many previous analyses in which a truncated power series representation is used for the form of the radius vector.

A form for the deformation mapping was proposed such that the functional relationship of the set of values of X^k to the set of values of x^k was isomorphic. This allowed each point (X^1,X^2,X^3) to be interpreted as a material point and consequently allowed the deformation gradient to be interpreted as a material deformation gradient. Then the mapped cross section in the current configuration is the same material surface as the corresponding unmapped cross section in the reference configuration. In Suhubi's analysis the mapped cross section will not in general be the same material surface as the unmapped cross section. The deformation gradient as defined by Suhubi has only a geometrical significance and does not in general represent a material transformation.

Finite director triads were defined at each point within the rod such that the transformation of the directors in deformation was identical to the transformation of material elements in deformation. The relationship between directors and base vectors was established. In this way it was possible to evaluate the deformed configuration of the directors directly from considerations of material transformation.

It was pointed out that the current configuration geometry is completely prescribed by the reference configuration geometry

and the deformation gradient and it was demonstrated how some parameters of the current configuration geometry may be calculated from them.

The form of the transformation tensor between director base vector systems defined on the rod axis and director base vector systems defined at any point in the corresponding cross section was investigated and a fundamental relationship between the components of the transformation tensor and the power series of the radius vector was developed. It was then demonstrated how the transformation tensor in the current configuration may be calculated from the corresponding transformation tensor in the reference configuration and the deformation gradient. The investigation of the form and properties of the transformation tensor was concluded with a demonstration of the frame-indifference property of the tensor.

The strain tensor was then evaluated in terms of the rod parameters and it was discovered that, in view of the fundamental relationships between the components of the transformation tensor and the power series expansion of the radius vector, the strain tensor may be completely expressed in terms of: (1) components of the transformation tensor, (2) the differential coefficients, $\frac{\partial x^{\underline{i}}}{\partial X^{\underline{i}}}$, obtained from the deformation mapping, and (3) the geometrical parameters of the rod axis. The differential coefficients, $\frac{\partial x^{\underline{i}}}{\partial X^{\underline{i}}}$, and the geometry of the rod axis are not functions of the power series expansion of the radius vector whereas the transformation tensor is a function of the power

series. However it had already been demonstrated that the transformation tensor, $\bar{\bar{\mu}}$, in the current configuration was a function of the transformation tensor, $\bar{\bar{M}}$, in the reference configuration and the deformation gradient. Since the transformation tensor, $\bar{\bar{M}}$, may be reduced to the identity tensor, $\bar{\bar{I}}$, without any loss in generality and the deformation gradient may be defined by considerations of material transformation, the transformation tensor, $\bar{\bar{\mu}}$, could be evaluated without calculating the power series expansion of the radius vector. Consequently it was shown how the strain tensor may be evaluated to the same degree of accuracy which would be obtained from using the full power series expansion of the radius vector without recourse to explicitly calculating the power series expansion.

It should be pointed out that in the doctoral thesis of Antman on Page 11 there is the note; "If we were not to assume that $\bar{r}$ was differentiable but only continuous, then these shifters, (i.e. the transformation tensor components), would take over as the fundamental kinematic variables." This point is however pursued no further by Antman and receives no mention in the subsequent paper of Antman and Warner.[4] It was not until after the independent investigation by the present author that this mention was discovered.

Thus an analysis for the strain in a rod continuum of finite dimensions is proposed which is both exact, as far as the Taylor power series is exact and relatively simple algebraically.

Finally the form of the deformation gradient in terms of the rod parameters was investigated. This led to the formulation of a set of nine equations which are sufficient to solve for the nine components of the inverse transformation tensor in terms of the power series expansion of the radius vector. Two equations of this set were seen to be identical to relationships expressed in the equivalent set of equations discovered in the investigation of the form of the transformation tensor, thus providing a check on the equations which had been derived by two independent methods.

REFERENCES

1. Leigh, D.C., "Nonlinear Continuum Mechanics" McGraw-Hill, 1968.

2. Hay, G.E., "The Finite Displacement of Thin Rods", Transactions of the American Mathematical Society. Vol. 51, 1942.

3. Ericksen, J.L., and Truesdell, C., "Exact Theory of Stress and Strain in Rods and Shells", Archive for Rational Mechanics and Analysis, Vol. 1, 1958.

4. Antman, S.S., and Warner, W.H., "Dynamical Theory of Hyperelastic Rods", Archive for Rational Mechanics and Analysis. Vol. 23, No. 19, 1968.

5. Green, A.E., Laws, N. and Naghdi, P.M., "Rods, Plates and Shells", Proceedings of Cambridge Philosophical Society, Vol. 64, No. 3, 1968.

6. Suhubi, E., "On the Foundations of the Theory of Rods" International Journal of Engineering Science, Vol. 6. 1968.

7. Oravas, G. AE., "Continuum Mechanics-Lecture Notes" Delivered at the McMaster University, Hamilton, Ontario, 1969-70.

8. Lichnerowicz, A. "Elements of Tensor Calculus" John Wiley and Sons Inc., New York, 1962.

9. Malvern, L.E., "Introduction to the Mechanics of a Continuous Medium" Prentice-Hall, 1969.

APPENDIX A

MAPPING

Sets:

The symbol X represents a *set*, class, collection or family which contains elements or objects $x_1, x_2, \ldots$ We can write this as,

$$X = \{x_1, x_2, \ldots\} \tag{A. 1}$$

The element x is in X, or is a member of X, or belongs to X. We designate this by:

$$x \in X \tag{A. 2}$$

Consider sets X and Y. The set X is a *subset* of Y if every x in X is also in Y. This relation is designated by:

$$X \subseteq Y \tag{A. 3}$$

Also Y is said to *include* or *contain* X, and we write $Y \supseteq X$. The *empty set* is designated by 0 and is contained in every set. If all elements of Y are in X, that is, $Y \subseteq X$, and all elements of X are in Y, that is, $X \subseteq Y$, then the sets Y and X

are said to *coincide* or to be equal.

$$Y = X$$

If $X \subseteq Y$ without being equal to Y, X is called a *proper subset* of Y, or we say Y contains X *properly*. We write,

$$X \subset Y \qquad Y \supset X \tag{A. 4}$$

Thus, $X \subset Y$ means that all elements of X are in Y and that there is at least one element in Y not belonging to X.

The *union* of two sets X and Y comprises all the elements of the two sets and is denoted by,

$$X \cup Y \tag{A. 5}$$

The *intersection* of two sets consists only of those elements which belong to both X and Y and is designated by,

$$X \cap Y \tag{A. 6}$$

Functions or Mappings; Functionals

A *function*, or *mapping*, consists of the following:

1. A set X called the *domain* of the function.
2. A set Y called the *range* of the function.
3. A *rule* or *correspondence* T which associates with each

element x of X a *single* element y of Y.

We designate the mapping by,

$$T: X \to Y \qquad \text{or} \qquad y = T(x) \qquad (A.\ 7)$$

We also say that y is the *image* of x under T. Y may contain elements other than those for which $T(x)$ is in Y; that is, $Y \supseteq T(X)$. Suppose D is a subset of X; then $E=T(D)$ is a subset of Y. The *inverse transformation* of the set E is designated by $T^{-1}(E)$ and is the set of elements of X whose images are in $E \subseteq Y$. D may be a proper subset of $T^{-1}(E)$.

Also

$$T^{-1}(Y) = X$$

where it must be remembered that the empty set is included in X. If no element of X has an image in E, then $T^{-1}(E) = 0$.

It should be noted that if E consists of a single element of Y, this need not be true of $T^{-1}(E)$; that is, T^{-1} need not define a function in the sense above from Y to X; there may be more than one element of X or there may be none. The mapping $T: X \to Y$ is called,

1. *One-one into* if $x_1 \neq x_2$ implies $T(x_1) \neq T(x_2)$ for all $x_1, x_2, \in X$.
2. *Onto* if $T(X) = Y$, that is, every $y \in Y$ corresponds to some $x \in X$.
3. *One-one* if *one-one into* and *onto*.

If $T:X \to Y$ is *one-one into*, then $T^{-1}(E)$, where E consists of a single element of Y, is a single element of X or is 0. If T is *one-one* then the *inverse* function $T^{-1}:Y \to X$ is defined; i.e., it is a function and is *one-one*, and we can write

$$x = T^{-1}(y)$$

Vector Spaces

A vector space (or directed space) consists of the following:

1. A set V of objects called *vectors*
2. A *field* F of scalars
3. A rule (or operation), called *vector addition*, which associates with each pair of vectors $\bar{u}$ and $\bar{v}$ in V, a vector $\bar{u} + \bar{v}$ in V, called the sum of $\bar{u}$ and $\bar{v}$, in such a way that:
   a) Addition is commutative, $\bar{u} + \bar{v} = \bar{v} + \bar{u}$
   b) Addition is associative, $\bar{u} + (\bar{v} + \bar{w}) = (\bar{u} + \bar{v}) + \bar{w}$
   c) There is a unique vector $\bar{0}$ in V, called the *zero vector*, such that $\bar{v} + \bar{0} = \bar{v}$
   d) For each vector $\bar{v}$ in V there is a unique vector $-\bar{v}$ in V such that $\bar{v} + (-\bar{v}) = \bar{0}$
4. A rule (or operation), called *scalar multiplication*, which associates with each scalar a in F and vector $\bar{v}$ in V a vector $a\bar{v}$ in V, called the product of a and $\bar{v}$, in such a

way that,

a) $1\bar{v} = \bar{v}$

b) $(ab)\bar{v} = a(b\bar{v})$

c) $a(\bar{u} + \bar{v}) = a\bar{u} + a\bar{v}$

d) $(a + b)\bar{v} = a\bar{v} + b\bar{v}$

n- Dimensional Vector Spaces

Let $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p$ be p non zero vectors in a vector space V. This system of vectors forms a *linearly independant* set of order p if it is impossible to find p numbers $a_1, a_2, \ldots, a_p$, not all zero, such that,

$$a_1\bar{v}_1 + a_2\bar{v}_2 + \ldots + a_p\bar{v}_p = \bar{0}$$

In the contrary case the given system of vectors is said to be *linearly dependant.*

Consider the set of all systems of linearly independant vectors in V. Two possibilities exist: either (1) there exist linearly independant systems of arbitrarily large order, or (2) the order of the linearly independant systems is bounded. In the second case it is possible to determine an integer n such that there exist linearly independant systems of order n but not of order n+1. If $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\}$ is any such system of order n, it will be called a basis of V, in conformity with the following definition: The *basis* or

directed base of a vector space V is any linearly independant system of vectors of maximum order.

Theorem*

For a system of vectors to constitute a basis of V it is necessary and sufficient that any vector of V can be expressed in one, and only one, way as a linear combination of the vectors of that system.

The number n is called the *dimension of the vector space under consideration*. We shall use V_n to denote an n-*dimensional vector space*.

Euclidean Vector Spaces

Consider first the vector space of elementary geometry. For each pair of vectors $\bar{u}$, $\bar{v}$ there is a process of multiplication which is called the *dot, scalar, or inner product*. If γ is the angle between the vectors $\bar{u}$ and $\bar{v}$, then the dot product in elementary geometry is $|\bar{u}||\bar{v}|\cos\gamma$. The dot product has the following properties:

1. $\bar{u}\cdot\bar{v} = \bar{v}\cdot\bar{u}$, commutative property.
2. $(a\bar{u})\cdot\bar{v} = a(\bar{u}\cdot\bar{v})$, associative property with respect to multiplication by a scalar a.
3. $\bar{u}\cdot(\bar{v}+\bar{w}) = \bar{u}\cdot\bar{v} + \bar{u}\cdot\bar{w}$, distributive property with respect to vector addition.
4. If $\bar{u}\cdot\bar{v} = 0$ for arbitrary $\bar{u}$, then $\bar{v} = \bar{0}$.

---

* For proof of this theorem see Lichnerowicz[8].

Consider, in general, a finite dimensional vector space V_n defined over the field of real numbers. Suppose there exists a rule of composition which assigns to every pair of vectors $\bar{u}$, $\bar{v}$ a correspondence with a real number $\bar{u}\cdot\bar{v}$ having properties 1 to 4. We then say that V_n is a *Euclidean* vectors space.

A vector space is said to be properly Eucidean if it is Euclidean and

5) $\bar{u}\cdot\bar{u} > 0$ for all $\bar{u} \neq \bar{o}$

The rules of composition 1 to 5 define a dot product in that space.

Definition of an Affine Space[8]

The points of the space E of elementary geometry define vectors - the position vectors of elementary vector analysis. These obviously satisfy the relations

$$\overrightarrow{AB} = - \overrightarrow{BA}$$

$$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$$

Choose an arbitrary point 0 in E, then each point A of E is associated with a displacement vector $\bar{a}$ defined by

$$\bar{a} = \overrightarrow{0A}$$

In general, consider a manifold of points, E, and suppose

that to each pair (A,B) of points of ξ taken in order there corresponds a vector of an n-dimensional vector space E_n denoted by $\overrightarrow{AB}$. Assume this correspondance to have the following properties:

a) $\overrightarrow{AB} = - \overrightarrow{BA}$

b) $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$

c) if 0 is an arbitrary point in E then, to every vector $\bar{a}$ of E_n there corresponds a unique point A such that,

$$\overrightarrow{0A} = \bar{a}$$

When these conditions hold we say the manifold E is an *affine point space* of n-dimensions.

Euclidean Point Spaces

Definition:

An affine point space which is associated with a Euclidean vector space is called a Euclidean point space.

Linear Transformations

Consider again the general concept of a mapping, T, of a set X to a set Y, that is $T:X \to Y$. Let X and Y be sets having an algebraic structure (i.e. certain laws of composition, together with a set of axioms satisfied by these operations) of the same type. A function $T:X \to Y$ which preserves

the given operations is called a *morphism*. A one-one morphism is called an *isomorphism*. A morphism $T:X \to X$, that is a morphism of a set into itself, is called an *endomorphism* of X, and a *one-one* endomorphism is called an *automorphism* of X.

When the algebraic structure is that of a vector space, a morphism is usually called a *linear transformation, linear operator*, or *tensor*. Thus a linear transformation T of a vector space V into another vector space W is defined by,

$$\text{a)} \quad T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$$

$$\text{b)} \quad T(a\bar{v}) = aT(\bar{v})$$

or in direct tensor notation,

$$\text{a)} \quad \bar{\bar{T}}\cdot(\bar{u} + \bar{v}) = \bar{\bar{T}}\cdot\bar{u} + \bar{\bar{T}}\cdot\bar{v}$$

$$\text{b)} \quad \bar{\bar{T}}\cdot(a\bar{v}) = a\bar{\bar{T}}\cdot\bar{v}$$

For all $\bar{u}$, $\bar{v}$ in V and a in F. The linearity of the relation is usually emphasized by the removal of the parentheses; thus we write,

$$\bar{w} = T(\bar{v}) = T\bar{v}$$

or

$$\bar{w} = \bar{\bar{T}}\cdot\bar{v} \qquad \text{in direct notation.}$$

Limits and Continuity

Consider vector spaces V and W, possibly infinite-dimensional, each possessing a dot product. Let a function or mapping $F:D \to W$ be defined, in general nonlinear, where D is a subset of V.

Let $\bar{v}_o \in D$ and $\bar{w}_o \in W$. We then say that F has the *limit* $\bar{w}_o$ as $\bar{v}$ tends toward $\bar{v}_o$, written,

$$\lim_{\bar{v} \to \bar{v}_o} \quad F(\bar{v}) = \bar{w}_o$$

if for each positive number ε, there is a positive number δ such that $\bar{v}_o \in D$ and $0 < |\bar{v} - \bar{v}_0| < \delta$ imply $|F(\bar{v}) - \bar{w}_o| < \varepsilon$.

The function $F:D \to W$ is said to be *continuous* at $\bar{v}_o \in D$ if the limit $\bar{w}_o$ exists and is $F(\bar{v}_o)$. The function F is said to be *continuous in* D if it is continuous at each vector of D. These definitions are precisely the usual ones when V and W are one-dimensional, that is, scalar spaces, and require only the notion of absolute value or distance (here defined by means of the dot product) in the domain and range to make sense.

APPENDIX B

DOUBLE TENSOR FIELDS

The Tensor Product of two Spaces[8]

Consider two vector spaces E_a and E_b of a and b dimensions respectively and associate with them a vector space of dimensionality ab denoted by $E_a \otimes E_b$. If $\bar{A}$ and $\bar{B}$ belong to E_a and E_b respectively we can set up a correspondence between the pair $(\bar{A},\bar{B})$ and an element of the vector space $E_a \otimes E_b$ denoted by $\bar{A} \otimes \bar{B}$ or $\bar{A}\bar{B}$. This correspondence has the following properties:

i) If $\bar{A}$, $\bar{A}_1$, $\bar{A}_2$ belong to E_a and $\bar{B}$, $\bar{B}_1$, $\bar{B}_2$ to E_b then the distributive law holds with respect to vector addition:

$$\bar{A}(\bar{B}_1 + \bar{B}_2) = \bar{A}\bar{B}_1 + \bar{A}\bar{B}_2$$

$$(\bar{A}_1 + \bar{A}_2)\bar{B} = \bar{A}_1\bar{B} + \bar{A}_2\bar{B}$$

ii) If α is an arbitrary scalar, the associative law holds:

$$\alpha\bar{A}\bar{B} = \bar{A}\alpha\bar{B} = \alpha(\bar{A}\bar{B})$$

iii) If $(\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_a)$ and $(\bar{B}_1, \bar{B}_2, \ldots, \bar{B}_b)$ are any two bases of E_a and E_b respectively the ab elements

$$\bar{A}_i \bar{B}_j \qquad (i = 1,2,\ldots,a;\ j = 1,2,\ldots,b)$$

of $E_a \otimes E_b$ form a basis in that space.

When these conditions hold we say that the vector space $E_a \otimes E_b$ is the tensor product of the vector space E_a and E_b and that the element $\bar{A}\bar{B}$ is the tensor product of the two vectors $\bar{A}$ and $\bar{B}$.

Tensor Products of Several Spaces

Consider the three vector spaces E_a, E_b, E_c with a, b, c dimensions respectively. If $\bar{A}$ belongs to E_a, $\bar{B}$ to E_b, $\bar{C}$ to E_c then the element $\bar{A}\bar{B}$ of $E_a \otimes E_b$ can be multiplied tensorially with the element $\bar{C}$ of E_c. The element $(\bar{A}\bar{B})\bar{C}$ of a vector space H is thus obtained. We assume that the same element of H is obtained by taking the tensor product of $\bar{A}$ with $\bar{B}\bar{C}$

i.e. $$(\bar{A}\bar{B})\bar{C} = \bar{A}(\bar{B}\bar{C}) = \bar{A}\bar{B}\bar{C}$$

which is the associative property of tensor products. The vector space H will be represented by $E_a \otimes E_b \otimes E_c$. In general we have the following definition.

Definition:

Each element of the vector space

$$E_a \otimes E_b \otimes E_c \otimes \ldots$$

constructed from the spaces E_a, E_b, E_c,... is called a tensor.

Reciprocal (or Dual) Base Vectors

If $\bar{G}_i$ (i = 1, 2, ..., a) is a basis of E_a we define reciprocal vectors $\bar{G}^j$ such that

$$\bar{G}_i \cdot \bar{G}^j = \delta_i^j \qquad \text{(B. 1)}$$

From this definition it may be seen that the vectors $\bar{G}^j$ (j = 1, 2, ..., a) are linearly independant and so the $\bar{G}^j$ form an alternative basis for E_a.

Analytical Expression for the Tensor Product of Two Vectors

For our purposes we shall consider,

$E_a = E_b$ = Euclidean three space, E_3:

Let $\bar{G}_i$ (i = 1,2,3) be a basis of E_3. We write the tensor product,

$$\bar{A}_i \, \bar{B}_i \equiv \bar{\bar{T}}$$

Now $\bar{A}_i$ and $\bar{B}_i$ may be expressed in terms of their components with respect to the $\bar{G}_i$ base vector system,

$$\bar{A}_i = A^L \bar{G}_L \quad ; \quad \bar{B}_i = B^M \bar{G}_M \qquad \text{(B. 2)}$$

or in terms of the reciprocal base system,

$$\bar{A}_i = A_N \bar{G}^N \quad ; \quad \bar{B}_i = B_P \bar{G}^P \qquad \text{(B. 3)}$$

so that $\bar{\bar{T}}$ may be expressed in a variety of ways;

$$\bar{\bar{T}} = A^L B^M \bar{G}_L \bar{G}_M = T^{LM} \bar{G}_L \bar{G}_M \qquad \text{(B. 4.i)}$$

where $T^{LM} = A^L B^M$

or $$\bar{\bar{T}} = A_N B_P \bar{G}^N \bar{G}^P = T_{NP} \bar{G}^N \bar{G}^P \qquad \text{(B. 4.ii)}$$

where $T_{NP} = A_N B_P$

or $$\bar{\bar{T}} = A^L B_P \bar{G}_L \bar{G}^P = T^L{}_P \bar{G}_L \bar{G}^P \qquad \text{(B. 4.iii)}$$

where $T^L{}_P = A^L B_P$

or $$\bar{\bar{T}} = A_N B^M \bar{G}^N \bar{G}_M = T_N{}^M \bar{G}^N \bar{G}_M \qquad \text{(B. 4.iv)}$$

where $T_N{}^M = A_N B^M$

The Identity Tensor

The fundamental or Identity Tensor, $\bar{\bar{I}}$, is defined such that,

$$\bar{A}\cdot\bar{\bar{I}} = \bar{A} = \bar{\bar{I}}\cdot\bar{A} \qquad \text{(B. 5)}$$

and $\bar{\bar{I}}$ is defined isotropic for all spaces.
such that, $\bar{\bar{I}}\ (\bar{R}) = \bar{\bar{I}}(\bar{r})$.

Now since $\bar{\bar{I}}\cdot\bar{G}_K = \bar{G}_K$

we must have the form of $\bar{\bar{I}}$ as,

$$\bar{\bar{I}} \equiv \bar{G}_i\ \bar{G}^i \qquad \text{(B. 6)}$$

and it may be easily demonstrated that $\bar{\bar{I}}$ has the alternate forms,

$$\bar{\bar{I}} = \bar{G}^i\ \bar{G}_i$$

$$= G^{ij}\ \bar{G}_i\ \bar{G}_j$$

$$= G_{ij}\ \bar{G}^i\ \bar{G}^j$$

Transformation Law Between Base Vectors of The Same Space

Consider an affine space spanned by parametric coördinates X^s, each point of which is associated with a Euclidean vector space whose base vectors are defined by,

$$\bar{G}_s \equiv \frac{\partial \bar{R}}{\partial X^s} \tag{B. 7}$$

where $\bar{R}$ is the radius vector to the point from an arbitrary fixed origin. Consider the affine space to be spanned by another system of parametric coördinates X'^K. New base vector systems for the Euclidean point spaces are defined by,

$$\bar{G}'_K \equiv \frac{\partial \bar{R}}{\partial X'^K} \tag{B. 8}$$

A vector of infinitesimal magnitude at a point in the affine space may be represented by,

$$dX^s \, \bar{G}_s \tag{B. 9}$$

Only vectors of infinitesimal magnitude may be represented as a product of the base vector, $\bar{G}_s$ and some multiple of X^s since in general the X^s coördinate is an arbitrary space curve and may only be regarded as straight over an infinitesimal length. The same vector may be represented in the $\bar{G}'_K$ base system as,

$$dX'^{K}\ \bar{G}'_{K} \tag{B.10}$$

Equating expressions (B. 9) and (B.10) for the same vector we have,

$$dX^{S}\ \bar{G}_{S} = dX'^{K}\ \bar{G}'_{K}$$

$$\therefore \quad \bar{G}_{S} = \frac{dX'^{K}}{dX^{S}}\ \bar{G}'_{K} \tag{B.11}$$

giving the transformation law between base vectors of the same space, $\bar{G}_{S}$ and $\bar{G}'_{K}$.

Transformation Law for Reciprocal Vectors

Since the identity tensor $\bar{\bar{I}}$ is defined as isotropic for all spaces we have,

$$\bar{\bar{I}} = \bar{G}^{T}\ \bar{G}_{T} = \bar{G}'^{A}\ \bar{G}'_{A}$$

Transforming the $\bar{G}_{T}$ according to equation (B.11) we have,

$$\bar{G}^{T}\ \frac{dX'^{B}}{dX^{T}}\ \bar{G}'_{B} = \bar{G}'^{A}\ \bar{G}'_{A}$$

Taking a scalar product with $\bar{G}'^{A}$ as postfactor with both sides

we have,

$$\bar{G}^{T} \frac{dX'^{B}}{dX^{T}} \delta^{A}_{B} = \bar{G}'^{A}$$

Therefore we have the transformation law between reciprocal base vectors as,

$$\bar{G}^{T} = \bar{G}'^{A} \frac{dX^{T}}{dX'^{A}}$$

Transformation of Tensors

In the tensor product,

$$\bar{\bar{T}} = \bar{A}_{i} \bar{B}_{i}$$

the vectors $\bar{A}_{i}$ and $\bar{B}_{i}$ may be similarly expressed in terms of their components with respect to the $\bar{G}'_{K}$ base system such that $\bar{\bar{T}}$ is referred to the base $\bar{G}'_{K}$.

$$\bar{\bar{T}} = T'^{KT} \bar{G}'_{K} \bar{G}'_{T} \qquad \text{(B.12.i)}$$

$$= T'_{uv} \bar{G}^{u} \bar{G}^{v} \qquad \text{(B.12.ii)}$$

$$= T'^{K}_{v} \bar{G}'_{K} \bar{G}'^{v} \qquad \text{(B.12.iii)}$$

$$= T'^{T}_{u} \bar{G}^{u} \bar{G}_{T} \qquad \text{(B.12.iv)}$$

Imposing invariance of the tensor $\bar{\bar{T}}$ with respect to distinct bases and equating expressions for $\bar{\bar{T}}$ from (B. 4) to expressions for $\bar{\bar{T}}$ from (B.12) we obtain the transformation laws between tensor components expressed in different base systems of the same space.

We have,

$$T'^{KT}\ \bar{G}'_K\ \bar{G}'_T = T^{LM}\ \bar{G}_L\ \bar{G}_M$$

or $$T'^{KT}\ \bar{G}'_K\ \bar{G}'_T = T^{LM}\ \frac{dX'^i}{dX^L}\ \bar{G}'_i\ \frac{dX'^j}{dX^M}\ \bar{G}'_j$$

from which we may write the relationship between the components as,

$$T'^{KT} = T^{LM}\ \frac{dX'^i}{dX^L}\ \delta^K_i\ \frac{dX'^j}{dX^M}\ \delta^T_j$$

or $$T'^{KT} = T^{LM}\ \frac{dX'^K}{dX^L}\ \frac{dX'^T}{dX^M} \qquad \text{(B.13.i)}$$

Similarly

$$T'_{uv} = T_{NP}\ \frac{dX^N}{dX'^u}\ \frac{dX^P}{dX'^v} \qquad \text{(B.13.ii)}$$

$$T'^K_{\ v} = T^L_{\ P}\ \frac{dX'^K}{dX^L}\ \frac{dX^P}{dX'^v} \qquad \text{(B.13.iii)}$$

and $$T'^{T}_{u} = T_{N}^{M} \frac{dX^{N}}{dX'^{u}} \frac{dX'^{T}}{dX^{M}} \qquad \text{(B.13.iv)}$$

These transformations of the tensor components may all be expressed in the transformation of the components of a tensor of arbitrary order,

$$T'^{K\ldots T}_{u\ldots v} = T_{N\ldots P}^{L\ldots M} \frac{dX'^{K}}{dX^{L}} \cdots \frac{dX'^{T}}{dX^{M}} \frac{dX^{N}}{dX'^{u}} \cdots \frac{dX^{P}}{dX'^{v}} \qquad \text{(B.14)}$$

The Covariant Derivative

In the derivative,

$$\frac{\partial \bar{\bar{T}}}{\partial X^{K}} = \frac{\partial}{\partial X^{K}} [T^{LM} \bar{G}_{L} \bar{G}_{M}]$$

the base vectors are functions of X^{K} and must be differentiated.

Thus we have,

$$\frac{\partial}{\partial X^{K}} [T^{LM} \bar{G}_{L} \bar{G}_{M}] = \frac{\partial T^{LM}}{\partial X^{K}} \bar{G}_{L} \bar{G}_{M} + T^{LM} \frac{\partial \bar{G}_{L}}{\partial X^{K}} \bar{G}_{M} + T^{LM} \bar{G}_{L} \frac{\partial \bar{G}_{M}}{\partial X^{K}}$$

which we may write as,

$$= \frac{\partial T^{LM}}{\partial X^K} \bar{G}_L \bar{G}_M + T^{PM} \frac{\partial \bar{G}_P}{\partial X^K} \bar{G}_M + T^{LQ} \bar{G}_L \frac{\partial \bar{G}_Q}{\partial X^K}$$

since L and M are only summation indices.

Consider the vector $\frac{\partial \bar{G}_P}{\partial X^K}$.

$$\frac{\partial \bar{G}_P}{\partial X^K} = \frac{\partial \bar{G}_P}{\partial X^K} \cdot \bar{\bar{I}}$$

$$= \frac{\partial \bar{G}_P}{\partial X^K} \cdot \bar{G}^L \bar{G}_L$$

$$= \left[\frac{\partial G_P}{\partial X^K} \cdot \bar{G}^L \right] \bar{G}_L$$

The quantity $\frac{\partial \bar{G}_P}{\partial X^K} \cdot \bar{G}^L$ is called the Christoffel symbol of the second kind which defines the nature of the connection between distinct neighbouring point in vector spaces and is denoted,

$$\frac{\partial \bar{G}_P}{\partial X^K} \cdot \bar{G}^L \equiv \left\{ \begin{matrix} L \\ P \; K \end{matrix} \right\} \qquad \text{(B.15)}$$

or $$\frac{\partial \bar{G}_P}{\partial X^K} \cdot \bar{G}^L \, \bar{G}_L \equiv \left\{ \begin{matrix} L \\ P \; K \end{matrix} \right\} \bar{G}_L$$

$$\therefore \quad \frac{\partial \bar{G}_P}{\partial X^K} \equiv \left\{ \begin{matrix} L \\ P \; K \end{matrix} \right\} \bar{G}_L \tag{B.16}$$

Now consider

$$\bar{G}_P \cdot \bar{G}^L = \delta_P^L$$

differentiating w.r.t. X^K,

$$\frac{\partial \bar{G}_P}{\partial X^K} \cdot \bar{G}^L + \bar{G}_P \cdot \frac{\partial \bar{G}^L}{\partial X^K} = 0$$

$$\therefore \quad \frac{\partial \bar{G}^L}{\partial X^K} \cdot \bar{G}_P = - \frac{\partial \bar{G}_P}{\partial X^K} \cdot \bar{G}^L = - \left\{ \begin{matrix} L \\ P \; K \end{matrix} \right\}$$

or $$= - \left\{ \begin{matrix} L \\ Q \; K \end{matrix} \right\} \delta_P^Q = - \left\{ \begin{matrix} L \\ Q \; K \end{matrix} \right\} \bar{G}^Q \cdot \bar{G}_P$$

$$\therefore \quad \frac{\partial \bar{G}^L}{\partial X^K} = - \left\{ \begin{matrix} L \\ Q \; K \end{matrix} \right\} \bar{G}^Q \tag{B.17}$$

giving the corresponding differential coefficient of the reciprocal vector $\bar{G}^L$.

Thus by (B.16) we have,

$$\frac{\partial \bar{\bar{T}}}{\partial X^K} = \frac{\partial T^{LM}}{\partial X^K} \bar{G}_L \bar{G}_M + T^{PM} \left\{ \begin{matrix} L \\ P\ K \end{matrix} \right\} \bar{G}_L \bar{G}_M + T^{LQ} \bar{G}_L \left\{ \begin{matrix} M \\ Q\ K \end{matrix} \right\} \bar{G}_M$$

$$\therefore \frac{\partial \bar{\bar{T}}}{\partial X^K} = \left[\frac{\partial T^{LM}}{\partial X^K} + T^{PM} \left\{ \begin{matrix} L \\ P\ K \end{matrix} \right\} + T^{LQ} \left\{ \begin{matrix} M \\ Q\ K \end{matrix} \right\} \right] \bar{G}_L \bar{G}_M \qquad \text{(B.18)}$$

The quantity, $\frac{\partial T^{LM}}{\partial X^K} + T^{PM} \left\{ \begin{matrix} L \\ P\ K \end{matrix} \right\} + T^{LQ} \left\{ \begin{matrix} M \\ Q\ K \end{matrix} \right\}$

is called the *covariant derivative* of the second order tensor, $\bar{\bar{T}}$, and is denoted by $T^{LM}{}_{,K}$ such that,

$$\frac{\partial \bar{\bar{T}}}{\partial X^K} = \frac{\partial}{\partial X^K} (T^{LM} \bar{G}_L \bar{G}_M) = T^{LM}{}_{,K} \bar{G}_L \bar{G}_M$$

The covariant derivatives of tensors of higher order may be found by a similar process.

Double Tensor Fields[3]

We consider, as before, an affine space spanned by parametric coördinates X^K, with base vectors, $\bar{G}_K$, of a Euclidean point space defined at each point. We now consider the affine space to be mapped into a new affine space spanned by parametric coördinates x^k, each point of which is associated with a Euclidean vector space with base vectors $\bar{g}_k$. In general the dimensions of the two spaces may be arbitrary and independant but for our purpose we shall consider both of them to be three dimensional Euclidean. $\bar{R}$ is the position vector of the point in the X^K space and $\bar{r}$ is the position vector of the point in the x^k space.

As before we may establish the identities;

$$\bar{g}_\delta \equiv \frac{\partial \bar{r}}{\partial x^\delta} \tag{B.19}$$

$$\bar{g}'_k \equiv \frac{\partial \bar{r}}{\partial x'^k} \tag{B.20}$$

and the equations,

$$\bar{g}_\delta = \frac{dx'^k}{dx^\delta}\ \bar{g}'_k \tag{B.21}$$

$$\bar{g}^{t} = \frac{dx^{t}}{dx^{\prime a}}\ \bar{g}^{\prime a} \tag{B.22}$$

$$T^{\prime k\ldots t}_{u\ldots v} = T^{\ell\ldots m}_{n\ldots p}\ \frac{dx^{\prime k}}{dx^{\ell}}\ldots\frac{dx^{\prime t}}{dx^{m}}\ \frac{dx^{n}}{dx^{\prime u}}\ldots\frac{dx^{p}}{dx^{\prime v}} \tag{B.23}$$

and

$$T^{\ell M}{}_{,p} = \frac{\partial T^{\ell m}}{\partial x^{k}} + T^{pm}\left\{\begin{matrix}\ell\\ p\ k\end{matrix}\right\} T^{\ell q}\left\{\begin{matrix}m\\ q\ k\end{matrix}\right\} \tag{B.24}$$

A double tensor field is defined as a field at each point of which is defined a tensor product between a number of vectors which may belong to either of the two vector spaces associated with the point in its unmapped and mapped configurations respectively. Thus a double tensor field has defined at each point a quantity of the form,

$$T^{L\ldots M\ \ell\ldots m}_{N\ldots P\ n\ldots p}\ \bar{G}_{L}\ldots\bar{G}_{M}\ \bar{G}^{N}\ldots\bar{G}^{P}\ \bar{g}_{\ell}\ldots\bar{g}_{m}\ \bar{g}^{n}\ldots\bar{g}^{p} \tag{B.25}$$

in which the order of the vectors in the tensor product may be in general arbitrary and fixed only for each particular double tensor field.

The simplest example of a double tensor field is the deformation gradient, which is a field at each point of which is defined a tensor of the form,

$$\bar{\bar{F}} \equiv \frac{\partial \bar{r}}{\partial \bar{R}} = \frac{\partial x^k}{\partial X^K} \bar{G}^K \bar{g}_k$$

where the two vector spaces in this case are;

1) That defined in the reference configuration of a continuum, and
2) That defined in the current configuration of the continuum which is the space into which the reference configuration is transformed during the deformation process.

Covariant Differentiation of Double Tensor Fields

Covariant differentiation of the double tensor field with respect to x^k and X^K may be defined in the usual way if we adjoin the convention that $\bar{R}$ is held constant when we differentiate with respect to x^k and vice versa. These partial covariant derivatives which we denote by the usual symbols "$_{,k}$" and "$_{,K}$" are double tensor fields of the type indicated by the number and position of their indices and the formal rules of ordinary covariant differentiation remain valid.

Consider the second order double tensor field,

$$\bar{\bar{T}} = T^{Lm}\, \bar{G}_L\, \bar{g}_m$$

The total covariant derivative is defined by,

$$\frac{\partial \bar{\bar{T}}}{\partial X^K} = \left[\frac{\partial \bar{\bar{T}}}{\partial X^K}\right]_{\bar{r}\ \text{constant}} + \left[\frac{\partial \bar{\bar{T}}}{\partial X^K}\right]_{\bar{R}\ \text{constant}} \tag{B.26}$$

$$= T^{Lm}{}_{,K}\, \bar{G}_L\, \bar{g}_m + \left[\frac{\partial \bar{\bar{T}}}{\partial x^k}\, \frac{\partial x^k}{\partial X^K}\right]_{\bar{R}\ \text{constant}}$$

where $T^{Lm}{}_{,K} \equiv \dfrac{\partial T^{Lm}}{\partial X^K} + T^{Pm} \left\{ \begin{matrix} L \\ P\ K \end{matrix} \right\}$

$$\therefore\quad \frac{\partial \bar{\bar{T}}}{\partial X^K} = T^{Lm}{}_{,K}\, \bar{G}_L \bar{g}_m + T^{Lm}{}_{,k}\, \frac{\partial x^k}{\partial X^K}\, \bar{G}_L\, \bar{g}_m$$

where $T^{Lm}{}_{,k} \equiv \dfrac{\partial T^{Lm}}{\partial x^k} + T^{Lq} \left\{ \begin{matrix} m \\ q\ k \end{matrix} \right\}$

$$\therefore\quad \frac{\partial \bar{\bar{T}}}{\partial X^K} = \left[T^{Lm}{}_{,K} + T^{Lm}{}_{,k}\, \frac{\partial x^k}{\partial X^K} \right] \bar{G}_L\, \bar{g}_m \tag{B.27}$$

The quantity $T^{Lm}{}_{,K} + T^{Lm}{}_{,k}\,\frac{\partial x^k}{\partial X^K}$ is called the *total covariant derivative* of the double tensor field $T^{Lm}\,\bar{G}_L\,\bar{g}_m$ and is denoted $T^{Lm}{}_{;K}$ such that

$$\frac{\partial \bar{\bar{T}}}{\partial X^K} = T^{Lm}{}_{;K}\,\bar{G}_L\,\bar{g}_m \tag{B.28}$$

We may easily generalise this result for a tensor of arbitrary order so that the total covariant derivative of a double tensor field whose components are $T^{\cdots}_{\cdots}$ is,

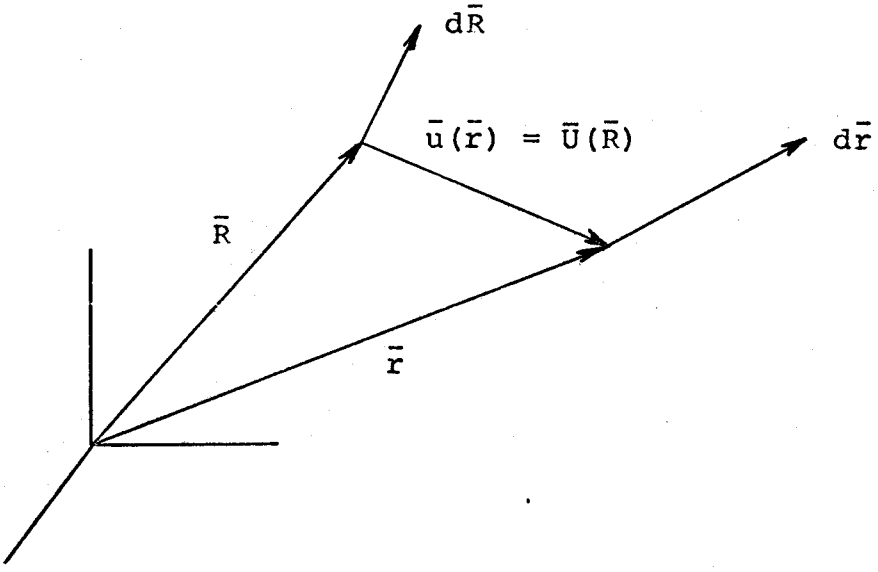
$$T^{\cdots}_{\cdots;K} \equiv T^{\cdots}_{\cdots,K} + T^{\cdots}_{\cdots,k}\,\frac{\partial x^k}{\partial X^K} \tag{B.29}$$

APPENDIX C

THE STRAIN MEASURE [7]

The quadratic strain measure:

$$d\delta^2 - dS^2 = d\bar{r}\cdot d\bar{r} - d\bar{R}\cdot d\bar{R} \qquad \text{(C. 1)}$$



<u>The Euler-Lagrange Strain Tensor</u>

$$d\delta^2 - dS^2 = \left(d\bar{R}\cdot\frac{\partial\bar{r}}{\partial\bar{R}}\right)\cdot\left(d\bar{R}\cdot\frac{\partial\bar{r}}{\partial\bar{R}}\right) - \left(d\bar{R}\cdot\frac{\partial\bar{R}}{\partial\bar{R}}\right)\cdot\left(d\bar{R}\cdot\frac{\partial\bar{R}}{\partial\bar{R}}\right)$$

$$= \left(d\bar{R}\cdot\frac{\partial\bar{r}}{\partial\bar{R}}\right)\cdot\left(\frac{\bar{r}\partial}{\partial\bar{r}}\cdot d\bar{R}\right) - \left(d\bar{R}\cdot\frac{\partial\bar{R}}{\partial\bar{R}}\right)\cdot\left(\frac{\bar{R}\partial}{\partial\bar{R}}\cdot d\bar{R}\right)$$

$$= d\bar{R} \cdot \left(\frac{\partial \bar{r}}{\partial \bar{R}} \cdot \frac{\bar{r}\partial}{\partial \bar{R}} \right) \cdot d\bar{R} - d\bar{R} \cdot \left(\frac{\partial \bar{R}}{\partial \bar{R}} \cdot \frac{\bar{R}\partial}{\partial \bar{R}} \right) \cdot d\bar{R}$$

$$= d\bar{R} \cdot \left[\left(\frac{\partial \bar{r}}{\partial \bar{R}} \cdot \frac{\bar{r}\partial}{\partial \bar{R}} \right) - \left(\frac{\partial \bar{R}}{\partial \bar{R}} \cdot \frac{\bar{R}\partial}{\partial \bar{R}} \right) \right] \cdot d\bar{R}$$

$$\therefore \quad d\delta^2 - dS^2 = d\bar{R}\, d\bar{R} : \left[\left(\frac{\partial \bar{r}}{\partial \bar{R}} \cdot \frac{\bar{r}\partial}{\partial \bar{R}} \right) - \left(\frac{\partial \bar{R}}{\partial \bar{R}} \cdot \frac{\bar{R}\partial}{\partial \bar{R}} \right) \right]$$

$$= d\bar{R}\, d\bar{R} : \left[\bar{\bar{F}} \cdot \bar{\bar{F}}_C - \bar{\bar{I}} \cdot \bar{\bar{I}}_C \right] \qquad \text{(C. 2)}$$

where, $\bar{\bar{F}} \equiv \frac{\partial \bar{r}}{\partial \bar{R}}$ is the deformation gradient (C. 3)

$$\therefore \quad d\delta^2 - dS^2 = d\bar{R}\, d\bar{R} : [\bar{\bar{C}} - \bar{\bar{I}}] \qquad \text{(C. 4)}$$

where, $\bar{\bar{C}} \equiv \bar{\bar{F}} \cdot \bar{\bar{F}}_C$, the Green deformation tensor (C. 5)

$$\therefore \quad d\delta^2 - dS^2 = d\bar{R}\, d\bar{R} : 2\bar{\bar{E}} \qquad \text{(C. 6)}$$

where,

$$\bar{\bar{E}} \equiv \frac{1}{2}[\bar{\bar{C}} - \bar{\bar{I}}], \text{ the Euler-Lagrange strain tensor} \qquad \text{(C. 7)}$$

Displacement Description of the Euler-Lagrange Strain Tensor

Consider the linear transformation,

$$\bar{r} = \bar{R} + \bar{U}(\bar{R}) \qquad \text{(C. 8)}$$

Then,

$$\bar{\bar{C}} \equiv \bar{\bar{F}} \cdot \bar{\bar{F}}_C = \frac{\partial \bar{r}}{\partial \bar{R}} \cdot \frac{\bar{r}\partial}{\partial \bar{R}}$$

$$= \left(\frac{\partial \bar{R}}{\partial \bar{R}} + \frac{\partial \bar{U}}{\partial \bar{R}}\right) \cdot \left(\frac{\partial \bar{R}}{\partial \bar{R}} + \frac{\partial \bar{U}}{\partial \bar{R}}\right)_C$$

$$\therefore \bar{\bar{C}} = \left(\frac{\partial \bar{R}}{\partial \bar{R}} + \frac{\partial \bar{U}}{\partial \bar{R}}\right) \cdot \left(\frac{\bar{R}\partial}{\partial \bar{R}} + \frac{\bar{U}\partial}{\partial \bar{R}}\right)$$

$$= \left(\bar{\bar{I}} + \frac{\partial \bar{U}}{\partial \bar{R}}\right) \cdot \left(\bar{\bar{I}} + \frac{\bar{U}\partial}{\partial \bar{R}}\right)$$

$$= \bar{\bar{I}} \cdot \bar{\bar{I}} + \frac{\partial \bar{U}}{\partial \bar{R}} \cdot \bar{\bar{I}} + \bar{\bar{I}} \cdot \frac{\bar{U}\partial}{\partial \bar{R}} + \frac{\partial \bar{U}}{\partial \bar{R}} \cdot \frac{\bar{U}\partial}{\partial \bar{R}}$$

$$\therefore \bar{\bar{C}} = \bar{\bar{I}} + \frac{\partial \bar{U}}{\partial \bar{R}} + \frac{\bar{U}\partial}{\partial \bar{R}} + \frac{\partial \bar{U}}{\partial \bar{R}} \cdot \frac{\bar{U}\partial}{\partial \bar{R}} \qquad \text{(C. 9)}$$

Therefore the displacement form for the Euler-Lagrange strain tensor.

$$\bar{\bar{E}} \equiv \frac{1}{2}[\bar{\bar{C}} - \bar{\bar{I}}] = \frac{1}{2}\left[\frac{\partial \bar{U}}{\partial \bar{R}} + \frac{\bar{U}\partial}{\partial \bar{R}} + \frac{\partial \bar{U}}{\partial \bar{R}} \cdot \frac{\bar{U}\partial}{\partial \bar{R}}\right] \tag{C.10}$$

The Euler Strain Tensor

$$d\delta^2 - dS^2 = d\bar{r}\cdot d\bar{r} - d\bar{R}\cdot d\bar{R}$$

$$= \left(d\bar{r}\cdot\frac{\partial \bar{r}}{\partial \bar{r}}\right)\cdot\left(d\bar{r}\cdot\frac{\partial \bar{r}}{\partial \bar{R}}\right) - \left(d\bar{r}\cdot\frac{\partial \bar{R}}{\partial \bar{r}}\right)\cdot\left(d\bar{r}\cdot\frac{\partial \bar{R}}{\partial \bar{r}}\right)$$

$$= d\bar{r}\, d\bar{r} : \left[\frac{\partial \bar{r}}{\partial \bar{r}} \cdot \frac{\bar{r}\partial}{\partial \bar{r}} - \frac{\partial \bar{R}}{\partial \bar{r}} \cdot \frac{\bar{R}\partial}{\partial \bar{r}}\right]$$

$$= d\bar{r}\, d\bar{r} : [\bar{\bar{I}}\cdot\bar{\bar{I}}_C - \bar{\bar{F}}^r\cdot\bar{\bar{F}}^r{}_C]$$

where the reciprocal, $\bar{\bar{F}}^r \equiv \frac{\partial \bar{R}}{\partial \bar{r}} \equiv \bar{\bar{F}}^{-1}$

$$= d\bar{r}\, d\bar{r} : [\bar{\bar{I}} - \bar{\bar{B}}^r] \tag{C.11}$$

where $\bar{\bar{B}}^r \equiv \bar{\bar{F}}^r\cdot\bar{\bar{F}}^r{}_C$, the Cauchy deformation tensor (C.12)

$$\therefore \quad dS^2 - dS^2 = d\bar{r}\, d\bar{r} : [2\bar{\bar{\varepsilon}}] \tag{C.13}$$

where $\bar{\bar{\varepsilon}} \equiv \frac{1}{2}[\bar{\bar{I}} - \bar{\bar{B}}^r]$, the Euler strain tensor (C.14)

Displacement Form of Euler Strain Tensor

We have the linear transformation,

$$\bar{r} = \bar{R} + \bar{u}(\bar{r}) \tag{C.15}$$

$$\therefore \quad \bar{R} = \bar{r} - \bar{u}(\bar{r})$$

Then

$$\bar{\bar{B}}^r \equiv \bar{\bar{F}}^r \cdot \bar{\bar{F}}^r_C = \frac{\partial \bar{R}}{\partial \bar{r}} \cdot \frac{\bar{R}\partial}{\partial \bar{r}}$$

$$= \left(\bar{\bar{I}} - \frac{\partial \bar{u}}{\partial \bar{r}}\right) \cdot \left(\bar{\bar{I}} - \frac{\partial \bar{u}}{\partial \bar{r}}\right)_C$$

$$= \left(\bar{\bar{I}} - \frac{\partial \bar{u}}{\partial \bar{r}}\right) \cdot \left(\bar{\bar{I}} - \frac{\bar{u}\partial}{\partial \bar{r}}\right)$$

$$\therefore \quad \bar{\bar{B}}^r = \bar{\bar{I}} - \frac{\partial \bar{u}}{\partial \bar{r}} - \frac{\bar{u}\partial}{\partial \bar{r}} + \frac{\partial \bar{u}}{\partial \bar{r}} \cdot \frac{\bar{u}\partial}{\partial \bar{r}} \tag{C.16}$$

and the displacement form for the Euler strain tensor,

$$\bar{\bar{\varepsilon}} \equiv \frac{1}{2}[\bar{\bar{I}} - \bar{\bar{B}}^{r}] = \frac{1}{2}\left[\frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\bar{u}\partial}{\partial \bar{r}} - \frac{\partial \bar{u}}{\partial \bar{r}} \cdot \frac{\bar{u}\partial}{\partial \bar{r}}\right] \tag{C.17}$$

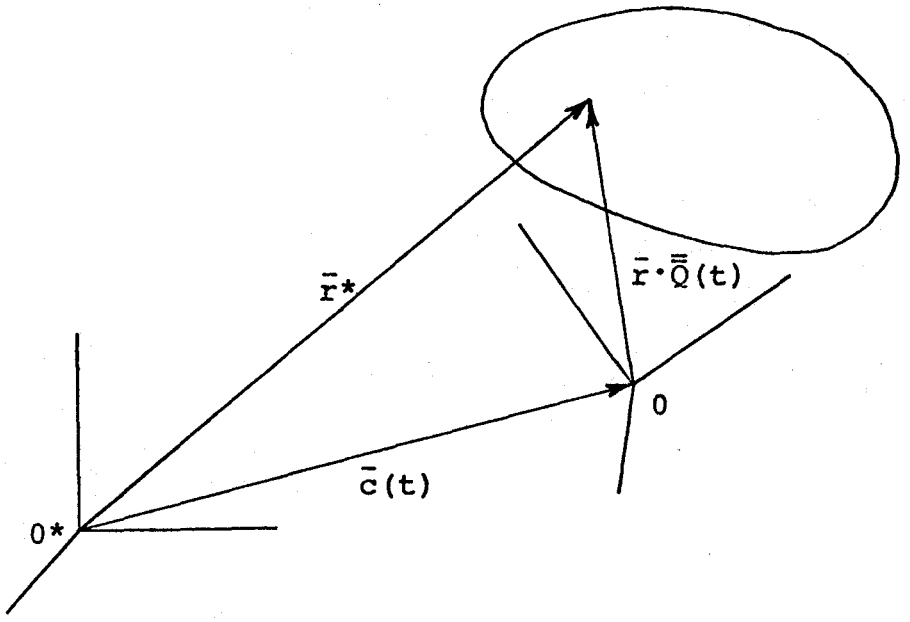
APPENDIX D

CHANGE OF FRAME AND FRAME - INDIFFERENCE

Change of Frame of Reference

A frame of reference representing an observer is a rigid frame endowed with a timing device. The most general functional relation between the place as referred to the preferred frame, $\{0^*, \bar{r}^*\}$, and to another frame of reference, $\{0, \bar{r}\}$, is expressed by the transformation,

$$\bar{r}^* = \bar{c}(t) + \bar{r}\cdot\bar{\bar{Q}}(t) \qquad \text{(D. 1)}$$



where $\bar{c}$ is the position vector of O in the frame of reference $\{O^*, \bar{r}\}$ and $\bar{\bar{Q}}$ is an orthogonal tensor which gives the orientation of the frame of reference $\{O, \bar{r}\}$ relative to the geometrically similar frame of reference $\{O^*, \bar{r}^*\}$ such that,

$$\bar{\bar{Q}}(t)\cdot\bar{\bar{Q}}_c(t) = \bar{\bar{I}} = \bar{\bar{Q}}_c(t)\cdot\bar{\bar{Q}}(t)$$

which may be proper or improper,

i.e. $$|\bar{\bar{Q}}| = \pm 1$$

Such a change of reference frame will be seen to preserve distances between particles. It is important to note here that we are not talking about changes of coördinate systems; coördinate systems can be changed within a Euclidean point space, but we are not doing this. The frames $\{O^* \bar{r}^*\}$ and $\{O, \bar{r}\}$ represent the same frame in two configurations of a rigid body motion.

Consider,

$$\bar{r}^* = \bar{c}(t) + \bar{r}\cdot\bar{\bar{Q}}(t) \qquad \text{(D. 1)}$$

for constant time.

differentiating w.r.t. $\bar{r}$,

$$\frac{\partial \bar{r}^*}{\partial \bar{r}} = \frac{\partial \bar{r}}{\partial \bar{r}} \cdot \bar{\bar{Q}}(t)$$

$$= \bar{\bar{I}} \cdot \bar{\bar{Q}}(t)$$

$$\therefore \quad \bar{\bar{Q}}(t) = \frac{\partial \bar{r}^*}{\partial \bar{r}} \qquad \text{(D. 2)}$$

differentiating (D. 1) w.r.t. $\bar{r}^*$,

$$\frac{\partial \bar{r}^*}{\partial \bar{r}^*} = \frac{\partial \bar{r}}{\partial \bar{r}^*} \cdot \bar{\bar{Q}}(t)$$

or

$$\bar{\bar{I}} = \frac{\partial \bar{r}}{\partial \bar{r}^*} \cdot \bar{\bar{Q}}(t)$$

$$\therefore \quad \bar{\bar{I}} \cdot \bar{\bar{Q}}_c(t) = \frac{\partial \bar{r}}{\partial \bar{r}^*} \cdot \bar{\bar{Q}}(t) \cdot \bar{\bar{Q}}_c(t)$$

$$= \frac{\partial \bar{r}}{\partial \bar{r}^*} \cdot \bar{\bar{I}}$$

$$\therefore \quad \bar{\bar{Q}}_c(t) = \frac{\partial \bar{r}}{\partial \bar{r}^*} \qquad \text{(D. 3)}$$

Conservation of Distance for the Transformation

We have $$d\bar{r}^* = d\bar{r} \cdot \frac{\partial \bar{r}^*}{\partial \bar{r}}$$

$\therefore$ by (D. 2) $$d\bar{r}^* = d\bar{r} \cdot \bar{\bar{Q}}(t) \qquad \text{(D. 4)}$$

$$\therefore \quad d\bar{r}^* \cdot d\bar{r}^* = (d\bar{r} \cdot \bar{\bar{Q}}) \cdot (d\bar{r} \cdot \bar{\bar{Q}})$$

$$= d\bar{r} \cdot \bar{\bar{Q}} \cdot \bar{\bar{Q}}_c \cdot d\bar{r}$$

$$= d\bar{r} \cdot \bar{\bar{I}} \cdot d\bar{r}$$

$$\therefore \quad d\bar{r}^* \cdot d\bar{r}^* = d\bar{r} \cdot d\bar{r} \qquad \text{(D. 5)}$$

Demonstrating that the transformation conserves distances.

Transformation of Vectors

The frames $\{O^*, \bar{r}^*\}$ and $\{O, \bar{r}\}$ have a relative orientation determined by the tensor $\bar{\bar{Q}}$. Therefore any vector whose representation in the $\{O, \bar{r}\}$ frame of reference is $\bar{u}$ will appear to an observer, whose frame of reference is the $\{O^*, \bar{r}^*\}$ frame, to have a relative orientation which is actually the orientation of the $\{O^*, \bar{r}^*\}$ frame. If the same vector is designated in the $\{O^*, \bar{r}^*\}$ frame by $\bar{v}^*$ then we have the relationship,

$$\bar{v}^* = \bar{v} \cdot \bar{\bar{Q}} \qquad \text{(D. 6)}$$

Base vectors $\bar{g}_i^*$ and $\bar{g}_i$ defined in the $\{O^*, \bar{r}^*\}$ and $\{O, \bar{r}\}$ frames resepctively will also have a relative orientation determined by $\bar{\bar{Q}}$ and so,

$$\bar{g}_i^* = \bar{g}_i \cdot \bar{\bar{Q}} \qquad \text{(D. 7)}$$

Similarly the reciprocal base vectors $\bar{g}^{i*}$ and $\bar{g}^{i}$ will have a relative orientation determined by $\bar{\bar{Q}}$ and therefore,

$$\bar{g}^{i*} = \bar{g}^{i} \cdot \bar{\bar{Q}} \qquad \text{(D. 8)}$$

Transformation of Tensors

The relationship between the representation of a second order tensor in the two frames of reference may be found as follows. Consider the second order tensor,

$$\bar{\bar{T}}^* = \bar{A}_i^* \, \bar{B}_i^*$$

$$= (\bar{A}_i \cdot \bar{\bar{Q}})(\bar{B}_i \cdot \bar{\bar{Q}}) \qquad \text{by (D. 6)}$$

$$= \bar{\bar{Q}}_C \cdot \bar{A}_i \; \bar{B}_i \cdot \bar{\bar{Q}}$$

$$\therefore \quad \bar{\bar{T}}^* = \bar{\bar{Q}}_C \cdot \bar{\bar{T}} \cdot \bar{\bar{Q}} \qquad \text{(D. 9)}$$

which is the rule of correspondance between the representations of the same second order tensor in the frames of reference $\{O^*, \bar{r}^*\}$ and $\{O, \bar{r}\}$ respectively.

Consequently vectors $\bar{v}^*$ and $\bar{v}$ related by equation (D. 6) and tensors $\bar{\bar{T}}^*$ and $\bar{\bar{T}}$ related by equation (D. 9) are called *Frame-Invariant* or *Frame-Indifferent*, since $\bar{v}^*$ and $\bar{v}$ are the same vector and $\bar{\bar{T}}^*$ and $\bar{\bar{T}}$ are the same tensor represented in different frames of reference $\{O^*, \bar{r}^*\}$ and

$\{O, \bar{r}\}$ respectively.

Transformation of the Directed Derivative Operator

We have,

$$\frac{\partial}{\partial \bar{r}^*} = \frac{\partial \bar{r}}{\partial \bar{r}^*} \cdot \frac{\partial}{\partial \bar{r}} \qquad \text{by the chain rule of differentiation}$$

$$\therefore \quad \frac{\partial}{\partial \bar{r}^*} = \bar{\bar{Q}}_C \cdot \frac{\partial}{\partial \bar{r}} \qquad \text{by (D. 3)} \qquad \text{(D.10)}$$

Therefore directed derivative operators $\frac{\partial}{\partial \bar{r}^*}$ in $\{O^*, r^*\}$ and $\frac{\partial}{\partial \bar{r}}$ in $\{O, \bar{r}\}$ are frame-covariant if related by equation (D.10). Expanding equation (D.10) we have,

$$\bar{g}^{i*} \frac{\partial}{\partial x^i} = \bar{\bar{Q}}_C \cdot \bar{g}^j \frac{\partial}{\partial x^j}$$

$$= \bar{g}^{j*} \frac{\partial}{\partial x^j} \qquad \text{by (D. 8)}$$

$$\therefore \quad \frac{\partial}{\partial x^i} = \delta^j_i \frac{\partial}{\partial x^j}$$

or $$\frac{\partial}{\partial x^{i*}} = \frac{\partial}{\partial x^{i}} \qquad (D.11)$$

Transformation of the Deformation Gradient, $\bar{\bar{F}}$

We assume[9] that the two frames of reference occupy the same position in the reference configuration such that,

$$\bar{R}^* = \bar{R} \qquad \text{at} \quad t = t_0 \qquad (D.12)$$

Then

$$\bar{\bar{F}} \equiv \frac{\partial \bar{r}}{\partial \bar{R}} = \bar{G}^i \frac{\partial x^j}{\partial X^i} \bar{g}_j$$

and

$$\bar{\bar{F}}^* = \frac{\partial \bar{r}^*}{\partial \bar{R}} = \bar{G}^i \frac{\partial x^{j*}}{\partial X^i} \bar{g}^*_j$$

Now

$$\bar{\bar{F}}^* = \frac{\partial \bar{r}^*}{\partial \bar{R}}$$

$$= \frac{\partial \bar{r}}{\partial \bar{R}} \cdot \frac{\partial \bar{r}^*}{\partial \bar{r}}$$

$$\therefore \qquad \bar{\bar{F}}^* = \bar{\bar{F}} \cdot \bar{\bar{Q}} \qquad \text{by (D. 2)} \qquad (D.13)$$

giving the transformation of the double tensor field, $\bar{\bar{F}}$, under change of frame of reference.

Now we have by (D.13)

$$\frac{\partial \bar{r}^*}{\partial \bar{R}} = \frac{\partial \bar{r}}{\partial \bar{R}} \cdot \bar{\bar{Q}}$$

or

$$\bar{G}^i \frac{\partial x^{j*}}{\partial X^i} \bar{g}^*_j = \bar{G}^p \frac{\partial x^q}{\partial X^p} \bar{g}_q \cdot \bar{\bar{Q}}$$

$$= \bar{G}^p \frac{\partial x^q}{\partial X^p} \bar{g}^*_q \qquad \text{by (D.7)}$$

$\therefore$

$$\bar{G}^i \frac{\partial x^{j*}}{\partial X^i} = \bar{G}^p \frac{\partial x^q}{\partial X^p} \delta^q_j$$

or

$$\frac{\partial x^{j*}}{\partial X^i} = \delta^p_i \frac{\partial x^j}{\partial X^p}$$

$\therefore$

$$\frac{\partial x^{j*}}{\partial X^i} = \frac{\partial x^j}{\partial X^i}$$

and since the frames coincide in the reference configuration,

$$X^{i*} = X^i \qquad \text{at} \quad t = t_0$$

$$\therefore \qquad \frac{\partial x^{j*}}{\partial X^{i*}} = \frac{\partial x^{j}}{\partial X^{i}} \qquad \text{(D.14)}$$

Transformation of Christoffel Symbols

We have,

$$\left\{ \begin{matrix} k \\ \ell \ m \end{matrix} \right\}^{*} = \frac{\partial \bar{g}_{\ell}^{*}}{\partial x^{m*}} \cdot \bar{g}^{k*}$$

Now under the change of frame

$$\bar{g}_{\ell}^{*} = \bar{g}_{\ell} \cdot \bar{\bar{Q}} \qquad \text{by (D. 7)}$$

$$\bar{g}^{k*} = \bar{g}^{k} \cdot \bar{\bar{Q}} \qquad \text{by (D. 8)}$$

and $$\frac{\partial}{\partial x^{m*}} = \frac{\partial}{\partial x^{m}} \qquad \text{by (D.11)}$$

$$\therefore \qquad \left\{ \begin{matrix} k \\ \ell \ m \end{matrix} \right\}^{*} = \frac{\partial}{\partial x^{m}} (\bar{g}_{\ell} \cdot \bar{\bar{Q}}) \cdot (\bar{g}^{k} \cdot \bar{\bar{Q}})$$

$$= \frac{\partial \bar{g}_\ell}{\partial x^m} \cdot \bar{\bar{Q}} \cdot (\bar{\bar{Q}}_C \cdot \bar{g}^k)$$

$$= \frac{\partial \bar{g}_\ell}{\partial x^m} \cdot (\bar{\bar{Q}} \cdot \bar{\bar{Q}}_C) \cdot \bar{g}^k$$

$$= \frac{\partial \bar{g}_\ell}{\partial x^m} \cdot \bar{\bar{I}} \cdot \bar{g}^k$$

$$= \frac{\partial \bar{g}_\ell}{\partial x^m} \cdot \bar{g}^k$$

$$\therefore \quad \left\{ {k \atop \ell \; m} \right\}^* = \left\{ {k \atop \ell \; m} \right\} \tag{D.15}$$