SOME APPLICATIONS OF SELF-AFFINE SETS TO WAVELET THEORY

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A Thesis
Submitted to the School of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree
Doctor of Philosophy

McMaster University

DOCTOR OF PHILOSOPHY (2011) McMaster University
(Mathematics) Hamilton, Ontario

TITLE: Some Applications of Self-Affine Sets to Wavelet Theory

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NUMBER OF PAGES: viii, 180

Abstract

In this thesis, we study several applications of self-affine sets to wavelet theory. Five major topics are considered here: wavelet sets (scaling sets), multiwavelet sets (generalized scaling sets), self-affine tiles, integral self-affine multi-tiles, self-affine sets. We divide the thesis into six chapters to discuss these topics.

In Chapter 1, the dimension function of a self-affine generalized scaling set associated with an $n \times n$ integral expansive dilation A is studied. More specifically, we consider the dimension function of an A-dilation generalized scaling set K assuming that K is a self-affine tile satisfying $BK = (K + d_1) \bigcup (K + d_2)$, where $B = A^t$ and A is an $n \times n$ integral expansive matrix with $|\det A| = 2$. We show that it must be a constant in dimension $n = 1$ or 2 and it is bounded by $2|K|$ for any n. This result shows that all A-dilation self-affine scaling sets must be A-dilation MRA scaling sets in dimensions one and two.

There exist results on the connection between the theory of wavelets and the theory of integral self-affine tiles and in particular, on the construction of wavelet bases using integral self-affine tiles. However, there are many non-integral self-affine tiles which can also yield wavelet basis. In Chapter 2 and Chapter 3, we give a complete characterization of all two dimensional A-dilation scaling sets K such that K is a self-affine tile satisfying $BK = (K + d_1) \bigcup (K + d_2)$ for some $d_1, d_2 \in \mathbb{R}^2$, where A is an $n \times n$ integral expansive matrix with $|\det A| = 2$. In Chapter 2, we deal with a particular case where $0 \in \{d_1, d_2\}$, i.e. a self-affine tile K satisfies $BK = K \bigcup (K + d)$ for some $d \in \mathbb{R}^2$. Chapter 3 is devoted to the general case with $d_1, d_2 \in \mathbb{R}^2$. Moreover, we give a sufficient condition for a self-affine tile, possibly non-integral, to be an MRA scaling set in Chapter 3.

Gabardo and Yu first considered using integral self-affine tiles in the Fourier domain to

construct wavelet sets and they produced a class of compact wavelet sets with certain self-similarity properties. In Chapter 4, we generalize their results to the integral self-affine multi-tiles setting. We characterize some analytic properties of integral self-affine multi-tiles under certain conditions. We also consider the problem of constructing (multi)wavelet sets using integral self-affine multi-tiles.

Suppose that a measurable $\mathbb{Z}^n$-tiling set $K \subset \mathbb{R}^n$ is an integral self-affine multi-tile associated with an $n \times n$ integral expansive matrix B. To our knowledge, no one considered how to represent an integral self-affine $\mathbb{Z}^n$-tiling set as the disjoint union of prototiles. In Chapter 5, we provide an algorithm to decompose K into disjoint pieces K_j which satisfy $K = \bigcup K_j$ such that the collection of the sets K_j is an integral self-affine collection associated with matrix B and the number of pieces K_j is minimal. Using this algorithm, we can determine whether a given measurable $\mathbb{Z}^n$-tiling set $K \subset \mathbb{R}^n$ is an integral self-affine multi-tile associated with any given $n \times n$ integral expansive matrix B. Furthermore, the minimal decomposition we provide is unique.

Let B be an $n \times n$ real expanding matrix and $\mathcal{D}$ be a finite subset of $\mathbb{R}^n$. The self-affine set $K = K(B, \mathcal{D})$ is the unique compact set satisfying the set equation $BK = \bigcup_{d \in \mathcal{D}} (K + d)$. In Chapter 6, we not only consider the problem how to compute the Lebesgue measure of self-affine sets $K(B, \mathcal{D})$, but also consider the Hausdorff measure for those with zero Lebesgue measure under the assumption that $K(B, \mathcal{D})$ is a self-similar set. In the case where $\mathrm{card}(\mathcal{D}) = |\det B|$, we relate the Lebesgue measure of $K(B, \mathcal{D})$ to the upper Beurling density of the associated measure $\mu = \lim_{s \to \infty} \sum_{\ell_0, \ldots, \ell_{s-1} \in \mathcal{D}} \delta_{\ell_0 + B\ell_1 + \cdots + B^{s-1}\ell_{s-1}}$. If, on the other hand, $\mathrm{card}(\mathcal{D}) < |\det B|$ and B is a similarity matrix, we relate the Hausdorff measure $\mathcal{H}^s(K)$, where s is the similarity dimension of K, to a corresponding notion of upper density for the measure μ.

Acknowledgements

First of all, I would like to take this opportunity to express my sincere gratitude to my supervisor, Professor Jean-Pierre Gabardo, for his constant and patient guidance, support and encouragement. The accomplishments of this Ph.D. dissertation should be attributed to his insightful suggestions, many excellent ideas throughout the thesis. I would especially like to thank him for proofreading my thesis and for much wise advice.

I would like to thank Professor Stanley Alama, Professor Nicholas Kevlahan, Professor Walter Craig and the external examiner Yang Wang for reading my thesis, and for their questions and suggestions made.

I would also like to express my appreciation to Professor Stanley Alama, Lia Bronsard, Eric Sawyer, Romyar Sharifi, Patrick Speissegger, Walter Craig who had taught me during my course of study at McMaster University. I would like to thank Professor Hans Boden, Nicholas Kevlahan, Tina Thorogood, Julie Fogarty, Kenneth Moyle and more for their various forms of help and support. Many thanks go to Feng Su, Jizhan Hong, Mengfai Lim for their helpful discussion throughout this thesis.

Finally, I wish to express my thanks to my family. It was their love, support and encouragement that drove me to work hard persistently for so many years.

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Introduction

In this thesis, our main goal is to study self-affine (generalized) scaling sets associated with an $n \times n$ integral expansive matrix, integral self-affine multi-tiles and self-affine sets. Before starting the discussion of these topics, we will provide some basic concepts firstly that will be used frequently later. For any function $f \in L^1(\mathbb{R}^n) \bigcap L^2(\mathbb{R}^n)$, its Fourier transform is defined by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\ f(x)\ dx, \tag{0.0.1}$$

where $x \cdot \xi$ is the standard inner product of the vectors $x, \xi \in \mathbb{R}^n$. The inverse Fourier transform will be denoted by $\mathcal{F}^{-1}$. Equality and inclusion between two measurable sets $E, F \subseteq \mathbb{R}^n$ will always be understood modulo sets of zero Lebesgue measure. So, for example, $E = F$ means that their symmetric difference i.e. $(F \setminus E) \bigcup (E \setminus F)$ has zero Lebesgue measure. Essentially disjoint between two measurable sets $E, F \subseteq \mathbb{R}^n$ means that the Lebesgue measure of $E \bigcap F$ is zero. The Lebesgue measure of a measurable set $K \subset \mathbb{R}^n$ is denoted by $|K|$.

Wavelets and Wavelet Sets

As a relatively new subject, wavelet analysis was introduced in the early 1980s. It has attracted much attention of researchers from many different fields including mathematics, physics, engineering and so on because it has a wide range of applications in signal analysis, image analysis, communications systems, differential equations, integral equations and even genomics analysis. Like Fourier analysis, wavelet analysis deals with expansion of functions in terms of a set of basis functions. Unlike Fourier analysis, however, wavelet

analysis expands functions not in terms of trigonometric polynomials but in terms of wavelets, which are generated in form of translations and dilations of a fixed function called the mother wavelet. The wavelets obtained in this way have special scaling properties. They are localized in time and frequency, permitting a closer connection between the function being represented and their coefficients. Greater numerical stability in reconstruction and manipulation is ensured. One of hot topics in wavelet analysis is the construction of various kinds of wavelet bases, which can be reduced to the construction of the (mother) wavelet.

Suppose A is an $n \times n$ real expansive matrix, i.e. a matrix with real entries whose eigenvalues are all of modulus greater than one. An A-dilation wavelet is a measurable function $\psi \in L^2(\mathbb{R}^n)$ such that the set

$$\{|\det A|^{\frac{j}{2}} \psi(A^j \cdot -k) : \ j \in \mathbb{Z}, \ k \in \mathbb{Z}^n\} \tag{0.0.2}$$

forms an orthonormal basis for $L^2(\mathbb{R}^n)$. Around 1987, Mallat [43] introduced the idea of Multiresolution Analysis (MRA) with associated dilation $A = 2$ in dimension one that provides a tool for the construction of wavelets. The construction of an orthonormal wavelet basis in $L^2(\mathbb{R})$ was reduced to the construction of a scaling function in a corresponding MRA. Later, Daubechies [16] used this result to construct compactly supported wavelet of arbitrary smoothness. Y. Meyer [44] gave a similar generalization of MRA in higher dimension by considering matrix dilations. If an A-dilation wavelet $\psi \in L^2(\mathbb{R}^n)$ is associated with an MRA, then ψ is called an A-dilation MRA wavelet.

However, not every A-dilation wavelet is associated with an MRA. It has been known [3, 30] that $|\det A| = 2$ is necessary and sufficient for the existence of an MRA wavelet. Dai, Larson and Speegle [14] proved the existence of a single wavelet of the form $\mathcal{F}^{-1}(\chi_Q)$ for some measurable set $Q \subseteq \mathbb{R}^n$ and this for any real expansive matrix A, where χ_Q is the characteristic function of Q. A measurable set $Q \subseteq \mathbb{R}^n$ is called an A-dilation wavelet set if $\mathcal{F}^{-1}(\chi_Q)$ is an A-dilation wavelet. If an A-dilation wavelet $\mathcal{F}^{-1}(\chi_Q)$ is associated with an MRA, then the set $Q \subseteq \mathbb{R}^n$ is called an A-dilation MRA wavelet set. Moreover, Dai, Larson and Speegle [14] provided a criterion for a measurable set $Q \subseteq \mathbb{R}^n$ to be an

A-dilation wavelet set.

Proposition 0.0.1. A measurable set $Q \subseteq \mathbb{R}^n$ is an A-dilation wavelet set if and only if Q satisfies the following two conditions:

(i) $\bigcup_{j\in\mathbb{Z}} B^jQ = \mathbb{R}^n$ *and* $B^iQ \bigcap B^jQ = \emptyset$ *for* $i \neq j$,

(ii) $\bigcup_{k\in\mathbb{Z}^n} (Q+k) = \mathbb{R}^n$ *and* $Q \bigcap (Q+\ell) = \emptyset$ *for* $0 \neq \ell \in \mathbb{Z}^n$.

The condition (ii) of Proposition 0.0.1 is equivalent to the following condition

$$\sum_{k\in\mathbb{Z}^n} \chi_Q(\xi + k) = 1 \text{ a.e. } \xi \in \mathbb{R}^n.$$

A measurable subset Q of $\mathbb{R}^n$ is called a $\mathbb{Z}^n$-tiling set if it satisfies the criterion (ii). The wavelets of form $\mathcal{F}^{-1}(\chi_Q)$ are a particular class of minimally supported frequency (MSF) wavelets (see [20, 34]). They are analogs of Shannon wavelet and lack of smoothness in the Fourier domain. However, they have good localization property in the frequency domain and they may have good potential for applications just like Shannon wavelet or Haar wavelet. Many researchers including Baggett, Medina and Merrill [5], Dai, Larson and Speegle [15], Benedetto and Leon [7, 8], Gabardo and Yu [24] and so on (e.g. see [9, 49, 36]) contributed their work to construct concrete examples of wavelet sets in $\mathbb{R}^n$ with better properties.

Multiwavelets and Multiwavelet Sets

The multiwavelet theory runs in parallel with the wavelet theory provided in the previous subsection. It adds a little bit of complexity, but not much, and results can be stated in greater generality. Mostly, it complicates the notation. Suppose A is an $n \times n$ real expansive matrix. A finite set $\psi = \{\psi^1, \ldots, \psi^M\} \subset L^2(\mathbb{R}^n)$ is called an A-dilation orthonormal multiwavelet if the system

$$\left\{ |\det A|^{\frac{j}{2}} \psi^\ell(A^j \cdot -k) : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \ell = 1, \ldots, M \right\}$$

is an orthonormal basis for $L^2(\mathbb{R}^n)$. If an A-dilation orthonormal multiwavelet Ψ consists of a single element $\psi \in L^2(\mathbb{R}^n)$, then we call ψ an A-dilation orthonormal wavelet. For the general theory and characterization of orthonormal multiwavelets, we refer the readers to [4, 22, 26, 12]. Dai, Larson and Speegle [14] asserted that every expansive dilation matrix has an MSF wavelet with the form $\mathcal{F}^{-1}(\chi_Q)$ for some measurable set $Q \subset \mathbb{R}^n$. As an extension of MSF wavelets, MSF multiwavelets are defined and characterized analogously in [7, 12]. It leads to the concept of multiwavelet set. A measurable set Q is called an A-dilation multiwavelet set of order M if $Q = \bigcup_{i=1}^{M} Q_i$ for some essentially disjoint sets $Q_i \subset \mathbb{R}^n$, $1 \leq i \leq M$, with the property that the finite collection $\left\{\mathcal{F}^{-1}(\chi_{Q_1}), \ldots, \mathcal{F}^{-1}(\chi_{Q_M})\right\}$ is an A-dilation orthonormal multiwavelet. A multiwavelet set of order 1 is called a wavelet set. Bownik, Rzeszotnik and Speegle [12] characterized all multiwavelet sets using the following theorem.

Theorem 0.0.2. *A measurable set Q is an A-dilation multiwavelet set of order M if and only if*

(i) $\sum_{k\in\mathbb{Z}^n} \chi_Q(\xi + k) = M$ *a.e.* $\xi \in \mathbb{R}^n$,

(ii) $\sum_{j\in\mathbb{Z}} \chi_Q(B^j\xi) = 1$ *a.e.* $\xi \in \mathbb{R}^n$.

The condition (ii) above is equivalent to the condition (i) of Proposition 0.0.1. If $M = 1$, then Theorem 0.0.2 is reduced to the criterion given by Dai, Larson and Speegle [14] for a measurable set $Q \subset \mathbb{R}^n$ to be an A-dilation wavelet set.

Scaling Sets and Generalized Scaling Sets

Suppose A is an $n \times n$ real expansive matrix and let $B = A^t$. The relationship between scaling function and wavelet function reflects the connection between scaling sets and wavelet sets. Bownik, Rzeszotnik and Speegle [12] clarified the relation between (generalized) scaling sets and (multi)wavelet sets. A measurable set K is called an A-dilation

scaling set (resp. MRA scaling set) if $Q = BK \setminus K$ is an A-dilation wavelet set (resp. MRA wavelet set). For fixed $M \in \mathbb{N}$, a measurable set $K \subset \mathbb{R}^n$ is called a generalized scaling set of order M associated with a dilation matrix A if $BK \setminus K$ is an A-dilation multiwavelet set of order M. The A-dilation (generalized) scaling set introduced by Bownik, Rzeszotnik and Speegle [12], can be defined in several equivalent ways. The following result, see ([12], Proposition 3.2), can also serve as a definition.

Lemma 0.0.3. *A measurable set $K \subseteq \mathbb{R}^n$ is an A-dilation generalized scaling set of order M if and only if $K = \bigcup_{j=1}^{\infty} B^{-j}Q$ for some A-dilation multiwavelet set Q of order M.*

Since $Q = BK \setminus K$, if K and Q are as in the previous lemma, it follows that an A-dilation generalized scaling set can be associated with a unique A-dilation multiwavelet set and vice-versa. We now state the following criterion of Bownik, Rzeszotnik and Speegle [12] for a measurable set $K \subseteq \mathbb{R}^n$ to be a generalized scaling set of order M associated with an $n \times n$ integral expansive dilation matrix A.

Theorem 0.0.4. *Let A be an $n \times n$ integral expanding matrix with $|\det A| = q$. A measurable set $K \subset \mathbb{R}^n$ is an A-dilation generalized scaling set of order M if and only if*

(i) $|K| = \frac{M}{q-1}$,

(ii) $K \subset BK$,

(iii) $\lim_{m\to\infty} \chi_K(B^{-m}\xi) = 1$ *for a.e.* $\xi \in \mathbb{R}^n$,

(iv) $\sum_{d\in\mathcal{D}} D(\xi + B^{-1}d) = D(B\xi) + M$ *a.e., where* $D(\xi) = \sum_{k\in\mathbb{Z}^n} \chi_K(\xi + k)$ *and* $\mathcal{D}$ *is a complete set of coset representatives for the group* $\mathbb{Z}^n/B\mathbb{Z}^n$.

As we mentioned before, $|\det A| = 2$ is necessary and sufficient for the existence of an MRA wavelet associated with the dilation A. Bownik, Rzeszotnik and Speegle [12] characterized a measurable set $K \subseteq \mathbb{R}^n$ to be an A-dilation MRA scaling set using the following lemma.

Lemma 0.0.5 ([12]). *Let A be an $n \times n$ real expansive matrix with $|\det A| = 2$ and K be an A-dilation generalized scaling set. Then K is associated with an MRA if and only if $D(\xi) := \sum_{k\in\mathbb{Z}^n} \chi_K(\xi + k) = 1$ a.e. $\xi \in \mathbb{R}^n$.*

Using Lemma 0.0.5 and the definition of scaling sets, we can get the following result.

Lemma 0.0.6. *A wavelet set Q is an A-dilation MRA wavelet set if and only if the set $K := \bigcup_{j=1}^{\infty} B^{-j}Q$ is a $\mathbb{Z}^n$-tiling set. Such a set K is called an A-dilation MRA scaling set.*

From Lemma 0.0.5 and Theorem 0.0.4, we obtain an equivalent criterion for a measurable set $K \subseteq \mathbb{R}^n$ to be an MRA scaling set associated with an $n \times n$ integral expansive dilation A with $|\det A| = 2$.

Corollary 0.0.7. *Let A be an $n \times n$ integral expansive matrix with $|\det A| = 2$. A measurable set $K \subset \mathbb{R}^n$ is an A-dilation MRA scaling set if and only if*

(i) K is a $\mathbb{Z}^n$-tiling set,

(ii) $K \subset BK$,

(iii) $\lim_{m\to\infty} \chi_K(B^{-m}\xi) = 1$ for a.e. $\xi \in \mathbb{R}^n$.

Dimension Functions of Wavelets

Suppose A is an $n \times n$ real expansive matrix and let $B = A^t$. The dimension function of an A-dilation orthonormal multiwavelet $\Psi = \{\psi^1, \ldots, \psi^M\} \subset L^2(\mathbb{R}^n)$ is defined by

$$D_\Psi(\xi) = \sum_{\ell=1}^{M} \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}^n} |\hat{\psi}^\ell(B^j(\xi + k))|^2, \ \xi \in \mathbb{R}^n. \tag{0.0.3}$$

The (multi)wavelet dimension function, which was firstly introduced and investigated by Auscher in [1], is an important tool in the theory of wavelets and it has been extensively studied by many researchers for the class of integer dilations [4, 12, 45, 33] and for the class of real dilations [10, 11]. Its importance is due to the fact that it can be used to

prove that certain wavelets are associated with a multiresolution analysis (MRA [16, 43]). Initially, Lemarié-Rieusset [41, 42] used the wavelet dimension function to show that all compactly supported wavelets are associated with an MRA in the one-dimensional dyadic case. After that, Gripenberg [26] and Wang [52] independently used it to characterize all wavelets which arise from an MRA and they proved that a wavelet ψ is an MRA wavelet if and only if its dimension function $D_\psi(\xi)$ is equal to 1 for almost every $\xi \in \mathbb{R}$. Baggett, Medina and Merrill [4, 5] made a systematic study of the properties of the multiwavelet dimension function associated with an $n \times n$ integral expansive matrix A, including the case where the wavelets are not associated with an MRA. They showed that the dimension function D_Ψ of the multiwavelet Ψ satisfies the following consistency equation

$$\sum_{d\in\mathcal{D}} D_\Psi(\xi + B^{-1}d) = D_\Psi(B\xi) + L, \text{ for } a.e.\ \xi \in \mathbb{R}^n, \tag{0.0.4}$$

where $\mathcal{D}$ is a complete set of coset representatives for the group $\mathbb{Z}^n/B\mathbb{Z}^n$. Furthermore, they exploited this consistency equation to provide a constructive procedure for producing all A-dilation wavelet sets in $\mathbb{R}^n$.

Self-Affine Tiles

A self-affine tile in $\mathbb{R}^n$ is a measurable set K with positive Lebesgue measure satisfying the set-valued equation $AK = \bigcup_{d\in\mathcal{D}} (K + d)$, where A is an expanding $n \times n$ real matrix with $|\det A| = m$ is an integer and $\mathcal{D} = \{d_1, d_2, \ldots, d_m\} \subseteq \mathbb{R}^n$ is a set of m digits.

In the case, where A is an $n \times n$ integral expanding matrix with $|\det A| = m$ and a set of m digits $\mathcal{D} = \{d_1, d_2, \ldots, d_m\} \subseteq \mathbb{Z}^n$, the set K is called an integral self-affine tile in $\mathbb{R}^n$.

The theory of self-affine tiles has been studied by many authors. Particularly, Lagarias and Wang [37, 38, 39] studied in detail the structure and tiling properties of self-affine tiles. These problems are connected with the construction of orthonormal wavelet basis in $\mathbb{R}^n$. Gröchenig and Madych [29] revealed an interesting connection between the theory of compactly supported wavelet bases and the theory of integral self-affine tiles. They

showed that Haar-type wavelet bases which can be constructed from an MRA always have an associated scaling function which is the characteristic function of an integral self-affine tile K and that such a tile gives a scaling function if and only if K is a $\mathbb{Z}^n$-tiling set in $\mathbb{R}^n$. Gröchenig and Hass [27], Lagarias and Wang [40] proved, for example, that any dilation A yields Haar-type wavelets in dimension 1 and dimension 2 respectively using Gröchenig and Madych's result [29]. We should mention that these authors only consider integral self-affine tiling sets in their work, i.e. the matrix A is an integral expansive matrix and the associated digits are in $\mathbb{Z}^n$. Gabardo and Yu [24] also contributed their work to construct wavelet sets using integral self-affine tiles.

To our knowledge, until now, there is no result in the literature on the connection between the theory of wavelets and the theory of non-integral self-affine tiles. However, there are many wavelets which come from non-integral self-affine tiles. one interesting example is the well-known Shannon wavelet whose corresponding scaling set is $K = [-\frac{1}{2}, \frac{1}{2}]$ with associated dilation factor $A = 2$. Then $BK = (K - \frac{1}{2}) \bigcup (K + \frac{1}{2})$, where $B = A^t = 2$, which shows that K is a self-affine tile but not an integral self-affine tile. Gu and Han [30] showed that there exists A-dilation MRA wavelets for any $n \times n$ integral expansive matrix A with $|\det A| = 2$ and they also gave some concrete examples of MRA wavelet (scaling) sets in $\mathbb{R}^2$. Among their examples, some of them are non-integral self-affine tiles. For example, for the matrix $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$, one is given in [30] and its corresponding scaling set is the unite square $K = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$. Then obviously, $BK = (K + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}) \bigcup (K + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix})$, where $B = A^t$, and again K is not an integral self-affine tile. Therefore, non-integral self-affine tiles are useful to construct new examples of wavelet sets. A measurable set K is called an A-dilation self-affine (generalized) scaling set if K is an A-dilation (generalized) scaling set and K is a self-affine tile which satisfies $BK = \bigcup_{d \in \mathcal{D}} K + d$ for some digit set $\mathcal{D}$, where $B = A^t$.

In Chapter 1, 2 and 3, we will always assume that A is an $n \times n$ integral expansive matrix with $|\det A| = 2$. In Chapter 2 and Chapter 3 of this thesis, we characterize all one and two dimensional self-affine scaling sets with a set of associated digits of the form $\{d_1, d_2\} \subset \mathbb{R}^n$, for an $n \times n$ integral expansive matrix A with $|\det A| = 2$, where $n = 1, 2$.

Before this, we characterize the properties of the dimension function of all A-dilation generalized scaling sets in $\mathbb{R}^n$ which satisfy $BK = (K + d_1) \bigcup (K + d_2)$ with $d_1, d_2 \in \mathbb{R}^n$, where $B = A^t$ in Chapter 1.

In Section 1.1, we provide some preliminary results needed for later subsections and define the dimension function of a generalized scaling set associated with an $n \times n$ integral expansive matrix. In Section 1.2 and Section 1.3, we prove that the dimension function of any A-dilation self-affine generalized scaling set K is a constant and is equal to the Lebesgue measure of the set K in $\mathbb{R}$ and $\mathbb{R}^2$ respectively. This result shows that all A-dilation self-affine scaling sets must be A-dilation MRA scaling sets in dimensions one and two. In Section 1.4, we consider our problem in arbitrary dimension and prove that the dimension function of a self-affine generalized scaling set is bounded by twice its Lebesgue measure.

In view of the results in Chapter one, the characterization of all A-dilation self-affine scaling sets in dimension one and dimension two is reduced to the characterization of all A-dilation self-affine MRA scaling sets in Chapter 2 and Chapter 3. In Chapter 2, we consider our problem in a particular case where the digit set with the form $\{0, d\}$, where $d \in \mathbb{R}^2$. In this case, the inclusion $K \subset BK$ holds automatically. In section 2.1, we show some preliminary results needed for later subsections. The remaining part of this chapter is devoted to characterizing all digit sets $\mathcal{D} = \{0, d\} \subset \mathbb{R}^n$ such that the associated self-affine set K tiles $\mathbb{R}^n$ using $\mathbb{Z}^n$- translation for a given integral expansive matrix A with $|\det A| = 2$ in the case $n = 1$ and $n = 2$. Furthermore, we determine which of such digit sets $\mathcal{D}$ yields a scaling set. The case $n = 1$ is considered in section 2.2 and the case $n = 2$, which has to be divided into several cases, is considered in sections 2.3 to 2.8.

As a complement to the work in Chapter 2, the main purpose of Chapter 3 is to characterize all self-affine MRA scaling sets with associated digit set $\mathcal{D} = \{d_1, d_2\} \subset \mathbb{R}^n$, $n = 1, 2$. The main difference with Chapter 2 is that we consider the general case, i.e., we do not assume here that $0 \in \mathcal{D}$. In Section 3.1, we give a sufficient condition on the digit set $\mathcal{D} = \{d_1, d_2\} \subset \mathbb{R}^n$ for the set $K = K(B, \mathcal{D})$ to be an A-dilation MRA scaling set. In section 3.2, we characterize the inclusion $K \subset BK$ of Corollary 0.0.7 under the

condition that K is a self-affine tile in terms of the digit set $\mathcal{D}$. Here, 0 is not necessarily one of digits. In particular, if $0 \in \mathcal{D}$, then the inclusion $K \subset BK$ holds automatically. However, the inclusion $K \subset BK$ is not obvious if $0 \notin \mathcal{D}$. In this section, we give an equivalent condition for the inclusion $K \subset BK$ whether $0 \notin \mathcal{D}$ or not and provide an algorithm to verify which digit set $\mathcal{D}$ makes the inclusion hold. In section 3.3, we describe all A-dilation self-affine scaling sets in dimension one. In the last section of this chapter, we give a complete characterization of all A-dilation self-affine scaling sets in $\mathbb{R}^2$.

Integral Self-Affine Multi-Tiles

A finite collection of essentially disjoint sets $K_i \subseteq \mathbb{R}^n, 1 \leq i \leq M$, that are either compact or empty, is said to be a (B, Γ)-self-affine collection if there is a matrix $B \in M_n(\mathbb{Z})$ with all eigenvalues of modulus greater than one and finite (possibly empty) sets $\Gamma_{ij} \subseteq \mathbb{Z}^n,\ i, j = 1, \ldots, M$, such that

$$BK_i = \bigcup_{j=1}^{M} (\Gamma_{ij} + K_j) \text{ for } i = 1, \ldots, M, \tag{0.0.5}$$

and for any $i, j, k \in \{1, \ldots, M\}$

$$(\beta + K_i) \bigcap (\gamma + K_j) = \emptyset \text{ for } \beta \in \Gamma_{ki},\ \gamma \in \Gamma_{kj} \text{ and } i \neq j \text{ or } \beta \neq \gamma. \tag{0.0.6}$$

The set $\Gamma := \{\Gamma_{ij}\}_{1 \leq i,j \leq M}$ is called a digit set. Γ is called a standard digit set if for each $j \in \{1, ..., M\}$, $\mathcal{D}_j := \bigcup_{i=1}^{M} \Gamma_{ij}$ is a complete set of coset representatives for the group $\mathbb{Z}^n / B\mathbb{Z}^n$. This definition is the analogue of that of the standard digit set introduced in [29, 38].

If $\Lambda \subseteq \mathbb{Z}^n$, a measurable set $K \subset \mathbb{R}^n$ is said to Λ-tile $\mathbb{R}^n$ or to be a Λ-tiling set, if

$$\bigcup_{\ell \in \Lambda} (\ell + K) = \mathbb{R}^n \text{ and } K \bigcap (\ell + K) = \emptyset \text{ for } 0 \neq \ell \in \Lambda. \tag{0.0.7}$$

Let $K = \bigcup_{i=1}^{M} K_i$. If K Λ-tiles $\mathbb{R}^n$ and a finite collection of sets $\{K_i \subset \mathbb{R}^n, 1 \leq i \leq M\}$ form a (B, Γ)-self-affine collection, we call K an integral self-affine Λ-tiling set with M

prototiles. Particularlly, in the case where $M = 1$, K is an integral self-affine Λ-tiling set with one tile, (0.0.5) has exactly one solution [35]. In such a case, we say that K is an integral self-affine tile. Thus integral self-affine tiles are a special class of integral self-affine multi-tiles. Self-affine tiles have been extensively studied (see e.g. [27, 37, 38, 39]) and they are showed to be closely related to wavelet theory [29, 27, 40]. In view of this, it is of interest to study integral self-affine multi-tiles and their connection with wavelet theory.

In contrast to the one tile situation, there may exist several solutions when some prototiles are allowed to be empty. The study of self-affine multi-tiles is not well developed because of their complicated structure. Gröchenig and Hass in [28] firstly provided a detailed description on integral self-affine multi-tiles. They constructed and classified the general solutions of (0.0.5) and derived a necessary condition for a solution of (0.0.5) to be a $\mathbb{Z}^n$-tiling set. Furthermore, they established the relationship between the theory of multiwavelets and the theory of integral self-affine multi-tiles. Their results extend the work in [29] to higher multiplicity. Similar to the result in [29], Gröchenig and Hass [28] showed that a multiresolution analysis [16, 43] for $L^2(\mathbb{R}^n)$ is generated by the characteristic functions χ_{K_i}, $i = 1, 2, \ldots, M$ if and only if $K = \bigcup_{i=1}^{M} K_i$ is a self-affine $\mathbb{Z}^n$-tiling set with M prototiles and each prototile $K_i, 1 \leq i \leq M$ has positive Lebesgue measure. Flaherty and Wang in [21] showed how integral self-affine multi-tiles can be used to construct Haar-type multiwavelets. As an another application to wavelet theory, we consider the problem of constructing (multi)wavelet sets using integral self-affine multi-tiles in Chapter 4.

Gabardo and Yu [24] first considered using integral self-affine tiles in the Fourier domain to construct wavelet sets. The wavelet sets they constructed are finite unions of integer translates of integral self-affine tiles. However, many wavelet sets are derived from integral self-affine multi-tiles, not integral self-affine tiles. For example, all wavelet sets given in ([30], section 3) are finite unions of integer translates of integral self-affine collection and they cannot be written as finite unions of integer translates of integral self-affine tiles. Another interesting example is the well-known Shannon wavelet whose

corresponding wavelet set is $Q = [-1, -\frac{1}{2}] \bigcup [\frac{1}{2}, 1]$, which is a finite union of an integral self-affine collection $\{K_1, K_2\}$, where $K_1 := [-\frac{1}{2}, 0]$ and $K_2 := [0, \frac{1}{2}]$, i.e. $Q = (K_1 + 1) \bigcup (K_2 - 1)$. Obviously, $K = K_1 \bigcup K_2$ is a self-affine $\mathbb{Z}$-tiling set with 2 prototiles. Thus integral self-affine multi-tiles can provide a nice framework in which the known examples of wavelet sets appear naturally and also constitute a new tool with the help of which new examples can be created. The main purpose of Chapter 4 in this thesis is to construct wavelet sets using integral self-affine multi-tiles.

In Section 4.1, we characterize some properties of integral self-affine multi-tiles. In section 4.2, we provide some sufficient conditions for constructing wavelet sets using integral self-affine multi-tiles. Comparing to the case of integral self-affine tiles, it is more restrictive to construct wavelet sets using integral self-affine multi-tiles. In section 4.3, we consider constructing multiwavelet sets using integral self-affine multi-tiles.

Indeed, the representation of an integral self-affine multi-tile is not unique. For example, in dimension one, if $K = [-\frac{1}{4}, \frac{3}{4}]$ associated with $B = 2$, then K is not only an integral self-affine $\mathbb{Z}$-tiling set with 3 prototiles, but also an integral self-affine $\mathbb{Z}$-tiling set with 4 prototiles. We will give a detailed explanation for this example in section 5.2. The main goal of Chapter 5 in this thesis is to provide a method to decompose an integral self-affine multi-tile K which is a $\mathbb{Z}^n$-tiling set into different essentially disjoint pieces K_j which satisfy $K = \bigcup K_j$ such that the collection of sets K_j is an integral self-affine collection associated with some given expansive matrix $B \in M_n(\mathbb{Z})$. Moreover, the representation of integral self-affine multi-tiles is unique and the number of prototiles is minimal.

In Section 5.1, we provide a method to represent a $\mathbb{Z}^n$-tiling set which is an integral self-affine multi-tile as a union of prototiles with the least number of pieces. Using this algorithm, we can determine whether or not a given measurable $\mathbb{Z}^n$-tiling set $K \subset \mathbb{R}^n$ is an integral self-affine multi-tile associated with any given $n \times n$ integral expansive matrix B. In section 5.2, we give some examples to illustrate our algorithm. Moreover, we construct some examples showing that some wavelet sets cannot be constructed by integral self-affine multi-tiles as was done in Chapter 4.

Self-Affine Sets

Let B be an $n \times n$ real expanding matrix and $\mathcal{D}$ be a finite subset of $\mathbb{R}^n$. The self-affine set $K = K(B, \mathcal{D})$ is the unique compact set satisfying the set equation $BK = \bigcup_{d \in \mathcal{D}} (K + d)$.

It is easy to see that $K(B, \mathcal{D})$ has zero Lebesgue measure if $m < |\det B|$. For the case $m = |\det B|$, Lagarias and Wang [37] provided some equivalent conditions for the Lebesgue measure of $K(B, \mathcal{D})$ to be positive. In the particular case where B is an integral matrix and the digit set $\mathcal{D} \subset \mathbb{Z}^n$ is a complete set of coset representatives for $\mathbb{Z}^n / B\mathbb{Z}^n$, Lagarias and Wang [38] showed that the Lebesgue measure of $K(B, \mathcal{D})$ is a positive integer. Furthermore, Gabardo and Yu [25] gave an algorithm to evaluate the Lebesgue measure of any such self-affine set. A self-affine set in $\mathbb{R}^n$ is called a self-affine tile if it has positive Lebesgue measure and the number of elements in digit set $\mathcal{D}$ equals to $|\det B|$. The structure and tiling properties of self-affine tiles have been studied in great detail since these problems are closely ralated to the construction of orthonormal wavelet basis in $\mathbb{R}^n$ (see, e.g., [27, 37, 38, 39, 29, 40]). For the case $m > |\det B|$, the situation becomes more complicated because of the overlapping of pieces $K + d$ for some $d \in \mathcal{D}$. He, Lau and Rao [31] considered the case where the self-affine set $K(B, \mathcal{D})$ with $\mathcal{D} \subset \mathbb{Z}^n$, has positive Lebesgue measure under the assumption that B is an integral matrix and they proved that the Lebesgue measure of $K(B, \mathcal{D})$ is a rational number.

However, the problem of how to compute the exact Lebesgue measure of self-affine set $K(B, \mathcal{D})$ for the more general case where $\mathcal{D} \subset \mathbb{R}^n$ is still open, to our knowledge. Moreover, some self-affine sets with zero Lebesgue measure may have positive Hausdorff measure associated for an appropriate Hausdorff dimension. For example, in dimension one, the middle Cantor set K which is a self-affine set and satisfies $3K = K \bigcup (K + 2)$ has Lebesgue measure zero, but its Hausdorff measure $\mathcal{H}^\alpha(K) = 1$ for the Hausdorff dimension $\alpha := \dim_H K = \log_3^2$. Ayer and Strichartz [2] gave an algorithm to compute the exact Hausdorff measure of a class of linear Cantor sets in dimension one. However, no similar result was obtained in higher dimension, even for self-similar sets. In Chapter 6, we not only consider the problem how to compute the Lebesgue measure of self-affine sets

$K(B, \mathcal{D})$, but also consider the Hausdorff measure for those with zero Lebesgue measure under the assumption that $K(B, \mathcal{D})$ is a self-similar set. Furthermore, we provide a connection between the Lebesgue or Hausdorff measure of $K(B, \mathcal{D})$ and an appropriate notion of upper density for the measure

$$\mu = \lim_{s\to\infty} \sum_{\ell_0,\ldots,\ell_{s-1}\in\mathcal{D}} \delta_{\ell_0+B\ell_1+\cdots+B^{s-1}\ell_{s-1}}, \tag{0.0.8}$$

where δ_x denotes the Dirac measure at x.

In section 6.1, we first consider the problem of computing the Lebesgue measure of the self-affine set $K(B, \mathcal{D})$ when $m = |\det B|$ and relate it to the upper Beurling density of μ. In section 6.2, we consider the case where $m < |\det B|$ and B is a similarity with scaling factor $\rho > 1$. We prove a result analogous to the one proved in section 2 relating the s-Hausdorff measure of $K(B, \mathcal{D})$ to a certain upper density of μ, where μ is defined by (0.0.8) and s is the similarity dimension.

Throughout this thesis, We will denote by $M_n(\mathbb{Z})$ (or $M_n(\mathbb{R})$) the set of $n \times n$ matrices with integer (or real) entries and denote by $M_n^{(2)}(\mathbb{Z})$ the set of all $n \times n$ integral expansive matrices with determinant equal to 2 or -2.

Chapter 1

Dimension Functions of Self-Affine Scaling Sets

In this chapter, we will restrict our discussion to the self-affine A-dilation generalized scaling sets K, where $A \in M_n^{(2)}(\mathbb{Z})$ is expansive, i.e. the set K is an A-dilation generalized scaling set and it satisfies the set equation $BK = (K + d_1) \bigcup (K + d_2)$, for some vectors $d_1, d_2 \in \mathbb{R}^n$. We are devoted to characterizing the properties of the dimension function of all A-dilation generalized scaling sets in $\mathbb{R}^n$ which satisfy a self-affine equation.

1.1 Definitions and Preliminary Results

In the following, we will deduce, for the reader's convenience, the definition of dimension function of an A-dilation generalized scaling set K as provided by Bownik, Rzeszotnik and Speegle in Section 3 of [12]. Given an A-dilation generalized scaling set K of order M, we can define an A-dilation multiwavelet set $Q := BK \setminus K$ of order M. By the definition of multiwavelet set, we can find essentially disjoint sets Q_i, $i = 1, \ldots, M$ with $Q = \bigcup_{i=1}^{M} Q_i$ and such that the set $\Psi := \left\{\mathcal{F}^{-1}(\chi_{Q_1}), \ldots, \mathcal{F}^{-1}(\chi_{Q_M})\right\}$ is an A-dilation orthonormal multiwavelet. Thus the dimension function of Ψ is well defined and, using (0.0.3), we

have

$$D_\Psi(\xi) = \sum_{\ell=1}^{M}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^n} |\chi_{Q_\ell}(B^j(\xi+k))|^2 = \sum_{\ell=1}^{M}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^n} \chi_{Q_\ell}(B^j(\xi+k)) \text{ a.e. } \xi\in\mathbb{R}^n. \quad (1.1.1)$$

By the relationship between generalized scaling set K and multiwavelet set Q, the set K can also be represented by $K = \bigcup_{j=1}^{\infty} B^{-j}Q$. Therefore, it follows from (1.1.1) and the properties of multiwavelet set that

$$D_\Psi(\xi) = \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^n} \chi_Q(B^j(\xi+k)) = \sum_{k\in\mathbb{Z}^n} \chi_K(\xi+k) \text{ a.e. } \xi\in\mathbb{R}^n.$$

Motivated by this last identity, we have the following definition.

Definition 1.1.1. The dimension function of an A-dilation generalized scaling set K is the function $D:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$$D(\xi) = \sum_{k\in\mathbb{Z}^n} \chi_K(\xi+k),\ \xi\in\mathbb{R}^n.$$

Lemma 1.1.2 below characterizes the condition (i) of Theorem 0.0.2 in the Fourier domain and it will be used in later sections.

Lemma 1.1.2. *Let $K\subseteq\mathbb{R}^n$ be a measurable set. Then $\sum_{j\in\mathbb{Z}^n}\chi_K(x+j) = L$ a.e. $\xi\in\mathbb{R}^n$ if and only if $\hat{\chi}_K(j) = L\delta_{j,0}$.*

Proof. Since

$$\int_{[0,1]^n}\sum_{s\in\mathbb{Z}^n}\chi_K(x+s)e^{-2\pi ix\cdot j}dx = \sum_{s\in\mathbb{Z}^n}\int_{[0,1]^n+s}\chi_K(x)e^{-2\pi ix\cdot j}dx = \int_{\mathbb{R}^n}\chi_K(x)e^{-2\pi ix\cdot j}dx = \hat{\chi}_K(j),$$

then we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}^n}\chi_K(x+j) = L \ \text{ a.e. } x\in\mathbb{R}^n &\iff L\delta_{j,0} = \textstyle\int_{[0,1]^n}\sum_{s\in\mathbb{Z}^n}\chi_K(x+s)e^{-2\pi ix\cdot j}dx,\ j\in\mathbb{Z}^n\\ &\iff \hat{\chi}_K(j) = L\delta_{j,0},\ j\in\mathbb{Z}^n,\end{aligned}$$

proving our claim. □

It follows from Lemma 1.1.2 that if the dimension function of an A-dilation generalized scaling set of order M is a constant, then it is equal to the measure of the set K. That is, we have $D(\xi) = \sum_{j\in\mathbb{Z}^n} \chi_K(\xi + j) = \frac{M}{q-1}$ *a.e.* $\xi \in \mathbb{R}^n$ using (i) of Theorem 0.0.4. Under this case, the condition (iv) in Theorem 0.0.4 holds automatically.

In the following, our goal will be to consider the dimension function of a generalized scaling set associated with an expansive dilation $A \in M_n^{(2)}(\mathbb{Z})$, which is a self-affine tile. For short, we call such dimension function the dimension function of an A-dilation generalized self-affine scaling set.

Assume that K is an A-dilation generalized self-affine saling set, then K satisfies the set equation $BK = (K + d_1)\bigcup(K + d_2)$, for some $d_1,\ d_2 \in \mathbb{R}^n$ and K satisfies (i), (ii), (iii) and (iv) of Theorem 0.0.4. Moreover, (iv) of Theorem 0.0.4 can also be written as

$$\sum_{d\in\mathcal{D}} D(B^{-1}(\xi + d)) = D(\xi) + L \text{ a.e. } \xi \in \mathbb{R}^n. \tag{1.1.2}$$

Then, since $\mathcal{D}$ is a complete set of coset representatives for $\mathbb{Z}^n/B\mathbb{Z}^n$, we have

$$\begin{aligned}\sum_{d\in\mathcal{D}} D(B^{-1}(\xi + d)) &= \sum_{d\in\mathcal{D}}\sum_{k\in\mathbb{Z}^n} \chi_K(B^{-1}(\xi + d) + k)\\ &= \sum_{d\in\mathcal{D}}\sum_{k\in\mathbb{Z}^n} \chi_{BK}(\xi + d + Bk)\\ &= \sum_{k\in\mathbb{Z}^n} \chi_{BK}(\xi + k).\end{aligned}$$

Since K satisfies that $BK = (K + d_1)\bigcup(K + d_2)$, for some $d_1,\ d_2 \in \mathbb{R}^n$, (1.1.3) implies that

$$\sum_{d\in\mathcal{D}} D(B^{-1}(\xi + d)) = \sum_{k\in\mathbb{Z}^n} \chi_{K+d_1}(\xi + k) + \sum_{k\in\mathbb{Z}^n} \chi_{K+d_2}(\xi + k). \tag{1.1.3}$$

It follows from (1.1.2) and (1.1.3) that

$$\sum_{k\in\mathbb{Z}^n} \chi_K(\xi - d_1 + k) + \sum_{k\in\mathbb{Z}^n} \chi_K(\xi - d_2 + k) = \sum_{k\in\mathbb{Z}^n} \chi_K(\xi + k) + L \text{ a.e. } \xi \in \mathbb{R}^n. \tag{1.1.4}$$

Since $D(\xi)$ is $\mathbb{Z}^n$-periodic, it can be expanded as the Fourier series $D(\xi) = \sum_{k\in\mathbb{Z}^n} a_k\ e^{2\pi ik\cdot\xi}$,

where

$$\begin{aligned} a_k &= \int_{[0,1]^n} D(\xi)\ e^{-2\pi ik\cdot\xi}\ d\xi = \int_{[0,1]^n} \sum_{\ell\in\mathbb{Z}^n} \chi_K(\xi+\ell)\ e^{-2\pi ik\cdot\xi}\ d\xi \\ &= \sum_{\ell\in\mathbb{Z}^n} \int_{[0,1]^n} \chi_K(\xi+\ell)\ e^{-2\pi ik\cdot\xi}\ d\xi = \int_{\mathbb{R}^n} \chi_K(\xi)\ e^{-2\pi ik\cdot\xi}\ d\xi = \hat{\chi}_K(k). \end{aligned}$$

From (1.1.4) and (1.1.5), we obtain

$$\sum_{k\in\mathbb{Z}^n} \hat{\chi}_K(k)\ e^{2\pi ik\cdot\xi}\ (e^{-2\pi ik\cdot d_1} + e^{-2\pi ik\cdot d_2}) = \sum_{k\in\mathbb{Z}^n} \hat{\chi}_K(k)\ e^{2\pi ik\cdot\xi} + L \text{ a.e. } \xi\ \in\mathbb{R}^n, \quad (1.1.5)$$

or equivalently,

$$(e^{-2\pi ik\cdot d_1} + e^{-2\pi ik\cdot d_2})\hat{\chi}_K(k) = \hat{\chi}_K(k) + L\delta_{k,0},\ k\in\mathbb{Z}^n. \quad (1.1.6)$$

We deduce from (1.1.6) that $\hat{\chi}_K(0) = L$ and

$$e^{-2\pi ik\cdot d_1} + e^{-2\pi ik\cdot d_2} = 1 \text{ for } k\in\mathbb{Z}^n\setminus\{0\} \text{ such that } \hat{\chi}_K(k)\neq 0. \quad (1.1.7)$$

Let $T = \Big\{k\in\mathbb{Z}^n\setminus\{0\} : \hat{\chi}_K(k)\neq 0\Big\}$. Then for $k\in T$, we have

$$\begin{cases} k\cdot d_1 \in \frac{1}{6}+\mathbb{Z}, \\ k\cdot d_2 \in -\frac{1}{6}+\mathbb{Z}, \end{cases} \text{ or } \begin{cases} k\cdot d_1 \in -\frac{1}{6}+\mathbb{Z}, \\ k\cdot d_2 \in \frac{1}{6}+\mathbb{Z}. \end{cases}$$

On the other hand, (1.1.3) implies that $\sum_{d\in\mathcal{D}} D(B^{-1}(\xi+d)) = \sum_{k\in\mathbb{Z}^n} \chi_{BK}(\xi+j)$ is also $\mathbb{Z}^n$-periodic. Suppose its Fourier series is $\sum_{k\in\mathbb{Z}^n} b_k\ e^{2\pi ik\cdot\xi}$, then as before, we get $b_k = \hat{\chi}_{BK}(k) = 2\hat{\chi}_K(B^tk)$. Then (1.1.2) is also equivalent to the following equation

$$\sum_{k\in\mathbb{Z}^n} 2\hat{\chi}_K(B^tk)\ e^{2\pi ik\cdot\xi} = \sum_{k\in\mathbb{Z}^n} \hat{\chi}_K(k)\ e^{2\pi ik\cdot\xi} + L \text{ a.e. } \xi\in\mathbb{R}^n. \quad (1.1.8)$$

From (1.1.8), we have $\hat{\chi}_K(0) = L$ and

$$2\hat{\chi}_K(B^tk) = \hat{\chi}_K(k) \text{ for } k\in\mathbb{Z}^n\setminus\{0\}. \quad (1.1.9)$$

(1.1.9) implies that $\hat{\chi}_K(B^tk)\neq 0$ for $k\in T$, and iteratively that, $\hat{\chi}_K((B^t)^mk)\neq 0$ for any $m\geq 0$ and $k\in T$. Thus, if $k\in T$, we have

$$\begin{cases} (B^t)^mk\cdot d_1 \in \frac{1}{6}+\mathbb{Z}, \\ (B^t)^mk\cdot d_2 \in -\frac{1}{6}+\mathbb{Z}, \end{cases} \text{ or } \begin{cases} (B^t)^mk\cdot d_1 \in -\frac{1}{6}+\mathbb{Z}, \\ (B^t)^mk\cdot d_2 \in \frac{1}{6}+\mathbb{Z}, \end{cases} \quad (1.1.10)$$

for any $m\geq 0$.

1.2 The dimension function of self-affine generalized scaling sets in $\mathbb{R}$

We first consider here one dimensional problem. In dimension one, since $|\det A| = 2$, we have two possibilities: $A = 2$ or $A = -2$.

Theorem 1.2.1. *Let $A \in M_1^{(2)}(\mathbb{Z})$ and K be a self-affine tile satisfying the set-valued equation $BK = (K + d_1) \bigcup (K + d_2)$ for some $d_1,\ d_2 \in \mathbb{R}$. If K is a generalized scaling set, then its dimension function $D(\xi)$ is a constant, moreover, $D(\xi) = |K|$.*

Proof. $A \in M_1^{(2)}(\mathbb{Z})$ implies that $A = 2$ or $A = -2$. Here we only consider the case $A = 2$. The proof of the case $A = -2$ is similar. Assume that $|K| = L$. To prove that $D(\xi) = |K| = L$ *a.e.* $\xi \in \mathbb{R}$, we only need to prove that $\hat{\chi}_K(j) = 0$ for $j \in \mathbb{Z} \setminus \{0\}$ since $\hat{\chi}_K(0) = |K| = L$ by Lemma 1.1.2. Assume that there exists $j_0 \in \mathbb{Z} \setminus \{0\}$ such that $\hat{\chi}_K(j_0) \neq 0$. Since K is a generalized scaling set and K is a self-affine tile satisfying $BK = (K + d_1) \bigcup (K + d_2)$ for some $d_1,\ d_2 \in \mathbb{R}$, we have, using (1.1.10), that

$$\begin{cases} 2^m j_0 d_1 \in \frac{1}{6} + \mathbb{Z}, \\ 2^m j_0 d_2 \in -\frac{1}{6} + \mathbb{Z}, \end{cases} \quad \text{or} \quad \begin{cases} 2^m j_0 d_1 \in -\frac{1}{6} + \mathbb{Z}, \\ 2^m j_0 d_2 \in \frac{1}{6} + \mathbb{Z}, \end{cases}$$

for any $m \geq 0$. In particular, $j_0 \cdot d_1 \in \frac{1}{6} + \mathbb{Z}$ or $j_0 \cdot d_1 \in -\frac{1}{6} + \mathbb{Z}$, but in that case, if $m = 1$, $2j_0 \cdot d_1 \notin \frac{1}{6} + \mathbb{Z}$ and $2j_0 \cdot d_1 \notin -\frac{1}{6} + \mathbb{Z}$, which is a contadiction. This proves our statement. □

1.3 The dimension function of self-affine generalized scaling sets in $\mathbb{R}^2$

In this section, we deal with the two dimensional case. Consider the following matrices:

$$C_1 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix},\ C_2 = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix},\ C_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},\ C_4 = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}.$$

Two $n \times n$ integral matrices A and $\widetilde{A}$ are called integrally similar if there exists an $n \times n$ integral matrix P with $|\det P| = 1$ such that $P^{-1}AP = \widetilde{A}$. Lagarias and Wang [40]

completely classified all integral expansive matrices $A \in M_2^{(2)}(\mathbb{Z})$ and they showed that there are exactly six integrally similar classes of such integral matrices. Representatives from each of these classes are given by $C_1,\ C_2,\ C_3,\ -C_3,\ C_4,\ -C_4$, respectively.

Proposition 1.3.1 ([40])**.** Let $A \in M_2^{(2)}(\mathbb{Z})$ be expansive. If $\det A = -2$, then A is integrally similar to C_1. If $\det A = 2$, then A is integrally similar to one of the matrices $C_2,\ \pm C_3,\ \pm C_4$.

Theorem 1.3.2. *Let $A \in M_2^{(2)}(\mathbb{Z})$ be expanding and K be a self-affine tile satisfying the set equation $BK = (K+d_1)\bigcup(K+d_2)$ for some $d_1,\ d_2 \in \mathbb{R}^2$. If K is a generalized scaling set, then its dimension function $D(\xi)$ is a constant, moreover, $D(\xi) = |K|$, where $D(\xi)$ is defined in Definition 1.1.1.*

Proof. Since $A \in M_2^{(2)}(\mathbb{Z})$, the characteristic polynomial of matrix B, where $B = A^t$, has a general form $\lambda^2 + b\lambda + c = 0$, where $b = -tr(B) \in \mathbb{Z}$ and $c = \det B = 2$ or -2. Assume that $|K| = L$. To prove that $D(\xi) = |K| = L$ *a.e.* $\xi \in \mathbb{R}^2$, we only need to prove that $\hat{\chi}_K(j) = 0$ for $j \in \mathbb{Z} \setminus \{0\}$ since $\hat{\chi}_K(0) = |K| = L$. Assume that there exists $j_0 \in \mathbb{Z}^2 \setminus \{0\}$ such that $\hat{\chi}_K(j_0) \neq 0$. By assumption, (1.1.10) holds, i.e.

$$\begin{cases} (B^t)^m j_0 \cdot d_1 \in \frac{1}{6} + \mathbb{Z}, \\ (B^t)^m j_0 \cdot d_2 \in -\frac{1}{6} + \mathbb{Z}, \end{cases} \quad \text{or} \quad \begin{cases} (B^t)^m j_0 \cdot d_1 \in -\frac{1}{6} + \mathbb{Z}, \\ (B^t)^m j_0 \cdot d_2 \in \frac{1}{6} + \mathbb{Z}, \end{cases} \tag{1.3.1}$$

holds for any $m \geq 0$. Since two similar matrices have the same eigenvalues, Proposition 1.3.1 implies that for any expansive matrix $A \in M_2^{(2)}(\mathbb{Z})$, the matrix B has two distinct eigenvalues. Let $\lambda_0, \lambda_1,\ \lambda_0 \neq \lambda_1$, be the eigenvalues of matrix B and ψ_0, ψ_1 be the corresponding eigenvectors respectively, i.e. $B\psi_0 = \lambda_0\psi_0$ and $B\psi_1 = \lambda_1\psi_1$. Then $\mathbb{C}^2 = span\{\psi_0, \psi_1\}$ and thus $d_1 \in \mathbb{R}^2$ can be represented as a linear combination of ψ_0 and ψ_1, i.e. there exists some $c_0, c_1 \in \mathbb{C}$ such that $d_1 = c_0\psi_0 + c_1\psi_1$, and

$$B^m d_1 = c_0 B^m \psi_0 + c_1 B^m \psi_1 = c_0\lambda_0^m\psi_0 + c_1\lambda_1^m\psi_1 = \lambda_0^m c_0\psi_0 + \lambda_1^m c_1\psi_1. \tag{1.3.2}$$

From (1.3.2), we obtain

$$(B^t)^m j_0 \cdot d_1 = j_0 \cdot B^m d_1 = \overline{\lambda_0^m}\overline{c_0} j_0 \cdot \psi_0 + \overline{\lambda_1^m}\overline{c_1} j_0 \cdot \psi_1. \tag{1.3.3}$$

Let $a_0 = \overline{c_0} j_0 \cdot \psi_0$, $a_1 = \overline{c_1} j_0 \cdot \psi_1$ and $r_m = (B^t)^m j_0 \cdot d_1$. From (1.3.1) and (1.3.3), we have

$$r_m = a_0 \overline{\lambda_0^m} + a_1 \overline{\lambda_1^m} \in \frac{1}{6} + \mathbb{Z} \text{ or } -\frac{1}{6} + \mathbb{Z} \text{ for any } m \geq 0. \tag{1.3.4}$$

Thus for each $m \geq 0$, there exists some $\delta_m \in \{-1, 1\}$ and some $k_m \in \mathbb{Z}$ such that

$$r_m = \frac{\delta_m + 6k_m}{6}. \tag{1.3.5}$$

Using the representation of the characteristic polynomial of the matrix B, we obtain

$$\begin{aligned} \lambda_i^2 + b\lambda_i + c = 0 &\iff \lambda_i^2 = -b\lambda_i - c, \quad i = 0, 1 \\ &\iff \lambda_i^{2+m} = -b\lambda_i^{m+1} - c\lambda_i^m, \text{ for any } m \geq 0, \quad i = 0, 1. \end{aligned} \tag{1.3.6}$$

For any $m \geq 0$, it follows from (1.3.4), (1.3.5) and (1.3.6) that

$$r_{m+2} \;=\; -br_{m+1} - cr_m = \frac{-b\delta_{m+1} - c\delta_m}{6} - bk_{m+1} - ck_m \in \pm\frac{1}{6} + \mathbb{Z}. \tag{1.3.7}$$

Since $b, c, k_m, k_{m+1} \in \mathbb{Z}$, (1.3.7) implies that for any $m \geq 0$,

$$-b\delta_{m+1} = \pm 1 + c\delta_m \mod 6, \text{ where } \delta_m, \delta_{m+1} \in \{-1, 1\}. \tag{1.3.8}$$

Since two similar matrices have the same characteristic polynomial, it follows from Proposition 1.3.1 that the characteristic polynomial of matrix $B \in M_2^{(2)}(\mathbb{Z})$ can be expressed as $\lambda^2 - 2 = 0$ if $\det B = -2$ and $\lambda^2 + b\lambda + 2 = 0$, where $b \in \{0, -1, 1, -2, 2\}$ if $\det B = 2$.

Case 1: $\det A = -2$. In this case, $b = 0, c = -2$. (1.3.8) cannot hold for any $m \geq 0$ since $0 \neq \pm 1 - 2\delta_m \mod 6$ for for any $m \geq 0$ and $\delta_m \in \{-1, 1\}$.

Case 2: $\det A = 2$. In this case, $b \in \{0, -1, 1, -2, 2\}$ and $c = 2$. It follows from (1.3.8) that the possible values for b are -1 and 1. Suppose that $r_0 = \frac{\delta_0 + 6k_0}{6}$, $r_1 = \frac{\delta_1 + 6k_1}{6}$, where $\delta_0, \delta_1 \in \{-1, 1\}$ and $k_0, k_1 \in \mathbb{Z}$. We obtain, using (1.3.8), that

$$r_2 \in \pm\frac{1}{6} + \mathbb{Z} \iff -b\delta_1 - 2\delta_0 = \pm 1 \mod 6, \tag{1.3.9}$$

and

$$r_3 \in \pm\frac{1}{6} + \mathbb{Z} \iff (b^2 - 2)\delta_1 + 2b\delta_0 = \pm 1 \mod 6, \tag{1.3.10}$$

for some $\delta_0, \delta_1 \in \{-1, 1\}$.
If $b = 1$, (1.3.9) implies that $\delta_0 = -1, \delta_1 = 1$ or $\delta_0 = 1, \delta_1 = -1$. Substituting these values into the right hand side of (1.3.10), we get $(b^2 - 2)\delta_1 + 2b\delta_0 = 3$ or -3. Hence, $r_3 \notin \pm\frac{1}{6} + \mathbb{Z}$, which contradicts (1.3.4).
If $b = -1$, (1.3.9) implies that $\delta_0 = \delta_1 = 1$ or $\delta_0 = \delta_1 = -1$. Substituting these values into the right hand side of (1.3.10), we get $(b^2 - 2)\delta_1 + 2b\delta_0 = -3$ or 3. Hence, $r_3 \notin \pm\frac{1}{6} + \mathbb{Z}$, which again yields a contradiction. This proves our claim. □

From Theorem 1.2.1 and Theorem 1.3.2, we have the following corollary.

Corollary 1.3.3. *Let $A \in M_n^{(2)}(\mathbb{Z})$ be expansive and K be a self-affine tile satisfying the set equation $BK = (K + d_1) \bigcup (K + d_2)$ for some $d_1,\ d_2 \in \mathbb{R}^n$, where $n = 1, 2$. If K is an A-dilation scaling set, then K must be an A-dilation MRA scaling set.*

1.4 The dimension function of self-affine generalized scaling sets in $\mathbb{R}^n$

Unlike the one or two dimensional case, we do not know, in dimension $n \geq 3$, if the dimension function of a self-affine generalized scaling set must be constant. However, we will show that, when n is arbitrary, the dimension function is always bounded above by $2|K|$.

Theorem 1.4.1. *Let $A \in M_n^{(2)}(\mathbb{Z})$ be expansive and K be a self-affine tile satisfying the set equation $BK = (K + d_1) \bigcup (K + d_2)$ for some $d_1,\ d_2 \in \mathbb{R}^n$. If K is a generalized scaling set, then its dimension function satisfies $D(\xi) \leq 2|K|$.*

Proof. Assume that $|K| = L$. Let $T = \{j \in \mathbb{Z}^n \setminus \{0\} : \hat{\chi}_K(j) \neq 0\}$. Let

$$\Lambda = \{j \in \mathbb{Z}^n : (d_1 \cdot j, d_2 \cdot j) \in \mathbb{Z}^2\},$$

$$\Gamma_k = \{j \in \mathbb{Z}^n : (d_1 \cdot j, d_2 \cdot j) \in \left(\frac{k}{6}, -\frac{k}{6}\right) + \mathbb{Z}^2\}, k = 0, 1, 2, 3, 4, 5.$$

Then $\Gamma_0 = \Lambda$, $\Gamma_1 = j_1 + \Lambda$, $\Gamma_5 = -j_1 + \Lambda$, where $j_1 \in \Gamma_1$, and $T \subseteq \Gamma_1 \bigcup \Gamma_5$. Since Λ is a subgroup of $\mathbb{Z}^n$, there exists an integral matrix C such that $\Lambda = C\mathbb{Z}^n$. From the discussion above and that in section 2, we have, using (1.1.10) with $m = 0$, that

$$\begin{aligned} D(\xi) &= \sum_{j\in\mathbb{Z}^n} \hat{\chi}_K(j)\ e^{2\pi i j\cdot\xi} \\ &= \hat{\chi}_K(0) + \sum_{j\in\mathbb{Z}^n\setminus\{0\}} \hat{\chi}_K(j)\ e^{2\pi i j\cdot\xi} \\ &= L + \sum_{j\in\mathbb{Z}^n} \hat{\chi}_K(j_1+Cj)\ e^{2\pi i(j_1+Cj)\cdot\xi} + \sum_{j\in\mathbb{Z}^n} \hat{\chi}_K(-j_1-Cj)\ e^{-2\pi i(j_1+Cj)\cdot\xi} \\ &= L + e^{2\pi i j_1\cdot\xi}\sum_{j\in\mathbb{Z}^n} \hat{\chi}_K(j_1+Cj)\ e^{2\pi i Cj\cdot\xi} + e^{-2\pi i j_1\cdot\xi}\sum_{j\in\mathbb{Z}^n} \hat{\chi}_K(-j_1-Cj)\ e^{-2\pi i Cj\cdot\xi} \\ &= L + Re\{2e^{2\pi i j_1\cdot\xi}\sum_{j\in\mathbb{Z}^n} \hat{\chi}_K(j_1+Cj)\ e^{2\pi i Cj\cdot\xi}\}. \end{aligned} \tag{1.4.1}$$

The last equality is obtained from the fact that $\hat{\chi}_K(-\xi) = \overline{\hat{\chi}_K(\xi)}$, $\xi \in \mathbb{R}^n$. Let

$$\Lambda^* = \{x \in \mathbb{R}^n : Cj\cdot x \in \mathbb{Z},\ \text{for all } j \in \mathbb{Z}^n\},$$

and

$$S(\xi) = \sum_{j\in\mathbb{Z}^n} \hat{\chi}_K(j_1+Cj)\ e^{2\pi i Cj\cdot\xi}.$$

Then for any $x \in \Lambda^*$, $S(\xi + x) = S(\xi)$. The fact that $j_1 \in \Gamma_1$ implies that $6j_1 \in \Lambda$ and thus $e^{2\pi i 6 j_1\cdot x} = 1$, if $x \in \Lambda^*$, which implies that

$$e^{2\pi i j_1\cdot x} \in \{e^{\frac{k\pi i}{3}}, k = 0,1,2,3,4,5\}.$$

By the definition of Λ, $d_1, d_2 \in \Lambda^*$. Hence, $e^{\frac{\pi i}{3}}, e^{-\frac{\pi i}{3}} \in \{e^{2\pi i j_1\cdot x}, x \in \Lambda^*\}$ using the definition of Γ_1. Therefore, we have

$$\begin{aligned} \left\{e^{2\pi i j_1\cdot \ell x} : \ell \in \mathbb{Z}, x \in \Lambda^*\right\} &= \left\{e^{2\pi i j_1\cdot \ell x} : 0 \le \ell \le 5, \ell \in \mathbb{Z}, x \in \Lambda^*\right\} \\ &= \left\{e^{\frac{k\pi i}{3}}, 0 \le k \le 5, k \in \mathbb{Z}\right\}. \end{aligned} \tag{1.4.2}$$

Let $z = 2e^{2\pi i j_1\cdot\xi}\ S(\xi)$. Consider the functions $D(\xi + \ell x)$, where $x \in \Lambda^*$ and $\ell \in \{0,1,2,3,4,5\}$. By (1.4.1), we have

$$\begin{aligned} D(\xi+\ell x) &= L + Re\{2e^{2\pi i j_1\cdot(\xi+\ell x)}\ S(\xi+\ell x)\} = L + Re\{2e^{2\pi i j_1\cdot\xi}\ S(\xi)\ e^{2\pi i j_1\cdot \ell x}\} \\ &= L + Re\{z\ e^{2\pi i j_1\cdot \ell x}\}. \end{aligned}$$

From the definition of $D(\xi)$, $D(\xi) \geq 0$ a.e. $\xi \in \mathbb{R}^n$. Hence, we deduce from (1.4.2) that

$$L + Re\{z\, e^{\frac{\pi i \ell}{3}}\} \geq 0 \text{ for each } \ell \in \{0, 1, 2, 3, 4, 5\}. \tag{1.4.3}$$

(1.4.3) implies that $|Re(z)| \leq L$. Thus we obtain that $D(\xi) \leq L + |Re(z)| \leq 2L = 2|K|$ for $\xi \in \mathbb{R}^n$. □

From Theorem 1.4.1, we can get the following result.

Corollary 1.4.2. *Let $A \in M_n^{(2)}(\mathbb{Z})$ be expansive and K be a self-affine A-dilation scaling set satisfying the set equation $BK = (K + d_1) \bigcup (K + d_2)$ for some $d_1,\ d_2 \in \mathbb{R}^n$. If K is not an A-dilation MRA scaling set, then its dimension function $D(\xi)$ will take the values 0, 1 and 2.*

We know that the equality (1.1.10) is necessary for the dimension function of a self-affine generalized scaling set to be a constant. In higher dimension, it is not easy to find an example of a self-affine generalized A-dilation scaling set, where $A \in M_n^{(2)}(\mathbb{Z})$, satisfying (1.1.10). Below, we find, using Matlab, a self-affine tile K associated with an expansive matrix $B \in M_n^{(2)}(\mathbb{Z})$ in dimension 4 which satisfies (1.1.10). However, it is a pity that we can not prove whether or not this self-affine tile is an A-dilation generalized scaling set, where $A = B^t$. In particular, we do not know if the inclusion $K \subset BK$ holds or not.

Example 1.4.3. Let K be a self-affine set satisfying $BK = (K+d_1) \bigcup (K+d_2)$ associated with the matrix $B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -1 & -1 & -1 \end{pmatrix}$ and the digits $d_1 = \frac{1}{6}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $d_2 = \frac{1}{6}\begin{pmatrix} -1 \\ -1 \\ -2 \\ -2 \end{pmatrix}$.

For this example, it is easy for us to prove that the matrix $B \in M_n^{(2)}(\mathbb{Z})$ is expansive and the digit set $\mathcal{D} := \{6d_1, 6d_2\}$ is a complete set of coset representatives for the group $\mathbb{Z}^4/B\mathbb{Z}^4$. Let $K(B, \mathcal{D})$ be the set satisfying $BK = \bigcup_{d \in \mathcal{D}} K + d$. Thus, $|K(B, \mathcal{D})|$ is a positive integer by Theorem 1.1 in [38]. Therefore, $|K| = \frac{1}{6}|K(B, \mathcal{D})| > 0$ and K is a self-affine tile. If we take $j = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$, we can also check that for any $m \geq 0$,

$$(B^t)^m j \cdot d_1 \in \frac{1}{6} + \mathbb{Z} \text{ and } (B^t)^m j \cdot d_2 \in -\frac{1}{6} + \mathbb{Z}.$$

To summarize, if an A-dilation generalized scaling set is a self-affine tile, then its corresponding dimension function must be bounded in $\mathbb{R}^n$ and particularly, it is a constant in $\mathbb{R}$ and in $\mathbb{R}^2$. On the other hand, if we remove the self-affine condition, the corresponding dimension function can be arbitrary large, as some examples in ([12], section 5) illustrate.

Chapter 2

Self-Affine Scaling Sets in $\mathbb{R}^2$, Part I

The main goal of this chapter is to characterize all two dimensional self-affine scaling sets with a set of associated digits of the form $\{0, d\}$, where $d \in \mathbb{R}^2$, for an integral expansive matrix A with $|\det A| = 2$. We will deal with the case where 0 is not one of the digits in the next chapter.

2.1 Notations and Preliminary Results

Lemma 2.1.1 below characterizes a $\mathbb{Z}^n$-tiling set in the frequency domain and it will be used in later subsections.

Lemma 2.1.1. *A measurable set $K \subseteq \mathbb{R}^n$ is a $\mathbb{Z}^n$-tiling set if and only if $\hat{\chi}_K(j) = \delta_{j,0}$, $j \in \mathbb{Z}^n$.*

Proof. The fact that $K \subseteq \mathbb{R}^n$ is a $\mathbb{Z}^n$-tiling set is equivalent to $\sum_{j\in\mathbb{Z}^n} \chi_K(\cdot + j) = 1$ almost everywhere. Since

$$\begin{aligned} \int_{[0,1]^n} \sum_{s\in\mathbb{Z}^n} \chi_K(x+s)\, e^{-2\pi i x\cdot j}\, dx &= \sum_{s\in\mathbb{Z}^n} \int_{[0,1]^n+s} \chi_K(x)\, e^{-2\pi i x\cdot j}\, dx \\ &= \int_{\mathbb{R}^n} \chi_K(x)\, e^{-2\pi i x\cdot j}\, dx \\ &= \hat{\chi}_K(j), \end{aligned}$$

We have

$$\begin{aligned} \sum_{j\in\mathbb{Z}^n} \chi_K(\cdot + j) = 1 &\iff \delta_{j,0} = \int_{[0,1]^n} \sum_{s\in\mathbb{Z}^n} \chi_K(x+s)\, e^{-2\pi i x\cdot j}\, dx,\ j \in \mathbb{Z}^n \\ &\iff \hat{\chi}_K(j) = \delta_{j,0},\ j \in \mathbb{Z}^n, \end{aligned}$$

proving our claim. □

In this chapter, we limit our discussion to the self-affine sets K satisfying $BK = K \bigcup (K + d)$. Then condition (ii) in Corollary 0.0.7 holds automatically. And in that case, condition (iii) of Corollary 0.0.7 is equivalent to $\bigcup_{m\geq 1} B^m K = \mathbb{R}^n$ "everywhere", as we show next. For the sake of convenience, we will denote by $K(B, \mathcal{D})$ the unique compact set K, which satisfies $BK = K \bigcup (K + d)$, where $\mathcal{D} = \{0, d\} \subset \mathbb{R}^n$. We also let

$$\mathcal{D}_m := \Big\{ \sum_{j=0}^{m-1} B^j d_j : d_j \in \mathcal{D}, j \geqslant 0 \Big\} \text{ for } m \geqslant 1 \text{ and } \mathcal{D}_\infty = \bigcup_{m=1}^{\infty} \mathcal{D}_m.$$

Lemma 2.1.2. *Let B be an $n \times n$ integral expansive matrix and let K be the unique compact set satisfying $BK = \bigcup_{d\in\mathcal{D}} K + d$, where $0 \in \mathcal{D} \subset \mathbb{R}^n$ and card$(\mathcal{D}) = |\det B|$. Then the following four statements are equivalent.*

(a) K contains a neighborhood of 0.

(b) $\bigcup_{m\geq 1} B^m K = \mathbb{R}^n$, where the equality is understood to hold everywhere.

(c) $\lim_{m\to\infty} \chi_K(B^{-m}\xi) = 1$ for a.e. $\xi \in \mathbb{R}^n$.

(d) K is a $\mathcal{D}_\infty-$tiling set.

Proof. $(a) \Rightarrow (b)$ is clear since B is expansive. $(b) \Rightarrow (c)$ is obvious. Since we assume that $0 \in \mathcal{D}$, we have the inclusion $K \subseteq BK$. Thus (c) is equivalent to saying that $\bigcup_{m\geq 1} B^m K = \mathbb{R}^n$ almost everywhere. $(c) \Rightarrow (d)$: suppose that $\bigcup_{m\geq 1} B^m K = \mathbb{R}^n$ almost everywhere. Since $B^m K = K + \mathcal{D}_m$, then we have, using the definition of $\mathcal{D}_\infty$, that

$$\bigcup_{m\geq 1} B^m K = \bigcup_{m\geq 1} K + \mathcal{D}_m = K + \mathcal{D}_\infty = \mathbb{R}^n.$$

The sets $K + d,\ d \in \mathcal{D}_m$, are pairwise disjoint a.e. for any $m \geq 1$, since the number of elements in $\mathcal{D}_m$ is $|\det B|^m$ and $|K| > 0$. Therefore, K is a $\mathcal{D}_\infty$-tiling set. $(d) \Rightarrow (a)$: let V be an open and bounded set containing 0. Let

$$S = \Big\{ s \in \mathcal{D}_\infty : |V \bigcap (K + s)| > 0 \Big\}.$$

If $s \in S$, then $s \in \overline{V} - K$ and thus $S + K \subset (\overline{V} - K) + K$, a compact set. Since $|K| > 0$ and the sets $s + K, s \in \mathcal{D}_\infty$ are essentially disjoint, we have $\text{card}(S) \leq \frac{|(\overline{V}-K)+K|}{|K|} < \infty$. Since we assume that $0 \in \mathcal{D}$, we have the inclusion $\mathcal{D}_m \subseteq \mathcal{D}_{m+1}$ for any $m \geq 1$. This implies that there exists some $M \geq 1$ such that $S \subseteq \mathcal{D}_M$ and thus $V \subseteq B^M K$ a.e., which implies that $|V \bigcap (B^M K)^c| = 0$, where $(B^M K)^c$ denotes the complement of the set $B^M K$. Since $V \bigcap (B^M K)^c$ is open, we obtain that $V \subset B^M K$. This proves that K contains a neighborhood of 0. □

The following lemma which can be obtained from Chapter 4 concerns some properties of self-affine set $K(B, \mathcal{D})$ under the condition that the digit set $\mathcal{D} = \{0, d\} \subset \mathbb{Z}^n$ is a complete set of coset representatives for $\mathbb{Z}^n / B\mathbb{Z}^n$. This lemma also characterizes condition (c) of Lemma 2.1.2.

Lemma 2.1.3. *Let $\mathcal{D} = \{0, d\} \subset \mathbb{Z}^n$ be a complete set of coset representatives for $\mathbb{Z}^n / B\mathbb{Z}^n$ and $K(B, \mathcal{D})$ be a $\mathbb{Z}^n$-tiling set, then $K(B, \mathcal{D})$ contains a neighborhood of 0 if and only if $K \bigcap \mathbb{Z}^n = \{0\}$.*

Gabardo and Yu [25] gave a characterization for the integral points in $K(B, \mathcal{D})$.

Lemma 2.1.4. *Let $\mathcal{D} = \{0, d\} \subset \mathbb{Z}^n$ be a complete set of coset representatives for $\mathbb{Z}^n / B\mathbb{Z}^n$, then $K(B, \mathcal{D}) \bigcap \mathbb{Z}^n = -S$, where $S := \bigcup_{m \geq 1} \Big\{ k \in \mathbb{Z}^n : (I - B^m)k = \sum_{i=0}^{m-1} B^i d_i, d_i \in \mathcal{D} \Big\}$.*

Note that for the integral self-affine tiles case, if $K(B, \mathcal{D})$ is a $\mathbb{Z}^n$-tiling set, then $K(B, \mathcal{D})$ contains a neighborhood of 0 if and only if $S = \{0\}$, where the set S is defined in Lemma 2.1.4. Vince provided an algorithm for computing the set S in [51].

Our next result shows that, for a self-affine tile K, the property of being an MRA scaling set is equivalent to certain tiling properties of K.

Theorem 2.1.5. *Let $A \in M_n^{(2)}(\mathbb{Z})$ be expanding and K be a self-affine set associated with digit set $\mathcal{D} = \{0, d\} \subset \mathbb{R}^n$. Then K is an A-dilation MRA scaling set if and only if K is a $\mathbb{Z}^n$-tiling set and K is a $\mathcal{D}_\infty$-tiling set.*

Proof. Suppose K is an A-dilation MRA scaling set. This implies that K is a $\mathbb{Z}^n$-tiling set from Corollary 0.0.7. It is left to prove K is a $\mathcal{D}_\infty$-tiling set. Let $Q = BK \setminus K$. Then Q is an A-dilation MRA wavelet set, and, in particular, $\bigcup_{m\in\mathbb{Z}} B^m Q = \mathbb{R}^n$. Since $\bigcup_{m\in\mathbb{Z}} B^m Q = \bigcup_{m\in\mathbb{Z}} B^m K$ and $K \subset BK$, we have $\bigcup_{m\geq 1} B^m K = \mathbb{R}^n$, which is equivalent to saying that K is a $\mathcal{D}_\infty$-tiling set by Lemma 2.1.2.

Conversely, suppose K is a $\mathbb{Z}^n$-tiling set and also a $\mathcal{D}_\infty$-tiling set. Let $Q = BK\setminus K = K+d$. In order to prove that K is an A-dilation MRA scaling set, we only need to prove Q is an A-dilation wavelet set. Since K is a $\mathbb{Z}^n$-tiling set, it follows that

$$\bigcup_{\ell\in\mathbb{Z}^n} Q + \ell = \bigcup_{\ell\in\mathbb{Z}^n} K + d + \ell = \mathbb{R}^n, \tag{2.1.1}$$

and for $0 \neq \ell \in \mathbb{Z}^n$,

$$Q \bigcap (Q + \ell) = (K + d) \bigcap (K + d + \ell) = K \bigcap (K + \ell) = \emptyset. \tag{2.1.2}$$

We deduce from (2.1.1) and (2.1.2) that Q is a $\mathbb{Z}^n$-tiling set. It is obvious that the sets in the collection $\{B^m Q\}_{m\in\mathbb{Z}}$ are essentially disjoint. Since $K \subseteq BK$ and K is a $\mathcal{D}_\infty$-tiling set, we have

$$\bigcup_{m\in\mathbb{Z}} B^m Q = \bigcup_{m\in\mathbb{Z}} B^m (BK \setminus K) = \bigcup_{m\geq 1} B^m K = \bigcup_{m\geq 1} K + \mathcal{D}_m = K + \mathcal{D}_\infty = \mathbb{R}^n. \tag{2.1.3}$$

From (2.1.1), (2.1.2) and (2.1.3), it follows that Q is a wavelet set. □

Combining Theorem 2.1.5 and Lemma 2.1.2, we get the following corollary.

Corollary 2.1.6. *Let $A \in M_n^{(2)}(\mathbb{Z})$ be expanding. Then $K(B, \mathcal{D})$ is an A-dilation MRA scaling set if and only if $K(B, \mathcal{D})$ is a $\mathbb{Z}^n$-tiling set and $K(B, \mathcal{D})$ contains a neighborhood of 0.*

Two $n\times n$ integral matrices A and $\widetilde{A}$ are called integrally similar if there exists an $n\times n$ integral matrix P with $|\det P| = 1$ such that $P^{-1}AP = \widetilde{A}$. The following lemma shows how MRA scaling sets associated with integrally similar dilation matrices are related.

Lemma 2.1.7. *Let A be an expanding matrix in $M_n^{(2)}(\mathbb{Z})$ and suppose that A is integrally similar to $\widetilde{A}$ using the matrix S, i.e. $S^{-1}AS = \widetilde{A}$, where S is an integral matrix with $|\det S| = 1$. Then K is an A-dilation MRA scaling set if and only if PK is an $\widetilde{A}$-dilation MRA scaling set, where $P = S^t$.*

Proof. Clearly, we only need to prove one direction of the equivalence, so we will assume that K is an A-dilation MRA scaling set. Let $B = A^t$ and $\widetilde{B} = (\widetilde{A})^t$. By assumption, $P^{-1}\widetilde{B}P = B$. Let $\widetilde{K} = PK$. It is enough to prove that $\widetilde{K}$ is a $\mathbb{Z}^n$-tiling set and that the set $\widetilde{Q} := \widetilde{B}\widetilde{K} \setminus \widetilde{K}$ is a wavelet set corresponding to the dilation $\widetilde{A}$. Since the matrix P is an integral matrix with $|\det P| = 1$, it satisfies that $P\mathbb{Z}^n = \mathbb{Z}^n$, and, using the fact that K is a $\mathbb{Z}^n$-tiling set, we have

$$\begin{aligned} \sum_{j\in\mathbb{Z}^n} \chi_{\widetilde{K}}(\xi + j) &= \sum_{j\in\mathbb{Z}^n} \chi_{PK}(\xi + j) = \sum_{j\in\mathbb{Z}^n} \chi_K(P^{-1}\xi + P^{-1}j) \\ &= \sum_{j\in\mathbb{Z}^n} \chi_K(P^{-1}\xi + j) = 1 \text{ for } a.e.\ \xi \in \mathbb{R}^n. \end{aligned} \tag{2.1.4}$$

It remains to prove that $\widetilde{Q}$ is a wavelet set associated with $\widetilde{A}$. Let $Q = BK \setminus K$. Since Q is an MRA wavelet set associated with A, then Q satisfies the two conditions of Proposition 0.0.1. Furthermore, $\widetilde{Q} = \widetilde{B}PK \setminus PK = PBK \setminus PK = PQ$ and $\widetilde{Q}$ is a $\mathbb{Z}^n$-tiling set by a proof similar to the one in (2.1.4). By (i) of Proposition 0.0.1, we get

$$\bigcup_{m\in\mathbb{Z}} (\widetilde{B})^m \widetilde{Q} = \bigcup_{m\in\mathbb{Z}} (\widetilde{B})^m PQ = \bigcup_{m\in\mathbb{Z}} PB^m Q = P \bigcup_{m\in\mathbb{Z}} B^m Q = \mathbb{R}^n, \tag{2.1.5}$$

and, if $m \neq \ell$,

$$(\widetilde{B})^m \widetilde{Q} \bigcap (\widetilde{B})^\ell \widetilde{Q} = P(B^m Q \bigcap B^\ell Q) = \emptyset. \tag{2.1.6}$$

This shows that $\widetilde{Q}$ is an $\widetilde{A}$-dilation wavelet set. □

Before ending this section, we will state some results in number theory [46] that will be used in section 6.

Definition 2.1.8. Let $[a_0, a_1, \cdots, a_n \cdots]$ be the continued fraction associated with $\sqrt{d}$, where d is a positive integer and d is not a perfect square, i.e. $\sqrt{d} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$. Define

$$\begin{aligned} p_0 &= a_0 & q_0 &= 1 \\ p_1 &= a_0 a_1 + 1 & q_1 &= a_1 \\ p_k &= a_k p_{k-1} + p_{k-2} & q_k &= a_k q_{k-1} + q_{k-2} \ for\ k \geq 2. \end{aligned}$$

Then $\frac{p_k}{q_k}$ is called the k-th convergent of the simple continued fraction expansion of $\sqrt{d}$.

Theorem 2.1.9 ([46])**.** *Let d be a positive integer that is not perfect square. Let $\frac{p_k}{q_k}$ denote the k-th convergent of the simple continued fraction of $\sqrt{d}$, $k = 1, 2, 3 \cdots$, and let n be the period length of this continued fraction. Then, when n is even, the positive solutions of the diophantine equation $x^2 - dy^2 = 1$ are $x = p_{jn-1}$, $y = q_{jn-1}$, $j \geq 1$, and the diophantine equation $x^2 - dy^2 = -1$ has no solutions. When n is odd, the positive solutions of $x^2 - dy^2 = 1$ are $x = p_{2jn-1}$, $y = q_{2jn-1}$, $j \geq 1$, and the solutions of $x^2 - dy^2 = -1$ are $x = p_{(2j-1)n-1}$, $y = q_{(2j-1)n-1}$, $j \geq 1$.*

Note that in order to find all integer solutions of $x^2 - dy^2 = \pm 1$, where d is a positive integer that is not perfect square, it is enough to find all its positive integer solutions. In the following, we will use Theorem 2.1.9 to give the positive solutions of Pell's equation $x^2 - 2y^2 = \pm 1$.

Example 2.1.10. Consider the equations $x^2 - 2y^2 = \pm 1$.

Here $d = 2 > 0$ is not a perfect square integer. $\sqrt{2} = [1, \bar{2}]$. Thus $n = 1$ is odd and $a_0 = 1$, $a_i = 2$ for $i \geq 1$. So

$$\begin{aligned} p_0 &= 1 & q_0 &= 1 \\ p_1 &= a_1 a_0 + 1 = 3 & q_1 &= a_1 = 2 \\ p_2 &= a_2 p_1 + p_0 = 7 & q_2 &= a_2 q_1 + q_0 = 5 \\ p_3 &= a_3 p_2 + p_1 = 17 & q_3 &= a_3 q_2 + q_1 = 12 \\ &\vdots \end{aligned}$$

So the positive solutions of $x^2 - 2y^2 = 1$ are $x = p_{2j-1},\ y = q_{2j-1},\ j \geq 1$. and the positive solutions of $x^2 - 2y^2 = -1$ are $x = p_{2j-2},\ y = q_{2j-2},\ j \geq 1$.

By the discussion in Chapter 1, $K(B, \mathcal{D})$ is an MRA scaling set if it is an A-dilation scaling set. Therefore, the characterization of all self-affine scaling sets $K(B, \mathcal{D})$ is reduced to the characterization of all self-affine MRA scaling sets. From Corollary 0.0.7, the condition that $K(B, \mathcal{D})$ is a $\mathbb{Z}^n$-tiling set is necessary for $K(B, \mathcal{D})$ to be an A-dilation MRA scaling set. Thus in order to characterize all self-affine scaling sets $K(B, \mathcal{D})$, we just need to find all digit sets $\mathcal{D} = \{0, d\} \subset \mathbb{R}^n$ such that the self-affine tile $K(B, \mathcal{D})$ is a $\mathbb{Z}^n$-tiling set.

2.2 Self-affine MRA scaling sets in $\mathbb{R}$

We first consider here one dimensional problem, which is very easy to solve. And provides some insight into high dimensional case. In dimension one, $|\det A| = 2$ has two possibilities: $A = 2$ or $A = -2$.

If $A = B = 2$, K is a self-affine set satisfying $2K = K \bigcup (K + d)$. Then $K = [0, d]$ for $d > 0$ or $K = [d, 0]$ for $d < 0$. For K to be a $\mathbb{Z}$-tiling set, we need $d = 1$ or $d = -1$. However, the set $K = [0, 1]$ and $K = [-1, 0]$ do not contain a neighborhood of 0, there is no digit set $\mathcal{D} = \{0, d\}$ such that K is a scaling set by Corollary 2.1.6.

If $A = B = -2$, K satisfies the set valued equation $-2K = K \bigcup (K + d)$. Then $K = [-\frac{2}{3}d, \frac{1}{3}d]$ for $d > 0$ or $K = [\frac{1}{3}d, -\frac{2}{3}d]$ for $d < 0$. For K to be a $\mathbb{Z}$-tiling set, we need $d = 1$ or $d = -1$. Moreover, the sets $K = [-\frac{2}{3}, \frac{1}{3}]$ and $K = [-\frac{1}{3}, \frac{2}{3}]$ both contain a neighborhood of 0. Hence they are both MRA scaling sets by Corollary 2.1.6.

2.3 Self-affine MRA scaling sets in $\mathbb{R}^2$

In this section, our goal is to find all digit sets $\mathcal{D} = \{0, d\} \subset \mathbb{R}^2$ such that the self-affine tile $K(B, \mathcal{D})$ is a $\mathbb{Z}^2$-tiling set and, among these, which digit set $\mathcal{D}$ yields an MRA scaling set.

Lagarias and Wang [40] completely classified all integral expansive matrices $B \in M_2^{(2)}(\mathbb{Z})$ and they showed that there are exactly six integrally similar classes of such integral matrices. Representatives from each of these classes are given by $C_1,\ C_2,\ C_3,\ -C_3,\ C_4,\ -C_4$ respectively, where $C_1,\ C_2,\ C_3,\ C_4$ are defined in section 1 of Chapter 1. By Lemma 2.1.7 and Proposition 1.3.1, it follows that in order to find all digit sets $\mathcal{D} = \{0, d\} \subset \mathbb{R}^2$ such that the self-affine tile $K(B, \mathcal{D})$ is a $\mathbb{Z}^2$-tiling set or an MRA scaling set associated with a given integral expansive matrix $A \in M_2^{(2)}(\mathbb{Z})$, it suffices to solve the problem for each one of the six representative matrices listed above.

Let C be a 2×2 real invertible matrix commuting with B, i.e. $CB = BC$. Then, if $K = K(B, \mathcal{D})$, we have $BCK = CBK = CK \bigcup (CK + Cd)$. This implies that $CK(B, \mathcal{D})$ is also a self-affine tiling set associated with the digit set $C\mathcal{D} = \{0, Cd\}$. If $K(B, \mathcal{D})$ is a $\mathbb{Z}^2$-tiling set , then $|\det C| = 1$ is a necessary condition for the self-affine set $K(B, C\mathcal{D}) = CK(B, \mathcal{D})$ to be a $\mathbb{Z}^2$-tiling set.

It has been proved in [40] that the digit set $\mathcal{D}_0 = \{\binom{0}{0}, \binom{1}{0}\}$ is one set which makes the self-affine tile $K(B, \mathcal{D}_0)$ be a $\mathbb{Z}^2$-tiling set for each representative class. To deal with the case of a general digit set $\mathcal{D} = \{0, d\} \subset \mathbb{R}^2$, we will show that we can always express d in the form $d = C\binom{1}{0}$, where C is a real-invertible matrix commuting with B. Since $K(B, \mathcal{D}) = K(B, C\mathcal{D}_0) = CK(B, \mathcal{D}_0)$, the matrix C has to satisfy $|\det C| = 1$ if $K(B, \mathcal{D})$ is a $\mathbb{Z}^2$-tiling set.

Throughout the rest of this chapter, we will assume that C is a 2×2 real matrix with $|\det C| = 1$ which commutes with B and we will let $\mathcal{D}_0 = \{\binom{0}{0}, \binom{1}{0}\}$ and $\mathcal{D} = C\mathcal{D}_0 = \{\binom{0}{0}, C\binom{1}{0}\}$. From Corollary 2.1.6, we notice that, if $K(B, \mathcal{D})$ is $\mathbb{Z}^2$-tiling set, then $K(B, \mathcal{D})$ is an A-dilation MRA scaling set if and only if $K(B, \mathcal{D}_0)$ is an A-dilation MRA scaling set since $CK(B, \mathcal{D}_0)$ contains a neighborhood of 0 if and only if $K(B, \mathcal{D}_0)$ does. In the following, we will let A be equal to each of the six representative matrices above and, for each of them, we will determine all matrices C that commutes with $B = A^t$ such that $K(B, C\mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set. We will also, for each of six representative matrices, determine whether or not $K(B, \mathcal{D}_0)$ and thus $K(B, C\mathcal{D}_0)$ contains a neighborhood of 0. This will allow us to determine all matrices C as above such that $K(B, C\mathcal{D}_0)$ is an A-

dilation MRA scaling set. Notice that if C is an integral matrix with $|\det C| = 1$, then $K(B, \mathcal{D}) = CK(B, \mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set.

Our goal is to determine the digit d for which $\mathcal{D} = \{0, d\}$ yields a $\mathbb{Z}^2$-tiling set or a scaling set for each of the matrices $C_i, i = 1, \ldots, 6$. The general result we obtain is summarized in the following table.

	a $\mathbb{Z}^2$-tiling set	a scaling set
C_1	$d = \binom{a}{c}$ with $a, c \in \mathbb{Z}$ and $a^2 - 2c^2 = \pm 1$	N/A
C_2	$d = \binom{1}{0}$ and $\binom{-1}{0}$	$d = \binom{1}{0}$ and $\binom{-1}{0}$
C_3	$d = \binom{1}{0}$, $\binom{-1}{0}$, $\binom{0}{1}$ and $\binom{0}{-1}$	N/A
$-C_3$	$d = \binom{1}{0}$, $\binom{-1}{0}$, $\binom{0}{1}$ and $\binom{0}{-1}$	$d = \binom{1}{0}$, $\binom{-1}{0}$, $\binom{0}{1}$ and $\binom{0}{-1}$
C_4	$d = \binom{1}{0}$, $\binom{-1}{0}$, $\binom{1}{1/2}$ and $\binom{-1}{-1/2}$	$d = \binom{1}{0}$, $\binom{-1}{0}$, $\binom{1}{1/2}$ and $\binom{-1}{-1/2}$
$-C_4$	$d = \binom{1}{0}$, $\binom{-1}{0}$, $\binom{1}{1/2}$ and $\binom{-1}{-1/2}$	$d = \binom{1}{0}$, $\binom{-1}{0}$, $\binom{1}{1/2}$ and $\binom{-1}{-1/2}$

Letting $d = C\binom{1}{0}$, then $K(B, \mathcal{D})$ satisfies

$$BK = K \bigcup (K + d). \tag{2.3.1}$$

The set valued equation (2.3.1) is equivalent to the following equation for the characteristic functions of K and BK:

$$\chi_{BK}(x) = \chi_K(x) + \chi_K(x - d). \tag{2.3.2}$$

Taking the Fourier transform on both sides of (2.3.2), we get

$$2\hat{\chi}_K(B^t\xi) = \hat{\chi}_K(\xi) + e^{-2\pi i d\cdot\xi}\hat{\chi}_K(\xi) = (1 + e^{-2\pi i d\cdot\xi})\hat{\chi}_K(\xi). \tag{2.3.3}$$

Letting $B^t\xi = \xi'$, this yields

$$\hat{\chi}_K(\xi) \quad = \tfrac{1+e^{-2\pi i d\cdot(B^t)^{-1}\xi}}{2}\hat{\chi}_K((B^t)^{-1}\xi). \tag{2.3.4}$$

Using (2.3.4) iteratively, we can express $\hat{\chi}_K$ as the infinite product

$$\hat{\chi}_K(\xi) = \hat{\chi}_K(0)\prod_{j=1}^{\infty}\frac{1 + e^{-2\pi i d\cdot(B^t)^{-j}\xi}}{2}. \tag{2.3.5}$$

Note that $\frac{1+e^{-2\pi id\cdot(B^t)^{-j}\xi}}{2} = 1 + \frac{e^{-2\pi id\cdot(B^t)^{-j}\xi}-1}{2}$, and

$$|\frac{e^{-2\pi id\cdot(B^t)^{-j}\xi}-1}{2}| = |\sin(\pi d\cdot(B^t)^{-j}\xi)| \leq |\pi d\cdot(B^t)^{-j}\xi| \leq \pi|\xi|\|d\|\|(B^t)^{-j}\|. \quad (2.3.6)$$

Since B is expansive, there exists $j_0 \in \mathbb{N}$ such that $\|(B^t)^{-j}\| < \rho$ for some $0 < \rho < 1$ and for any $j > j_0$. Let $a_j(\xi) = \frac{e^{-2\pi id\cdot(B^t)^{-j}\xi}-1}{2}$. Then for each fixed $\xi \in \mathbb{R}^n$, we have, using (2.3.6), that

$$\sum_{j=1}^{\infty}|a_j(\xi)| \leq \sum_{j=1}^{\infty}\pi|\xi|\|d\|\|(B^t)^{-j}\| \leq \pi|\xi|\|d\|(\sum_{j=1}^{j_0}\|(B^t)^{-j}\| + \sum_{j=j_0+1}^{\infty}\rho^{-j}). \quad (2.3.7)$$

(2.3.7) shows that the series $\sum_{j=1}^{\infty}|a_j(\xi)|$ converges for any given $\xi \in \mathbb{R}^n$. Since the infinite product $\prod_{j=1}^{\infty}1+|\frac{e^{-2\pi id\cdot(B^t)^{-j}\xi}-1}{2}| = \prod_{j=1}^{\infty}1+|a_j(\xi)|$ converges if and only if the series $\sum_{j=1}^{\infty}|a_j(\xi)|$ converges, then (2.3.5) is well defined. Suppose that there is a matrix C as above such that $K(B,\mathcal{D}) = CK(B,\mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set, then $\hat{\chi}_K(0) = |K| = 1$. By Lemma 2.1.1 and formula (2.3.5), we have, letting $e_1 = \binom{1}{0}$,

$$\hat{\chi}_K(m) = \prod_{j=1}^{\infty}\frac{1+e^{-2\pi iCe_1\cdot(B^t)^{-j}m}}{2} = \delta_{m,0},\ m \in \mathbb{Z}^2. \quad (2.3.8)$$

Since C commutes with B, then (2.3.8) can be written as

$$\hat{\chi}_K(m) = \prod_{j=1}^{\infty}\frac{1+e^{-2\pi ie_1\cdot(B^t)^{-j}C^tm}}{2} = \delta_{m,0},\ m \in \mathbb{Z}^2. \quad (2.3.9)$$

Since $\frac{1+e^{-2\pi ie_1\cdot(B^t)^{-j}C^tm}}{2} = 1 + a_j(m)$ and $\lim_{j\to\infty}|a_j(m)| = 0$ for each $m \in \mathbb{Z}^2$, then there must exist some $j \geq 1$ such that $\frac{1+e^{-2\pi ie_1\cdot(B^t)^{-j}C^tm}}{2} = 0$ if $\hat{\chi}_K(m) = 0$ for any given $0 \neq m \in \mathbb{Z}^2$. Hence, From (2.3.9), it follows that, for $0 \neq m \in \mathbb{Z}^2$,

$$\hat{\chi}_K(m) = 0 \iff e_1\cdot(B^t)^{-j}C^tm \in \frac{1}{2} + \mathbb{Z} \text{ for some } j \geq 1. \quad (2.3.10)$$

2.4 The case $A = C_1$

Theorem 2.4.1. *Let $A = C_1 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$. Then, $K(B, \mathcal{D})$ is a $\mathbb{Z}^2$-tiling set if and only if $\mathcal{D} = \{0, d\} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ c \end{pmatrix}\}$, where $a, c \in \mathbb{Z}$ and $a^2 - 2c^2 = \pm 1$. Among these digit sets, no one makes $K(B, \mathcal{D})$ be an A-dilation MRA scaling set.*

Proof. If $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, then $B = A^t = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ and a matrix C commuting with B has the general form $C = \begin{pmatrix} a & 2c \\ c & a \end{pmatrix}$. In this case, $K(B, \mathcal{D}_0)$ is the unit square,

$$K(B, \mathcal{D}_0) = [0, 1] \times [0, 1].$$

It is obvious that $K(B, \mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set, but it does not contain a neighborhood of 0. So $K(B, \mathcal{D}_0)$ is not an A-dilation MRA scaling set by Corollary 2.1.6. The fact that $|\det C| = 1$ is equivalent to $a^2 - 2c^2 = \pm 1$. This last equation is called Pell's equation and its integral solutions can be obtained from Example 2.1.10. As we discussed before, all integral matrices C that we obtain from these solutions have the property that $K(B, \mathcal{D}) = CK(B, \mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set but not an A-dilation MRA scaling set. We now turn our attention to the case where C is not an integral matrix. Note that

$$B^{-1} = \begin{pmatrix} 0 & 1 \\ 1/2 & 0 \end{pmatrix}, \ B^{-2} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = \frac{1}{2} I.$$

If j is even, i.e. $j = 2\ell$, then $(B^t)^{-j} = (B^t)^{-2\ell} = (\frac{1}{2})^\ell I$.
If j is odd, i.e. $j = 2\ell + 1$, then $(B^t)^{-j} = (B^t)^{-2\ell-1} = (\frac{1}{2})^\ell \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$.
Let $m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \mathbb{Z}^2 \setminus \{0\}$, by (2.3.10), we get

$$\begin{aligned} \hat{\chi}_K(m) = 0 \iff & \begin{cases} (\frac{1}{2})^\ell (am_1 + cm_2) \in \frac{1}{2} + \mathbb{Z} \text{ for some } \ell \geq 1, \text{ or,} \\ (\frac{1}{2})^{\ell+1} (2cm_1 + am_2) \in \frac{1}{2} + \mathbb{Z} \text{ for some } \ell \geq 0 \end{cases} \\ \iff & \begin{cases} (\frac{1}{2})^\ell (am_1 + cm_2) \in \frac{1}{2} + \mathbb{Z} \text{ for some } \ell \geq 1, \text{ or,} \\ (\frac{1}{2})^\ell (2cm_1 + am_2) \in \frac{1}{2} + \mathbb{Z} \text{ for some } \ell \geq 1. \end{cases} \end{aligned} \tag{2.4.1}$$

Taking $m = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $m = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in (2.4.1), we obtain that a and c belong to $\mathbb{Q}$. So C is a rational matrix. Let $a = \frac{k}{s}, c = \frac{p}{q}$, where $(k, s) = 1$, $(p, q) = 1$ and $k,\ s,\ p,\ q \in \mathbb{Z}$. Then

$$a^2 - 2c^2 = \pm 1 \iff \frac{k^2 \pm s^2}{s^2} = \frac{2p^2}{q^2}. \tag{2.4.2}$$

Note that

$$(k,s)=1 \Rightarrow (k^2,s^2)=1 \Rightarrow (k^2 \pm s^2, s^2)=1.$$

If $2|q$, then $(p,q)=1 \Rightarrow (p^2,\frac{q^2}{2})=1$. From formula (2.4.2), we get that $k^2 \pm s^2 = p^2$ and $s^2=\frac{q^2}{2}$, which implies that $|\frac{q}{s}|=\sqrt{2}$, a contradiction since $q,\ s \in \mathbb{Z}$. So $2 \nmid q$. In this case, $(2,q)=1$ and $(p,q)=1$ implies that $(2p^2,q^2)=1$. By (2.4.2), we get $k^2 \pm s^2 = 2p^2$ and $s^2=q^2$, which implies that $q=\pm s$. The fact that $2 \nmid q$ implies that both q and s are odd. Since $2|k^2 \pm s^2$ and s is odd, k must be odd. Replacing p by $-p$ if $q=-s$, we can thus write $a=\frac{k}{s},\ c=\frac{p}{s}$, where $k,\ s \in 2\mathbb{Z}+1,\ p \in \mathbb{Z},\ s \neq 1$ and $(k,s)=(p,s)=1$. Therefore, (2.4.1) can be written as

$$\hat{\chi}_K(m)=0 \Longleftrightarrow \begin{cases} (\frac{1}{2})^\ell(\frac{k}{s}m_1+\frac{p}{s}m_2) \in \frac{1}{2}+\mathbb{Z} \text{ for some } \ell \geq 1, \text{ or,} \\ (\frac{1}{2})^\ell(\frac{2p}{s}m_1+\frac{k}{s}m_2) \in \frac{1}{2}+\mathbb{Z} \text{ for some } \ell \geq 1. \end{cases} \tag{2.4.3}$$

Taking $m_1=2,\ m_2=0$, we obtain that $(\frac{1}{2})^\ell(\frac{k}{s}m_1+\frac{p}{s}m_2)=(\frac{1}{2})^\ell(\frac{2k}{s}) \notin \frac{1}{2}+\mathbb{Z}$ for any $\ell \geq 1$ and $(\frac{1}{2})^\ell(\frac{2p}{s}m_1+\frac{k}{s}m_2)=(\frac{1}{2})^\ell(\frac{4p}{s}) \notin \frac{1}{2}+\mathbb{Z}$ for any $\ell \geq 1$. This shows that we can not find a non-integral matrix C such that $CK(B,\mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set, i.e. there is no non-integral digit set $\mathcal{D}=\{0,d\}$ such that $K(B,\mathcal{D})$ is a $\mathbb{Z}^2$-tiling set (resp. an A-dilation MRA scaling set) for the matrices A integrally similar to C_1. □

2.5 The case $A=C_2$

Theorem 2.5.1. *Let $A=C_2=\begin{pmatrix}0 & -1\\ 2 & 0\end{pmatrix}$. Then the following three statements are equivalent:*

(a) $K(B,\mathcal{D})$ is a $\mathbb{Z}^2$-tiling set,

(b) $K(B,\mathcal{D})$ is an A-dilation MRA scaling set,

(c) $\mathcal{D}=\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\}$ or $\mathcal{D}=\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}-1\\0\end{pmatrix}\}$.

Proof. If $A=\begin{pmatrix}0 & -1\\ 2 & 0\end{pmatrix}$, then $B=A^t=\begin{pmatrix}0 & 2\\ -1 & 0\end{pmatrix}$ and a matrix C commuting with B has the general form $C=\begin{pmatrix}a & -2c\\ c & a\end{pmatrix}$. In this case, $K(B,\mathcal{D}_0)$ is the unit square,

$$K(B,\mathcal{D}_0)=[-\frac{2}{3},\frac{1}{3}]\times[-\frac{1}{3},\frac{2}{3}].$$

It is obvious that $K(B,\mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set which contains a neighborhood of 0. So $K(B,\mathcal{D}_0)$ is an A-dilation MRA scaling set by Corollary 2.1.6. Since $|\det C| = 1$ is equivalent to $a^2+2c^2=1$, the only integral solutions are $a=\pm1, c=0$. Then the only integral matrices C that we obtain from these solutions are $\pm I$. As we discussed before, these integral matrices $C=\pm I$ have the property that $K(B,\mathcal{D})=CK(B,\mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set and also an A-dilation MRA scaling set. It remains to prove that there is no non-integral matrix C such that $K(B,\mathcal{D})=CK(B,\mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set and thus an A-dilation MRA scaling set. Note that

$$B^{-1}=\begin{pmatrix}0 & -1\\ \frac{1}{2} & 0\end{pmatrix},\ B^{-2}=\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}=-\frac{1}{2}I.$$

If j is even, i.e. $j=2\ell$, then $(B^t)^{-j}=(B^t)^{-2\ell}=(-\frac{1}{2})^\ell I$.
If j is odd, i.e. $j=2\ell+1$, then $(B^t)^{-j}=(B^t)^{-2\ell-1}=(-\frac{1}{2})^\ell\begin{pmatrix}0 & \frac{1}{2}\\ -1 & 0\end{pmatrix}$.
Let $m=\begin{pmatrix}m_1\\ m_2\end{pmatrix}\in\mathbb{Z}^2\setminus\{0\}$. Using (2.3.10), it follows that

$$\hat{\chi}_K(m)=0\Longleftrightarrow\begin{cases}(-\frac{1}{2})^\ell(am_1+cm_2)\in\frac{1}{2}+\mathbb{Z}\text{ for some }\ell\geq1,\ \text{or,}\\ (-\frac{1}{2})^{\ell+1}(2cm_1-am_2)\in\frac{1}{2}+\mathbb{Z}\text{ for some }\ell\geq0.\end{cases}\tag{2.5.1}$$

Taking $m=\begin{pmatrix}1\\0\end{pmatrix}$ and $m=\begin{pmatrix}1\\1\end{pmatrix}$ in (2.5.1), we obtain that a and c belong to $\mathbb{Q}$. So C is a rational matrix. Let $a=\frac{k}{s}, c=\frac{p}{q}$, where $(k,s)=1,\ (p,q)=1$ and $k,\ s,\ p,\ q\in\mathbb{Z}$. Then

$$a^2+2c^2=\pm1\Longleftrightarrow\frac{\pm s^2-k^2}{s^2}=\frac{2p^2}{q^2}.\tag{2.5.2}$$

Note that

$$(k,s)=1\Rightarrow(k^2,s^2)=1\Rightarrow(\pm s^2-k^2,s^2)=1.$$

If $2|q$, then $(p,q)=1\Rightarrow(p^2,\frac{q^2}{2})=1$. From Formula (2.5.2), we get that $\pm s^2-k^2=p^2$ and $s^2=\frac{q^2}{2}$, which implies that $|\frac{q}{s}|=\sqrt{2}$, a contradiction since $q,\ s\in\mathbb{Z}$. So $2\nmid q$. In this case, $(2,q)=1$ and $(p,q)=1$ implies that $(2p^2,q^2)=1$. By (2.5.2), we get $\pm s^2-k^2=2p^2$ and $s^2=q^2$, which implies that $q=\pm s$. The fact that $2\nmid q$ implies both q and s are odd. Since $2|\pm s^2-k^2$ and s is odd, k must be odd. Replacing p by $-p$ if $q=-s$, we can thus

write $a = \frac{k}{s},\ c = \frac{p}{s}$, where $k,\ s \in 2\mathbb{Z}+1,\ p \in \mathbb{Z},\ s \neq 1$ and $(k,s) = (p,s) = 1$. Therefore, (2.5.1) is reduces to

$$\hat{\chi}_K(m) = 0 \Longleftrightarrow \begin{cases} (-\frac{1}{2})^\ell(\frac{k}{s}m_1 + \frac{p}{s}m_2) \in \frac{1}{2} + \mathbb{Z} \text{ for some } \ell \geq 1, \text{ or,} \\ (-\frac{1}{2})^\ell(\frac{2p}{s}m_1 - \frac{k}{s}m_2) \in \frac{1}{2} + \mathbb{Z} \text{ for some } \ell \geq 1. \end{cases} \tag{2.5.3}$$

Taking $m_1 = 2,\ m_2 = 0$, we obtain that $(-\frac{1}{2})^\ell(\frac{k}{s}m_1 + \frac{p}{s}m_2) = (-\frac{1}{2})^\ell(\frac{2k}{s}) \notin \frac{1}{2} + \mathbb{Z}$ for any $\ell \geq 1$ and $(-\frac{1}{2})^\ell(\frac{2p}{s}m_1 + \frac{k}{s}m_2) = (-\frac{1}{2})^\ell(\frac{4p}{s}) \notin \frac{1}{2} + \mathbb{Z}$ for any $\ell \geq 1$. This shows that we can not find a non-integral matrix C such that $CK(B, \mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set (and thus an A-dilation MRA scaling set). □

2.6 The case $A = C_3$

Theorem 2.6.1. *Let $A = C_3 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Then, $K(B, \mathcal{D})$ is a $\mathbb{Z}^2$-tiling set if and only if $\mathcal{D} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$, $\mathcal{D} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\}$, $\mathcal{D} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$, $\mathcal{D} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\}$. However, $K(B, \mathcal{D})$ is never an A-dilation MRA scaling set.*

Proof. If $A = C_3 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, then $B = A^t = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Under this case, $K(B, \mathcal{D}_0)$ is the well-known "twin dragon", which was proved to be a $\mathbb{Z}^2$-tiling set in [29]. It was showed [24] that $K(B, \mathcal{D}_0) \bigcap \mathbb{Z}^2 = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$. Thus $K(B, \mathcal{D}_0)$ does not contain a neighborhood of 0 from Lemma 2.1.3. So $K(B, \mathcal{D}_0)$ is not an A-dilation MRA scaling set by Corollary 2.1.6. A matrix C commuting with B must have the form $C = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and the condition $|\det C| = 1$ is equivalent to $a^2 + b^2 = 1$. The all integral solutions are $a = \pm 1, b = 0$ or $a = 0, b = \pm 1$. Arguments similar to those used for the case where $A = C_1$ show that $K(B, \mathcal{D}) = CK(B, \mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set but not an A-dilation MRA scaling set for these integral matrices C. We will now prove that there is no non-integral matrix C such that $K(B, \mathcal{D}) = CK(B, \mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set and thus an A-dilation MRA scaling set. We have

$$B^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, B^{-2} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, B^{-3} = \frac{1}{4}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, B^{-4} = -\frac{1}{4}I.$$

If $j = 4\ell$, then $(B^t)^{-j} = (B^t)^{-4\ell} = (-\frac{1}{4})^\ell I$.

If $j = 4\ell + 1$, then $(B^t)^{-j} = (B^t)^{-4\ell-1} = (-\frac{1}{4})^\ell \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

If $j = 4\ell + 2$, then $(B^t)^{-j} = (B^t)^{-4\ell-2} = (-\frac{1}{4})^\ell \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
If $j = 4\ell + 3$, then $(B^t)^{-j} = (B^t)^{-4\ell-3} = (-\frac{1}{4})^\ell \frac{1}{4}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$.
Suppose there is a non-integral matrix C such that $K(B, \mathcal{D}) = CK(B, \mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set. Let $m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \mathbb{Z}^2 \setminus \{0\}$. Then using (2.3.10), we have

$$\hat{\chi}_K(m) = 0 \iff \begin{cases} (-\frac{1}{4})^\ell (am_1 - bm_2) \in \frac{1}{2} + \mathbb{Z} \text{ for some } \ell \geq 1, \text{ or,} \\ (-\frac{1}{4})^\ell \frac{1}{2}[(a+b)m_1 + (a-b)m_2] \in \frac{1}{2} + \mathbb{Z} \text{ for some } \ell \geq 0, \text{ or,} \\ (-\frac{1}{4})^\ell \frac{1}{2}(bm_1 + am_2) \in \frac{1}{2} + \mathbb{Z} \text{ for some } \ell \geq 0, \text{ or,} \\ (-\frac{1}{4})^\ell \frac{1}{4}[(b-a)m_1 + (a+b)m_2] \in \frac{1}{2} + \mathbb{Z} \text{ for some } \ell \geq 0. \end{cases} \tag{2.6.1}$$

Write $a = \cos\theta, b = \sin\theta$ for $\theta \in [0, 2\pi)$. Take $m_1 = 1,\ m_2 = 0$. By Formula (2.6.1), we obtain

$$\begin{aligned} \hat{\chi}_K(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = 0 \iff & \begin{cases} (-\frac{1}{4})^\ell \cos\theta \in \frac{1}{2} + \mathbb{Z} \text{ for some } \ell \geq 1, \text{ or,} \\ (-\frac{1}{4})^\ell \frac{1}{2}(\cos\theta + \sin\theta) \in \frac{1}{2} + \mathbb{Z} \text{ for some } \ell \geq 0, \text{ or,} \\ (-\frac{1}{4})^\ell \frac{1}{2}\sin\theta \in \frac{1}{2} + \mathbb{Z} \text{ for some } \ell \geq 0, \text{ or,} \\ (-\frac{1}{4})^\ell \frac{1}{4}(\sin\theta - \cos\theta) \in \frac{1}{2} + \mathbb{Z} \text{ for some } \ell \geq 0 \end{cases} \\ \iff & \begin{cases} \cos\theta \in (-4)^\ell(\frac{1}{2} + \mathbb{Z}) \text{ for some } \ell \geq 1, \text{ or,} \\ \cos\theta + \sin\theta \in (-4)^\ell(2\mathbb{Z} + 1) \text{ for some } \ell \geq 0, \text{ or,} \\ \sin\theta \in (-4)^\ell(2\mathbb{Z} + 1) \text{ for some } \ell \geq 0, \text{ or,} \\ \sin\theta - \cos\theta \in (-4)^\ell(4\mathbb{Z} + 2) \ \textit{for some } \ell \geq 0. \end{cases} \end{aligned}$$

Since $|(-4)^\ell(\frac{1}{2} + k)| \geq 2$ for any $\ell \geq 1$ and $k \in \mathbb{Z}$, $\cos\theta \notin (-4)^\ell(\frac{1}{2} + \mathbb{Z})$ for any $\ell \geq 1$ and the 1^{st} case can not happen.
If $\cos\theta + \sin\theta \in (-4)^\ell(2\mathbb{Z} + 1)$ for some $\ell \geq 0$, then $\ell = 0$ and $\cos\theta + \sin\theta = \pm 1$, which implies that $\cos\theta = 1$ and $\sin\theta = 0$ or $\cos\theta = 0$ and $\sin\theta = 1$. This yields the integral matrices $C = I$ or $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
If $\sin\theta \in (-4)^\ell(2\mathbb{Z}+1)$ for some $\ell \geq 0$, then the only possibilities are $\ell = 0$ and $\sin\theta = \pm 1$. In this case, the matrix C is still an integral matrix.
Since $|(-4)^\ell(4k + 2)| \geq 2$ for any $\ell \geq 0$ and $k \in \mathbb{Z}$, $\sin\theta - \cos\theta \notin (-4)^\ell(4\mathbb{Z} + 2)$ for any $\ell \geq 0$ and the last case does not have any solution. Hence, there is no non-integral digit set $\mathcal{D}$ such that the self-affine tile $K(B, \mathcal{D})$ is a $\mathbb{Z}^2$-tiling set. □

2.7 The case $A = -C_3$

Theorem 2.7.1. *Let* $A = -C_3 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$. *Then the following three statements are equivalent:*

(a) $K(B, \mathcal{D})$ *is a* $\mathbb{Z}^2$*-tiling set,*

(b) $K(B, \mathcal{D})$ *is an* A*-dilation MRA scaling set,*

(c) $\mathcal{D}$ *is one of the following four digit sets:* $\mathcal{D} = \{\binom{0}{0}, \binom{1}{0}\}$, $\mathcal{D} = \{\binom{0}{0}, \binom{-1}{0}\}$, $\mathcal{D} = \{\binom{0}{0}, \binom{0}{1}\}$ *or* $\mathcal{D} = \{\binom{0}{0}, \binom{0}{-1}\}$.

Proof. If $A = -C_3 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$, then $B = A^t = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$. In this case, $K(B, \mathcal{D}_0)$ is the well-known "twin dragon", which was proved to be a $\mathbb{Z}^2$-tiling set and to satisfy the condition $K(B, \mathcal{D}_0) \bigcap \mathbb{Z}^2 = \{\binom{0}{0}\}$ in [24]. Thus $K(B, \mathcal{D}_0)$ contains a neighborhood of 0 by Lemma 2.1.3. So $K(B, \mathcal{D}_0)$ is an A-dilation MRA scaling set by Corollary 2.1.6. A matrix C commuting with B has the form $C = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $|\det C| = 1$ if and only if $a^2 + b^2 = 1$. The integer solutions of $a^2 + b^2 = 1$ are $a = \pm 1, b = 0$ and $a = 0, b = \pm 1$. By arguments similar to those used in the case where $A = C_2$, $K(B, \mathcal{D}) = CK(B, \mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set and also an A-dilation MRA scaling set for these integral matrices C. Since the right-hand side of (2.3.10) holds for some matrix B if and only if it holds for $-B$, the same proof as for the case where $A = C_3$ shows that there is no non-integral digit set $\mathcal{D}$ such that the self-affine tile $K(B, \mathcal{D})$ is a $\mathbb{Z}^2$-tiling set (or an A-dilation MRA scaling set). □

2.8 The case $A = C_4$

As in the preceding cases, in order to find all digit sets in the form of $\mathcal{D} = \{0, d\} \subset \mathbb{R}^2$ such that $K(B, \mathcal{D})$ is a $\mathbb{Z}^2$-tiling set or an A-dilation MRA scaling set, we need to find a suitable expression for the entries of the matrices $(B^t)^{-j}, j \geq 1$ and matrices C which commute with B and satisfy $|\det C| = 1$. The lemma below gives a useful form for the upper entries of the negative powers of B^t, where $B = A^t = C_4^t = \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$.

Lemma 2.8.1. *Let* $(B^t)^{-j} = \begin{pmatrix} a_1^{(j)} & a_2^{(j)} \\ * & * \end{pmatrix}$. *Then* $a_1^{(j)} = \frac{\ell_1^{(j)}}{2^j}$, $a_2^{(j)} = \frac{\ell_2^{(j)}}{2^j}$, *where* $\ell_1^{(j)}$, $\ell_2^{(j)} \in 2\mathbb{Z}+1$ *for* $j \geq 1$ *with* $\frac{\ell_1^{(j)}-\ell_2^{(j)}}{2} \in 2\mathbb{Z}+1$, $\frac{\ell_1^{(j)}+\ell_2^{(j)}}{4} \in 2\mathbb{Z}+1$ *for* $j \geq 3$.

Proof. Since $(B^t)^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 0 \end{pmatrix}$ and $(B^t)^{-2} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$, we have

$$(a_1^{(1)}, a_2^{(1)}) = (\frac{1}{2}, \frac{1}{2}), \quad (a_1^{(2)}, a_2^{(2)}) = (-\frac{1}{4}, \frac{1}{4}) = (-\frac{1}{2^2}, \frac{1}{2^2}),$$

which shows that $\ell_1^{(j)}$, $\ell_2^{(j)} \in 2\mathbb{Z}+1$ for $j = 1, 2$. For $j = 3$, $(B^t)^{-3} = \begin{pmatrix} -\frac{3}{8} & -\frac{1}{8} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$, which gives

$$(a_1^{(3)}, a_2^{(3)}) = (-\frac{3}{8}, -\frac{1}{8}) = (-\frac{3}{2^3}, -\frac{1}{2^3}). \tag{2.8.1}$$

From (2.8.1), $\ell_1^{(3)} = -3$, $\ell_2^{(3)} = -1$, $\frac{\ell_1^{(3)}-\ell_2^{(3)}}{2} = -1$ and $\frac{\ell_1^{(3)}+\ell_2^{(3)}}{4} = -1$ are all odd, so our claim holds for $j = 3$. Suppose the statement is true for $3 \leq j \leq m$, then for $j = m+1$, we get

$$\begin{aligned} (a_1^{(m+1)}, a_2^{(m+1)}) &= (a_1^{(m)}, a_2^{(m)}) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 0 \end{pmatrix} \\ &= (\frac{\ell_1^{(m)}}{2^m}, \frac{\ell_2^{(m)}}{2^m}) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 0 \end{pmatrix} \\ &= (\frac{\ell_1^{(m)} - 2\ell_2^{(m)}}{2^{m+1}}, \frac{\ell_1^{(m)}}{2^{m+1}}). \end{aligned} \tag{2.8.2}$$

By (2.8.2) and our assumption, we get $\ell_1^{(m+1)} = \ell_1^{(m)} - 2\ell_2^{(m)}$, $\ell_2^{(m+1)} = \ell_1^{(m)}$, showing that $\ell_1^{(m+1)}$, $\ell_2^{(m+1)}$ are both odd. Moreover,

$$\frac{\ell_1^{(m+1)} - \ell_2^{(m+1)}}{2} = -\ell_2^{(m)} \in 2\mathbb{Z}+1, \ \frac{\ell_1^{(m+1)} + \ell_2^{(m+1)}}{4} = \frac{2(\ell_1^{(m)} - \ell_2^{(m)})}{4} = \frac{\ell_1^{(m)} - \ell_2^{(m)}}{2} \in 2\mathbb{Z}+1,$$

which proves our claim. □

Remark 2.8.2. By the proof of Lemma 2.8.1, we get $\ell_1^{(j)} = \ell_1^{(j-1)} - 2\ell_2^{(j-1)}$, $\ell_2^{(j)} = \ell_1^{(j-1)}$ for any $j > 1$ and thus we have $\ell_1^{(j)} - \ell_2^{(j)} = -2\ell_2^{(j-1)}$ for any $j > 1$ and $\ell_1^{(j)} + \ell_2^{(j)} = -4\ell_2^{(j-2)}$ for any $j > 2$.

Define $D_{jj'} := \ell_1^{(j+1)}\ell_2^{(j'-1)} - \ell_1^{(j')}\ell_2^{(j)}$ for $j \geq 1, j' > 1$ and $j' \neq j+1$. The following lemmas give some information about $D_{jj'}$, which will be used later.

Lemma 2.8.3. $D_{jj} = -2^{j-1}$.

Proof. By the definition of $D_{jj'}$, if $j = j'$, then $j \geq 2$ since $j' > 1$. If $j = 2$, then

$$D_{22} = \ell_1^{(3)}\ell_2^{(1)} - \ell_1^{(2)}\ell_2^{(2)} = -2 = -2^{2-1}.$$

So the statement is true for $j = 2$. By Remark 2.8.2, we have, for any $j > 1$, that

$$\begin{aligned} D_{jj} &= \ell_1^{(j+1)}\ell_2^{(j-1)} - \ell_1^{(j)}\ell_2^{(j)} \\ &= (\ell_1^{(j)} - 2\ell_2^{(j)})\ell_2^{(j-1)} - (\ell_1^{(j-1)} - 2\ell_2^{(j-1)})\ell_2^{(j)} \\ &= \ell_1^{(j)}\ell_2^{(j-1)} - \ell_1^{(j-1)}\ell_2^{(j)}. \end{aligned} \tag{2.8.3}$$

Assume that $D_{jj} = -2^{j-1}$. Then by (2.8.3) and our assumption, we obtain

$$\begin{aligned} D_{j+1,j+1} &= \ell_1^{(j+1)}\ell_2^{(j)} - \ell_1^{(j)}\ell_2^{(j+1)} \\ &= \ell_1^{(j)}\ell_2^{(j)} - 2\ell_2^{(j)}\ell_2^{(j)} - \ell_1^{(j-1)}\ell_2^{(j+1)} + 2\ell_2^{(j-1)}\ell_2^{(j+1)} \\ &= 2\ell_1^{(j)}\ell_2^{(j-1)} - 2\ell_1^{(j-1)}\ell_2^{(j)} \\ &= 2D_{jj} = -2^j. \end{aligned}$$

□

Lemma 2.8.4. $D_{j+k,j} \in (2\mathbb{Z}+1)2^{j-1}$ *for any* $j \geq 3$ *and* $k \geq 1$.

Proof. By the definition of $D_{j,j'}$ and Remark 2.8.2, if $j \geq 3$ and $k \geq 1$, we have

$$\begin{aligned} D_{j+k,j} &= \ell_1^{(j+k+1)}\ell_2^{(j-1)} - \ell_1^{(j)}\ell_2^{(j+k)} \\ &= \ell_1^{(j+k)}\ell_2^{(j-1)} - 2\ell_2^{(j+k)}\ell_2^{(j-1)} - \ell_1^{(j-1)}\ell_2^{(j+k)} + 2\ell_2^{(j-1)}\ell_2^{(j+k)} \\ &= \ell_1^{(j+k)}\ell_2^{(j-1)} - \ell_1^{(j-1)}\ell_2^{(j+k)} \\ &= \ell_1^{(j+k-1)}\ell_2^{(j-1)} - 2\ell_2^{(j+k-1)}\ell_2^{(j-1)} - \ell_1^{(j-2)}\ell_2^{(j+k)} + 2\ell_2^{(j-2)}\ell_2^{(j+k)} \\ &= -2\ell_1^{(j+k-2)}\ell_2^{(j-1)} + 2\ell_1^{(j+k-1)}\ell_2^{(j-2)}. \end{aligned} \tag{2.8.4}$$

If $k = 1$, it follows from (2.8.4) and Lemma 2.8.3 that

$$\begin{aligned} D_{j+1,j} &= -2\ell_1^{(j-1)}\ell_2^{(j-1)} + 2\ell_1^{(j)}\ell_2^{(j-2)} \\ &= 2D_{j-1,j-1} \\ &= 2\cdot(-2^{j-2}) \\ &= -2^{j-1} \in (2\mathbb{Z}+1)2^{j-1}. \end{aligned}$$

So the statement is true for $k = 1$. Suppose $D_{j+k,j} \in (2\mathbb{Z}+1)2^{j-1}$ for all $1 \le k \le s$. Then by (2.8.4) and our assumption, we obtain

$$\begin{aligned} D_{j+s+1,j} &= -2\ell_1^{(j+s-1)}\ell_2^{(j-1)} + 2\ell_1^{(j+s)}\ell_2^{(j-2)} \\ &= -2\ell_1^{(j+s-2)}\ell_2^{(j-1)} + 4\ell_2^{(j+s-2)}\ell_2^{(j-1)} + 2\ell_1^{(j+s-1)}\ell_2^{(j-2)} - 4\ell_2^{(j+s-1)}\ell_2^{(j-2)} \\ &= -2\ell_1^{(j+s-2)}\ell_2^{(j-1)} + 2\ell_1^{(j+s-1)}\ell_2^{(j-2)} + 4\ell_1^{(j+s-3)}\ell_2^{(j-1)} - 4\ell_1^{(j+s-2)}\ell_2^{(j-2)} \\ &= D_{j+s,j} - 2D_{j+s-1,j} \in (2\mathbb{Z}+1)2^{j-1}, \end{aligned}$$

as claimed. □

Lemma 2.8.5. *$D_{j,j+k} \in (2\mathbb{Z}+1)2^j$ for $k \ge 2$ and $j > 1$.*

Proof. By the definition of $D_{j,j'}$ and Remark 2.8.2, if $k \ge 2$ and $j > 1$, we have

$$\begin{aligned} D_{j,j+k} &= \ell_1^{(j+1)}\ell_2^{(j+k-1)} - \ell_1^{(j+k)}\ell_2^{(j)} \\ &= \ell_1^{(j)}\ell_2^{(j+k-1)} - 2\ell_2^{(j)}\ell_2^{(j+k-1)} - \ell_1^{(j+k-1)}\ell_2^{(j)} + 2\ell_2^{(j+k-1)}\ell_2^{(j)} \\ &= \ell_1^{(j)}\ell_2^{(j+k-1)} - \ell_1^{(j+k-1)}\ell_2^{(j)} \\ &= \ell_1^{(j-1)}\ell_2^{(j+k-1)} - 2\ell_2^{(j-1)}\ell_2^{(j+k-1)} - \ell_1^{(j+k-2)}\ell_2^{(j)} + 2\ell_2^{(j+k-2)}\ell_2^{(j)} \\ &= -2\ell_2^{(j+k-1)}\ell_2^{(j-1)} + 2\ell_2^{(j+k-2)}\ell_2^{(j)} \\ &= -2\ell_1^{(j+k-2)}\ell_2^{(j-1)} + 2\ell_1^{(j+k-3)}\ell_2^{(j)}. \end{aligned} \tag{2.8.5}$$

If $k = 2$, we have, using (2.8.5), (2.8.3) and Lemma 2.8.3, that

$$\begin{aligned} D_{j,j+2} &= -2\ell_1^{(j)}\ell_2^{(j-1)} + 2\ell_1^{(j-1)}\ell_2^{(j)} \\ &= -2D_{j,j} = 2^j \in (2\mathbb{Z}+1)2^j. \end{aligned}$$

So the statement is true for $k = 2$. Suppose $D_{j,j+k} \in (2\mathbb{Z}+1)2^j$ for all $2 \le k \le s$. Then by (2.8.5) and our assumption, we get

$$\begin{aligned} D_{j,j+s+1} &= -2\ell_1^{(j+s-1)}\ell_2^{(j-1)} + 2\ell_1^{(j+s-2)}\ell_2^{(j)} \\ &= -2\ell_1^{(j+s-2)}\ell_2^{(j-1)} + 4\ell_2^{(j+s-2)}\ell_2^{(j-1)} + 2\ell_1^{(j+s-3)}\ell_2^{(j)} - 4\ell_2^{(j+s-3)}\ell_2^{(j)} \\ &= -2\ell_1^{(j+s-2)}\ell_2^{(j-1)} + 2\ell_1^{(j+s-3)}\ell_2^{(j)} + 4\ell_1^{(j+s-3)}\ell_2^{(j-1)} - 4\ell_1^{(j+s-4)}\ell_2^{(j)} \\ &= D_{j,j+s} - 2D_{j,j+s-1} \in (2\mathbb{Z}+1)2^j. \end{aligned}$$

□

If $A = C_4 = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$, then $B = A^t = \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$. It is easily checked that the general form of a matrix C commuting with B is $C = \begin{pmatrix} a & -2c \\ c & a-c \end{pmatrix}$. Lemma 2.8.6 below shows that when such a matrix C also has the property that $CK(B, \mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set, its entries must all be rational numbers.

Lemma 2.8.6. *Suppose $K(B, \mathcal{D}) = CK(B, \mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set, where $C = \begin{pmatrix} a & -2c \\ c & a-c \end{pmatrix}$. Then, the matrix C is a rational matrix.*

Proof. By the representation for matrices $(B^t)^{-j}$ obtained in Lemma 2.8.1, it follows that

$$\begin{aligned} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (B^t)^{-j} C^t \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} &= (1,0) \begin{pmatrix} a_1^{(j)} & a_2^{(j)} \\ * & * \end{pmatrix} \begin{pmatrix} a & c \\ -2c & a-c \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \\ &= \frac{\ell_1^{(j)}}{2^j}(am_1 + cm_2) + \frac{\ell_2^{(j)}}{2^j}[-2cm_1 + (a-c)m_2]. \end{aligned} \tag{2.8.6}$$

Taking $m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively, we deduce from (2.8.6) and (2.3.10) that there is some $j \geq 1$ and $j' \geq 1$ such that

$$\frac{a\ell_1^{(j)}}{2^j} - \frac{2c\ell_2^{(j)}}{2^j} \in \frac{1}{2} + \mathbb{Z}, \tag{2.8.7}$$

$$\frac{c\ell_1^{(j')}}{2^{j'}} + \frac{(a-c)\ell_2^{(j')}}{2^{j'}} \in \frac{1}{2} + \mathbb{Z}. \tag{2.8.8}$$

We can rewrite (2.8.7) and (2.8.8) in matrix form as

$$\begin{pmatrix} \ell_1^{(j)} & -2\ell_2^{(j)} \\ \ell_2^{(j')} & \ell_1^{(j')} - \ell_2^{(j')} \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{2.8.9}$$

where $v_1 \in 2^{j-1}(2\mathbb{Z}+1)$ and $v_2 \in 2^{j'-1}(2\mathbb{Z}+1)$. If $j' = 1$, then $\ell_1^{(j')} = \ell_2^{(j')} = 1$. In that case (2.8.9) yields $a = v_2 \in \mathbb{Q}$, which implies that $c \in \mathbb{Q}$ since $\ell_2^{(j)} \neq 0$. If $j' > 1$, then $\ell_2^{(j')} = \ell_1^{(j'-1)}$, $\ell_1^{(j')} - \ell_2^{(j')} = -2\ell_2^{(j'-1)}$ by Remark 2.8.2. From (2.8.7) and (2.8.8), we get $j' \neq j+1$, since, otherwise, both $\frac{a\ell_1^{(j)} - 2c\ell_2^{(j)}}{2^{j+1}} \in \frac{1}{2} + \mathbb{Z}$ and $\frac{a\ell_1^{(j)} - 2c\ell_2^{(j)}}{2^j} \in \frac{1}{2} + \mathbb{Z}$ would hold, which is impossible. Next we will prove that

$$\det \begin{pmatrix} \ell_1^{(j)} & -2\ell_2^{(j)} \\ \ell_2^{(j')} & \ell_1^{(j')} - \ell_2^{(j')} \end{pmatrix} = \det \begin{pmatrix} \ell_1^{(j)} & -2\ell_2^{(j)} \\ \ell_1^{(j'-1)} & -2\ell_2^{(j'-1)} \end{pmatrix} \neq 0 \text{ for } j' \neq j+1 \text{ and } j' > 1.$$

Assume If it were not the case, we would have $\frac{\ell_2^{(j)}}{\ell_1^{(j)}} = \frac{\ell_2^{(j'-1)}}{\ell_1^{(j'-1)}}$. Note that $j \neq 1$, since otherwise, we would get $j' = 2$, i.e. $j' = j+1$. Similarly, $j' \neq 2$. So $j > 1$ and $j' > 2$. By Remark 2.8.2,

$$\det \begin{pmatrix} \ell_1^{(j)} & -2\ell_2^{(j)} \\ \ell_1^{(j'-1)} & -2\ell_2^{(j'-1)} \end{pmatrix} = 0 \Longleftrightarrow \frac{\ell_1^{(j)}}{\ell_1^{(j-1)}} = \frac{\ell_1^{(j')}}{\ell_1^{(j'-1)}} \text{ for some } j,\ j' > 1 \text{ and } j \neq j'.$$

WLOG, we assume $j < j'$, i.e. $j' = j + k$ for some $k \geq 1$. Let $q = \frac{\ell_1^{(j)}}{\ell_1^{(j-1)}}$, then

$$\begin{aligned} \frac{\ell_1^{(j+1)}}{\ell_1^{(j-1)}} &= \frac{\ell_1^{(j)} - 2\ell_2^{(j)}}{\ell_1^{(j-1)}} = \frac{\ell_1^{(j)} - 2\ell_1^{(j-1)}}{\ell_1^{(j-1)}} = q - 2, \\ \frac{\ell_1^{(j+1)}}{\ell_1^{(j)}} &= \frac{\ell_1^{(j+1)}}{\ell_1^{(j-1)}} \cdot \frac{\ell_1^{(j-1)}}{\ell_1^{(j)}} = \frac{q-2}{q}. \end{aligned} \tag{2.8.10}$$

Let $a_i = \frac{\ell_1^{(j+i)}}{\ell_1^{(j+i-1)}}$, $i \geq 0$. Then $a_0 = q$ and by an argument similar to (2.8.10), we get

$$\begin{aligned} a_1 &= \frac{\ell_1^{(j+1)}}{\ell_1^{(j)}} = \frac{a_0 - 2}{a_0} \\ a_2 &= \frac{\ell_1^{(j+2)}}{\ell_1^{(j+1)}} = \frac{a_1 - 2}{a_1} \\ &\vdots \\ a_k &= \frac{\ell_1^{(j+k)}}{\ell_1^{(j+k-1)}} = \frac{a_{k-1} - 2}{a_{k-1}}. \end{aligned}$$

Now we will prove that for any $k \geq 1$, $a_k = \frac{aq+b}{cq+d}$ for some $a, c \in 2\mathbb{Z}+1$ and $b, d \in \mathbb{Z}$ (with a, b, c, d depending on k) by induction. For $k = 1$, $a_1 = \frac{a_0-2}{a_0} = \frac{q-2}{q}$. So our claim is true for $k = 1$. Suppose that $a_k = \frac{aq+b}{cq+d}$ for some $a, c \in 2\mathbb{Z}+1$ and $b, d \in \mathbb{Z}$. Then we have

$$a_{k+1} = \frac{a_k - 2}{a_k} = \frac{\frac{aq+b}{cq+d} - 2}{\frac{aq+b}{cq+d}} = \frac{(a-2c)q + (b-2d)}{aq+b},$$

where $a - 2c, a \in 2\mathbb{Z}+1$ and $b - 2d, b \in \mathbb{Z}$ since $a, c \in 2\mathbb{Z}+1$ and $b, d \in \mathbb{Z}$ by our assumption, which proves our claim.

Next we will prove that for any $k \geq 1$, $a_k = a_0 \Longleftrightarrow q^2 - q + 2 = 0$. If $k = 1$, then

$$a_1 = a_0 \Longleftrightarrow \frac{q-2}{q} = q \Longleftrightarrow q^2 - q + 2 = 0.$$

So the statement is true for $k = 1$. Suppose that $a_k = \frac{aq+b}{cq+d}$, where $a, c \in 2\mathbb{Z}+1$ and $b, d \in \mathbb{Z}$, satisfies the statement. Then we have

$$\begin{aligned} a_k = a_0 &\iff \frac{aq+b}{cq+d} = q \iff cq^2 + (d-a)q - b = 0 \\ &\iff q^2 + \frac{d-a}{c}q - \frac{b}{c} = 0. \end{aligned} \tag{2.8.11}$$

By assumption,

$$a_k = a_0 \iff q^2 - q + 2 = 0. \tag{2.8.12}$$

It follows from (2.8.11) and (2.8.12) that $q^2 + \frac{d-a}{c}q - \frac{b}{c} = q^2 - q + 2$, thus $\frac{d-a}{c} = -1$ and $-\frac{b}{c} = 2$. By the relationship between a_k and a_{k+1}, we obtain

$$a_{k+1} = \frac{a_k - 2}{a_k} = \frac{\frac{aq+b}{cq+d} - 2}{\frac{aq+b}{cq+d}} = \frac{(a-2c)q + (b-2d)}{aq+b},$$

and

$$a_{k+1} = a_0 \iff \frac{(a-2c)q + (b-2d)}{aq+b} = q \iff aq^2 + (b-a+2c)q + (2d-b) = 0.$$

Since $\frac{d-a}{c} = -1$ and $-\frac{b}{c} = 2$, we obtain that $b - a + 2c = -a$ and $2d - b = 2a$. Thus we have $aq^2 + (b-a+2c)q + (2d-b) = 0 \iff aq^2 - aq + 2a = 0 \iff q^2 - q + 2 = 0$. This shows that $a_k = a_0 \iff q^2 - q + 2 = 0$ for any $k \geq 1$. Since this last equation has no rational solution, this contradicts the fact that $q = \frac{\ell_1^{(j)}}{\ell_1^{(j-1)}} \in \mathbb{Q}$. So we can not find $j' > j$ such that $\frac{\ell_1^{(j-1)}}{\ell_1^{(j)}} = \frac{\ell_1^{(j'-1)}}{\ell_1^{(j')}}$. Therefore the rational matrix $\begin{pmatrix} \ell_1^{(j)} & -2\ell_2^{(j)} \\ \ell_1^{(j')} & -2\ell_2^{(j'-1)} \end{pmatrix}$ is invertible. This shows that $a,\ c \in \mathbb{Q}$ by (2.8.9), i.e. the matrix $C = \begin{pmatrix} a & -2c \\ c & a-c \end{pmatrix}$ is a rational matrix. □

If $K(B, \mathcal{D}) = CK(B, \mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set, where C is defined as in Lemma 2.8.6, then $|\det C| = 1$, i.e. $a(a-c) + 2c^2 = (a - \frac{c}{2})^2 + \frac{7}{4}c^2 = 1$ and $a, c \in \mathbb{Q}$ by Lemma 2.8.6. The only integral solutions of $(a - \frac{c}{2})^2 + \frac{7}{4}c^2 = 1$ are $a = \pm 1, c = 0$. Now we consider the rational and non-integral solutions of $(a - \frac{c}{2})^2 + \frac{7}{4}c^2 = 1$. Let

$$a - \frac{c}{2} = \cos\theta,\ \frac{\sqrt{7}}{2}c = \sin\theta.$$

The fact that $a,\ c \in \mathbb{Q}$ implies that $\cos\theta \in \mathbb{Q}$. Let $\cos\theta = \frac{p}{q}$ where $p,\ q \in \mathbb{Z}$ and $(p,q) = 1$. Since $c = \frac{2\sin\theta}{\sqrt{7}} \in \mathbb{Q}$, we can let $\sin\theta = \frac{k}{w}\sqrt{7}$, where $k,\ w \in \mathbb{Z}$ and $(k,w) = 1$. Then we have

$$\sin^2\theta + \cos^2\theta = 1 \Longleftrightarrow \frac{7k^2}{w^2} + \frac{p^2}{q^2} = 1 \Longleftrightarrow \frac{7k^2}{w^2} = \frac{q^2 - p^2}{q^2}. \tag{2.8.13}$$

Note that

$$(p,q) = 1 \Rightarrow (p^2, q^2) = 1 \Rightarrow (q^2 - p^2, q^2) = 1.$$

If $7|w$, $(k,w) = 1 \Rightarrow (k^2, \frac{w^2}{7}) = 1$. By (2.8.13), we get $q^2 - p^2 = k^2$ and $q^2 = \frac{w^2}{7}$. This last equality implies that $|\frac{w}{q}| = \sqrt{7}$, which contradicts the fact that $q, w \in \mathbb{Z}$. So $7 \nmid w$ and thus $(7k^2, w^2) = 1$. Using (2.8.13) again, we get $q^2 - p^2 = 7k^2$ and $q = \pm w$. Replacing p by $-p$ if $q = -w$, thus we have

$$c = \frac{2}{\sqrt{7}} \cdot \frac{k}{w}\sqrt{7} = \frac{2k}{w},\ a = \frac{c}{2} + \frac{p}{w} = \frac{k+p}{w},$$

and

$$C = \begin{pmatrix} (k+p)/w & -4k/w \\ 2k/w & (p-k)/w \end{pmatrix}, \tag{2.8.14}$$

where $p, k, w \in \mathbb{Z}, (p,w) = (k,w) = 1$, $w \neq 1$ (otherwise, $a, c \in \mathbb{Z}$) and $p^2 + 7k^2 = w^2$.

Lemma 2.8.7. *Suppose $K(B,\mathcal{D}) = K(B, C\mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set, where C is defined as in (2.8.14). Then $w = 2^s t$ for some $s \geq 2$, $t \in 2\mathbb{Z}+1$, $p = u + 2v$, $k = 2v - u$ for some $u, v \in 2\mathbb{Z}+1$.*

Proof. By Lemma 2.8.1 and (2.8.14), we have

$$\begin{aligned} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (B^t)^{-j} C^t \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} &= (1,0) \begin{pmatrix} \frac{\ell_1^{(j)}}{2^j} & \frac{\ell_2^{(j)}}{2^j} \\ * & * \end{pmatrix} \begin{pmatrix} \frac{k+p}{w} & \frac{2k}{w} \\ -\frac{4k}{w} & \frac{p-k}{w} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \\ &= \frac{\ell_1^{(j)}}{2^j}\left[\frac{(k+p)m_1 + 2km_2}{w}\right] + \frac{\ell_2^{(j)}}{2^j}\left[\frac{-4km_1 + (p-k)m_2}{w}\right] \end{aligned} \tag{2.8.15}$$

Note that for any $x \in \mathbb{Z}$, $x^2 \mod 4 = 0$ or 1 depending on whether x is even or odd. Considering the equation $p^2 + 7k^2 = w^2$ modulo $4\mathbb{Z}$, it follows that both $p,\ k$ are odd if

w is even and that k is even and p is odd if w is odd.

Case 1: $w \neq 1$ is odd, k is even, p is odd.

Take $m = \binom{1}{0}$. By (2.3.10) and (2.8.15), there exists some $j \geq 1$ such that

$$\binom{1}{0} \cdot (B^t)^{-j} C^t \binom{1}{0} = \frac{\ell_1^{(j)}}{2^j} \frac{k+p}{w} - \frac{\ell_2^{(j)}}{2^j} \frac{4k}{w} \in \frac{1}{2} + \mathbb{Z}.$$

Since $(k+p)\ell_1^{(j)} - 4k\ell_2^{(j)} \in 2\mathbb{Z}+1$ for any $j \geq 1$, we cannot have $\frac{\ell_1^{(j)}}{2^j}\frac{k+p}{w} - \frac{\ell_2^{(j)}}{2^j}\frac{4k}{w} \in \frac{1}{2} + \mathbb{Z}$ if $j > 1$. So the only possibility is $j = 1$. Then we should have

$$\frac{\ell_1^{(1)}}{2} \frac{k+p}{w} - \frac{\ell_2^{(1)}}{2} \frac{4k}{w} = \frac{p-3k}{2w} \in \frac{1}{2} + \mathbb{Z}. \tag{2.8.16}$$

(2.8.16) implies that $w|(p-3k)$. Since

$$p^2 + 7k^2 = w^2 \Longleftrightarrow (p-3k)(p+3k) = w^2 - 16k^2,$$

it follows that $w|(w^2 - 16k^2)$ and thus $w|k^2$, contradicting $(w,k) = 1$.

Case 2: w is even and both $p,\ k$ are odd.

Let $w = 2^s t$, where $t \in 2\mathbb{Z}+1$, and $s \geq 1$. Take $m = \binom{0}{2^{s-1}}$. It follows from (2.3.10) and (2.8.15) that there is an integer $j \geq 1$ satisfying

$$\begin{aligned} \binom{1}{0} \cdot (B^t)^{-j} C^t \binom{0}{2^{s-1}} &= \frac{\ell_1^{(j)}}{2^j} \frac{2^s k}{2^s t} + \frac{\ell_2^{(j)}}{2^j} \frac{(p-k)2^{s-1}}{2^s t} \\ &= \frac{[\ell_1^{(j)} k + \frac{p-k}{2} \ell_2^{(j)}]}{2^j t} \in \frac{1}{2} + \mathbb{Z}. \end{aligned} \tag{2.8.17}$$

If $j = 1$, then $\ell_1^{(1)} = \ell_2^{(1)} = 1$. By (2.8.17), we have

$$\frac{[\ell_1^{(1)} k + \frac{p-k}{2} \ell_2^{(1)}]}{2t} = \frac{p+k}{4t} \in \frac{1}{2} + \mathbb{Z}.$$

Hence, there is $u \in 2\mathbb{Z}+1$ such that $p + k = 2u$. Since p and k are both odd, we have $p - k = 2^\ell v$ for some $\ell \geq 2$ and $v \in 2\mathbb{Z}+1$. Hence,

$$p^2 + 7k^2 = w^2 \Longleftrightarrow p^2 - k^2 = w^2 - 8k^2 \Longleftrightarrow 2^{\ell+1} uv = 2^{2s} t^2 - 8k^2,$$

and the fact that $8|2^{\ell+1}$ implies that $8|2^{2s}$, we obtain that $s \geq 2$. Therefore, $2^{\ell-2}uv = 2^{2s-3}t^2 - k^2$. Since $2^{2s-3}t^2 - k^2 \in 2\mathbb{Z}+1$ if $s \geq 2$, we deduce that $\ell = 2$, i.e. that $p-k = 4v$. Similarly, taking $m = \binom{2^{s-2}}{0}$, we obtain from (2.3.10), (2.8.15) and the results above that there is some $j \geq 1$ such that

$$\binom{1}{0} \cdot (B^t)^{-j} C^t \binom{2^{s-2}}{0} = \frac{[\ell_1^{(j)} \frac{p+k}{4} - \ell_2^{(j)} k]}{2^j t} = \frac{u\ell_1^{(j)} - 2\ell_2^{(j)} k}{2^{j+1} t} \in \frac{1}{2} + \mathbb{Z}. \tag{2.8.18}$$

However, (2.8.18) cannot hold for any $j \geq 1$, since $u\ell_1^{(j)} - 2\ell_2^{(j)} k$ is odd. So (2.8.17) has to hold for some $j > 1$, i.e. $\frac{[\ell_1^{(j)} k + \frac{p-k}{2}\ell_2^{(j)}]}{2^j t} \in \frac{1}{2} + \mathbb{Z}$ for some $j > 1$, which implies that $\ell_1^{(j)} k + \frac{p-k}{2}\ell_2^{(j)} \in 2\mathbb{Z}$. Since $\ell_1^{(j)}, \ell_2^{(j)}$ and k, p are all odd, it follows that $p - k = 2u$ for some $u \in 2\mathbb{Z}+1$ and thus $p + k = 2^\ell v$ for some $\ell \geq 2,\ v \in 2\mathbb{Z}+1$. Then, as before, we have

$$p^2 + 7k^2 = w^2 \Longleftrightarrow p^2 - k^2 = w^2 - 8k^2 \Longleftrightarrow 2^{\ell+1}uv = 2^{2s}t^2 - 8k^2,$$

implying that $s \geq 2$. Let now $m = \binom{2^{s-2}}{0}$. From (2.3.10) and the results above, there is an integer $j \geq 1$ such that

$$\binom{1}{0} \cdot (B^t)^{-j} C^t \binom{2^{s-2}}{0} = \frac{[\ell_1^{(j)} \frac{p+k}{4} - \ell_2^{(j)} k]}{2^j t} \in \frac{1}{2} + \mathbb{Z}. \tag{2.8.19}$$

If $j = 1$, then from (2.8.19), we have $\frac{p-3k}{4t} \in 2\mathbb{Z}+1$, which implies that there exists $\tilde{u} \in 2\mathbb{Z}+1$ such that $p - 3k = 4\tilde{u}$. This forces $p + 3k = 2\tilde{v}$ for some $\tilde{v} \in 2\mathbb{Z}+1$ since $p = \tilde{v} + 2\tilde{u}$ in that case and p is odd. On the other hand,

$$\begin{aligned} p^2 + 7k^2 = w^2 &\iff (p+3k)(p-3k) = 2^{2s}t^2 - 16k^2 \\ &\iff 8\tilde{u}\tilde{v} = 2^{2s}t^2 - 16k^2. \end{aligned} \tag{2.8.20}$$

(2.8.20) implies that $\tilde{u}\tilde{v} = 2^{2s-3}t^2 - 2k^2 \in 2\mathbb{Z}$, which is a contradiction. So $j \neq 1$ and thus $\frac{[\ell_1^{(j)} 2^{\ell-2} v - \ell_2^{(j)} k]}{2^j t} \in \frac{1}{2} + \mathbb{Z}$ for some $j > 1$, i.e. $\ell_1^{(j)} 2^{\ell-2} v - \ell_2^{(j)} k \in 2\mathbb{Z}$, which implies that $\ell = 2$, i.e. $p + k = 4v$ for some $v \in 2\mathbb{Z}+1$. To summarize, we have $w = 2^s t, t \in 2\mathbb{Z}+1$, $s \geq 2$ and $p = u + 2v,\ k = 2v - u$ for some $u, v \in 2\mathbb{Z}+1$. □

Remark 2.8.8. By Lemma 2.8.6 and Lemma 2.8.7, if $K(B, \mathcal{D}) = CK(B, \mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set, where C is non-integral, then

$$C = \begin{pmatrix} a & -2c \\ c & a-c \end{pmatrix} = \begin{pmatrix} (k+p)/w & -4k/w \\ 2k/w & (p-k)/w \end{pmatrix} = \begin{pmatrix} 4v/2^s t & -4(2v-u)/2^s t \\ 2(2v-u)/2^s t & 2u/2^s t \end{pmatrix},$$

where $u, v, t \in 2\mathbb{Z}+1$, $s \geq 2$ and $s \in \mathbb{Z}$. And in this case, we have

$$\begin{aligned} p^2+7k^2 = w^2 &\iff (u+2v)^2+7(2v-u)^2 = (2^s t)^2 \\ &\iff u^2-3uv+4v^2 = 2^{2s-3}t^2, \end{aligned} \tag{2.8.21}$$

and

$$\begin{aligned} & \binom{1}{0} \cdot (B^t)^{-j} C^t \binom{m_1}{m_2} \\ = \; & \frac{\ell_1^{(j)}}{2^j} \frac{4vm_1+2(2v-u)m_2}{2^s t} + \frac{\ell_2^{(j)}}{2^j} \frac{-4(2v-u)m_1+2um_2}{2^s t}. \end{aligned} \tag{2.8.22}$$

The following two lemmas show that the conditions in Lemma 2.8.7 are not sufficient for $K(B, \mathcal{D})$ to be a $\mathbb{Z}^2$-tiling set. In the following remark, we list some formulas about $\ell_1^{(j)}, \ell_2^{(j)}$, which will be frequently used in Lemma 2.8.10.

Remark 2.8.9. By Remark 2.8.2, we obtain, if $j > 2$, that

$$\ell_1^{(j)} + \ell_2^{(j-1)} = \ell_1^{(j-1)} - \ell_2^{(j-1)} = -2\ell_2^{(j-2)}.$$

And if $j > 4$, then

$$\ell_1^{(j)} - \ell_2^{(j-1)} \;=\; \ell_1^{(j-1)} - 3\ell_2^{(j-1)} = \ell_1^{(j-2)} - 2\ell_2^{(j-2)} - 3\ell_1^{(j-2)} = -2(\ell_1^{(j-2)} + \ell_2^{(j-2)}) = 8\ell_2^{(j-4)}$$

Lemma 2.8.10. *Let matrix C be defined as in Remark 2.8.8, then $K(B, \mathcal{D}) = CK(B, \mathcal{D}_0)$ is not a $\mathbb{Z}^2$-tiling set if $s > 2$.*

Proof. If $s > 2$, we have, using (2.8.21) and the fact that $u, v, t \in 2\mathbb{Z}+1$, that

$$\begin{aligned} u^2-3uv+4v^2 = 2^{2s-3}t^2 &\iff (u-v)(u-2v) = 2^{2s-3}t^2 - 2v^2 \\ &\implies u-v \in 2(2\mathbb{Z}+1). \end{aligned} \tag{2.8.23}$$

$$\begin{aligned} u^2-3uv+4v^2 = 2^{2s-3}t^2 &\iff (u+v)(u-4v) = 2^{2s-3}t^2 - 8v^2 \\ &\implies \begin{cases} u+v \in 8(2\mathbb{Z}+1), & s > 3 \\ u+v \in 32\mathbb{Z}, & s = 3. \end{cases} \end{aligned} \tag{2.8.24}$$

Assume $K(B,\mathcal{D}) = CK(B,\mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set, then $\hat{\chi}_K(m) = \delta_{m,0}$, $m \in \mathbb{Z}^2$ by Lemma 2.1.1. Take $m = \binom{2^{s-3}}{0}$. By (2.8.22) and (2.3.10), there is some $j \geq 1$ which satisfies

$$\binom{1}{0} \cdot (B^t)^{-j} C^t \binom{2^{s-3}}{0} = \frac{v\ell_1^{(j)} - (2v-u)\ell_2^{(j)}}{2^{j+1}t} = \frac{v\ell_1^{(j+1)} + u\ell_2^{(j)}}{2^{j+1}t} \in \frac{1}{2} + \mathbb{Z}.$$

Hence, there exists some $r \in \mathbb{Z}$ and some $j \geq 1$ such that

$$v\ell_1^{(j+1)} + u\ell_2^{(j)} = 2^j t(2r+1). \tag{2.8.25}$$

Take $m = \binom{0}{2^{s-3}}$. By (2.8.22) and (2.3.10), there is some $j' \geq 1$ which satisfies

$$\binom{1}{0} \cdot (B^t)^{-j'} C^t \binom{0}{2^{s-3}} = \frac{(2v-u)\ell_1^{(j')} + u\ell_2^{(j')}}{2^{j'+2}t} \in \frac{1}{2} + \mathbb{Z}.$$

If $j' = 1$, then $\ell_1^{(j')} = \ell_2^{(j')} = 1$ and the above equation would imply that $v \in 2\mathbb{Z}$, a contradiction. There exists thus some $r' \in \mathbb{Z}$ and some $j' > 1$ such that

$$v\ell_1^{(j')} + u\ell_2^{(j'-1)} = 2^{j'} t(2r'+1). \tag{2.8.26}$$

From (2.8.25) and (2.8.26), it follows that $j' \neq j+1$ and that

$$(v-u)D_{jj'} = 2^j t(2r+1)(\ell_1^{(j')} + \ell_2^{(j'-1)}) - 2^{j'} t(2r'+1)(\ell_1^{(j+1)} + \ell_2^{(j)}), \tag{2.8.27}$$

$$(v+u)D_{jj'} = 2^j t(2r+1)(\ell_2^{(j'-1)} - \ell_1^{(j')}) - 2^{j'} t(2r'+1)(\ell_2^{(j)} - \ell_1^{(j+1)}). \tag{2.8.28}$$

Case 1: if $j' = j$, then $j \geq 2$ since $j' > 1$. Furthermore, $D_{jj} = -2^{j-1}$ by Lemma 2.8.3. We have, using (2.8.27) and Remark 2.8.9, that

$$\begin{aligned} v-u &= -2t(2r+1)(\ell_1^{(j)} + \ell_2^{(j-1)}) + 2t(2r'+1)(\ell_1^{(j+1)} + \ell_2^{(j)}) \\ &= \begin{cases} 4t(2r+1)\ell_2^{(j-2)} - 4t(2r'+1)\ell_2^{(j-1)}, & j \geq 3 \\ -4t(2r'+1), & j = 2. \end{cases} \end{aligned}$$

It is obvious that $v-u \in 4\mathbb{Z}$ for any $j \geq 2$, which contradicts the fact that $v-u \in 2(2\mathbb{Z}+1)$. Case 2: if $j' < j$, then $j \geq 3$ and by Remark 2.8.9, (2.8.27) can be rewritten as

$$(v-u)D_{jj'} = \begin{cases} 2^{j'+1} t\left[(2r'+1)\ell_2^{(j-1)} - 2^{j-j'}(2r+1)\ell_2^{(j'-2)}\right], & j' \geq 3 \\ 8t(2r'+1)\ell_2^{(j-1)}, & j' = 2. \end{cases} \tag{2.8.29}$$

If $j' \geq 3$, Lemma 2.8.4 implies that $D_{jj'} \in (2\mathbb{Z}+1)2^{j'-1}$, so $u-v \in 4(2\mathbb{Z}+1)$ by (2.8.29), which is a contradiction by (2.8.23). If $j'=2$, then

$$D_{j2} = \ell_1^{(j+1)}\ell_2^{(1)} - \ell_1^{(2)}\ell_2^{(j)} = \ell_1^{(j+1)} + \ell_2^{(j)} = -2\ell_2^{(j-1)}. \tag{2.8.30}$$

(2.8.29) and (2.8.30) show that $u-v \in 4(2\mathbb{Z}+1)$ if $j'=2$, which is impossible by (2.8.23).
Case 3: if $j' > j$, then $j' \geq j+2$. It follows from (2.8.28), we should have

$$\begin{aligned}(v+u)D_{jj'} &= -2^j t(2r+1)(\ell_1^{(j')} - \ell_2^{(j'-1)}) + 2^{j'}t(2r'+1)(\ell_1^{(j+1)} - \ell_2^{(j)}) \\ &= 2^j t\left[2^{j'-j}(2r'+1)(\ell_1^{(j)} - 3\ell_2^{(j)}) - (2r+1)(\ell_1^{(j'-1)} - 3\ell_2^{(j'-1)})\right].\end{aligned} \tag{2.8.31}$$

If $j=1$, then $j' \geq 3$ and (2.8.31) can be reduced to

$$-(v+u)(\ell_2^{(j'-1)} + \ell_1^{(j')}) = 2t\left[-2^{j'}(2r'+1) + 2(2r+1)(\ell_1^{(j'-2)} + \ell_2^{(j'-2)})\right]. \tag{2.8.32}$$

If $j'=3$, this reduces to

$$2(u+v) = 8t[(2r+1) - 2(2r'+1)].$$

This would imply that $u+v \in 4(2\mathbb{Z}+1)$, which is impossible by (2.8.24).
If $j'=4$, (2.8.32) yields

$$2(u+v) = -2^5 t(2r'+1),$$

and thus that $u+v \in 16(2\mathbb{Z}+1)$, which is also impossible by (2.8.24).
If $j' \geq 5$, we have, using Remark 2.8.9 and Remark 2.8.2, that

$$2(u+v)\ell_2^{(j'-2)} = 2t\left[-2^{j'}(2r'+1) - 8(2r+1)\ell_2^{(j'-4)}\right],$$

which implies that $u+v \in 8(2\mathbb{Z}+1)$. This shows that $s>3$ by (2.8.24).
If $j=2$, then $j' \geq 4$ and we obtain from (2.8.31) that

$$(v+u)(-3\ell_2^{(j'-1)} - \ell_1^{(j')}) = 4t[-2^{j'}(2r'+1) - (2r+1)(\ell_1^{(j'-1)} - 3\ell_2^{(j'-1)})]. \tag{2.8.33}$$

If $j'=4$, this yields

$$(v+u)(-3\ell_2^{(3)} - \ell_1^{(4)}) = 4(u+v) = -2^6 t(2r'+1),$$

and thus that $u+v \in 16(2\mathbb{Z}+1)$, which is impossible by (2.8.24).

If $j' \geq 5$, then we have, using Remark 2.8.9, that

$$4(u+v)\ell_2^{(j'-3)} = 4t\left[-2^{j'}(2r'+1) - 8(2r+1)\ell_2^{(j'-4)}\right],$$

which implies that $u+v \in 8(2\mathbb{Z}+1)$. This shows again that $s>3$ by (2.8.24).

If $j=3$, then $j' \geq 5$ and using (2.8.31) and Remark 2.8.2, we obtain that

$$\begin{aligned}(v+u)(\ell_1^{(j')} - \ell_2^{(j'-1)}) &= 2^3 t\left[2^{j'-3}(2r'+1)(\ell_1^{(3)} - 3\ell_2^{(3)}) - (2r+1)(\ell_1^{(j'-1)} - 3\ell_2^{(j'-1)})\right] \\ &= -2^3 t(2r+1)(\ell_1^{(j')} - \ell_2^{(j'-1)}),\end{aligned}$$

i.e. we have $v+u = -2^3 t(2r+1) \in 8(2\mathbb{Z}+1)$, which implies that $s>3$ by (2.8.24).

If $j>3$, then $j'>5$ and it follows from (2.8.31) and Remark 2.8.9 that

$$(v+u)D_{jj'} = 2^{j+3}t\left[2^{j'-j}(2r'+1)\ell_2^{(j-3)} - (2r+1)\ell_2^{(j'-4)}\right].$$

Since $D_{jj'} \in (2\mathbb{Z}+1)2^j$ by Lemma 2.8.5, it follows that $u+v \in 8(2\mathbb{Z}+1)$, which also implies that $s>3$ by (2.8.24).

We deduce that $s \geq 4$ by the above three cases if we assume that $s>2$. Taking $m = \binom{2^{s-4}}{0}$, then by (2.8.22) and (2.3.10), there exists some $j \geq 1$ which satisfies

$$\binom{1}{0} \cdot (B^t)^{-j} C^t \binom{2^{s-4}}{0} = \frac{v\ell_1^{(j)} - (2v-u)\ell_2^{(j)}}{2^{j+2}t} = \frac{v\ell_1^{(j+1)} + u\ell_2^{(j)}}{2^{j+2}t} \in \frac{1}{2} + \mathbb{Z}.$$

If $j=1$, we get $u-v \in 4(2\mathbb{Z}+1)$ contradicting (2.8.23). That is, there exists some $h \in \mathbb{Z}$ and some $j>1$ such that

$$v\ell_1^{(j+1)} + u\ell_2^{(j)} = 2^{j+1}t(2h+1). \tag{2.8.34}$$

Taking now $m = \binom{0}{2^{s-4}}$, then there exists some $j' \geq 1$ which satisfies

$$\binom{1}{0} \cdot (B^t)^{-j'} C^t \binom{0}{2^{s-4}} = \frac{(2v-u)\ell_1^{(j')} + u\ell_2^{(j')}}{2^{j'+3}t} \in \frac{1}{2} + \mathbb{Z}.$$

If $j'=1$, we get $v \in 4(2\mathbb{Z}+1)$, contradicting the fact that $v \in 2\mathbb{Z}+1$. If $j'=2$, we get $u-v \in 8(2\mathbb{Z}+1)$, contradicting (2.8.23). Hence, there exists some $h' \in \mathbb{Z}$ and some $j' \geq 3$ such that

$$v\ell_1^{(j')} + u\ell_2^{(j'-1)} = 2^{j'+1}t(2h'+1). \tag{2.8.35}$$

The equalities (2.8.34) and (2.8.35) show that $j' \neq j+1$. Furthermore, using Remark 2.8.9, we have

$$\begin{aligned}(v-u)D_{jj'} &= 2^{j+1}t(2h+1)(\ell_1^{(j')}+\ell_2^{(j'-1)}) - 2^{j'+1}t(2h'+1)(\ell_1^{(j+1)}+\ell_2^{(j)}) \\ &= -2^{j+2}t(2h+1)\ell_2^{(j'-2)} + 2^{j'+2}t(2h'+1)\ell_2^{(j-1)}. \end{aligned} \tag{2.8.36}$$

If $j'=j$, then $D_{jj}=-2^{j-1}$ by Lemma 2.8.3. Thus $u-v \in 8\mathbb{Z}$, which contradicts (2.8.23). If $j'<j$, then $j \geq 4$ and from (2.8.36), we get

$$(v-u)D_{jj'} = 2^{j'+2}t\left[(2h'+1)\ell_2^{(j-1)} - 2^{j-j'}(2h+1)\ell_2^{(j'-2)}\right].$$

Since $D_{jj'} \in (2\mathbb{Z}+1)2^{j'-1}$ by Lemma 2.8.4, we obtain that $u-v \in 8(2\mathbb{Z}+1)$, which is again impossible by (2.8.23).
If $j'>j$, then $j' \geq j+2$ and, in this case, we have

$$(v-u)D_{jj'} = -2^{j+2}t\left[(2h+1)\ell_2^{(j'-2)} - 2^{j'-j}(2h'+1)\ell_2^{(j-1)}\right].$$

Since $D_{jj'} \in 2^j(2\mathbb{Z}+1)$ by Lemma 2.8.5, so we obtain that $u-v \in 4(2\mathbb{Z}+1)$, which contradicts (2.8.23). This proves that $K(B,\mathcal{D}) = K(B,C\mathcal{D}_0)$ is not a $\mathbb{Z}^2$-tiling set if $s>2$. □

Lemma 2.8.11. *Let matrix C be defined as in Remark 2.8.8, then $K(B,\mathcal{D}) = CK(B,\mathcal{D}_0)$ is not a $\mathbb{Z}^2$-tiling set if $s=2$ with $t \neq \pm 1$.*

Proof. If $s=2$ and $t \neq \pm 1$, then (2.8.21) can be written as

$$p^2+7k^2 = w^2 \iff u^2-3uv+4v^2 = 2t^2. \tag{2.8.37}$$

Assume $K(B,\mathcal{D}) = CK(B,\mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set, then $\hat{\chi}_K(m) = \delta_{m,0}$, $m \in \mathbb{Z}^2$ by Lemma 2.1.1. Taking $m = \binom{1}{0}$, we obtain by (2.8.22) and (2.3.10), the existence of $j \geq 1$ which satisfies the following:

$$\binom{1}{0} \cdot (B^t)^{-j}C^t\binom{1}{0} = \frac{v\ell_1^{(j)}}{2^j t} - \frac{(2v-u)\ell_2^{(j)}}{2^j t} = \frac{v(\ell_1^{(j)}-\ell_2^{(j)}) - (v-u)\ell_2^{(j)}}{2^j t} \in \frac{1}{2}+\mathbb{Z}.$$

There exists thus $r \in \mathbb{Z}$ and some $j \geq 1$ such that

$$v(\ell_1^{(j)} - \ell_2^{(j)}) - (v-u)\ell_2^{(j)} = 2^{j-1}t(2r+1). \tag{2.8.38}$$

Since $\ell_1^{(j)}, \ell_2^{(j)}, u$ and v are all odd, it follows that $v(\ell_1^{(j)} - \ell_2^{(j)}) - (v-u)\ell_2^{(j)} \in 2\mathbb{Z}$ and, obviously, $j \neq 1$. If $j = 2$, then $\ell_1^{(2)} = -1,\ \ell_2^{(2)} = 1$ and (2.8.38) yields $u - 3v = 2t(2r+1)$ for some $r \in \mathbb{Z}$. This means that $t|(u-3v)$. Using (2.8.37), we get

$$u^2 - 3uv + 4v^2 = 2t^2 \Longleftrightarrow u(u-3v) = 2t^2 - 4v^2.$$

This implies that $t|v^2$ and thus $(t,v) \neq 1$, since $t \neq \pm 1$. Furthermore, we get that $(t,k) = (t, 2v-u) \neq 1$, which contradicts the fact that $(w,k) = 1$, where $w = 4t$. Suppose now that $j \geq 3$. Since $\frac{\ell_1^{(j)} - \ell_2^{(j)}}{2} \in 2\mathbb{Z}+1$ for $j \geq 3$ by Lemma 2.8.1, it follows from (2.8.38) that $\frac{v-u}{2} \in 2\mathbb{Z}+1$, i.e. $v - u = 2\tau$ for some $\tau \in 2\mathbb{Z}+1$. The equality $u^2 - 3uv + 4v^2 = 2t^2$ which is equivalent to $(u-v)(u-2v) = 2(t^2-v^2)$ implies that $t^2 - v^2 \in 2\mathbb{Z}+1$, contradicting $t, v \in 2\mathbb{Z}+1$. □

Theorem 2.8.12. *Let $A = C_4 = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$ and $\mathcal{D} = \{0, d\}$. Then $K(B, \mathcal{D})$ is a $\mathbb{Z}^2$-tiling set if and only if $\mathcal{D} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$, $\mathcal{D} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\}$, $\mathcal{D} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}\}$, $\mathcal{D} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix}\}$. Furthermore, for each of these digit sets, $K(B, \mathcal{D})$ is an A-dilation MRA scaling set.*

Proof. If $A = C_4 = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$, then $B = A^t = \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$. In [40], $K(B, \mathcal{D}_0)$ was proved to be a $\mathbb{Z}^2$-tiling set using Cohen's Condition. By the definition of S in Lemma 2.1.4, we get $S = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$. Then Lemma 2.1.4 gives $K(B, \mathcal{D}_0) \bigcap \mathbb{Z}^2 = -S = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$. Thus $K(B, \mathcal{D}_0)$ contains a neighborhood of 0 by Lemma 2.1.3. So $K(B, \mathcal{D}_0)$ is an A-dilation MRA scaling set by Corollary 2.1.6. The general form of a matrix C commuting with B is $C = \begin{pmatrix} a & -2c \\ c & a-c \end{pmatrix}$. The condition $|\det C| = 1$ yields $a(a-c) + 2c^2 = (a - \frac{c}{2})^2 + \frac{7}{4}c^2 = 1$. The integral solutions of $(a - \frac{c}{2})^2 + \frac{7}{4}c^2 = 1$ are $a = \pm 1, c = 0$. As we discussed, $K(B, \mathcal{D}) = CK(B, \mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set and also an A-dilation MRA scaling set for integral matrices $C = \pm I$. Suppose there is a non-integral matrix C such that $K(B, \mathcal{D}) = CK(B, \mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set. Then matrix C is a rational matrix by Lemma 2.8.6. It follows from Remark 2.8.8 that

$$C = \begin{pmatrix} a & -2c \\ c & a-c \end{pmatrix} = \begin{pmatrix} (k+p)/w & -4k/w \\ 2k/w & (p-k)/w \end{pmatrix} = \begin{pmatrix} 4v/2^s t & -4(2v-u)/2^s t \\ 2(2v-u)/2^s t & 2u/2^s t \end{pmatrix},$$

where $u, v, t \in 2\mathbb{Z}+1$, $s \geq 2$, $s \in \mathbb{Z}$ and $p^2 + 7k^2 = w^2$. From Lemma 2.8.10 and Lemma 2.8.11, we can not find a non-integral matrix C such that $K(B, \mathcal{D}) = CK(B, \mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set if $s > 2$ or if $s = 2$ and $t \neq \pm 1$. If $s = 2$ and $t = \pm 1$, then $w = 2^s t = \pm 4$ and there are eight triples $(p, k, w) = (\pm 3, \pm 1, \pm 4)$ which satisfy the equation $p^2 + 7k^2 = w^2$. We consider each of these cases separately.
If $(p, k, w) = \pm(3, 1, 4)$, then $a = 1,\ c = \frac{1}{2}$ and the matrix $C^t = \begin{pmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix}$. We have

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (B^t)^{-j} C^t \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = (1,0)(B^t)^{-j} \begin{pmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = (1,0)(B^t)^{-j} \begin{pmatrix} m_1 + \frac{1}{2}m_2 \\ -m_1 + \frac{1}{2}m_2 \end{pmatrix}. \quad (2.8.39)$$

Taking $j = 1$, we have, using (2.8.39), that

$$\begin{aligned} (1,0)(B^t)^{-1} \begin{pmatrix} m_1 + \frac{1}{2}m_2 \\ -m_1 + \frac{1}{2}m_2 \end{pmatrix} &= (1,0) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} m_1 + \frac{1}{2}m_2 \\ -m_1 + \frac{1}{2}m_2 \end{pmatrix} \\ &= \tfrac{1}{2}(m_1 + \tfrac{1}{2}m_2) + \tfrac{1}{2}(-m_1 + \tfrac{1}{2}m_2) \\ &= \tfrac{1}{2}m_2 \end{aligned}$$

So $\hat{\chi}_K(m) = 0$ if $m_2 \in 2\mathbb{Z}+1$ by (2.3.10). If m_2 is even, say $m_2 = 2k_2,\ k_2 \in \mathbb{Z}$, then

$$C^t m = \begin{pmatrix} m_1 + \frac{1}{2}m_2 \\ -m_1 + \frac{1}{2}m_2 \end{pmatrix} = \begin{pmatrix} m_1 + k_2 \\ -m_1 + k_2 \end{pmatrix} \in \mathbb{Z}^2 \setminus \{0\} \text{ if } m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \mathbb{Z}^2 \setminus \{0\}.$$

The fact that $K_0 := K(B, \mathcal{D}_0)$ is a $\mathbb{Z}^2$-tiling set is equivalent to that $\hat{\chi}_{K_0}(m) = \delta_{m,0}$, $m \in \mathbb{Z}^2$. Then $\hat{\chi}_K(m) = \hat{\chi}_{CK_0}(m) = \hat{\chi}_{K_0}(C^t m) = 0$ if $m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \mathbb{Z}^2 \setminus \{0\}$ and $m_2 \in 2\mathbb{Z}$. Thus we proves that $\hat{\chi}_K(m) = \delta_{m,0}$, i.e. $K(B, \mathcal{D}) = CK_0$ is a $\mathbb{Z}^2$-tiling set, and, therefore, $K(B, \mathcal{D})$ is an A-dilation MRA scaling set since K_0 is.
If $(p, k, w) = \pm(3, 1, -4)$, then $a = -1,\ c = -\frac{1}{2}$ and the matrix $C^t = \begin{pmatrix} -1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$ which is the negative of the corresponding matrix C^t in the previous case. Using the same arguments, we can thus prove that $K(B, \mathcal{D}) = CK_0$ is also an A-dilation MRA scaling set.

If $(p,k,w)=\pm(-3,1,-4)$, then $a=\frac{1}{2},\ c=-\frac{1}{2}$ and the matrix $C^t=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ 1 & 1\end{pmatrix}$. We have

$$\begin{aligned}\begin{pmatrix}1\\0\end{pmatrix}\cdot(B^t)^{-j}C^t\begin{pmatrix}m_1\\m_2\end{pmatrix} &= (1,0)\begin{pmatrix}\frac{\ell_1^{(j)}}{2^j} & \frac{\ell_2^{(j)}}{2^j}\\ * & *\end{pmatrix}\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ 1 & 1\end{pmatrix}\begin{pmatrix}m_1\\m_2\end{pmatrix}\\ &= (1,0)\begin{pmatrix}\frac{\ell_1^{(j)}}{2^j} & \frac{\ell_2^{(j)}}{2^j}\\ * & *\end{pmatrix}\begin{pmatrix}\frac{1}{2}(m_1-m_2)\\ m_1+m_2\end{pmatrix}\\ &= \frac{\ell_1^{(j)}}{2^j}[\frac{1}{2}(m_1-m_2)]+\frac{\ell_2^{(j)}}{2^j}(m_1+m_2)\\ &= \frac{m_1(\ell_1^{(j)}+2\ell_2^{(j)})}{2^{j+1}}+\frac{m_2(2\ell_2^{(j)}-\ell_1^{(j)})}{2^{j+1}}. \qquad (2.8.40)\end{aligned}$$

Taking $m=\begin{pmatrix}0\\1\end{pmatrix}$, we obtain from (2.8.40) that

$$(1,0)(B^t)^{-j}C^t\begin{pmatrix}0\\1\end{pmatrix}=\frac{2\ell_2^{(j)}-\ell_1^{(j)}}{2^{j+1}}=-\frac{\ell_1^{(j+1)}}{2^{j+1}} \qquad (2.8.41)$$

It is obvious that $-\frac{\ell_1^{(j+1)}}{2^{j+1}}\notin\frac{1}{2}+\mathbb{Z}$ for any $j\geq 1$. So $K(B,\mathcal{D})=CK_0$ is not a $\mathbb{Z}^2$-tiling set. If $(p,k,w)=\pm(-3,1,4)$, then $a=-\frac{1}{2},\ c=\frac{1}{2}$ and the matrix $C^t=\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\\ -1 & -1\end{pmatrix}$ which is the negative of the corresponding matrix C^t in the previous case. Using the same method, we can then show that $K=CK_0$ is not a $\mathbb{Z}^2$-tiling set. □

Let $C=\begin{pmatrix}1 & -1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$. From the discussion for the representative class $A=C_4=\begin{pmatrix}0 & -1\\ 2 & 1\end{pmatrix}$, we get $K(B,\mathcal{D})=\pm CK_0=CK(B,\mathcal{D}_0)$ is an A-dilation MRA scaling set. That is, there are two non-integral digit sets $\mathcal{D}=\{0,d\}$, where $d=\pm C\begin{pmatrix}1\\0\end{pmatrix}=\pm\begin{pmatrix}1\\\frac{1}{2}\end{pmatrix}$ such that $K(B,\mathcal{D})$ is an A-dilation MRA scaling set. This is a very interesting $\mathbb{Z}^2$-tiling set. Intuitively, we might think digit set $\mathcal{D}=\{0,d\}$ should be a subset of $\mathbb{Z}^2$ if $K(B,\mathcal{D})$ is a $\mathbb{Z}^2$-tiling set. However, the examples we give here show that there are non-integral digit set $\mathcal{D}=\{0,d\}$ such that $K(B,\mathcal{D})$ is a $\mathbb{Z}^2$-tiling set. Note that $K(B,\mathcal{D})$, where $\mathcal{D}=\pm\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\\frac{1}{2}\end{pmatrix}\}$, is not only a $\mathbb{Z}^2$-tiling set, but also a $C\mathbb{Z}^2$-tiling set since $K(B,\mathcal{D})=CK_0$ and K_0 is a $\mathbb{Z}^2$-tiling set. The set $K(B,\mathcal{D})$ and its translates $K(B,\mathcal{D})+\begin{pmatrix}0\\1\end{pmatrix}$, $K(B,\mathcal{D})+\begin{pmatrix}1\\1\end{pmatrix}$, where $\mathcal{D}=\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\\frac{1}{2}\end{pmatrix}\}$, are depicted in Figure 2.1. The set $K(B,\mathcal{D})$ and its translates $K(B,\mathcal{D})+C\begin{pmatrix}1\\0\end{pmatrix}$, $K(B,\mathcal{D})+C\begin{pmatrix}1\\1\end{pmatrix}$, where $\mathcal{D}=\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\\frac{1}{2}\end{pmatrix}\}$, are depicted in Figure 2.2. The corresponding wavelet set $Q:=BK\setminus K=K+\begin{pmatrix}1\\\frac{1}{2}\end{pmatrix}$ and its translate $Q+\begin{pmatrix}1\\0\end{pmatrix}$ are depicted in Figure 2.3. The sets $BQ, Q, B^{-1}Q, B^{-2}Q$ are depicted in Figure 2.4.

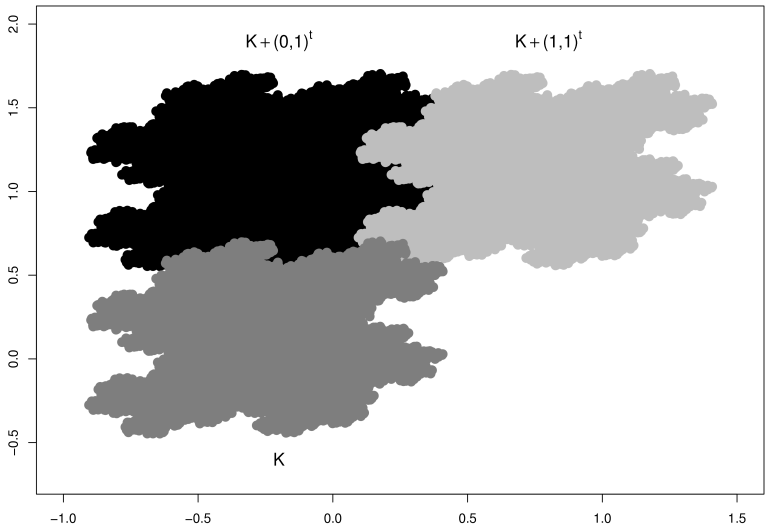


Figure 2.1: K and its $\mathbb{Z}^2$-translation:$K + \binom{0}{1}$, $K + \binom{1}{1}$ in Section 2.8.

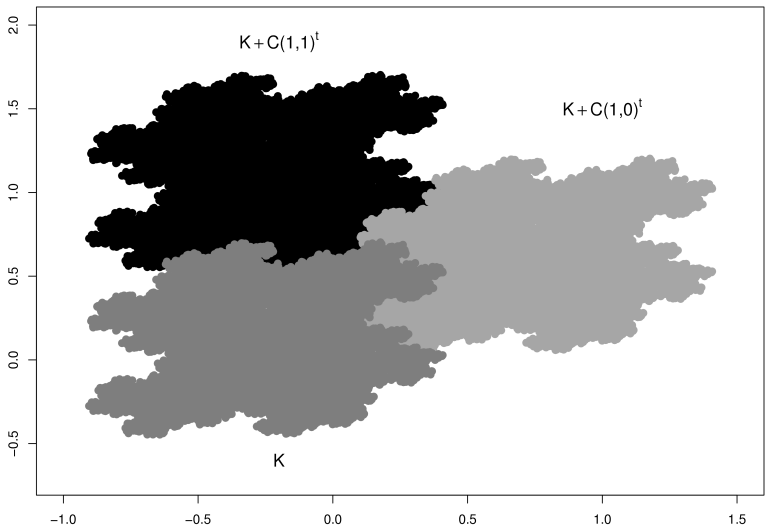


Figure 2.2: K and its $C\mathbb{Z}^2$-translation: $K + C\binom{1}{0}$, $K + C\binom{1}{1}$ in Section 2.8.

2.9 The case $A = -C_4$

If $A = -C_4 = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}$, then $B = A^t = \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix}$. The same argument to the matrix C_4, we can show that there are four digit sets $\mathcal{D} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$, $\mathcal{D} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\}$, $\mathcal{D} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}\}$ and $\mathcal{D} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix}\}$ such that $K(B, \mathcal{D})$ is an A-dilation MRA scaling set.

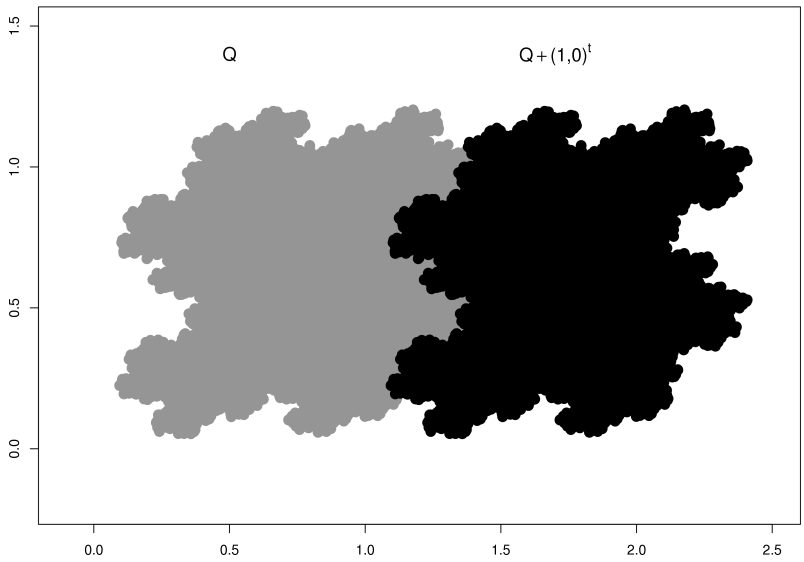


Figure 2.3: The wavelet set $Q = BK \setminus K$ and its $\mathbb{Z}^2$-translation: $Q + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in Section 2.8.

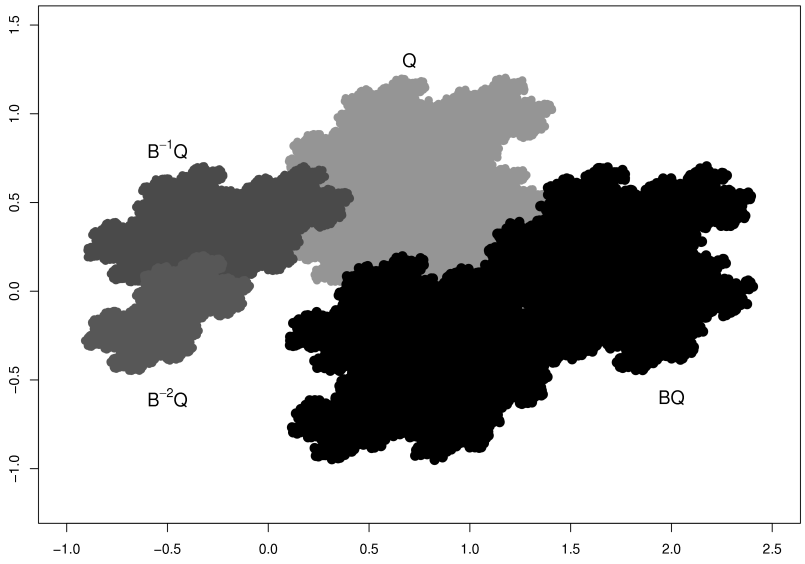


Figure 2.4: B-dilation of the set Q: $B^{-2}Q, B^{-1}Q, Q, BQ$ in Section 2.8.

Chapter 3

Self-Affine Scaling Sets in $\mathbb{R}^2$, Part II

In last chapter, we characterized all one and two dimensional self-affine scaling sets with a set of associated digits of the form $\{0, d\}$, where $d \in \mathbb{R}^2$ and d is not necessarily in $\mathbb{Z}^2$, for an integral expansive matrix A with $|\det A| = 2$. As a complement to the work in Chapter 2, the main purpose of this chapter is to characterize all self-affine scaling sets with associated $n \times n$ integral expanding matrix A with $|\det A| = 2$ and associated digit set $\mathcal{D} = \{d_1, d_2\} \subset \mathbb{R}^n$, $n = 1, 2$. We point out that we have shown in Chapter 1 that if such a scaling set can be constructed in dimension 1 or 2, then it must necessarily be an MRA scaling set. Hence, the characterization of A-dilation scaling sets is reduced to the characterization of A-dilation MRA scaling sets in dimension one and dimension two. The main difference with Chapter 2 is that we do not assume here that $0 \in \mathcal{D}$. Let K be a self-affine tiling set in $\mathbb{R}^n$ associated with the matrix B and the digit set $\mathcal{D} = \{d_1, d_2\} \subset \mathbb{R}^n$, i.e. $K = K(B, \mathcal{D})$ satisfies the set-valued equation

$$BK = (K + d_1) \bigcup (K + d_2). \tag{3.0.1}$$

(3.0.1) determines a unique compact set K, given explicitly by

$$K(B, \mathcal{D}) := \Big\{ \sum_{j=1}^{\infty} B^{-j} \ell_j : \ \ell_j \in \mathcal{D} \Big\}. \tag{3.0.2}$$

3.1 A sufficient condition for $K(B, \mathcal{D})$ to be an A-dilation MRA scaling set

Throughout the rest of this paper, $K(B, \mathcal{D})$ denotes the self-affine set K associated with the matrix B and the digit set $\mathcal{D}$ which satisfies equation (3.0.1). Let

$$\mathcal{D}_m := \Big\{ \sum_{j=0}^{m-1} B^j d_j : d_j \in \mathcal{D} \Big\} \text{ for } m \geqslant 1. \tag{3.1.1}$$

Then for any $m \geqslant 1$, by (3.0.1) and (3.1.1), we have $B^m K = K + \mathcal{D}_m$.

Theorem 3.1.1. *Let $A \in M_n^{(2)}(\mathbb{Z})$ be expanding and K be a self-affine tile associated with the matrix B and the digit set $\mathcal{D} = \{d_1, d_2\}$. Suppose that K is a $\mathbb{Z}^n$-tiling set and that there exists a sequence $\{\varepsilon_j\}_{j=1}^{\infty}$, $\varepsilon_j \in \mathcal{D}$, such that $\sum_{j=1}^{\infty} B^{-j}\varepsilon_j = 0$. Define $\widetilde{\mathcal{D}}_m := \mathcal{D}_m - \sum_{j=1}^{m} B^{m-j}\varepsilon_j$. If $\widetilde{\mathcal{D}}_m \subseteq \widetilde{\mathcal{D}}_{m+1}$ for every $m \geq 1$ and K is a $\widetilde{\mathcal{D}}_\infty$-tiling set, where $\widetilde{\mathcal{D}}_\infty := \bigcup_{j \geqslant 1} \widetilde{\mathcal{D}}_j$, then K is an A-dilation MRA scaling set.*

Proof. Suppose that there exists a sequence $\{\varepsilon_j\}_{j=1}^{\infty}$, $\varepsilon_j \in \mathcal{D} = \{d_1, d_2\}$, such that $\sum_{j=1}^{\infty} B^{-j}\varepsilon_j = 0$ and $\widetilde{\mathcal{D}}_m \subseteq \widetilde{\mathcal{D}}_{m+1}$, where $\widetilde{\mathcal{D}}_m = \mathcal{D}_m - \sum_{j=1}^{m} B^{m-j}\varepsilon_j$. Then, for any $m \geq 1$,

$$\begin{aligned}
\widetilde{\mathcal{D}}_m \subseteq \widetilde{\mathcal{D}}_{m+1} &\implies K + \widetilde{\mathcal{D}}_m \subseteq K + \widetilde{\mathcal{D}}_{m+1} \\
&\implies K + \mathcal{D}_m - \sum_{j=1}^{m} B^{m-j}\varepsilon_j \subseteq K + \mathcal{D}_{m+1} - \sum_{j=1}^{m+1} B^{m+1-j}\varepsilon_j \\
&\implies B^m K - \sum_{j=1}^{m} B^{m-j}\varepsilon_j \subseteq B^{m+1}K - \sum_{j=1}^{m+1} B^{m+1-j}\varepsilon_j \\
&\implies K - \sum_{j=1}^{m} B^{-j}\varepsilon_j \subseteq BK - \sum_{j=1}^{m+1} B^{1-j}\varepsilon_j \\
&\implies K \subseteq BK - (\sum_{j=1}^{m+1} B^{1-j}\varepsilon_j - \sum_{j=1}^{m} B^{-j}\varepsilon_j).
\end{aligned}$$

$\sum_{j=1}^{\infty} B^{-j}\varepsilon_j = 0$ implies that $\lim_{m\to\infty}(\sum_{j=1}^{m} B^{1-j}\varepsilon_j - \sum_{j=1}^{m} B^{-j}\varepsilon_j) = 0$, i.e, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $K \subseteq BK + B(0,\varepsilon)$ if $m \geqslant N$. Letting $\varepsilon \longrightarrow 0$, we get $K \subseteq BK$. Let V contain a neighborhood of 0 and $\overline{V+K}$ be compact. Define

$$S := \{s \in \widetilde{\mathcal{D}}_\infty : |(K+V)\bigcap(K+s)| > 0\}.$$

Since $S \subset (K+V) - K$, S is bounded and so is $\bigcup_{s\in S} K + s$. Hence, there exists $r > 0$ such that

$$\overline{V+K} \subseteq \bigcup_{s\in S} K + s \subseteq B(0,r). \tag{3.1.2}$$

Since $K + s,\ s \in S$ are essentially disjoint, then the number of elements in S is at most $\frac{|B(0,r)|}{|K|}$. The inclusion $\widetilde{\mathcal{D}}_m \subseteq \widetilde{\mathcal{D}}_{m+1}$ shows that there is $m \in \mathbb{N}$ such that $S \subseteq \widetilde{\mathcal{D}}_m$. It follows from (3.1.2) that

$$V + K \subseteq K + \widetilde{\mathcal{D}}_m = K + \mathcal{D}_m - \sum_{j=1}^{m} B^{m-j}\varepsilon_j = B^m K - \sum_{j=1}^{m} B^{m-j}\varepsilon_j. \tag{3.1.3}$$

The equation $\sum_{j=1}^{\infty} B^{-j}\varepsilon_j = 0$ is equivalent to $\sum_{j=1}^{m} B^{-j}\varepsilon_j = -\sum_{j=m+1}^{\infty} B^{-j}\varepsilon_j$. Thus we have, using (3.0.2), that

$$\sum_{j=1}^{m} B^{m-j}\varepsilon_j = -\sum_{j=m+1}^{\infty} B^{m-j}\varepsilon_j = -\sum_{j=1}^{\infty} B^{-j}\varepsilon_{j+m} \in -K. \tag{3.1.4}$$

From (3.1.3) and (3.1.4), we obtain that

$$V - \sum_{j=1}^{m} B^{m-j}\varepsilon_j \subseteq V + K \subseteq B^m K - \sum_{j=1}^{m} B^{m-j}\varepsilon_j. \tag{3.1.5}$$

(3.1.5) implies that $V \subseteq B^m K$, so K contains a neighborhood of 0. Since B is expansive and $K \subseteq BK$, we have $\mathbb{R}^n = \bigcup_{m\in\mathbb{Z}} B^m K = \lim_{m\to\infty} B^m K$. This shows that $\lim_{m\to\infty} \chi_K(B^{-m}x) = 1$ *a.e.* $x \in \mathbb{R}^n$. By assumption, K is a $\mathbb{Z}^n$-tiling set. Hence K is an A-dilation MRA scaling set since it satisfies the three conditions of Lemma 0.0.7. □

The conditions in Theorem 3.1.1 are not necessary. It depends on the choice of the sequence $\{\varepsilon_j\}_{j=1}^{\infty}, \varepsilon_j \in \mathcal{D}$. The example below illustrates this situation. However, Theorem 3.1.1 provides some conditions to verify the inclusion $K \subset BK$ and when the set K contains a neighborhood of 0.

Example 3.1.2. In dimension two, consider the self-affine set $K(B, \mathcal{D})$ associated with the matrix $B = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ and the digit set $\mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \right\}$. Then $K(B, \mathcal{D})$ is an A-dilation MRA scaling set and

$$K(B, \mathcal{D}) = conv\left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right\}.$$

Take $\varepsilon_1 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$ and $\varepsilon_j = \begin{cases} \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, & \text{if } j \text{ is odd and } j \geq 3, \\ \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, & \text{if } j \text{ is even and } j \geq 4. \end{cases}$ It is easy to see that $\varepsilon_j \in \mathcal{D}$ for $j \geq 1$ and the sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ satisfies that $\sum_{j=1}^{\infty} B^{-j}\varepsilon_j = 0$. Define $\widetilde{\mathcal{D}}_m$ as in Theorem 3.1.1. However,

$$\widetilde{\mathcal{D}}_1 = \mathcal{D}_1 - \varepsilon_1 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\},$$

$$\widetilde{\mathcal{D}}_2 = \mathcal{D}_2 - (B\varepsilon_1 + \varepsilon_2) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}.$$

Obviously, $\widetilde{\mathcal{D}}_1 \nsubseteq \widetilde{\mathcal{D}}_2$.

For Example 3.1.2, if we take $\varepsilon_1 = \varepsilon_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$ and $\varepsilon_j = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$ for $j \geq 3$, then obviously, $\varepsilon_j \in \mathcal{D}$ for $j \geq 1$ and the sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ satisfies that $\sum_{j=1}^{\infty} B^{-j}\varepsilon_j = 0$. Define $\mathcal{D}_m$ as in (3.1.1) and define $\widetilde{\mathcal{D}}_m$ as in Theorem 3.1.1. Then we have

$$\mathcal{D}_1 = \left\{ \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \right\}, \ \mathcal{D}_2 = B\mathcal{D}_1 + \mathcal{D}_1 = \left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right\},$$

$$\mathcal{D}_3 = B\mathcal{D}_2 + \mathcal{D}_1 = \left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\}.$$

It follows that $\mathcal{D}_2 \subset \mathcal{D}_3$. Suppose $\mathcal{D}_{m-1} \subset \mathcal{D}_m$ for $m \geq 3$. Then, by the definition of $\mathcal{D}_m$ and by assumption, we have

$$\mathcal{D}_m = B\mathcal{D}_{m-1} + \mathcal{D}_1 \subset B\mathcal{D}_m + \mathcal{D}_1 = \mathcal{D}_{m+1}. \tag{3.1.6}$$

This shows that $\mathcal{D}_m \subset \mathcal{D}_{m+1}$ holds for any $m \geq 2$. Moreover, we have

$$\varepsilon_1 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \text{ and } \sum_{j=1}^{m} B^{m-j}\varepsilon_j = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \text{ for m} \geq 2. \tag{3.1.7}$$

Thus, by the definition of $\widetilde{\mathcal{D}}_m$, we get

$$\widetilde{\mathcal{D}}_1 = \mathcal{D}_1 - \varepsilon_1 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}, \widetilde{\mathcal{D}}_2 = \mathcal{D}_2 - (B\varepsilon_1 + \varepsilon_2) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}.$$

Obviously, $\widetilde{\mathcal{D}}_1 \subsetneqq \widetilde{\mathcal{D}}_2$. For any $m \geq 2$, it follows from (3.1.6) and (3.1.7), we obtain that

$$\widetilde{\mathcal{D}}_m = \mathcal{D}_m - \sum_{j=1}^{m} B^{m-j}\varepsilon_j = \mathcal{D}_m - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \subset \mathcal{D}_{m+1} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \mathcal{D}_{m+1} - \sum_{j=1}^{m+1} B^{m+1-j}\varepsilon_j = \widetilde{\mathcal{D}}_{m+1}.$$

This proves that $\widetilde{\mathcal{D}}_m \subset \widetilde{\mathcal{D}}_{m+1}$ for any $m \geq 1$. It is easy to compute that $\widetilde{\mathcal{D}}_\infty = \bigcup_{m=1}^{\infty} \widetilde{\mathcal{D}}_m = \mathbb{Z}^2$. Therefore, the set K is also a $\widetilde{\mathcal{D}}_\infty$-tiling set.

Example 3.1.2 illustrates that for some cases, we may choose the sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ with $\varepsilon_j \in \mathcal{D}$ for $j \geq 1$ satisfying $\sum_{j=1}^{\infty} B^{-j}\varepsilon_j = 0$ such that the conditions in Theorem 3.1.1 become necessary. However, we have no idea on how to choose such a sequence.

3.2 Characterization of the inclusion $K \subset BK$

As one of the necessary and sufficient conditions for a measurable set K to be an A-dilation MRA scaling set, the inclusion $K \subset BK$ plays a key role in the theory of scaling set. In this section, we will derive an equivalent condition for that inclusion assuming that K is a self-affine tile which satisfies equation (3.0.1). The fact that $K \subset BK$ implies that $B^{-m}K \subset K$ for any $m \geq 1$ and thus that $0 \in K$, since B is expansive. It follows from

(3.0.2) that there exists a sequence $\{\varepsilon_j\}_{j=1}^{\infty},\ \varepsilon_j \in \mathcal{D} = \{d_1, d_2\}$ such that $0 = \sum_{j=1}^{\infty} B^{-j}\varepsilon_j$. Define

$$\widetilde{\mathcal{D}}_m := \mathcal{D}_m - \sum_{j=1}^{m} B^{m-j}\varepsilon_j,\ m \geq 1. \tag{3.2.1}$$

Theorem 3.2.1. *$K \subseteq BK$ if and only if $\widetilde{\mathcal{D}}_m \subseteq \widetilde{\mathcal{D}}_{m+1} + H$ for every $m \geq 1$, where H is a compact subset of $\mathbb{R}^n$ and $\widetilde{\mathcal{D}}_m$ is defined as (3.2.1). Moreover, if $K \subseteq BK$, we can let $H = K - K$.*

Proof. Suppose $\widetilde{\mathcal{D}}_m$ satisfies that $\widetilde{\mathcal{D}}_m \subseteq \widetilde{\mathcal{D}}_{m+1} + H$ for $m \geq 1$. By the definition of $\widetilde{\mathcal{D}}_m$, for any $m \geq 1$, we have

$$\begin{aligned}
\widetilde{\mathcal{D}}_m \subseteq \widetilde{\mathcal{D}}_{m+1} + H \implies & K + \widetilde{\mathcal{D}}_m \subseteq K + \widetilde{\mathcal{D}}_{m+1} + H \\
\implies & K + \mathcal{D}_m - \sum_{j=1}^{m} B^{m-j}\varepsilon_j \subseteq K + \mathcal{D}_{m+1} - \sum_{j=1}^{m+1} B^{m+1-j}\varepsilon_j + H \\
\implies & B^m K - \sum_{j=1}^{m} B^{m-j}\varepsilon_j \subseteq B^{m+1}K - \sum_{j=1}^{m+1} B^{m+1-j}\varepsilon_j + H \\
\implies & K - \sum_{j=1}^{m} B^{-j}\varepsilon_j \subseteq BK - \sum_{j=1}^{m+1} B^{1-j}\varepsilon_j + B^{-m}H \\
\implies & K \subseteq BK - (\sum_{j=1}^{m+1} B^{1-j}\varepsilon_j - \sum_{j=1}^{m} B^{-j}\varepsilon_j) + B^{-m}H.
\end{aligned}$$

Given $\varepsilon > 0$, we have

$$\sum_{j=1}^{m} B^{-j}\varepsilon_j - \sum_{j=1}^{m+1} B^{-j+1}\varepsilon_j + B^{-m}H \subset B(0, \varepsilon)$$

if m is large enough, since $\sum_{j=1}^{\infty} B^{-j}\varepsilon_j = 0$, H is compact and B is expansive. So $K \subseteq BK + B(0, \varepsilon)$ for any $\varepsilon > 0$. Letting $\varepsilon \longrightarrow 0$, we obtain that $K \subseteq BK$.
Conversely, suppose that $K \subset BK$. For any $m \in \mathbb{N}$ and any $\eta_1, \eta_2 \cdots, \eta_m \in \mathcal{D}$, we have

the identity

$$\sum_{j=1}^{m} B^{m-j}(\eta_j - \varepsilon_j) = B^m \sum_{j=1}^{\infty} B^{-j}(\rho_j - \varepsilon_j), \tag{3.2.2}$$

where $\rho_j = \eta_j$ for $1 \leqslant j \leqslant m$ and $\rho_j = \varepsilon_j$ for $j \geqslant m+1$. From (3.2.2) and the fact that $\sum_{j=1}^{\infty} B^{-j}\varepsilon_j = 0$, it follows that $\sum_{j=1}^{m} B^{m-j}(\eta_j - \varepsilon_j) \in B^m K$. The inclusion $K \subseteq BK$ implies that $\sum_{j=1}^{m} B^{m-j}(\eta_j - \varepsilon_j) \in B^{m+1}K$. Hence, given any $m \geq 1$, there exists a sequence $\{\eta'_j\}_{j=1}^{\infty}$, where $\eta'_j \in \mathcal{D}$, such that

$$\begin{aligned}
\sum_{j=1}^{m} B^{m-j}(\eta_j - \varepsilon_j) &= B^{m+1} \sum_{j=1}^{\infty} B^{-j}\eta'_j \\
&= B^{m+1} \sum_{j=1}^{\infty} B^{-j}(\eta'_j - \varepsilon_j) \\
&= \sum_{j=1}^{m+1} B^{m+1-j}(\eta'_j - \varepsilon_j) + \sum_{j=1}^{\infty} B^{-j}(\eta'_{j+m+1} - \varepsilon_{j+m+1}). \quad (3.2.3)
\end{aligned}$$

Since $\sum_{j=1}^{\infty} B^{-j}(\eta'_{j+m+1} - \varepsilon_{j+m+1}) \in K - K$, we obtain that $\widetilde{\mathcal{D}}_m \subseteq \widetilde{\mathcal{D}}_{m+1} + (K - K)$ from (3.2.3). □

Remark 3.2.2. Note that in the proof of Theorem 3.2.1, we do not assume that $|K| > 0$. That is to say that Theorem 3.2.1 holds if K is a compact set satisfying equation $BK = \bigcup_{d \in \mathcal{D}} K + d$, where $\mathcal{D}_m, \widetilde{\mathcal{D}}_m$ are defined similarly.

Remark 3.2.3. In fact, Theorem 3.2.1 shows that $K \subseteq BK$ if and only if for any $m \in \mathbb{N}$ and any $\eta_1, \eta_2, \cdots, \eta_m \in \mathcal{D}$, there exists a sequence $\{\eta'_j\}_{j=1}^{\infty}$, where $\eta'_j \in \mathcal{D}$, such that

$$\sum_{j=1}^{m} B^{m-j}(\eta_j - \varepsilon_j) = \sum_{j=1}^{m+1} B^{m+1-j}(\eta'_j - \varepsilon_j) + \sum_{j=1}^{\infty} B^{-j}(\eta'_{j+m+1} - \varepsilon_{j+m+1}), \tag{3.2.4}$$

where the sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ satisfies $\sum_{j=1}^{\infty} B^{-j}\varepsilon_j = 0$, $\varepsilon_j \in \mathcal{D}$. Define

$$a_i = \begin{cases} 1, & \text{if } \varepsilon_{\mathrm{i}} = \mathrm{d}_1 \\ -1, & \text{if } \varepsilon_{\mathrm{i}} = \mathrm{d}_2. \end{cases} \tag{3.2.5}$$

Then for each $j \in \{1, 2, \ldots, m+1\}$, there exists $\delta_j, \delta'_j \in \{0, 1\}$ such that

$$\eta_j - \varepsilon_j = a_j\delta_j(d_2 - d_1), \quad \eta'_j - \varepsilon_j = a_j\delta'_j(d_2 - d_1).$$

And for $j \geq 1$, there exists $\delta''_{j+m+1} \in \{-1, 0, 1\}$ such that $\eta'_{j+m+1} - \varepsilon_{j+m+1} = \delta''_{j+m+1}(d_2 - d_1)$. Thus equation (3.2.4) can be written as

$$\sum_{j=1}^{m} B^{m-j}a_j\delta_j(d_2 - d_1) = \sum_{j=1}^{m+1} B^{m+1-j}a_j\delta'_j(d_2 - d_1) + \sum_{j=1}^{\infty} B^{-j}\delta''_{j+m+1}(d_2 - d_1). \tag{3.2.6}$$

Lemma 3.2.4. *Let K be a self-affine tile associated with an expanding matrix $B \in M_n^{(2)}(\mathbb{Z})$ and a digit set $\mathcal{D} = \{d_1, d_2\}$ satisfying (3.0.1). For any given $m \in \mathbb{Z}$, if $\sum_{j=m}^{\infty} B^{-j}\delta_j(d_2 - d_1) = 0$ for some $\delta_j \in \{-1, 0, 1\}$, then $\sum_{j=m}^{\infty} B^{-j}\delta_j = 0$.*

Proof. Let $C = \sum_{j=m}^{\infty} B^{-j}\delta_j$ and let $M = \{u \in \mathbb{R}^n : \ Cu = 0\}$, then M is a subspace of $\mathbb{R}^n$. Obviously, $0 \in M$ and by assumption, $d_2 - d_1 \in M$. Therefore, $\sum_{j=1}^{\infty} B^{-j}\varepsilon_j \in M$ if $\varepsilon_j \in \{0, d_2 - d_1\}$. Let

$$K' = \Big\{ \sum_{j=1}^{\infty} B^{-j}\varepsilon_j, \quad \varepsilon_j \in \{0, d_2 - d_1\} \Big\}.$$

Then $K' = K - \sum_{j=1}^{\infty} B^{-j}d_1$ and $K' \subseteq M$. This implies that $|K'| = |K| > 0$ since K is a self-affine tile, and $|M| \geq |K'| > 0$. M is a subspace of $\mathbb{R}^n$, hence, $M = \mathbb{R}^n$. This proves that $C = 0$. □

Remark 3.2.5. Lemma 3.2.4 can also be stated in the following way:
Let K be a self-affine tile associated with an expanding matrix $B \in M_n^{(2)}(\mathbb{Z})$ and a digit set $\mathcal{D} = \{d_1, d_2\}$ (in particular, $|K| > 0$). If $C(d_2 - d_1) = 0$ for some invertible matrix C which commutes with B, then $C = 0$.

Remark 3.2.6. By Remark 3.2.3, Remark 3.2.5 and (3.2.6), $K(B, \mathcal{D}) \subseteq BK(B, \mathcal{D})$ if and only if given $m \in \mathbb{N}$, for any sequence $(\delta_1, \delta_2, \ldots, \delta_m) \in \{0,1\}^m$, there exists a sequence $\{\delta'_j\}_{j=1}^{m+1} \in \{0,1\}^{m+1}$ and $\delta''_j \in \{-1, 0, 1\}$ for $j > m+1$, such that

$$\sum_{j=1}^{m} B^{m-j} a_j \delta_j = \sum_{j=1}^{m+1} B^{m+1-j} a_j \delta'_j + \sum_{j=1}^{\infty} B^{-j} \delta''_{j+m+1}, \tag{3.2.7}$$

where a_j, $j \in \mathbb{N}$ is defined as in (3.2.5).

If K is a self-affine tile satisfying $BK = (K + d_1) \bigcup (K + d_2)$, where $B \in M_n^{(2)}(\mathbb{Z})$ is expansive, then using Remark 3.2.6, we will characterize the digit set $\mathcal{D} := \{d_1, d_2\}$ for which the inclusion $K(B, \mathcal{D}) \subset BK(B, \mathcal{D})$ is satisfied. That is, if $K(B, \mathcal{D}) \subset BK(B, \mathcal{D})$, then $0 \in K(B, \mathcal{D})$ and it follows from (3.0.2) that there exists a sequence $\{\varepsilon_j\}_{j=1}^{\infty}$, where $\varepsilon_j \in \mathcal{D}$, such that $0 = \sum_{j=1}^{\infty} B^{-j} \varepsilon_j$. Define a_j as in (3.2.5), then sequences $\{\varepsilon_j\}_{j=1}^{\infty}$ and $\{a_j\}_{j=1}^{\infty}$ are uniquely determined by one another. If the sequence $\{a_j\}_{j=1}^{\infty}$ satisies (3.2.7) for any given $m \geq 1$ and any sequence $(\delta_1, \delta_2, \ldots, \delta_m) \in \{0,1\}^m$, then the relationship between $\{a_j\}_{j=1}^{\infty}$ and $\{\varepsilon_j\}_{j=1}^{\infty}$ and the fact that $\sum_{j=1}^{\infty} B^{-j} \varepsilon_j = 0$ will give us an equation relating the digits d_1 and d_2 which makes the inclusion $K(B, \mathcal{D}) \subseteq BK(B, \mathcal{D})$ possible. Therefore, the problem of characterizing the digit set $\mathcal{D} = \{d_1, d_2\}$ which satisfies the inclusion $K(B, \mathcal{D}) \subseteq BK(B, \mathcal{D})$ is reduced to finding sequences $\{a_j\}_{j=1}^{\infty}$ such that for any given $m \geq 1$ and any sequence $(\delta_1, \delta_2, \ldots, \delta_m) \in \{0,1\}^m$, the matrix identity (3.2.7) is satisfied, or, equivalently, such that the corresponding vector identity

$$\sum_{j=1}^{m} B^{m-j} a_j \delta_j z = \sum_{j=1}^{m+1} B^{m+1-j} a_j \delta'_j z + \sum_{j=1}^{\infty} B^{-j} \delta''_{j+m+1} z, \tag{3.2.8}$$

holds for any fixed $z \in \mathbb{R}^2$, with $z \neq 0$.

Let $d \in \mathbb{Z}^n \setminus \{0\}$ be a vector such that $\mathcal{D}_0 := \{0, d\}$ is a complete set of coset representatives for $\mathbb{Z}^n/B\mathbb{Z}^n$. Then d and Bd are linearly independent by Lemma 3.4.1 and thus $\{d,\ Bd\}$ is a basis for the Euclidean space $\mathbb{R}^2$. If the vector d satisfies (3.2.8) with $z = d$, then the vector $z = Bd$ also satisfies it and furthermore, any linear combination of d and Bd satisfies (3.2.8). Therefore, the inclusion $K(B, \mathcal{D}) \subseteq BK(B, \mathcal{D})$ holds if and only if the following identity is satisfied for the vector d and for all $m \geq 1$,

$$\sum_{j=1}^{m} B^{m-j} a_j \delta_j d = \sum_{j=1}^{m+1} B^{m+1-j} a_j \delta'_j d + \sum_{j=1}^{\infty} B^{-j} \delta''_{j+m+1} d. \tag{3.2.9}$$

Note that, since both $\sum_{j=1}^{m} B^{m-j} a_j \delta_j d$ and $\sum_{j=1}^{m+1} B^{m+1-j} a_j \delta'_j d$ are in $\mathbb{Z}^n$, the term $\sum_{j=1}^{\infty} B^{-j} \delta''_{j+m+1} d$ $\in \mathbb{Z}^n$ by (3.2.9). Let $K' := \Big\{ \sum_{j=1}^{\infty} B^{-j} \delta''_{j+m+1} d, \delta''_{j+m+1} \in \{-1, 0, 1\} \Big\}$. Then K' satisfies the set-valued equation

$$BK' = K' \bigcup (K' - d) \bigcup (K' + d). \tag{3.2.10}$$

In order to find digits d_1, d_2 which satisfy the inclusion $K(B, \mathcal{D}) \subseteq BK(B, \mathcal{D})$, it will be necessary to find the integer points of the set K'. Once this is solved, we will determine which sequence $\{a_j\}_{j=1}^{m+1} \in \{-1, 1\}^{m+1}$ satisfies (3.2.9) for any fixed $m \geq 1$ and for any $\{\delta_j\}_{j=1}^{m} \in \{0, 1\}^m$. Letting $m \longrightarrow \infty$, we can find all sequences $\{a_j\}_{j=1}^{\infty}$ satisfying (3.2.9) for any $m \geq 1$. In the last section, we use this method to characterize the digit sets $\mathcal{D} = \{d_1, d_2\}$ such that $K(B, \mathcal{D}) \subseteq BK(B, \mathcal{D})$.

To summarize the above, we have the following result

Proposition 3.2.7. Let $B \in M_n^{(2)}(\mathbb{Z})$ be an expanding $n \times n$ matrix and let K be a self-affine tile satisfying the set equation $BK = (K + d_1) \bigcup (K + d_2)$. Let $d \in \mathbb{Z}^n \setminus \{0\}$ be a vector such that $\mathcal{D}_0 := \{0, d\}$ is a complete set of coset representatives for $\mathbb{Z}^n/B\mathbb{Z}^n$. Then the inclusion $K(B, \mathcal{D}) \subseteq BK(B, \mathcal{D})$ holds if and only if for any $\delta_1, \ldots, \delta_m \in \{0, 1\}$, there exist $\delta'_1, \ldots, \delta'_m, \delta'_{m+1} \in \{0, 1\}$ and some $x \in K' \bigcap \mathbb{Z}^n$ such that

$$\sum_{j=1}^{m} B^{m-j} a_j \delta_j d = \sum_{j=1}^{m+1} B^{m+1-j} a_j \delta'_j d + x,$$

where $a_j,\ j \geq 1$ are defined by (3.2.5) and K' is defined in (3.2.10).

Remark 3.2.8. We will show that if a sequence $\{a_j\}_{j=1}^{\infty}$ as above can be constructed, then the corresponding digits d_1 and d_2 can be formed such that the inclusion $K(B,\mathcal{D}) \subseteq BK(B,\mathcal{D})$ holds. The particular case where $\{a_j\}_{j=1}^{\infty}$ is a constant sequence, corresponds to the situation where $0 \in \mathcal{D}$ and the inclusion is then obvious.

Gabardo and Yu [25, 24] gave a characterization for the integral points in $K(B,\mathcal{D})$ when the digit set $\mathcal{D}$ is a complete set of coset representatives for $\mathbb{Z}^n/B\mathbb{Z}^n$. The following lemma extends their result by characterizing the integral points of $K(B,\mathcal{D})$ when the digit set $\mathcal{D} \subset \mathbb{Z}^n$ contains a complete set of coset representatives for $\mathbb{Z}^n/B\mathbb{Z}^n$. In particular, Lemma 3.2.9 below gives an algorithm to find $K' \bigcap \mathbb{Z}^n$, where K' is defined above.

Lemma 3.2.9. *Let B be an $n \times n$ integral expansive matrix and K be a self-affine set which satisfies that $BK = \bigcup_{i=1}^{M} K + d_i$, where $\mathcal{D} := \{d_1, d_2, \cdots, d_M\} \subseteq \mathbb{Z}^n$ contains a complete set of coset representatives for $\mathbb{Z}^n/B\mathbb{Z}^n$. Then $K \bigcap \mathbb{Z}^n = -S$, where $S = \bigcup S_i$ and S_i is defined as follows:*

$$S_0 := \{k \in \mathbb{Z}^n : (I - B^m)k = \sum_{i=0}^{m-1} B^i d_i,\ d_i \in \mathcal{D},\ \text{for some m} \geqslant 1\},$$

$$S_1 := \{x \in \mathbb{Z}^n : B(-x) = d - s_0,\ \text{for some d} \in \mathcal{D} \text{ and some } \mathrm{s}_0 \in \mathrm{S}_0\},$$

$$S_2 := \{x \in \mathbb{Z}^n : B(-x) = d - s_1,\ \text{for some d} \in \mathcal{D},\ \text{and some } \mathrm{s}_1 \in \mathrm{S}_1\},$$

$$\vdots$$

Proof. Suppose $k \in S_0$ and $(I - B^m)k = \sum_{i=0}^{m-1} B^i d_i$ for some $m \geq 1$. Let $z \in K$ be defined by

$$z = B^{-1}d_{m-1} + B^{-2}d_{m-2} + \cdots + B^{-m}d_0 + B^{-(m+1)}d_{m-1} + \cdots + B^{-2m}d_0 + \cdots.$$

Then $B^m z = B^{m-1}d_{m-1} + B^{m-2}d_{m-2} + \cdots + d_0 + z$. Thus $(I - B^m)(-z) = d_0 + Bd_1 + \cdots + B^{m-1}d_{m-1}$ and we obtain $-z = k \in (-K) \bigcap \mathbb{Z}^n$. Therefore, $-S_0 \subseteq K \bigcap \mathbb{Z}^n$. For any

$x \in S_1 \subseteq \mathbb{Z}^n$, there is, by definition, some $s_0 \in S_0,\ d \in \mathcal{D}$ such that $-x = -B^{-1}s_0 + B^{-1}d$. Since $-s_0 \in K \bigcap \mathbb{Z}^n$, there exists a sequence $\{\varepsilon_j\}_{j=1}^{\infty}$, where $\varepsilon_j \in \mathcal{D}$, such that $-s_0 = \sum_{j=1}^{\infty} B^{-j}\varepsilon_j$. Thus $-x = B^{-1}d + B^{-1}\sum_{j=1}^{\infty} B^{-j}\varepsilon_j \in K$. This implies that $-S_1 \subseteq K \bigcap \mathbb{Z}^n$. Similarly, we can prove that $-S_i \subseteq K \bigcap \mathbb{Z}^n$ for $i \geq 2$. So $-S \subseteq K \bigcap \mathbb{Z}^n$.

Conversely, let $z \in K \bigcap \mathbb{Z}^n$. Since $BK = \bigcup_{i=1}^{M} K + d_i$, then $Bz = d_1 + x_1$ for some $d_1 \in \mathcal{D}$ and $x_1 \in K$. Thus $x_1 = Bz - d_1 \in K \bigcap \mathbb{Z}^n$. Then $Bx_1 = d_2 + x_2$ for some $d_2 \in \mathcal{D}$ and $x_2 \in K$, thus $x_2 = Bx_1 - d_2 \in K \bigcap \mathbb{Z}^n$. By induction, we have $x_N = Bx_{N-1} - d_N \in K \bigcap \mathbb{Z}^n$ for any $N \geqslant 1$, where $x_0 = z$. Since K is compact, then $K \bigcap \mathbb{Z}^n$ is finite. Therefore, there must exist two integers $m \geqslant 1,\ N \geqslant 1$ such that $x_N = x_{N+m}$. That is, we have

$$\begin{aligned}
Bz = d_1 + x_1 &\iff -x_1 = d_1 + B(-z)\\
Bx_1 = d_2 + x_2 &\iff -x_2 = d_2 + B(-x_1)\\
&\vdots\\
Bx_{N-1} = d_N + x_N &\iff -x_N = d_N + B(-x_{N-1})\\
&\vdots\\
Bx_{N+m-1} = d_{N+m} + x_{N+m} &\iff -x_{N+m} = d_{N+m} + B(-x_{N+m-1}).
\end{aligned}$$

The fact that $x_N = x_{N+m}$ implies that

$$\begin{aligned}
-x_N = -x_{N+m} &= d_{N+m} + B(-x_{N+m-1})\\
&= d_{N+m} + Bd_{N+m-1} + B^2(-x_{N+m-2})\\
&\vdots\\
&= d_{N+m} + Bd_{N+m-1} + B^2 d_{N+m-2} + \cdots + B^{m-1}d_{N+1} + B^m(-x_N).
\end{aligned}$$

Then we get $(I - B^m)(-x_N) = d_{N+m} + Bd_{N+m-1} + \cdots + B^{m-1}d_{N+1}$, which implies that $-x_N \in S_0$. By the above argument and the definition of S_i, we have

$$-x_N \in S_0 \Longrightarrow -x_{N-1} \in S_1 \Longrightarrow -x_{N-2} \in S_2 \Longrightarrow \cdots \Longrightarrow -z \in S_N.$$

Hence, $-z \in S$ and $K \bigcap \mathbb{Z}^n \subseteq -S$. So $K \bigcap \mathbb{Z}^n = -S$. □

Remark 3.2.10. If $K \subseteq \mathbb{R}^n$ is an integral self-affine tile associated with an $n \times n$ expanding matrix B and a digit set $\mathcal{D} = \{d_1, d_2, \cdots, d_m\}$, i.e. $BK = \bigcup_{i=1}^{m} K + d_i$, where $\mathcal{D}$ is a complete set of coset representatives for $\mathbb{Z}^n/B\mathbb{Z}^n$, then the set $K - K$ satisfies that $B(K-K) = \bigcup_{\ell \in \mathcal{D}-\mathcal{D}} (K-K) + \ell$. Note that $\mathcal{D} - \mathcal{D}$ contains a complete set of coset representatives for $\mathbb{Z}^n/B\mathbb{Z}^n$ since $\mathcal{D}$ is a complete set of coset representatives for $\mathbb{Z}^n/B\mathbb{Z}^n$. Thus using Lemma 3.2.9, we can compute $(K-K)\bigcap \mathbb{Z}^n$, which is the set W defined in section 3 of [25].

3.3 Self-affine MRA scaling sets in $\mathbb{R}$

In this section, we will give a brief description of the self-affine MRA scaling set in dimension one. The condition $|\det A| = 2$ yields only two possibilities in dimension one: $A = 2$ or $A = -2$.

Lemma 3.3.1. *Assume $K \subset \mathbb{R}$ is a self-affine tile satisfying $BK = (K + d_1)\bigcup(K + d_2)$, where $B = 2$ or -2, then the set K is an interval in $\mathbb{R}$. Moreover, if $B = 2$, $K = [d_1, d_2]$ and if $B = -2$, $K = [\frac{1}{3}d_1 - \frac{2}{3}d_2, -\frac{2}{3}d_1 + \frac{1}{3}d_2]$, where $d_1 < d_2$.*

It is easy to see that for K to be a $\mathbb{Z}-$tiling set, we need $d_2 - d_1 = 1$ whenever $B = 2$ or $B = -2$. Let $d_1 = a$, then $d_2 = 1+a$. By Lemma 0.0.7 and Lemma 3.3.1, $K = [a, 1+a]$ is 2-dilation scaling set if and only if $-1 < a < 0$; $K = [-\frac{1}{3}a - \frac{2}{3}, -\frac{1}{3}a + \frac{1}{3}]$ is a -2-dilation scaling set if and only if $-2 < a < 1$.

3.4 Self-affine MRA scaling sets in $\mathbb{R}^2$

In this section, we will turn our attention to the two dimensional case. We will characterize the digit sets $\mathcal{D}$ for which the inclusion $K(B, \mathcal{D}) \subseteq BK(B, \mathcal{D})$ holds and those for which $K(B, \mathcal{D})$ is an A-dilation MRA scaling set.

Lemma 3.4.1. *Let $B \in M_n^{(2)}(\mathbb{Z})$ be expanding, where $n \geq 2$, and K be a self-affine tile which satisfies $BK = (K + d_1) \bigcup (K + d_2)$, then $d_2 - d_1$ is not an eigenvector of the matrix B.*

Proof. By assumption, $BK = [K \bigcup K + (d_2 - d_1)] + d_1$, then $K = K' + \sum_{j=1}^{\infty} B^{-j} d_1$, where the set K' satisfies that $BK' = K' \bigcup K' + (d_2 - d_1)$. Let $d := d_2 - d_1$. Suppose that d is an eigenvector of the matrix B corresponding to the eigenvalue λ, i.e. $Bd = \lambda d$. By (3.0.2) and our assumption,

$$K' = \Big\{ \sum_{j=1}^{\infty} B^{-j} \varepsilon_j,\ \varepsilon_j \in \{0, d\} \} = \{ \sum_{j=1}^{\infty} \lambda^{-j} \varepsilon_j,\ \varepsilon_j \in \{0, d\} \Big\} \subset \Big\{ cd, c \in \mathbb{R} \Big\}. \quad (3.4.1)$$

Thus K' is contained in a 1-dimensional subspace of $\mathbb{R}^n$ with $n \geq 2$ and $|K'| = 0$. However, $|K'| = |K| > 0$ since K is a self-affine tile. Therefore, $d = d_2 - d_1$ cannot be an eigenvector of the matrix B. □

Let us define matrices $C_i,\ i = 1, 2, 3, 4$ by

$$C_1 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}, C_3 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, C_4 = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}.$$

Two $n \times n$ integral matrices A and $\widetilde{A}$ are called integrally similar if there exists an integral matrix P with $|\det P| = 1$ such that $P^{-1}AP = \widetilde{A}$. Lagarias and Wang [40] completely classified all integral expansive matrices $B \in M_2^{(2)}(\mathbb{Z})$ and showed that there are exactly six integrally similar classes of such integral matrices. Representatives from each of these classes are given by $C_1,\ C_2,\ C_3,\ -C_3,\ C_4,\ -C_4$, respectively. Indeed, we have showed in Chapter 2 that if A is an expanding matrix in $M_n^{(2)}(\mathbb{Z})$ and A is integrally similar to $\widetilde{A}$ using the matrix S, i.e. $S^{-1}AS = \widetilde{A}$, where S is an integral matrix with $|\det S| = 1$, then K is an A-dilation MRA scaling set if and only if PK is an $\widetilde{A}$-dilation MRA scaling set, where $P = S^t$. Therefore, in order to find all digit sets $\mathcal{D} = \{d_1, d_2\} \subset \mathbb{R}^2$ such that the self-affine set $K(B, \mathcal{D})$ is an MRA scaling set associated with a different expansive integral matrix $A \in M_2^{(2)}(\mathbb{Z})$, it suffices to solve the problem for each one of the six representative matrices listed above. Similarly, the problem of the inclusion $K \subset BK$ can be reduced to the 6 cases above.

3.4.1 The case $A = C_1$

Theorem 3.4.2. *Let $A = C_1 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ and K be a self-affine tile associated with the matrix B and the digit set $\mathcal{D} = \{d_1, d_2\} \subset \mathbb{R}^2$, where $B = A^t$. Let $d_1 = \begin{pmatrix} d_{11} \\ d_{12} \end{pmatrix}$, $d_2 = \begin{pmatrix} d_{21} \\ d_{22} \end{pmatrix}$, then $K(B, \mathcal{D}) \subset BK(B, \mathcal{D})$ if and only if the digits d_1, d_2 satisfy the equation*

$$\frac{1}{(d_{21}-d_{11})^2 - 2(d_{22}-d_{12})^2} \begin{pmatrix} d_{21}-d_{11} & -2(d_{22}-d_{12}) \\ -(d_{22}-d_{12}) & d_{21}-d_{11} \end{pmatrix} \begin{pmatrix} d_{11} & d_{21} \\ d_{12} & d_{22} \end{pmatrix} = \begin{pmatrix} \tilde{a} & 1+\tilde{a} \\ 0 & 0 \end{pmatrix}.$$

for some $-1 \le \widetilde{a} \le 0$. Moreover, $K(B, \mathcal{D})$ is an A-dilation MRA scaling set if and only if the digits d_1, d_2 satisfy the following conditions:

(i) $d_{21} - d_{11} \in \mathbb{Z}$, $d_{22} - d_{12} \in \mathbb{Z}$.

(ii) $(d_{21} - d_{11})^2 - 2(d_{22} - d_{12})^2 \in \{1, -1\}$.

(iii) $\begin{pmatrix} d_{21}-d_{11} & -2(d_{22}-d_{12}) \\ -(d_{22}-d_{12}) & d_{21}-d_{11} \end{pmatrix} \begin{pmatrix} d_{11} & d_{21} \\ d_{12} & d_{22} \end{pmatrix} = \varepsilon \begin{pmatrix} \tilde{a} & 1+\tilde{a} \\ 0 & 0 \end{pmatrix}$, *for some $-1 < \widetilde{a} < 0$ and $\varepsilon \in \{\pm 1\}$.*

Proof. If $A = C_1 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, then $B = A^t = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. Let $d := d_2 - d_1 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, Lemma 3.4.1 implies that $x_0 \neq \pm\sqrt{2} y_0$. Let C be a matrix commuting with B, then C has the general form $C = \begin{pmatrix} a & 2c \\ c & a \end{pmatrix}$ for some $a,\ c \in \mathbb{R}$. Moreover, $CK(B, \mathcal{D})$ is a self-affine tile associated with the matrix B and the digit set $C\mathcal{D}$ if C is invertible since

$$\begin{aligned} BCK = CBK &= (CK + Cd_1) \bigcup (CK + Cd_2) \\ &= (CK \bigcup CK + Cd) + Cd_1. \end{aligned} \tag{3.4.2}$$

Let $\mathcal{D}_0 = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$. It is known (e.g.[40]) that $K(B, \mathcal{D}_0)$ is the unit square,

$$K(B, \mathcal{D}_0) = conv\Big\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\Big\}, \tag{3.4.3}$$

where $conv(E)$ denotes the convex hull of E. Suppose that $Cd = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then we have

$$\begin{pmatrix} a & 2c \\ c & a \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{cases} ax_0 + 2cy_0 = 1 \\ cx_0 + ay_0 = 0 \end{cases} \Longleftrightarrow \begin{pmatrix} x_0 & 2y_0 \\ y_0 & x_0 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since $x_0^2 - 2y_0^2 \neq 0$, the matrix $\begin{pmatrix} x_0 & 2y_0 \\ y_0 & x_0 \end{pmatrix}$ is invertible and we obtain

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} x_0 & 2y_0 \\ y_0 & x_0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{x_0^2 - 2y_0^2} \begin{pmatrix} x_0 & -2y_0 \\ -y_0 & x_0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{x_0^2 - 2y_0^2} \begin{pmatrix} x_0 \\ -y_0 \end{pmatrix}.$$

This shows that given a self-affine tile $K(B,\mathcal{D})$ with associated digit set $\mathcal{D}=\{d_1,d_2\}$, where $d_2-d_1=\binom{x_0}{y_0}$, we can always find an invertible matrix $C:=\frac{1}{x_0^2-2y_0^2}\begin{pmatrix}x_0 & -2y_0\\ -y_0 & x_0\end{pmatrix}$ such that $CK(B,\mathcal{D})$ is a translation of the unite square $K(B,\mathcal{D}_0)$ by (3.4.2), i.e.

$$CK(B,\mathcal{D})=K(B,\mathcal{D}_0)+C\sum_{j=1}^{\infty}B^{-j}d_1. \tag{3.4.4}$$

It is obvious that $K(B,\mathcal{D})\subseteq BK(B,\mathcal{D})\Longleftrightarrow CK(B,\mathcal{D})\subseteq BCK(B,\mathcal{D})$ and we will now characterize when this last inclusion holds. Let $\widetilde{K}:=CK$, (3.4.4) implies that $\widetilde{K}$ is also a unit square. Let $C\sum_{j=1}^{\infty}B^{-j}d_1=\binom{\tilde{a}}{\tilde{b}}$. It follows from (3.4.3) and (3.4.4) that

$$\widetilde{K}=conv\Big\{\binom{\tilde{a}}{\tilde{b}},\binom{\tilde{a}}{1+\tilde{b}},\binom{1+\tilde{a}}{\tilde{b}},\binom{1+\tilde{a}}{1+\tilde{b}}\Big\}, \tag{3.4.5}$$

and, therefore,

$$B\widetilde{K}=conv\Big\{\binom{2\tilde{b}}{\tilde{a}},\binom{2(1+\tilde{b})}{\tilde{a}},\binom{2\tilde{b}}{1+\tilde{a}},\binom{2(1+\tilde{b})}{1+\tilde{a}}\Big\}. \tag{3.4.6}$$

The sets $\widetilde{K}$ and $B\widetilde{K}$ are depicted in Figure 3.1. (3.4.5) and (3.4.6) show that $\widetilde{K}\subseteq B\widetilde{K}$ if and only if $\tilde{a}=\tilde{b}$ and $-1\leq\widetilde{a}\leq 0$ (see Figure 3.1). In this case, (3.4.5) and (3.4.6) are reduced to

$$\widetilde{K}=conv\Big\{\binom{\tilde{a}}{\tilde{a}},\binom{\tilde{a}}{1+\tilde{a}},\binom{1+\tilde{a}}{\tilde{a}},\binom{1+\tilde{a}}{1+\tilde{a}}\Big\}. \tag{3.4.7}$$

$$B\widetilde{K}=conv\Big\{\binom{2\tilde{a}}{\tilde{a}},\binom{2(1+\tilde{a})}{\tilde{a}},\binom{2\tilde{a}}{1+\tilde{a}},\binom{2(1+\tilde{a})}{1+\tilde{a}}\Big\}. \tag{3.4.8}$$

Therefore, we have

$$B\widetilde{K}=\widetilde{K}+\binom{\tilde{a}}{0}\bigcup\widetilde{K}+\binom{1+\tilde{a}}{0}, \tag{3.4.9}$$

and we can let

$$Cd_1=\binom{\tilde{a}}{0},\quad Cd_2=\binom{1+\tilde{a}}{0}. \tag{3.4.10}$$

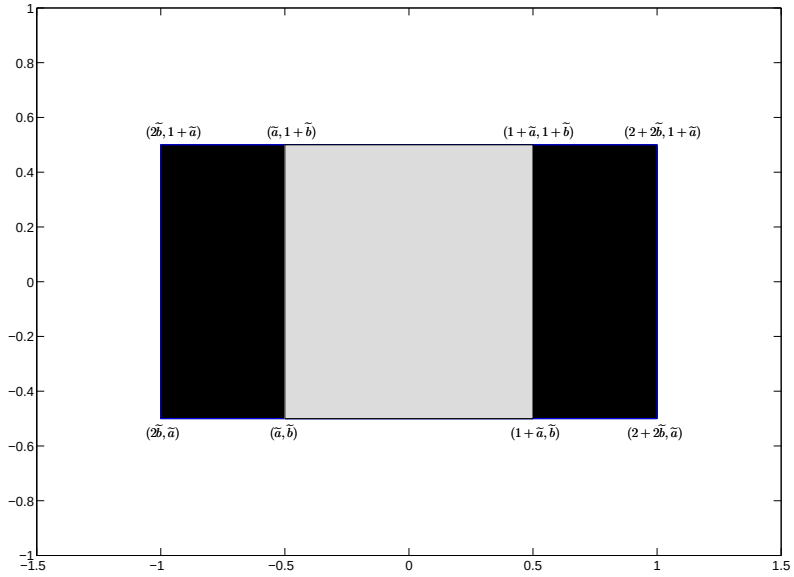


Figure 3.1: $\widetilde{K}$ and $B\widetilde{K}$ in Theorem 3.4.2.

Since $d_1 = \binom{d_{11}}{d_{12}}$ and $d_2 = \binom{d_{21}}{d_{22}}$, we obtain that $x_0 = d_{21} - d_{11}$, $y_0 = d_{22} - d_{12}$. By (3.4.10) and the above argument, we obtain that $K \subseteq BK$ if and only if the digits $d_1,\ d_2$ satisfy the equation

$$C\begin{pmatrix} d_{11} & d_{21} \\ d_{12} & d_{22} \end{pmatrix} = \begin{pmatrix} \tilde{a} & 1+\tilde{a} \\ 0 & 0 \end{pmatrix} \tag{3.4.11}$$

for some $-1 \le \widetilde{a} \le 0$, where $C = \frac{1}{(d_{21}-d_{11})^2 - 2(d_{22}-d_{12})^2}\begin{pmatrix} d_{21}-d_{11} & -2(d_{22}-d_{12}) \\ -(d_{22}-d_{12}) & d_{21}-d_{11} \end{pmatrix}$, which proves the first part of our claim. Assume that $K(B, \mathcal{D})$ is an A-dilation MRA scaling set, then $K(B, \mathcal{D})$ is a $\mathbb{Z}^2$-tiling set by Lemma 0.0.7. Thus $K(B, \mathcal{D}')$ is also a $\mathbb{Z}^2$-tiling set, where $\mathcal{D}' = \{0, d\}$ and $d = d_2 - d_1$. It has been proved in Theorem 2.4.1 that a digit $d = \binom{x_0}{y_0}$ such that $K(B, \mathcal{D}')$ is a $\mathbb{Z}^2$-tiling set must satisfy

$$x_0^2 - 2y_0^2 = \pm 1, \quad x_0,\ y_0 \in \mathbb{Z}. \tag{3.4.12}$$

Thus the matrix C is an integral matrix with $|\det C| = \pm 1$ and $K(B, \mathcal{D})$ is an A-dilation MRA scaling set if and only if $\widetilde{K} := CK(B, \mathcal{D})$ is one as well. It follows from (3.4.7)

that $\widetilde{K}$ contains a neighborhood of 0 if and only if $-1 < \widetilde{a} < 0$ and particularly, if $\widetilde{a} = 0$ or -1, $\widetilde{K} = \pm conv\Big\{\binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1}\Big\}$. Neither of these two sets is an A-dilation MRA scaling set. This also shows that no integral digit set $\mathcal{D} \subset \mathbb{Z}^2$ can be such that $K(B, \mathcal{D})$ is an A-dilation MRA scaling set. Based on the above results and (3.4.12), using Lemma 0.0.7, we get $K(B, \mathcal{D})$ is an A-dilation MRA scaling set if and only if the digits $d_1 = \binom{d_{11}}{d_{12}}$, $d_2 = \binom{d_{21}}{d_{22}}$ satisfy the following conditions:

(i) $d_{21} - d_{11} \in \mathbb{Z}$, $d_{22} - d_{12} \in \mathbb{Z}$,

(ii) $(d_{21} - d_{11})^2 - 2(d_{22} - d_{12})^2 = \pm 1$,

(iii) $\begin{pmatrix} d_{21} - d_{11} & -2(d_{22} - d_{12}) \\ -(d_{22} - d_{12}) & d_{21} - d_{11} \end{pmatrix} \begin{pmatrix} d_{11} & d_{21} \\ d_{12} & d_{22} \end{pmatrix} = \varepsilon \begin{pmatrix} \tilde{a} & 1 + \tilde{a} \\ 0 & 0 \end{pmatrix}$, for some $-1 < \widetilde{a} < 0$ and $\varepsilon \in \{\pm 1\}$.

□

By the discussion for the matrix C_1, it follows that there exist non-integral self-affine tiles which are A-dilation MRA scaling sets, where $A = C_1$. However, no integral self-affine tile is an MRA scaling set associated with this matrix as with $B = 2$ in one dimension. Example 3.4.3 below gives a non-integral self-affine tile which is an MRA scaling set associated with this matrix. This illustrates the usefulness of constructing wavelet sets using non-integral self-affine tiles.

Example 3.4.3. $A = C_1 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, then $B = A^t = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. Let

$$K = conv\Big\{\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\Big\} \tag{3.4.13}$$

Then by (3.4.13),

$$BK = conv\Big\{\begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}\Big\} \tag{3.4.14}$$

(3.4.13) and (3.4.14) show that

$$BK = (K + \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}) \bigcup (K + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}). \tag{3.4.15}$$

(3.4.15) implies that K is a self-affine tile associated with the matrix B and the digit set $\mathcal{D} = \{\binom{-\frac{1}{2}}{0}, \binom{\frac{1}{2}}{0}\}$. Also, by Theorem 3.4.2, the set K is an A-dilation MRA scaling set.

3.4.2 The case $A = C_2$

The discussion for the matrix C_2 is similar to that of the matrix C_1. For completeness, we will write the details below.

Theorem 3.4.4. *Let $A = C_2 = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$ and K be a self-affine tile associated with the matrix $B = A^t = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$ and the digit set $\mathcal{D} = \{d_1, d_2\} \subset \mathbb{R}^2$. Let $d_1 = \begin{pmatrix} d_{11} \\ d_{12} \end{pmatrix}$, $d_2 = \begin{pmatrix} d_{21} \\ d_{22} \end{pmatrix}$, then $K(B, \mathcal{D}) \subset BK(B, \mathcal{D})$ if and only if the digits d_1, d_2 satisfy the equation*

$$\frac{1}{(d_{21}-d_{11})^2 + 2(d_{22}-d_{12})^2} \begin{pmatrix} d_{21}-d_{11} & 2(d_{22}-d_{12}) \\ -(d_{22}-d_{12}) & d_{21}-d_{11} \end{pmatrix} \begin{pmatrix} d_{11} & d_{21} \\ d_{12} & d_{22} \end{pmatrix} = \begin{pmatrix} -a_0 & 1-a_0 \\ 0 & 0 \end{pmatrix}.$$

for some $0 \le a_0 \le 1$. Moreover, $K(B, \mathcal{D})$ is an A-dilation MRA scaling set if and only if the digits d_1, d_2 satisfy the following condition:

$$\begin{pmatrix} d_{11} & d_{21} \\ d_{12} & d_{22} \end{pmatrix} = \varepsilon \begin{pmatrix} -a_0 & 1-a_0 \\ 0 & 0 \end{pmatrix}, \text{ for some } 0 \le a_0 \le 1 \text{ and } \varepsilon \in \{\pm 1\}.$$

Proof. If $A = C_2 = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$, then $B = A^t = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$. Let $d = d_2 - d_1 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. Then a matrix C commuting with B has the general form $C = \begin{pmatrix} a & -2c \\ c & a \end{pmatrix}$ for some $a,\ c \in \mathbb{R}$. The set $CK(B, \mathcal{D})$ is a self-affine tile associated with the matrix B and the digit set $C\mathcal{D}$ if C is invertible since

$$\begin{aligned} BCK = CBK &= (CK + Cd_1) \bigcup (CK + Cd_2) \\ &= (CK \bigcup CK + Cd) + Cd_1. \end{aligned} \tag{3.4.16}$$

If $Cd = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we have

$$\begin{pmatrix} a & -2c \\ c & a \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{cases} ax_0 - 2cy_0 = 1 \\ cx_0 + ay_0 = 0 \end{cases} \Longleftrightarrow \begin{pmatrix} x_0 & -2y_0 \\ y_0 & x_0 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since $d_1 \neq d_2$, i.e. $x_0^2 + 2y_0^2 \neq 0$, we have

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} x_0 & -2y_0 \\ y_0 & x_0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{x_0^2 + 2y_0^2} \begin{pmatrix} x_0 & 2y_0 \\ -y_0 & x_0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence, given a digit set $\mathcal{D} = \{d_1, d_2\}$, where $d_2 - d_1 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, we can always find an invertible matrix $C = \frac{1}{x_0^2+2y_0^2} \begin{pmatrix} x_0 & 2y_0 \\ -y_0 & x_0 \end{pmatrix}$ such that $CK(B, \mathcal{D})$ is a translation of $K(B, \mathcal{D}_0)$

by (3.4.16), where $\mathcal{D}_0 = \{\binom{0}{0}, \binom{1}{0}\}$, i.e.

$$CK(B,\mathcal{D}) = K(B,\mathcal{D}_0) + C\sum_{j=1}^{\infty} B^{-j}d_1. \tag{3.4.17}$$

It was proved in [40] that $K(B,\mathcal{D}_0)$ is the unit square

$$K(B,\mathcal{D}_0) = conv\Big\{\begin{pmatrix}\frac{1}{3}\\ -\frac{1}{3}\end{pmatrix}, \begin{pmatrix}\frac{1}{3}\\ \frac{2}{3}\end{pmatrix}, \begin{pmatrix}-\frac{2}{3}\\ \frac{2}{3}\end{pmatrix}, \begin{pmatrix}-\frac{2}{3}\\ -\frac{1}{3}\end{pmatrix}\Big\}. \tag{3.4.18}$$

By (3.4.17), $CK(B,\mathcal{D})$ is also a unit square and similarly as for the matrix C_1, we can characterize the inclusion $K(B,\mathcal{D}) \subseteq BK(B,\mathcal{D})$ by characterizing the inclusion $CK(B,\mathcal{D}) \subseteq BCK(B,\mathcal{D})$. Let $\widetilde{K} := CK$ and $C\sum_{j=1}^{\infty} B^{-j}d_1 = \binom{\tilde{a}}{\tilde{b}}$. (3.4.17) and (3.4.18) give

$$\widetilde{K} = conv\Big\{\begin{pmatrix}\frac{1}{3}+\tilde{a}\\ -\frac{1}{3}+\tilde{b}\end{pmatrix}, \begin{pmatrix}\frac{1}{3}+\tilde{a}\\ \frac{2}{3}+\tilde{b}\end{pmatrix}, \begin{pmatrix}-\frac{2}{3}+\tilde{a}\\ \frac{2}{3}+\tilde{b}\end{pmatrix}, \begin{pmatrix}-\frac{2}{3}+\tilde{a}\\ -\frac{1}{3}+\tilde{b}\end{pmatrix}\Big\}, \tag{3.4.19}$$

and therefore,

$$B\widetilde{K} = conv\Big\{\begin{pmatrix}-\frac{2}{3}+2\tilde{b}\\ -\frac{1}{3}-\tilde{a}\end{pmatrix}, \begin{pmatrix}\frac{4}{3}+2\tilde{b}\\ -\frac{1}{3}-\tilde{a}\end{pmatrix}, \begin{pmatrix}\frac{4}{3}+2\tilde{b}\\ \frac{2}{3}-\tilde{a}\end{pmatrix}, \begin{pmatrix}-\frac{2}{3}+2\tilde{b}\\ \frac{2}{3}-\tilde{a}\end{pmatrix}\Big\}. \tag{3.4.20}$$

The sets $\widetilde{K}$ and $B\widetilde{K}$ are depicted in Figure 3.2. Then $\widetilde{K} \subseteq B\widetilde{K}$ if and only if $\tilde{b} = -\tilde{a}$ and $0 \le \tilde{a} \le \frac{1}{3}$ (see Figure 3.2) and in this case,

$$\widetilde{K} = conv\Big\{\begin{pmatrix}\frac{1}{3}+\tilde{a}\\ -\frac{1}{3}-\tilde{a}\end{pmatrix}, \begin{pmatrix}\frac{1}{3}+\tilde{a}\\ \frac{2}{3}-\tilde{a}\end{pmatrix}, \begin{pmatrix}-\frac{2}{3}+\tilde{a}\\ \frac{2}{3}-\tilde{a}\end{pmatrix}, \begin{pmatrix}-\frac{2}{3}+\tilde{a}\\ -\frac{1}{3}-\tilde{a}\end{pmatrix}\Big\}. \tag{3.4.21}$$

$$B\widetilde{K} = conv\Big\{\begin{pmatrix}-\frac{2}{3}-2\tilde{a}\\ -\frac{1}{3}-\tilde{a}\end{pmatrix}, \begin{pmatrix}\frac{4}{3}-2\tilde{a}\\ -\frac{1}{3}-\tilde{a}\end{pmatrix}, \begin{pmatrix}\frac{4}{3}-2\tilde{a}\\ \frac{2}{3}-\tilde{a}\end{pmatrix}, \begin{pmatrix}-\frac{2}{3}-2\tilde{a}\\ \frac{2}{3}-\tilde{a}\end{pmatrix}\Big\}. \tag{3.4.22}$$

(3.4.21) and (3.4.22) show that

$$B\widetilde{K} = \widetilde{K} + \begin{pmatrix}-a_0\\ 0\end{pmatrix} \bigcup \widetilde{K} + \begin{pmatrix}1-a_0\\ 0\end{pmatrix}, \tag{3.4.23}$$

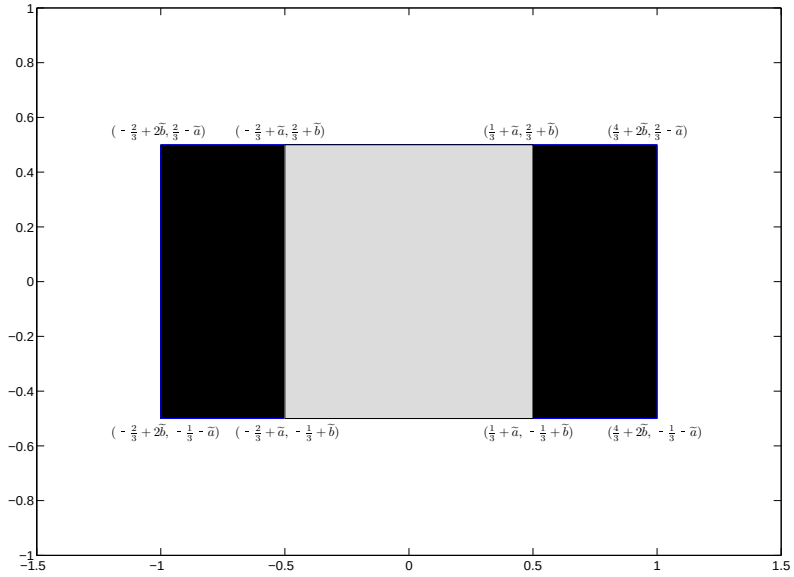


Figure 3.2: $\widetilde{K}$ and $B\widetilde{K}$ in Theorem 3.4.4.

where $a_0 = 3\tilde{a}$, and we can let

$$Cd_1 = \begin{pmatrix} -a_0 \\ 0 \end{pmatrix}, \quad Cd_2 = \begin{pmatrix} 1-a_0 \\ 0 \end{pmatrix}. \tag{3.4.24}$$

Since $d_1 = \begin{pmatrix} d_{11} \\ d_{12} \end{pmatrix}$, $d_2 = \begin{pmatrix} d_{21} \\ d_{22} \end{pmatrix}$, $x_0 = d_{21} - d_{11}$, $y_0 = d_{22} - d_{12}$. By (3.4.23) and the above argument, it follows that the inclusion $K \subseteq BK$ holds if and only if the digits d_1, d_2 satisfy the equation

$$C\begin{pmatrix} d_{11} & d_{21} \\ d_{12} & d_{22} \end{pmatrix} = \begin{pmatrix} -a_0 & 1-a_0 \\ 0 & 0 \end{pmatrix}, \tag{3.4.25}$$

for some $0 \le a_0 \le 1$, where $C = \frac{1}{(d_{21}-d_{11})^2+2(d_{22}-d_{12})^2}\begin{pmatrix} d_{21}-d_{11} & 2(d_{22}-d_{12}) \\ -(d_{22}-d_{12}) & d_{21}-d_{11} \end{pmatrix}$, which proves the first part of the statement in the theorem. It follows from (3.4.21) that $\widetilde{K}$ contains a neighborhood of 0 for any $0 \le a_0 \le 1$. Assume that $K(B, \mathcal{D})$ is an A-dilation MRA scaling set, then $K(B, \mathcal{D})$ is a $\mathbb{Z}^2$-tiling set by Lemma 0.0.7. Hence, $K(B, \mathcal{D}')$ is also a $\mathbb{Z}^2$-tiling set, where $\mathcal{D}' = \{0, d\}$ and $d = d_2 - d_1$. It has been proved in Theorem 2.5.1 that $d = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ if $K(B, \mathcal{D}')$ is a $\mathbb{Z}^2$-tiling set. Thus the matrix $C = I$ or $-I$. Then

$K(B,\mathcal{D})$ is an A-dilation MRA scaling set if and only if $\widetilde{K} := CK(B,\mathcal{D})$ is one as well. Combining (3.4.21), (3.4.25) and Lemma 0.0.7, we get $K(B,\mathcal{D})$ is an A-dilation MRA scaling set if and only if digits $d_1 = \binom{d_{11}}{d_{12}}$, $d_2 = \binom{d_{21}}{d_{22}}$ satisfy the condition:

$$\begin{pmatrix} d_{11} & d_{21} \\ d_{12} & d_{22} \end{pmatrix} = \varepsilon \begin{pmatrix} -a_0 & 1-a_0 \\ 0 & 0 \end{pmatrix}, \text{ where } 0 \leq \mathrm{a}_0 \leq 1 \text{ and } \varepsilon \in \{\pm 1\}.$$

□

3.4.3 The case $A = C_3, -C_3, C_4, -C_4$

When the matrix A is integrally similar to one of the matrices C_1 and C_2, the set $K(B,\mathcal{D})$ is simple enough that the inclusion $K(B,\mathcal{D}) \subseteq BK(B,\mathcal{D})$ can be verified by simple algebraic manipulations. However, this is not the cases for the remaining four matrices and we will need to use the method introduced in section 3 to find conditions for the inclusion $K(B,\mathcal{D}) \subseteq BK(B,\mathcal{D})$. We know, from the discussion in section 3 and by taking $d = \binom{1}{0}$ in (3.2.9), that the fact that $K(B,\mathcal{D}) \subseteq BK(B,\mathcal{D})$ implies the existence of a sequence $\{a_j\}_{j=1}^{\infty}$ with $a_j = \pm 1$, for all j, such that the inclusion

$$K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2 \tag{3.4.26}$$

holds for all $m \geq 1$, where

$$K_1^{(m)} := \Big\{ \sum_{j=1}^{m} B^{m-j} a_j \delta_j \tbinom{1}{0},\ \delta_j \in \{0,1\} \Big\},$$

$$K_2^{(m)} := \Big\{ \sum_{j=1}^{m+1} B^{m+1-j} a_j \delta'_j \tbinom{1}{0},\ \delta'_j \in \{0,1\} \Big\},$$

$$K_3 := \Big\{ \sum_{j=1}^{\infty} B^{-j} \delta'_{j+m+1} \tbinom{1}{0},\ \delta'_{j+m+1} \in \{-1,0,1\} \Big\}.$$

In the following, we will determine for which digit sets $\mathcal{D} := \{d_1, d_2\}$ the inclusion $K(B,\mathcal{D}) \subseteq BK(B,\mathcal{D})$ is satisfied by finding which sequence $\{a_j\}_{j=1}^{\infty}$ satisfies (3.4.26) for all $m \geq 1$. The strategy will be to sart with $m = 1$ and determine all the couples (a_1, a_2) with $a_i = \pm 1$, such that $K_1^{(1)} \subseteq K_2^{(1)} + K_3 \bigcap \mathbb{Z}^2$. Inductively, if a sequence

$(a_1, \ldots, a_{m+1})$ satisfies $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$, we then check whether or not the sequences $(a_1, a_2, \ldots, a_{m+1}, \pm 1)$ satisfy $K_1^{(m+1)} \subseteq K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$.

It is clear, from its definition, that K_3 is independent of m and $K_3 = -K_3$. So if $\{a_j\}_{j=1}^{\infty}$, where $a_j \in \{-1, 1\}$, satisfies (3.4.26) for any $m \geq 1$, then $\{-a_j\}_{j=1}^{\infty}$ also satisfies (3.4.26). Therefore, it suffices to consider sequence $\{a_j\}_{j=1}^{\infty}$, where $a_j \in \{-1, 1\}$ and $a_1 = 1$. On the other hand, the constant sequences $\pm\{1, 1, \ldots, 1, \ldots\}$ always satisfies (3.2.7) since we can take $\delta'_{j+1} = \delta_j$ for $1 \leq j \leq m$, $\delta'_1 = 0$ and $\delta'_{j+m+1} = 0$ for $j \geq 1$. In the following, we will concentrate on looking for non-constant sequences $\{a_j\}_{j=1}^{\infty}$ with $a_1 = 1$, where $a_j \in \{-1, 1\}$, such that it satisfies (3.4.26) for any $m \geq 1$, and then we provide conditions equivalent to the inclusion $K(B, \mathcal{D}) \subseteq BK(B, \mathcal{D})$.

Before discussing the matrix C_3, we show how to find $K_3 \bigcap \mathbb{Z}^2$ using Lemma 3.2.9.

Proposition 3.4.5. Let $B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Define $K_3 := \Big\{ \sum_{j=1}^{\infty} B^{-j}\delta_j,\ \delta_j \in \{\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\} \Big\}$. Then

$$K_3 \bigcap \mathbb{Z}^2 = \Big\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Big\}.$$

Proof. Since $\|B^{-1}\| = \frac{1}{\sqrt{2}}$, then K_3 is contained in the disk centered at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with radius $1 + \sqrt{2}$. Let

$$K_4 : \ = \Big\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Big\}.$$

Then $K_3 \bigcap \mathbb{Z}^2 \subseteq K_4 \bigcup (-K_4)$. Define $S_i, i \geq 0$ as in Lemma 3.2.9. By the definition of K_3 and S_i, $j \in K_3(S_i)$ if and only if $-j \in K_3(S_i)$. Therefore, it is enough to find explicitly the intersection $K_3 \bigcap K_4$, which we will do using the criterion in Lemma 3.2.9. Obviously, $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in S_0$. We have $(I - B)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, and

$$(I - B^6)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + B\begin{pmatrix} 0 \\ 0 \end{pmatrix} + B^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + B^3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + B^4\begin{pmatrix} 0 \\ 0 \end{pmatrix} + B^5\begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

$$(I - B^6)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + B\begin{pmatrix} 1 \\ 0 \end{pmatrix} + B^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + B^3\begin{pmatrix} 0 \\ 0 \end{pmatrix} + B^4\begin{pmatrix} -1 \\ 0 \end{pmatrix} + B^5\begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

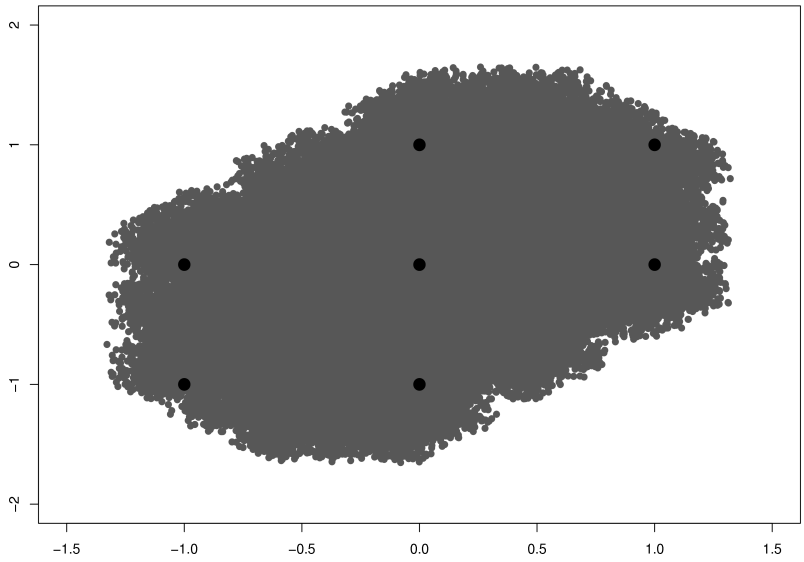


Figure 3.3: The set K_3 and its integer points in Proposition 3.4.5.

which implies that $\binom{0}{1}, \binom{1}{0}, \binom{1}{1} \in S_0$. So

$$R_0 := \left\{ \binom{-1}{-1}, \binom{-1}{0}, \binom{0}{-1}, \binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1} \right\} \subseteq S_0.$$

Next we will prove that R_0 is invariant in the sense that for $x \in R_0$, if $x = d + Bx'$ for some $d \in \{\binom{-1}{0}, \binom{0}{0}, \binom{1}{0}\}$ and some $x' \in \mathbb{Z}^2$, then $x' \in R_0$. In the following, we will list all possible representations of the elements in R_0 in the form $d + Bx'$, where $d \in \{\binom{-1}{0}, \binom{0}{0}, \binom{1}{0}\}$ and $x' \in \mathbb{Z}^2$. It is enough here to list all possible representations for the elements $\binom{-1}{-1}, \binom{-1}{0}, \binom{0}{-1}$ and $\binom{0}{0}$ in R_0 because of the symmetry of the digit set $\{\binom{-1}{0}, \binom{0}{0}, \binom{1}{0}\}$ and the set R_0.

$$\binom{-1}{-1} = \binom{0}{0} + B\binom{0}{-1}, \quad \binom{-1}{0} = \begin{cases} \binom{-1}{0} + B\binom{0}{0} \\ \binom{1}{0} + B\binom{-1}{-1}, \end{cases}$$

$$\binom{0}{0} = \binom{0}{0} + B\binom{0}{0}, \quad \binom{0}{-1} = \begin{cases} \binom{-1}{0} + B\binom{1}{0} \\ \binom{1}{0} + B\binom{0}{-1}. \end{cases}$$

These representations prove our claim. Let

$$R_0^- := \left\{\binom{-1}{-1}, \binom{-1}{0}, \binom{0}{-1}\right\}, \quad R_0^+ := \left\{\binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1}\right\}.$$

Then $R_0 = R_0^+ \bigcup R_0^-$. And let

$$R_1 := K_4 \setminus R_0^+ = \left\{\binom{0}{2}, \binom{2}{0}, \binom{1}{2}, \binom{2}{1}, \binom{1}{-1}, \binom{1}{-2}, \binom{2}{-1}\right\}.$$

Since, we have

$$\binom{0}{2} = \binom{0}{0} + B\binom{0}{0} + B^2\binom{-1}{0}, \quad \binom{2}{0} = \binom{0}{0} + B\binom{1}{1}, \quad \binom{1}{-1} = \binom{0}{0} + B\binom{1}{0},$$

$$\binom{1}{2} = \begin{cases} \binom{1}{0} + B\binom{0}{0} + B^2\binom{-1}{0}, \\ \binom{-1}{0} + B\binom{0}{0} + B^2\binom{0}{0} + B^3\binom{-1}{0}, \end{cases} \qquad \binom{1}{-2} = \begin{cases} \binom{1}{0} + B\binom{0}{0} + B^2\binom{1}{0}, \\ \binom{-1}{0} + B\binom{0}{0} + B^2\binom{1}{1}, \end{cases}$$

$$\binom{2}{1} = \begin{cases} \binom{1}{0} + B\binom{0}{1}, \\ \binom{-1}{0} + B\binom{1}{0} + B^2\binom{0}{0} + B^3\binom{-1}{0}, \\ \binom{-1}{0} + B\binom{-1}{0} + B^2\binom{0}{0} + B^3\binom{0}{0} + B^4\binom{-1}{0}, \end{cases}$$

$$\binom{2}{-1} = \begin{cases} \binom{1}{0} + B\binom{1}{0}, \\ \binom{-1}{0} + B\binom{1}{0} + B^2\binom{0}{1}, \\ \binom{-1}{0} + B\binom{-1}{0} + B^2\binom{1}{0} + B^3\binom{0}{0} + B^4\binom{-1}{0}, \\ \binom{-1}{0} + B\binom{-1}{0} + B^2\binom{-1}{0} + B^3\binom{0}{0} + B^4\binom{0}{0} + B^5\binom{-1}{0}. \end{cases}$$

By the above expressions of the elements in R_1, for any element $x \in R_1$, $x = \sum_{i=0}^{m-1} B^i d_i + B^m x_0$ for some $d_i \in \left\{\binom{-1}{0}, \binom{0}{0}, \binom{1}{0}\right\}$, $1 \le i \le m-1$ and some $x_0 \in R_0$. By the definition of S_0, if $x \in S_0$, then $x = d_0 + Bd_1 + \cdots + B^m d_m + B^{m+1}x$ for some $d_i \in \left\{\binom{-1}{0}, \binom{0}{0}, \binom{1}{0}\right\}$, $0 \le i \le m$. The representations of the elements in R_1 and the invariance property of the set R_0 show that $R_1 \bigcap S_0 = \emptyset$. Hence $S_0 = R_0$. It follows from the definition of S_1 and the representation of the elements in S_0 given before that $S_1 = S_0$.

Thus $\bigcup S_i = S_0$. Therefore, we get

$$K_3 \bigcap \mathbb{Z}^2 = \left\{ \binom{-1}{-1}, \binom{-1}{0}, \binom{0}{-1}, \binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1} \right\}.$$

□

Lemma 3.4.6 below is from [24] and it provides a representation for integer points associated with the matrix B and the digit set $\mathcal{D}$ assuming that $\mathcal{D} \subset \mathbb{Z}^n$ is a complete set of coset representatives for the group $\mathbb{Z}^n/B\mathbb{Z}^n$. Lemma 3.4.7 is an application of Lemma 3.4.6. We state here these two lemmas for the convenience of later use.

Lemma 3.4.6. *Suppose that $B \in M_n(\mathbb{Z})$ is expanding and $\mathcal{D} \subset \mathbb{Z}^n$ is a complete set of coset representatives for $\mathbb{Z}^n/B\mathbb{Z}^n$. Then every $k \in \mathbb{Z}^n$ has a unique representation*

$$k = \sum_{i=0}^{m-1} B^i d_i + B^m s, \ \text{for some } m \geq 0, \ d_i \in \mathcal{D} \text{ and } s \in S, \tag{3.4.27}$$

with the convention that the first term on the right-hand side of the above equality is 0 if $m = 0$, where $S = -K \bigcap \mathbb{Z}^n$. We say that $k \in \mathbb{Z}^n$ has a finite radix representation if $s = 0$ in the representation (3.4.27).

Lemma 3.4.7. *Let $B = C_3^t = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $\mathcal{D}' = \{\binom{0}{0}, \binom{1}{0}\}$. Then $\mathcal{D}'$ is a complete set of coset representatives for $\mathbb{Z}^2/B\mathbb{Z}^2$. Let K' be the self-affine tile associated with the matrix B and the digit set $\mathcal{D}'$. Let*

$$H = \left\{ \binom{2}{1}, \binom{2}{2}, \binom{1}{1}, \binom{2}{0}, \binom{0}{1}, \binom{1}{2} \right\}.$$

Then, no element $y \in H$, y has a finite radix representation.

Proof. For the matrix $B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $\mathcal{D}' = \{\binom{0}{0}, \binom{1}{0}\}$, it was showed in [24] that

$$S := -K' \bigcap \mathbb{Z}^2 = \left\{ \binom{0}{0}, \binom{0}{-1} \right\}.$$

However, for each element in H, we have

$$\begin{aligned}
\binom{2}{1} &= \binom{1}{0} + B\binom{1}{0} + B^2\binom{1}{0} + B^3\binom{0}{0} + B^4\binom{0}{-1},\\
\binom{2}{2} &= \binom{0}{0} + B\binom{0}{0} + B^2\binom{0}{0} + B^3\binom{1}{0} + B^4\binom{0}{0} + B^5\binom{0}{-1},\\
\binom{1}{1} &= \binom{0}{0} + B\binom{1}{0} + B^2\binom{1}{0} + B^3\binom{0}{0} + B^4\binom{0}{-1},\\
\binom{2}{0} &= \binom{0}{0} + B\binom{0}{0} + B^2\binom{1}{0} + B^3\binom{1}{0} + B^4\binom{0}{0} + B^5\binom{0}{-1},\\
\binom{1}{2} &= \binom{1}{0} + B\binom{0}{0} + B^2\binom{1}{0} + B^3\binom{0}{0} + B^4\binom{0}{-1},\\
\binom{0}{1} &= \binom{1}{0} + B\binom{1}{0} + B^2\binom{0}{0} + B^3\binom{0}{-1}.
\end{aligned}$$

This shows that for each $y \in H$, $s = \binom{0}{-1}$ using the representaition (3.4.27). So our claim holds. □

Theorem 3.4.8. *Let $A = C_3 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Then $K(B,\mathcal{D}) \subseteq BK(B,\mathcal{D})$ if and only if $0 \in \mathcal{D}$ and $K(B,\mathcal{D})$ cannot be an A-dilation scaling set for any digit set $\mathcal{D} \subset \mathbb{R}^2$.*

Proof. Here, $B = A^t = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. By the previous discussion, the inclusion $K(B,\mathcal{D}) \subseteq BK(B,\mathcal{D})$ holds if and only if (3.4.26) holds for all $m \geq 1$. In order to find conditions equivalent to the inclusion $K(B,\mathcal{D}) \subseteq BK(B,\mathcal{D})$, it is enough to find non-constant sequences $\{a_j\}_{j=1}^{\infty}$ with $a_1 = 1$, where $a_j \in \{-1, 1\}$, such that it satisfies (3.4.26) for all $m \geq 1$. From Proposition 3.4.5,

$$K_3 \bigcap \mathbb{Z}^2 = \left\{\binom{-1}{-1}, \binom{-1}{0}, \binom{0}{-1}, \binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1}\right\}.$$

If $m = 1$ and $(a_1, a_2) = (1, -1)$, then

$$K_1^{(1)} = \left\{a_1\delta_1\binom{1}{0},\ \delta_1 \in \{0,1\}\right\} = \left\{\delta_1\binom{1}{0},\ \delta_1 \in \{0,1\}\right\} = \left\{\binom{0}{0}, \binom{1}{0}\right\},$$

$$\begin{aligned}
K_2^{(1)} &= \left\{Ba_1\delta_1'\binom{1}{0} + a_2\delta_2'\binom{1}{0},\ \delta_1', \delta_2' \in \{0,1\}\right\} = \left\{B\delta_1'\binom{1}{0} - \delta_2'\binom{1}{0}, \delta_1', \delta_2' \in \{0,1\}\right\}\\
&= \left\{\binom{0}{0}, \binom{-1}{0}, \binom{1}{-1}, \binom{0}{-1}\right\}.
\end{aligned}$$

Obviously, $K_1^{(1)} \subseteq K_2^{(1)} + K_3 \bigcap \mathbb{Z}^2$.

If $m = 2$ and $(a_1, a_2, a_3) = (1, 1, -1)$, then

$$\begin{aligned} K_1^{(2)} &= \left\{ Ba_1\delta_1 \binom{1}{0} + a_2\delta_2 \binom{1}{0},\ \delta_1, \delta_2 \in \{0, 1\} \right\} \\ &= \left\{ B\delta_1 \binom{1}{0} + \delta_2 \binom{1}{0},\ \delta_1, \delta_2 \in \{0, 1\} \right\} = \left\{ \binom{0}{0}, \binom{1}{-1}, \binom{1}{0}, \binom{2}{-1} \right\}, \end{aligned}$$

$$\begin{aligned} K_2^{(2)} &= \left\{ B^2a_1\delta_1' \binom{1}{0} + Ba_2\delta_2' \binom{1}{0} + a_3\delta_3' \binom{1}{0},\ \delta_1', \delta_2', \delta_2' \in \{0, 1\} \right\} \\ &= \left\{ B^2\delta_1' \binom{1}{0} + B\delta_2' \binom{1}{0} - \delta_3' \binom{1}{0}, \delta_1', \delta_2', \delta_3' \in \{0, 1\} \right\} \\ &= \left\{ \binom{0}{0}, \binom{-1}{0}, \binom{1}{-1}, \binom{0}{-1}, \binom{0}{-2}, \binom{-1}{-2}, \binom{1}{-3}, \binom{0}{-3} \right\}. \end{aligned}$$

If $m = 2$ and $(a_1, a_2, a_3) = (1, -1, 1)$, then

$$K_1^{(2)} = \left\{ \binom{0}{0}, \binom{-1}{0}, \binom{1}{-1}, \binom{0}{-1} \right\},$$

$$K_2^{(2)} = \left\{ \binom{0}{0}, \binom{1}{0}, \binom{-1}{1}, \binom{0}{1}, \binom{0}{-2}, \binom{1}{-2}, \binom{-1}{-1}, \binom{0}{-1} \right\}.$$

If $m = 2$ and $(a_1, a_2, a_3) = (1, -1, -1)$, then

$$K_1^{(2)} = \left\{ \binom{0}{0}, \binom{-1}{0}, \binom{1}{-1}, \binom{0}{-1} \right\},$$

$$K_2^{(2)} = \left\{ \binom{0}{0}, \binom{-1}{0}, \binom{-1}{1}, \binom{-2}{1}, \binom{0}{-2}, \binom{-1}{-2}, \binom{-1}{-1}, \binom{-2}{-1} \right\}.$$

So $K_1^{(2)} \subseteq K_2^{(2)} + K_3 \bigcap \mathbb{Z}^2$ for any sequence $(a_1, a_2, a_3) \in \{1, -1\}^3$.

If $m = 3$ and $(a_1, a_2, a_3, a_4) = (1, 1, 1, -1)$, then

$$K_1^{(3)} = \left\{ \binom{0}{0}, \binom{1}{-1}, \binom{1}{0}, \binom{2}{-1}, \binom{0}{-2}, \binom{1}{-3}, \binom{1}{-2}, \binom{2}{-3} \right\},$$

$$\begin{aligned} K_2^{(3)} = \Big\{ &\binom{0}{0}, \binom{0}{-2}, \binom{1}{-1}, \binom{1}{-3}, \binom{-2}{-2}, \binom{-2}{-4}, \binom{-1}{-3}, \binom{-1}{-5}, \\ &\binom{-1}{0}, \binom{-1}{-2}, \binom{0}{-1}, \binom{0}{-3}, \binom{-3}{-2}, \binom{-3}{-4}, \binom{-2}{-3}, \binom{-2}{-5} \Big\}. \end{aligned}$$

For this case, $K_1^{(3)} \subseteq K_2^{(3)} + K_3 \bigcap \mathbb{Z}^2$.

If $m = 3$ and $(a_1, a_2, a_3, a_4) = (1, 1, -1, 1)$, then

$$K_1^{(3)} = \left\{ \binom{0}{0}, \binom{-1}{0}, \binom{1}{-1}, \binom{0}{-1}, \binom{0}{-2}, \binom{-1}{-2}, \binom{1}{-3}, \binom{0}{-3} \right\},$$

$$K_2^{(3)} = \Big\{\binom{0}{0}, \binom{0}{-2}, \binom{-1}{1}, \binom{-1}{-1}, \binom{-2}{-2}, \binom{-3}{-1}, \binom{-2}{-4}, \binom{-3}{-3},$$
$$\binom{1}{0}, \binom{0}{1}, \binom{1}{-2}, \binom{0}{-1}, \binom{-1}{-2}, \binom{-2}{-1}, \binom{-1}{-4}, \binom{-2}{-3}\Big\},$$

and, clearly, $K_1^{(3)} \subseteq K_2^{(3)} + K_3 \bigcap \mathbb{Z}^2$.

If $m = 3$ and $(a_1, a_2, a_3, a_4) = (1, 1, -1, -1)$, then

$$K_1^{(3)} = \Big\{\binom{0}{0}, \binom{-1}{0}, \binom{1}{-1}, \binom{0}{-1}, \binom{0}{-2}, \binom{-1}{-2}, \binom{1}{-3}, \binom{0}{-3}\Big\},$$

$$K_2^{(3)} = \Big\{\binom{0}{0}, \binom{0}{-2}, \binom{-1}{1}, \binom{-1}{-1}, \binom{-2}{-2}, \binom{-3}{-1}, \binom{-2}{-4}, \binom{-3}{-3},$$
$$\binom{-1}{0}, \binom{-2}{1}, \binom{-1}{-2}, \binom{-2}{-1}, \binom{-3}{-2}, \binom{-4}{-1}, \binom{-3}{-4}, \binom{-4}{-3}\Big\}.$$

In this case, $\binom{1}{-3} \in K_1^{(3)}$, but $\binom{1}{-3} \notin K_2^{(3)} + K_3 \bigcap \mathbb{Z}^2$, so $K_1^{(3)} \nsubseteq K_2^{(3)} + K_3 \bigcap \mathbb{Z}^2$.

If $m = 3$ and $(a_1, a_2, a_3, a_4) = (1, -1, 1, 1)$, then

$$K_1^{(3)} = \Big\{\binom{0}{0}, \binom{1}{0}, \binom{-1}{1}, \binom{0}{1}, \binom{0}{-2}, \binom{1}{-2}, \binom{-1}{-1}, \binom{0}{-1}\Big\},$$

$$K_2^{(3)} = \Big\{\binom{0}{0}, \binom{0}{2}, \binom{1}{-1}, \binom{1}{1}, \binom{-2}{-2}, \binom{-1}{-3}, \binom{-2}{0}, \binom{-1}{-1},$$
$$\binom{1}{0}, \binom{2}{-1}, \binom{1}{2}, \binom{2}{1}, \binom{-1}{-2}, \binom{0}{-3}, \binom{-1}{0}, \binom{0}{-1}\Big\}.$$

In this case, $K_1^{(3)} \subseteq K_2^{(3)} + K_3 \bigcap \mathbb{Z}^2$.

If $m = 3$ and $(a_1, a_2, a_3, a_4) = (1, -1, 1, -1)$, then

$$K_1^{(3)} = \Big\{\binom{0}{0}, \binom{1}{0}, \binom{-1}{1}, \binom{0}{1}, \binom{0}{-2}, \binom{1}{-2}, \binom{-1}{-1}, \binom{0}{-1}\Big\},$$

$$K_2^{(3)} = \Big\{\binom{0}{0}, \binom{0}{2}, \binom{1}{-1}, \binom{1}{1}, \binom{-2}{-2}, \binom{-1}{-3}, \binom{-2}{0}, \binom{-1}{-1},$$
$$\binom{-1}{0}, \binom{0}{-1}, \binom{-1}{2}, \binom{0}{1}, \binom{-3}{-2}, \binom{-2}{-3}, \binom{-3}{0}, \binom{-2}{-1}\Big\}.$$

In this case, $K_1^{(3)} \subseteq K_2^{(3)} + K_3 \bigcap \mathbb{Z}^2$.

If $m = 3$ and $(a_1, a_2, a_3, a_4) = (1, -1, -1, 1)$, then

$$K_1^{(3)} = \Big\{\binom{0}{0}, \binom{-1}{0}, \binom{-1}{1}, \binom{-2}{1}, \binom{0}{-2}, \binom{-1}{-2}, \binom{-1}{-1}, \binom{-2}{-1}\Big\},$$

$$K_2^{(3)} = \Big\{ \binom{0}{0}, \binom{0}{2}, \binom{-1}{1}, \binom{-1}{3}, \binom{-2}{-2}, \binom{-3}{-1}, \binom{-2}{0}, \binom{-3}{1}, \binom{1}{0}, \binom{0}{1}, \binom{1}{2}, \binom{0}{3}, \binom{-1}{-2}, \binom{-2}{-1}, \binom{-1}{0}, \binom{-2}{1} \Big\}.$$

In this case, $K_1^{(3)} \subseteq K_2^{(3)} + K_3 \bigcap \mathbb{Z}^2$.
If $m = 3$ and $(a_1, a_2, a_3, a_4) = (1, -1, -1, -1)$, then

$$K_1^{(3)} = \Big\{ \binom{0}{0}, \binom{-1}{0}, \binom{-1}{1}, \binom{-2}{1}, \binom{0}{-2}, \binom{-1}{-2}, \binom{-1}{-1}, \binom{-2}{-1} \Big\},$$

$$K_2^{(3)} = \Big\{ \binom{0}{0}, \binom{0}{2}, \binom{-1}{1}, \binom{-1}{3}, \binom{-2}{-2}, \binom{-3}{-1}, \binom{-2}{0}, \binom{-3}{1}, \binom{-1}{0}, \binom{-2}{1}, \binom{-1}{2}, \binom{-2}{3}, \binom{-3}{-2}, \binom{-4}{-1}, \binom{-3}{0}, \binom{-4}{1} \Big\}.$$

In this case, $\binom{0}{-2} \in K_1^{(3)}$, but $\binom{0}{-2} \notin K_2^{(3)} + K_3 \bigcap \mathbb{Z}^2$, so $K_1^{(3)} \nsubseteq K_2^{(3)} + K_3 \bigcap \mathbb{Z}^2$.
To summarize, if $m = 3$, the sequences $(1, a_2, a_3, a_4)$ which satisfy $K_1^{(3)} \subseteq K_2^{(3)} + K_3 \bigcap \mathbb{Z}^2$ are

$$(1,\ 1,\ 1,\ 1), (1,\ 1,\ 1,\ -1), (1,\ 1,\ -1,\ 1), (1,\ -1,\ 1,\ 1), (1,\ -1,\ -1,\ 1), (1,\ -1,\ 1,\ -1).$$

Using the strategy introduced at the beginning of this subsection, we can show which sequence satisfies $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$ and which does not, by using the information from the previous step. Here, we list the results for $m = 4, 5, 6$. For each $m \in \{4, 5, 6\}$, the sequences listed on the left-hand side satisfy $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$ and those listed on the right-hand side do not satisfy $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$. The vector after each sequence of the right-hand side is the one which belongs to $K_1^{(m)}$ but is not in $K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$.

This explains why the corresponding sequence does not satisfy (3.4.26).

$m = 4$, $(a_1,\ a_2,\ a_3,\ a_4,\ a_5)$;

$(1,\ 1,\ 1,\ 1,\ 1)$

$(1,\ 1,\ 1,\ 1,-1)$

$(1,\ 1,\ 1,-1,\ 1)$

$(1,\ 1,-1,\ 1,\ 1)$

$(1,\ 1,-1,\ 1,-1)$

$(1,-1,\ 1,\ 1,\ 1)$

$(1,-1,\ 1,\ 1,-1)$

$(1,-1,\ 1,-1,-1)$

$m = 4$, $(a_1,\ a_2,\ a_3,\ a_4,\ a_5)$

$(1,\ 1,\ 1,-1,-1)\ \binom{1}{-3}$

$(1,-1,\ 1,-1,\ 1)\ \binom{-2}{-3}$

$(1,-1,-1,\ 1,\ 1)\ \binom{-1}{-2}$

$(1,-1,-1,\ 1,-1)\ \binom{-2}{-2}$

$m = 5$, $(a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6)$;

$(1,\ 1,\ 1,\ 1,\ 1,\ 1)$

$(1,\ 1,\ 1,\ 1,\ 1,-1)$

$(1,\ 1,\ 1,\ 1,-1,\ 1)$

$(1,\ 1,\ 1,-1,\ 1,-1)$

$(1,\ 1,\ 1,-1,\ 1,\ 1)$

$(1,\ 1,-1,\ 1,\ 1,-1)$

$(1,\ 1,-1,\ 1,-1,-1)$

$(1,-1,\ 1,\ 1,\ 1,\ 1)$

$(1,-1,\ 1,\ 1,-1,-1)$

$m = 5$, $(a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6)$

$(1,\ 1,\ 1,\ 1,-1,-1)\ \binom{1}{-3}$

$(1,\ 1,-1,\ 1,\ 1,\ 1)\ \binom{-6}{-2}$

$(1,\ 1,-1,\ 1,-1,\ 1)\ \binom{-2}{-3}$

$(1,-1,\ 1,\ 1,\ 1,-1)\ \binom{3}{2}$

$(1,-1,\ 1,\ 1,-1,\ 1)\ \binom{-5}{-2}$

$(1,-1,\ 1,-1,-1,\ 1)\ \binom{-6}{-1}$

$(1,-1,\ 1,-1,-1,-1)\ \binom{-6}{-1}$

$m = 6$, $(a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7)$;

$(1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)$

$(1,\ 1,\ 1,\ 1,\ 1,\ 1,-1)$

$(1,\ 1,\ 1,-1,\ 1,\ 1,-1)$

$(1,\ 1,\ 1,\ 1,-1,\ 1,\ 1)$

$(1,\ 1,\ 1,\ 1,-1,\ 1,-1)$

$(1,\ 1,\ 1,\ 1,\ 1,-1,\ 1)$

$(1,\ 1,\ 1,-1,\ 1,-1,-1)$

$m = 6$, $(a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7)$

$(1,\ 1,\ 1,\ 1,\ 1,-1,-1)\ \binom{1}{-3}$

$(1,\ 1,\ 1,-1,\ 1,-1,\ 1)\ \binom{-2}{-3}$

$(1,\ 1,\ 1,-1,\ 1,\ 1,\ 1)\ \binom{-6}{-2}$

$(1,\ 1,-1,\ 1,\ 1,-1,\ 1)\ \binom{-5}{-2}$

$(1,\ 1,-1,\ 1,\ 1,-1,-1)\ \binom{-7}{6}$

$(1,\ 1,-1,\ 1,-1,-1,\ 1)\ \binom{-6}{-1}$

$(1,\ 1,-1,\ 1,-1,-1,-1)\ \binom{-6}{-1}$

$(1,-1,\ 1,\ 1,\ 1,\ 1,\ 1)\ \binom{6}{-1}$

$(1,-1,\ 1,\ 1,\ 1,\ 1,-1)\ \binom{6}{-1}$

$(1,-1,\ 1,\ 1,-1,-1,\ 1)\ \binom{-8}{3}$

$(1,-1,\ 1,\ 1,-1,-1,-1)\ \binom{-8}{3}$

Claim: For $m \geq 6$, sequences $\{a_j\}_{j=1}^{m+1}$ with $a_1 = 1$ which satisfy $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$ have the following forms.

$$
\begin{aligned}
& & (a_1, a_2, \ldots, a_{m-4}, a_{m-3}, a_{m-2}, a_{m-1}, a_m, a_{m+1}) \\
s_1 =& (1,\ 1, \ldots,\ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1) \\
s_2 =& (1,\ 1, \ldots,\ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ -1) \\
s_3 =& (1,\ 1, \ldots,\ 1,\ \ 1,\ -1,\ \ 1,\ \ 1,\ -1) \\
s_4 =& (1,\ 1, \ldots,\ 1,\ \ 1,\ \ 1,\ -1,\ \ 1,\ \ 1) \\
s_5 =& (1,\ 1, \ldots,\ 1,\ \ 1,\ \ 1,\ -1,\ \ 1,\ -1) \\
s_6 =& (1,\ 1, \ldots,\ 1,\ \ 1,\ \ 1,\ \ 1,\ -1,\ \ 1) \\
s_7 =& (1,\ 1, \ldots,\ 1,\ \ 1,\ -1,\ \ 1,\ -1,\ -1)
\end{aligned}
$$

Proof of claim: By computation, the claim holds for $m = 6$. Firstly, we will prove that sequences $s_i, 1 \leq i \leq 7$, satisfy $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$ for any $m \geq 6$. This is equivalent

to proving that for any $\delta_j \in \{0,1\}, 1 \leq j \leq m$, there exists $\delta'_j \in \{0,1\}, 1 \leq j \leq m+1$ and $\delta'_{j+m+1} \in \{-1,0,1\}, j \geq 1$ such that the following equation holds

$$\begin{aligned}
&B^{m-1}\delta_1 a_1\begin{pmatrix}1\\0\end{pmatrix} + B^{m-2}\delta_2 a_2\begin{pmatrix}1\\0\end{pmatrix} + \cdots + B^2\delta_{m-2}a_{m-2}\begin{pmatrix}1\\0\end{pmatrix} + B\delta_{m-1}a_{m-1}\begin{pmatrix}1\\0\end{pmatrix} + \delta_m a_m\begin{pmatrix}1\\0\end{pmatrix}\\
&= B^m\delta'_1 a_1\begin{pmatrix}1\\0\end{pmatrix} + B^{m-1}\delta'_2 a_2\begin{pmatrix}1\\0\end{pmatrix} + \cdots + B^4\delta'_{m-3}a_{m-3}\begin{pmatrix}1\\0\end{pmatrix} + B^3\delta'_{m-2}a_{m-2}\begin{pmatrix}1\\0\end{pmatrix} +\\
&B^2\delta'_{m-1}a_{m-1}\begin{pmatrix}1\\0\end{pmatrix} + B\delta'_m a_m\begin{pmatrix}1\\0\end{pmatrix} + \delta'_{m+1}a_{m+1}\begin{pmatrix}1\\0\end{pmatrix} + x \text{ for some } x \in K_3 \bigcap \mathbb{Z}^2. \qquad (3.4.28)
\end{aligned}$$

For $s_2 = (1,1,\ldots,1,1,-1)$, taking $\delta'_1 = \delta'_{m+1} = 0$, $\delta'_j = \delta_{j-1}$ for $2 \leq j \leq m$, then (3.4.28) holds if $x = \begin{pmatrix}0\\0\end{pmatrix}$ when $\delta_m = 0$ and $x = \begin{pmatrix}1\\0\end{pmatrix}$ when $\delta_m = 1$.
For $s_3 = (1,1,\ldots,1,-1,1,1,-1)$, taking $\delta'_1 = 0$, $\delta'_j = \delta_{j-1}$ for $2 \leq j \leq m-3$ and $\delta'_m = \delta_{m-1}$, then (3.4.28) can be reduced to the following equation

$$B^3\delta_{m-3}\begin{pmatrix}1\\0\end{pmatrix} - B^2\delta_{m-2}\begin{pmatrix}1\\0\end{pmatrix} + \delta_m\begin{pmatrix}1\\0\end{pmatrix} = -B^3\delta'_{m-2}\begin{pmatrix}1\\0\end{pmatrix} + B^2\delta'_{m-1}\begin{pmatrix}1\\0\end{pmatrix} - \delta'_{m+1}\begin{pmatrix}1\\0\end{pmatrix} + x$$

Letting $A = \left\{B^3\delta_{m-3}\begin{pmatrix}1\\0\end{pmatrix} - B^2\delta_{m-2}\begin{pmatrix}1\\0\end{pmatrix} + \delta_m\begin{pmatrix}1\\0\end{pmatrix}, \delta_{m-3}, \delta_{m-2}, \delta_m \in \{0,1\}\right\}$, we need to show that $A \subset -A + K_3 \bigcap \mathbb{Z}^2$. This is the case since

$$\begin{aligned}
A =& \left\{\delta_{m-3}\begin{pmatrix}-2\\-2\end{pmatrix} - \delta_{m-2}\begin{pmatrix}0\\-2\end{pmatrix} + \delta_m\begin{pmatrix}1\\0\end{pmatrix}, \delta_{m-3}, \delta_{m-2}, \delta_m \in \{0,1\}\right\}\\
=& \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}-2\\-2\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\end{pmatrix}, \begin{pmatrix}-1\\-2\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}\right\}\\
=& \left\{\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}-1\\-2\end{pmatrix} + \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix},\right.\\
&\left.\begin{pmatrix}-1\\0\end{pmatrix} + \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}-1\\-2\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}\right\}.
\end{aligned}$$

This proves that $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$, $m \geq 6$ for the sequence s_3.
For $s_4 = (1,1,\ldots,1,-1,1,1)$, taking $\delta'_1 = 0$, $\delta'_j = \delta_{j-1}$ for $2 \leq j \leq m-2$ and $\delta'_{m+1} = \delta_m$, then (3.4.28) can be reduced to the following equation

$$B^2\delta_{m-2}\begin{pmatrix}1\\0\end{pmatrix} - B\delta_{m-1}\begin{pmatrix}1\\0\end{pmatrix} = -B^2\delta'_{m-1}\begin{pmatrix}1\\0\end{pmatrix} + B\delta'_m\begin{pmatrix}1\\0\end{pmatrix} + x$$

Letting $A = \left\{B^2\delta_{m-2}\begin{pmatrix}1\\0\end{pmatrix} - B\delta_{m-1}\begin{pmatrix}1\\0\end{pmatrix}, \delta_{m-2}, \delta_{m-1} \in \{0,1\}\right\}$. To show $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$, $m \geq 6$ for the sequence s_4, it is enough to prove that $A \subset -A + K_3 \bigcap \mathbb{Z}^2$. This

is the case since

$$\begin{aligned} A &= \left\{\delta_{m-2}\binom{0}{-2} - \delta_{m-1}\binom{1}{-1}, \delta_{m-2}, \delta_{m-1} \in \{0,1\}\right\} \\ &= \left\{\binom{0}{0}, \binom{0}{-2}, \binom{-1}{1}, \binom{-1}{-1}\right\} \\ &= \left\{\binom{0}{0}+\binom{0}{0}, \binom{1}{-1}+\binom{-1}{-1}, \binom{0}{2}+\binom{-1}{-1}, \binom{0}{0}+\binom{-1}{-1}\right\}. \end{aligned}$$

For $s_5 = (1, 1, \ldots, 1, 1, -1, 1, -1)$, taking $\delta'_1 = 0$, $\delta'_j = \delta_{j-1}$ for $2 \leq j \leq m-2$, then (3.4.28) can be reduced to the following equation

$$B^2\delta_{m-2}\binom{1}{0} - B\delta_{m-1}\binom{1}{0} + \delta_m\binom{1}{0} = -B^2\delta'_{m-1}\binom{1}{0} + B\delta'_m\binom{1}{0} - \delta'_{m+1}\binom{1}{0} + x$$

Let $A = \left\{B^2\delta_{m-2}\binom{1}{0} - B\delta_{m-1}\binom{1}{0} + \delta_m\binom{1}{0}, \delta_{m-2}, \delta_{m-1}, \delta_m \in \{0,1\}\right\}$. In order to prove that $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$, $m \geq 6$ for the sequence s_5, we need to show that $A \subset -A + K_3 \bigcap \mathbb{Z}^2$. This is the case since

$$\begin{aligned} A =& \left\{\delta_{m-2}\binom{0}{-2} - \delta_{m-1}\binom{1}{-1} + \delta_m\binom{1}{0}, \delta_{m-2}, \delta_{m-1}, \delta_m \in \{0,1\}\right\} \\ =& \left\{\binom{0}{0}, \binom{0}{-2}, \binom{-1}{1}, \binom{1}{0}, \binom{-1}{-1}, \binom{1}{-2}, \binom{0}{1}, \binom{0}{-1}\right\} \\ =& \left\{\binom{0}{0}+\binom{0}{0}, \binom{0}{-1}+\binom{0}{0}, \binom{-1}{0}+\binom{0}{1}, \binom{1}{1}+\binom{0}{-1}, \right. \\ & \left. \binom{0}{0}+\binom{-1}{-1}, \binom{1}{-1}+\binom{0}{-1}, \binom{0}{0}+\binom{0}{1}, \binom{0}{0}+\binom{0}{-1}\right\}. \end{aligned}$$

For $s_6 = (1, 1, \ldots, 1, 1, 1, -1, 1)$, taking $\delta'_1 = 0$, $\delta'_j = \delta_{j-1}$ for $2 \leq j \leq m-1$, then (3.4.28) can be reduced to the following equation

$$B\delta_{m-1}\binom{1}{0} - \delta_m\binom{1}{0} = -B\delta'_m\binom{1}{0} + \delta'_{m+1}\binom{1}{0} + x$$

Let $A = \left\{B\delta_{m-1}\binom{1}{0} - \delta_m\binom{1}{0}, \delta_{m-1}, \delta_m \in \{0,1\}\right\}$. We need to show that $A \subset -A + K_3 \bigcap \mathbb{Z}^2$. This is the case since

$$\begin{aligned} A &= \left\{\delta_{m-1}\binom{1}{-1} - \delta_m\binom{1}{0}, \delta_{m-1}, \delta_m \in \{0,1\}\right\} \\ &= \left\{\binom{0}{0}, \binom{1}{-1}, \binom{-1}{0}, \binom{0}{-1}\right\} \\ &= \left\{\binom{0}{0}+\binom{0}{0}, \binom{1}{0}+\binom{0}{-1}, \binom{0}{0}+\binom{-1}{0}, \binom{0}{0}+\binom{0}{-1}\right\}. \end{aligned}$$

For $s_7 = (1, 1, \ldots, 1, -1, 1, -1, -1)$, taking $\delta'_1 = 0$, $\delta'_j = \delta_{j-1}$ for $2 \leq j \leq m-3$ and $\delta'_{m+1} = \delta_m$, then (3.4.28) can be reduced to the following equation

$$B^3\delta_{m-3}\binom{1}{0} - B^2\delta_{m-2}\binom{1}{0} + B\delta_{m-1}\binom{1}{0} = -B^3\delta'_{m-2}\binom{1}{0} + B^2\delta'_{m-1}\binom{1}{0} - B\delta'_m\binom{1}{0} + x$$

Letting $A = \left\{B^3\delta_{m-3}\binom{1}{0} - B^2\delta_{m-2}\binom{1}{0} + B\delta_{m-1}\binom{1}{0}, \delta_{m-3}, \delta_{m-2}, \delta_{m-1} \in \{0,1\}\right\}$, we need to show that $A \subset -A + K_3 \bigcap \mathbb{Z}^2$. This is the case since

$$\begin{aligned}
A =& \left\{\delta_{m-3}\tbinom{-2}{-2} - \delta_{m-2}\tbinom{0}{-2} + \delta_{m-1}\tbinom{1}{-1}, \delta_{m-3}, \delta_{m-2}, \delta_{m-1} \in \{0,1\}\right\} \\
=& \left\{\tbinom{0}{0}, \tbinom{-2}{-2}, \tbinom{0}{2}, \tbinom{1}{-1}, \tbinom{-2}{0}, \tbinom{-1}{-3}, \tbinom{1}{1}, \tbinom{-1}{-1}\right\} \\
=& \left\{\tbinom{0}{0} + \tbinom{0}{0}, \tbinom{-1}{-1} + \tbinom{-1}{-1}, \tbinom{1}{3} + \tbinom{-1}{-1}, \tbinom{0}{-2} + \tbinom{1}{1},\right. \\
& \left.\tbinom{-1}{1} + \tbinom{-1}{-1}, \tbinom{0}{-2} + \tbinom{-1}{-1}, \tbinom{0}{0} + \tbinom{1}{1}, \tbinom{0}{0} + \tbinom{-1}{-1}\right\}.
\end{aligned}$$

This proves that the sequences $s_i,\ 1 \leq i \leq 7$ satisfy the inclusion $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$ for any $m \geq 6$. Suppose the claim is true for $m \geq 6$, then we will show that it holds also

for $m+1$. We need to consider the following fourteen sequences.

$$
\begin{array}{rc}
 & (a_1, a_2, \ldots, a_{m-3}, a_{m-2}, a_{m-1}, a_m, a_{m+1}, a_{m+2}) \\
t_1 = & (1,\ 1, \ldots,\ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1) \\
t_2 = & (1,\ 1, \ldots,\ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ -1) \\
t_3 = & (1,\ 1, \ldots,\ 1,\ \ 1,\ \ 1,\ \ 1,\ -1,\ \ 1) \\
t_4 = & (1,\ 1, \ldots,\ 1,\ \ 1,\ \ 1,\ \ 1,\ -1,\ -1) \\
t_5 = & (1,\ 1, \ldots,\ 1,\ -1,\ \ 1,\ \ 1,\ -1,\ \ 1) \\
t_6 = & (1,\ 1, \ldots,\ 1,\ -1,\ \ 1,\ \ 1,\ -1,\ -1) \\
t_7 = & (1,\ 1, \ldots,\ 1,\ \ 1,\ -1,\ \ 1,\ \ 1,\ \ 1) \\
t_8 = & (1,\ 1, \ldots,\ 1,\ \ 1,\ -1,\ \ 1,\ \ 1,\ -1) \\
t_9 = & (1,\ 1, \ldots,\ 1,\ \ 1,\ -1,\ \ 1,\ -1,\ \ 1) \\
t_{10} = & (1,\ 1, \ldots,\ 1,\ \ 1,\ -1,\ \ 1,\ -1,\ -1) \\
t_{11} = & (1,\ 1, \ldots,\ 1,\ \ 1,\ \ 1,\ -1,\ \ 1,\ \ 1) \\
t_{12} = & (1,\ 1, \ldots,\ 1,\ \ 1,\ \ 1,\ -1,\ \ 1,\ -1) \\
t_{13} = & (1,\ 1, \ldots,\ 1,\ -1,\ \ 1,\ -1,\ -1,\ \ 1) \\
t_{14} = & (1,\ 1, \ldots,\ 1,\ -1,\ \ 1,\ -1,\ -1,\ -1)
\end{array}
$$

From the above argument, seven of them $(t_1, t_2, t_3, t_8, t_{10}, t_{11}, t_{12})$ satisfy $K_1^{(m+1)} \subseteq K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$. It is enough to prove the other remaining seven sequences $(t_4, t_5, t_6, t_7, t_9, t_{13}, t_{14})$ do not satisfy (3.4.26).

For the sequence $t_4 = (1, 1, \ldots, 1, -1, -1)$, we will show that $\binom{1}{-3} \in K_1^{(m+1)}$, but $\binom{1}{-3} \notin K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$. Indeed, taking $\delta_j = 0$ for $1 \leq j \leq m-2, \delta_{m-1} = \delta_m = 1, \delta_{m+1} = 0$, we obtain

$$B^2 a_{m-1}\binom{1}{0} + B a_m \binom{1}{0} = B^2\binom{1}{0} + B\binom{1}{0} = \binom{1}{-3},$$

which implies that $\binom{1}{-3} \in K_1^{(m+1)}$. If $\binom{1}{-3} \in K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$, then there exists some

$x \in K_3 \bigcap \mathbb{Z}^2$ and $\delta'_j \in \{0,1\}$, $1 \le j \le m+2$ such that

$$\binom{1}{-3} = B^{m+1}\delta'_1\binom{1}{0} + \cdots + B^2\delta'_m\binom{1}{0} - B\delta'_{m+1}\binom{1}{0} - \delta'_{m+2}\binom{1}{0} + x \tag{3.4.29}$$

Given any element $x \in K_3 \bigcap \mathbb{Z}^2$, δ'_{m+1} and δ'_{m+2} are determined since $\binom{1}{-3} - x + \delta'_{m+2}\binom{1}{0} + B\delta'_{m+1}\binom{1}{0}$ should belong to $B^2\mathbb{Z}^2$ by (3.4.29). For $x \in K_3 \bigcap \mathbb{Z}^2$, let $T_4(x) = B^{-2}(\binom{1}{-3} - x + \delta'_{m+2}\binom{1}{0} + B\delta'_{m+1}\binom{1}{0})$, where δ'_{m+1} and δ'_{m+2} are such that $T_4(x) \in \mathbb{Z}^2$. Then we have

$$T_4(\binom{1}{0}) = T_4(\binom{0}{1}) = T_4(\binom{0}{0}) = \binom{2}{1},\ T_4(\binom{1}{1}) = \binom{2}{0},$$
$$T_4(\binom{-1}{0}) = \binom{2}{2},\ T_4(\binom{0}{-1}) = T_4(\binom{-1}{-1}) = \binom{1}{1}.$$

This is equivalent to saying that

$$T_4(K_3 \bigcap \mathbb{Z}^2) = \left\{\binom{2}{1}, \binom{2}{2}, \binom{1}{1}, \binom{2}{0}\right\}.$$

Note that for any $y \in T_4(K_3 \bigcap \mathbb{Z}^2) \subset \mathbb{Z}^2$, y should have a finite radix representation if (3.4.29) holds, i.e. $s = \binom{0}{0} \in S$ in (3.4.27) for such y. Lemma 3.4.7 implies that $s = \binom{0}{-1}$ in (3.4.27) for $y \in T_4(K_3 \bigcap \mathbb{Z}^2) \subset \mathbb{Z}^2$. A contradiction is obtained. Therefore, the sequence t_4 does not satisfy $K_1^{(m+1)} \subseteq K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$ for any $m \ge 6$.

For $t_5 = (1,1,\ldots,1,-1,1,1,-1,1)$, we will show that $\binom{-5}{-2} \in K_1^{(m+1)}$, but $\binom{-5}{-2} \notin K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$. Taking $\delta_j = 0$ for $1 \le j \le m-4, \delta_{m-3} = \delta_{m-1} = \delta_{m+1} = 1, \delta_{m-2} = \delta_m = 0$, we obtain

$$B^4 a_{m-3}\binom{1}{0} + B^2 a_{m-1}\binom{1}{0} + a_{m+1}\binom{1}{0} = B^4\binom{1}{0} + B^2\binom{1}{0} - \binom{1}{0} = \binom{-5}{-2},$$

which implies that $\binom{-5}{-2} \in K_1^{(m+1)}$. If we assume that $\binom{-5}{-2} \in K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$, then there should exist some $x \in K_3 \bigcap \mathbb{Z}^2$ and $\delta'_j \in \{0,1\}$, $1 \le j \le m+2$ such that

$$\begin{aligned}\binom{-5}{-2} &= B^{m+1}\delta'_1\binom{1}{0} + \cdots + B^5\delta'_{m-3}\binom{1}{0} - B^4\delta'_{m-2}\binom{1}{0} + B^3\delta'_{m-1}\binom{1}{0} \\ &+ B^2\delta'_m\binom{1}{0} - B\delta'_{m+1}\binom{1}{0} + \delta'_{m+2}\binom{1}{0} + x\end{aligned}$$

Given any $x \in K_3 \bigcap \mathbb{Z}^2$, $\delta'_{m-2}, \delta'_{m-1}, \delta'_m, \delta'_{m+1}$ and δ'_{m+2} are determined since

$$\begin{aligned}&\binom{-5}{-2} - x + B^4\delta'_{m-2}\binom{1}{0} - B^3\delta'_{m-1}\binom{1}{0} - B^2\delta'_m\binom{1}{0} \\ &+B\delta'_{m+1}\binom{1}{0} - \delta'_{m+2}\binom{1}{0} \in B^5\mathbb{Z}^2.\end{aligned} \tag{3.4.30}$$

Let

$$T_5(x) = B^{-5}(\binom{-5}{-2} - x + B^4\delta'_{m-2}\binom{1}{0} - B^3\delta'_{m-1}\binom{1}{0} - B^2\delta'_m\binom{1}{0} + B\delta'_{m+1}\binom{1}{0} - \delta'_{m+2}\binom{1}{0}),$$

for $x \in K_3 \bigcap \mathbb{Z}^2$, where $\delta'_{m-2}, \delta'_{m-1}, \delta'_m, \delta'_{m+1}$ and δ'_{m+2} are such that $T_5(x) \in \mathbb{Z}^2$. Then $T_5(\binom{0}{1}) = \binom{0}{1}$, and

$$T_5(\binom{1}{0}) = T_5(\binom{0}{0}) = T_5(\binom{1}{1}) = T_5(\binom{-1}{0}) = T_5(\binom{0}{-1}) = T_5(\binom{-1}{-1}) = \binom{1}{1}.$$

That is, we have $T_5(K_3 \bigcap \mathbb{Z}^2) = \left\{\binom{1}{1}, \binom{0}{1}\right\}$. By Lemma 3.4.7, no y in $T_5(K_3 \bigcap \mathbb{Z}^2)$ can have a finite radix representation, contradicting (3.4.30).

For $t_6 = (1, 1, \ldots, 1, -1, 1, 1, -1, -1)$, we will show that $\binom{-7}{6} \in K_1^{(m+1)}$, but $\binom{-7}{6} \notin K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$. If we take $\delta_j = 0$ for $1 \leq j \leq m-5, \delta_{m-4} = \delta_{m-3} = \delta_{m-2} = \delta_{m+1} = 1, \delta_{m-1} = \delta_m = 0$, we obtain

$$B^5 a_{m-4}\binom{1}{0} + B^4 a_{m-3}\binom{1}{0} + B^3 a_{m-2}\binom{1}{0} + a_{m+1}\binom{1}{0} = B^5\binom{1}{0} + B^4\binom{1}{0} - B^3\binom{1}{0} - \binom{1}{0} = \binom{-7}{6},$$

which implies that $\binom{-7}{6} \in K_1^{(m+1)}$. If $\binom{-7}{6} \in K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$, then there exists some $x \in K_3 \bigcap \mathbb{Z}^2$ and $\delta'_j \in \{0, 1\}$, $1 \leq j \leq m+2$ such that

$$\begin{aligned} \binom{-7}{6} &= B^{m+1}\delta'_1\binom{1}{0} + \cdots + B^5\delta'_{m-3}\binom{1}{0} - B^4\delta'_{m-2}\binom{1}{0} + B^3\delta'_{m-1}\binom{1}{0} \\ &+ B^2\delta'_m\binom{1}{0} - B\delta'_{m+1}\binom{1}{0} - \delta'_{m+2}\binom{1}{0} + x \end{aligned}$$

Given any element $x \in K_3 \bigcap \mathbb{Z}^2$, $\delta'_{m-2}, \delta'_{m-1}, \delta'_m, \delta'_{m+1}$ and δ'_{m+2} are determined since

$$\begin{aligned} &\binom{-7}{6} - x + B^4\delta'_{m-2}\binom{1}{0} - B^3\delta'_{m-1}\binom{1}{0} - B^2\delta'_m\binom{1}{0} \\ &+B\delta'_{m+1}\binom{1}{0} + \delta'_{m+2}\binom{1}{0} \in B^5\mathbb{Z}^2. \end{aligned} \tag{3.4.31}$$

Let

$$T_6(x) = B^{-5}(\binom{-7}{6} - x + B^4\delta'_{m-2}\binom{1}{0} - B^3\delta'_{m-1}\binom{1}{0} - B^2\delta'_m\binom{1}{0} + B\delta'_{m+1}\binom{1}{0} + \delta'_{m+2}\binom{1}{0}),$$

for $x \in K_3 \bigcap \mathbb{Z}^2$, where $\delta'_{m-2}, \delta'_{m-1}, \delta'_m, \delta'_{m+1}$ and δ'_{m+2} are such that $T_6(x) \in \mathbb{Z}^2$. Then we obtain

$$T_6(\binom{1}{0}) = T_6(\binom{0}{1}) = T_6(\binom{0}{0}) = T_6(\binom{1}{1}) = T_6(\binom{-1}{0}) = T_6(\binom{0}{-1}) = T_6(\binom{-1}{-1}) = \binom{2}{0},$$

i.e. $T_6(K_3 \bigcap \mathbb{Z}^2) = \left\{ \binom{2}{0} \right\}$. Using Lemma 3.4.7, it follows that no $y \in T_6(K_3 \bigcap \mathbb{Z}^2)$ can have a finite radix representation, contradicting (3.4.31).

For $t_7 = (1, 1, \ldots, 1, 1, -1, 1, 1, 1)$, we will prove that $\binom{-6}{-2} \in K_1^{(m+1)}$, but $\binom{-6}{-2} \notin K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$. If we take $\delta_j = 0$ for $1 \leq j \leq m-4, \delta_{m-3} = \delta_{m-2} = 1, \delta_{m-1} = \delta_m = \delta_{m+1} = 0$, we obtain

$$B^4 a_{m-3} \binom{1}{0} + B^3 a_{m-2} \binom{1}{0} = B^4 \binom{1}{0} + B^3 \binom{1}{0} = \binom{-6}{-2},$$

which implies that $\binom{-6}{-2} \in K_1^{(m+1)}$. If $\binom{-6}{-2} \in K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$, then there exists some $x \in K_3 \bigcap \mathbb{Z}^2$ and $\delta'_j \in \{0, 1\}$, $1 \leq j \leq m+2$ such that

$$\begin{aligned} \binom{-6}{-2} &= B^{m+1}\delta'_1 \binom{1}{0} + \cdots + B^4 \delta'_{m-2} \binom{1}{0} - B^3 \delta'_{m-1} \binom{1}{0} + B^2 \delta'_m \binom{1}{0} + B\delta'_{m+1} \binom{1}{0} \\ &+ \delta'_{m+2} \binom{1}{0} + x \end{aligned}$$

Given any element $x \in K_3 \bigcap \mathbb{Z}^2$, $\delta'_{m-1}, \delta'_m, \delta'_{m+1}$ and δ'_{m+2} are determined by the previous equality since,

$$\binom{-6}{-2} - x + B^3 \delta'_{m-2} \binom{1}{0} - B^2 \delta'_m \binom{1}{0} - B\delta'_{m+1} \binom{1}{0} - \delta'_{m+2} \binom{1}{0} \in B^4 \mathbb{Z}^2. \tag{3.4.32}$$

Let

$$T_7(x) = B^{-4}(\binom{-6}{-2} - x + B^3 \delta'_{m-1} \binom{1}{0} - B^2 \delta'_m \binom{1}{0} - B\delta'_{m+1} \binom{1}{0} - \delta'_{m+2} \binom{1}{0}), \ x \in K_3 \bigcap \mathbb{Z}^2,$$

where $\delta'_{m-1}, \delta'_m, \delta'_{m+1}$ and δ'_{m+2} are such that $T_7(x) \in \mathbb{Z}^2$. Then we have

$$\begin{aligned} &T_7(\binom{1}{0}) = T_7(\binom{0}{1}) = T_7(\binom{1}{1}) = T_7(\binom{0}{-1}) = T_7(\binom{-1}{-1}) = \binom{2}{0}, \\ &T_7(\binom{0}{0}) = T_7(\binom{-1}{0}) = \binom{2}{1}. \end{aligned}$$

Hence, we get $T_7(K_3 \bigcap \mathbb{Z}^2) = \left\{ \binom{2}{0}, \binom{2}{1} \right\}$ and a contradiction follows for (3.4.32) using Lemma 3.4.7.

For $t_9 = (1, 1, \ldots, 1, 1, -1, 1, -1, 1)$, we will show that $\binom{-2}{-3} \in K_1^{(m+1)}$, but $\binom{-2}{-3} \notin K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$. If we take $\delta_j = 0$ for $1 \leq j \leq m-3, \delta_{m-2} = \delta_{m+1} = 1, \delta_{m-1} = \delta_m = 0$, we obtain

$$B^3 a_{m-2} \binom{1}{0} + a_{m+1} \binom{1}{0} = B^3 \binom{1}{0} - \binom{1}{0} = \binom{-2}{-3},$$

which implies that $\binom{-2}{-3} \in K_1^{(m+1)}$. If $\binom{-2}{-3} \in K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$, then there exists some $x \in K_3 \bigcap \mathbb{Z}^2$ and $\delta'_j \in \{0,1\}$, $1 \leq j \leq m+2$ such that

$$\begin{aligned}\binom{-2}{-3} &= B^{m+1}\delta'_1\binom{1}{0} + \cdots + B^4\delta'_{m-2}\binom{1}{0} - B^3\delta'_{m-1}\binom{1}{0} + B^2\delta'_m\binom{1}{0} - B\delta'_{m+1}\binom{1}{0} \\ &+ \delta'_{m+2}\binom{1}{0} + x\end{aligned}$$

Given any element $x \in K_3 \bigcap \mathbb{Z}^2$, $\delta'_{m-1}, \delta'_m, \delta'_{m+1}$ and δ'_{m+2} are determined by the previous equality since,

$$\binom{-2}{-3} - x + B^3\delta'_{m-2}\binom{1}{0} - B^2\delta'_m\binom{1}{0} + B\delta'_{m+1}\binom{1}{0} - \delta'_{m+2}\binom{1}{0} \in B^4\mathbb{Z}^2. \tag{3.4.33}$$

Let

$$T_9(x) = B^{-4}(\binom{-2}{-3} - x + B^3\delta'_{m-1}\binom{1}{0} - B^2\delta'_m\binom{1}{0} + B\delta'_{m+1}\binom{1}{0} - \delta'_{m+2}\binom{1}{0}),\ x \in K_3 \bigcap \mathbb{Z}^2,$$

where $\delta'_{m-1}, \delta'_m, \delta'_{m+1}$ and δ'_{m+2} are such that $T_9(x) \in \mathbb{Z}^2$. Then we have $T_9(\binom{-1}{0}) = \binom{0}{1}$ and

$$T_9(\binom{1}{0}) = T_9(\binom{0}{1}) = T_9(\binom{0}{0}) = T_9(\binom{1}{1}) = T_9(\binom{0}{-1}) = T_9(\binom{-1}{-1}) = \binom{1}{1}.$$

This gives us that $T_9(K_3 \bigcap \mathbb{Z}^2) = \left\{\binom{1}{1}, \binom{0}{1}\right\}$, contradicting (3.4.33) by Lemma 3.4.7. Thus, the sequence t_9 does not satisfy $K_1^{(m+1)} \subseteq K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$ for any $m \geq 6$.
For $t_{13} = (1, 1, \ldots, 1, -1, 1, -1, -1, 1)$, we will prove that $\binom{-6}{-1} \in K_1^{(m+1)}$, but $\binom{-6}{-1} \notin K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$. If we take $\delta_j = 0$ for $1 \leq j \leq m-4, \delta_{m-3} = \delta_{m-1} = \delta_m = \delta_{m+1} = 1, \delta_{m-2} = 0$, we obtain

$$B^4 a_{m-3}\binom{1}{0} + B^2 a_{m-1}\binom{1}{0} + B a_m\binom{1}{0} + a_{m+1}\binom{1}{0} = B^4\binom{1}{0} + B^2\binom{1}{0} - B\binom{1}{0} - \binom{1}{0} = \binom{-6}{-1},$$

which implies that $\binom{-6}{-1} \in K_1^{(m+1)}$. If $\binom{-6}{-1} \in K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$, then there exists some $x \in K_3 \bigcap \mathbb{Z}^2$ and $\delta'_j \in \{0,1\}$, $1 \leq j \leq m+2$ such that

$$\begin{aligned}\binom{-6}{-1} &= B^{m+1}\delta'_1\binom{1}{0} + \cdots + B^5\delta'_{m-3}\binom{1}{0} - B^4\delta'_{m-2}\binom{1}{0} + B^3\delta'_{m-1}\binom{1}{0} - B^2\delta'_m\binom{1}{0} \\ &- B\delta'_{m+1}\binom{1}{0} + \delta'_{m+2}\binom{1}{0} + x\end{aligned}$$

Given any element $x \in K_3 \bigcap \mathbb{Z}^2$, $\delta'_{m-2}, \delta'_{m-1}, \delta'_m, \delta'_{m+1}$ and δ'_{m+2} are determined since

$$\binom{-6}{-1} - x + B^4\delta'_{m-2}\binom{1}{0} - B^3\delta'_{m-1}\binom{1}{0} + B^2\delta'_m\binom{1}{0} + B\delta'_{m+1}\binom{1}{0} - \delta'_{m+2}\binom{1}{0} \in B^5\mathbb{Z}^2.$$

Let

$$T_{13}(x) = B^{-5}(\binom{-6}{-1} - x + B^4\delta'_{m-2}\binom{1}{0} - B^3\delta'_{m-1}\binom{1}{0} + B^2\delta'_m\binom{1}{0} + B\delta'_{m+1}\binom{1}{0} - \delta'_{m+2}\binom{1}{0}),$$

for $x \in K_3 \bigcap \mathbb{Z}^2$, where $\delta'_{m-2}, \delta'_{m-1}, \delta'_m, \delta'_{m+1}$ and δ'_{m+2} are such that $T_{13}(x) \in \mathbb{Z}^2$. Then we have

$$\begin{aligned}
&T_{13}(\binom{1}{0}) = T_{13}(\binom{0}{1}) = T_{13}(\binom{0}{0}) = T_{13}(\binom{0}{-1}) = T_{13}(\binom{-1}{-1}) = \binom{1}{1},\\
&T_{13}(\binom{1}{1}) = \binom{1}{2},\ T_{13}(\binom{-1}{0}) = \binom{0}{1}.
\end{aligned}$$

This implies that $T_{13}(K_3 \bigcap \mathbb{Z}^2) = \left\{\binom{1}{1}, \binom{1}{2}, \binom{0}{1}\right\}$. Using Lemma 3.4.7, we obtain a contradiction as before.

For $t_{14} = (1, 1, \ldots, 1, -1, 1, -1, -1, -1)$. It follows from the result for the sequence t_{13} that $\binom{-6}{-1} \in K_1^{(m+1)}$. In the following, we will prove that $\binom{-6}{-1} \notin K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$. If we assume that $\binom{-6}{-1} \in K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$, then there should exist some $x \in K_3 \bigcap \mathbb{Z}^2$ and $\delta'_j \in \{0, 1\}$, $1 \leq j \leq m+2$ such that

$$\begin{aligned}
\binom{-6}{-1} &= B^{m+1}\delta'_1\binom{1}{0} + \cdots + B^5\delta'_{m-3}\binom{1}{0} - B^4\delta'_{m-2}\binom{1}{0} + B^3\delta'_{m-1}\binom{1}{0} - B^2\delta'_m\binom{1}{0}\\
&- B\delta'_{m+1}\binom{1}{0} - \delta'_{m+2}\binom{1}{0} + x
\end{aligned}$$

Given any $x \in K_3 \bigcap \mathbb{Z}^2$, $\delta'_{m-2}, \delta'_{m-1}, \delta'_m, \delta'_{m+1}$ and δ'_{m+2} are determined since

$$\binom{-6}{-1} - x + B^4\delta'_{m-2}\binom{1}{0} - B^3\delta'_{m-1}\binom{1}{0} + B^2\delta'_m\binom{1}{0} + B\delta'_{m+1}\binom{1}{0} + \delta'_{m+2}\binom{1}{0} \in B^5\mathbb{Z}^2.$$

Let

$$T_{14}(x) = B^{-5}(\binom{-6}{-1} - x + B^4\delta'_{m-2}\binom{1}{0} - B^3\delta'_{m-1}\binom{1}{0} + B^2\delta'_m\binom{1}{0} + B\delta'_{m+1}\binom{1}{0} + \delta'_{m+2}\binom{1}{0}),$$

for $x \in K_3 \bigcap \mathbb{Z}^2$, where $\delta'_{m-2}, \delta'_{m-1}, \delta'_m, \delta'_{m+1}$ and δ'_{m+2} are such that $T_{14}(x) \in \mathbb{Z}^2$. Then we have

$$\begin{aligned}
&T_{14}(\binom{1}{0}) = T_{14}(\binom{0}{1}) = T_{14}(\binom{1}{1}) = T_{14}(\binom{0}{-1}) = T_{14}(\binom{-1}{-1}) = \binom{1}{1},\\
&T_{14}(\binom{0}{0}) = T_{14}(\binom{-1}{0}) = \binom{0}{1}.
\end{aligned}$$

Thus $T_{14}(K_3 \bigcap \mathbb{Z}^2) = \left\{\binom{0}{1}, \binom{1}{1}\right\}$, which again yields a contradiction by Lemma 3.4.7. Hence, the sequence t_{14} does not satisfy $K_1^{(m+1)} \subseteq K_2^{(m+1)} + K_3 \bigcap \mathbb{Z}^2$ for any $m \geq 6$. The induction is done, so the claim is true. By the claim, only the constant sequences $\pm(1, 1, \ldots, 1, 1 \ldots)$ satisfy $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$ for any $m \geq 1$. Note that if $a_j = 1$ for all $j \geq 1$, then $\varepsilon_j = d_1$ and $\sum_{j=1}^{\infty} B^{-j} d_1 = 0$, which implies that $d_1 = \binom{0}{0}$. The case $a_j = -1$ for all $j \geq 1$ yields $d_2 = \binom{0}{0}$ in the same way. That is, $0 \in \mathcal{D}$ is necessary and sufficient for the inclusion $K(B, \mathcal{D}) \subset BK(B, \mathcal{D})$. From the result for the matrix C_3 in Theorem 2.6.1, $K(B, \mathcal{D})$ can not be an A-dilation MRA scaling set for any digit set $\mathcal{D} = \{d_1, d_2\}$. □

The search algorithm for the proofs of Theorem 3.4.9, 3.4.10 and 3.4.11 was implemented in Matlab. We list the results for the remaining three representative classes without detailed proof as they are very similar to the one we gave for the case where $A = C_3$. For each $m \geq 1$, the corresponding sequences $\{a_j\}_{j=1}^{m+1}$, where $a_1 = 1$ and $a_j \in \{-1, 1\}, j > 1$, listed on the left-hand side satisfy $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$ and those listed on the right-hand side do not satisfy $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$. The vector after each sequence on the right-hand side is the one which belongs to $K_1^{(m)}$ but not to $K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$. This explains why the corresponding sequence does not satisfy (3.4.26). Moreover, for $m > 1$, the choice of the sequences listed below is based on the information from the previous step using the strategy introduced before.

Theorem 3.4.9. *Let* $A = -C_3 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$ *and let* $\mathcal{D} = \{d_1, d_2\}$ *with* $d_1, d_2 \in \mathbb{R}^2$. *Then*

(i) $K(B, \mathcal{D}) \subseteq BK(B, \mathcal{D})$ *if and only if* $0 \in \mathcal{D}$,

(ii) $K(B, \mathcal{D})$ *is an* A*-dilation scaling set if and only if* $\mathcal{D}$ *is one of the 4 digit sets* $\mathcal{D} = \{\binom{0}{0}, \binom{1}{0}\}$, $\mathcal{D} = \{\binom{0}{0}, \binom{-1}{0}\}$, $\mathcal{D} = \{\binom{0}{0}, \binom{0}{1}\}$, $\mathcal{D} = \{\binom{0}{0}, \binom{0}{-1}\}$.

Proof. Here, we only prove (i). If $0 \in \mathcal{D}$, (ii) follows from Theorem 2.7.1. By the definition of K_3, K_3 is the same for the matrix B and $-B$. In this case, $B = A^t = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} = -C_3^t$. Then, using Proposition 3.4.5, we have

$$K_3 \bigcap \mathbb{Z}^2 = \left\{\binom{-1}{-1}, \binom{-1}{0}, \binom{0}{-1}, \binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1}\right\}.$$

In the following, we list the sequences $\{a_j\}_{j=1}^{m+1}$ with $a_1 = 1$ and $a_j \in \{-1, 1\}$ for $j > 1$ which do and do not satisfy $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$ for each $m = 1,\ 2$.

$$\begin{array}{lll} m=1,\ (a_1,\ a_2); & m=2,\ (a_1,\ a_2,\ a_3); & m=2,\ (a_1,\ a_2,\ a_3); \\ (1,\ \ 1) & (1,\ \ 1,\ \ 1) & (1,\ -1,\ \ 1)\ \binom{-1}{1} \\ (1,\ -1) & (1,\ \ 1,\ -1) & (1,\ -1,\ -1)\ \binom{-2}{1} \end{array}$$

If $m \geq 1$, sequences $\{a_j\}_{j=1}^{m+1}$ with $a_1 = 1$ which satisfy $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$ have the following forms.

$$\begin{array}{ll} m \geq 1 & (a_1, a_2, \ldots, a_{m-1}, a_m, a_{m+1}) \\ & (1,\ \ 1, \ldots,\ \ 1,\ \ 1,\ \ 1) \\ & (1,\ \ 1, \ldots,\ \ 1,\ \ 1,\ -1) \end{array}$$

Therefore, only the constant sequences $\pm(1,\ \ 1, \ldots,\ \ 1,\ \ 1 \ldots)$ satisfy $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$ for all $m \geq 1$. As in the proof of Theorem 3.4.8, it follows that $0 \in \mathcal{D}$ is necessary and sufficient for the inclusion $K(B, \mathcal{D}) \subset BK(B, \mathcal{D})$. □

Theorem 3.4.10. *Let* $A = C_4 = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$ *and let* $\mathcal{D} = \{d_1, d_2\}$ *with* $d_1, d_2 \in \mathbb{R}^2$. *Then*

(i) $K(B, \mathcal{D}) \subseteq BK(B, \mathcal{D})$ *if and only if* $0 \in \mathcal{D}$,

(ii) $K(B, \mathcal{D})$ *is an* A*-dilation scaling set if and only if* $\mathcal{D}$ *is one of the 4 digit sets* $\mathcal{D} = \{\binom{0}{0}, \binom{1}{0}\}$, $\mathcal{D} = \{\binom{0}{0}, \binom{-1}{0}\}$, $\mathcal{D} = \{\binom{0}{0}, \binom{1}{\frac{1}{2}}\}$, $\mathcal{D} = \{\binom{0}{0}, \binom{-1}{-\frac{1}{2}}\}$.

Proof. Here, we only prove (i). If $0 \in \mathcal{D}$, (ii) follows from Theorem 2.8.12. In this case, $B = A^t = \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$. Since $\|B^{-1}\| = \frac{\sqrt{3+\sqrt{5}}}{2}$ and $\|B^{-2}\| = \frac{1}{\sqrt{2}} < 1$, then K_3 is contained in the disk centered at 0 with radius $(\sqrt{2}+1)(1 + \frac{\sqrt{3+\sqrt{5}}}{\sqrt{2}}) < 7$. Then, using the same method as in the proof of Proposition 3.4.5, we have

$$K_3 \bigcap \mathbb{Z}^2 = \left\{\binom{-1}{-1}, \binom{-1}{0}, \binom{0}{-1}, \binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1}\right\}.$$

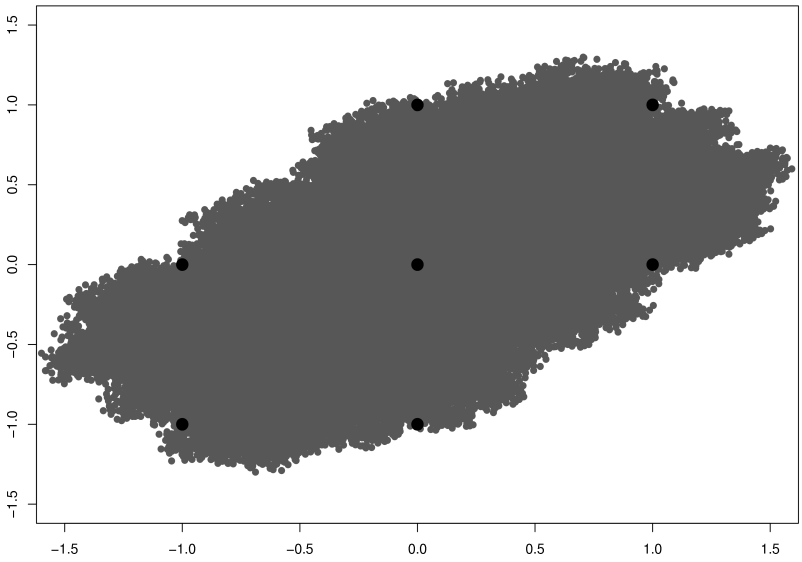


Figure 3.4: The set K_3 and its integer points in Theorem 3.4.10.

The set K_3 and its integer points (black points) are depicted in Figure 3.4. In the following, we list the sequences $\{a_j\}_{j=1}^{m+1}$ with $a_1 = 1$ which do and do not satisfy $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$ for each $1 \leq m \leq 6$.

$m = 1$, $(a_1,\ a_2)$;

$(1,\ \ 1)$

$(1,\ -1)$

$m = 2$, $(a_1,\ a_2,\ a_3)$

$(1,\ \ 1,\ \ 1)$

$(1,\ \ 1,\ -1)$

$(1,\ -1,\ \ 1)$

$(1,\ -1,\ -1)$

$m = 3$, $(a_1,\ a_2,\ a_3,\ a_4)$;

$(1,\ \ 1,\ \ 1,\ \ 1)$

$(1,\ \ 1,\ \ 1,\ -1)$

$(1,\ \ 1,\ -1,\ \ 1)$

$(1,\ \ 1,\ -1,\ -1)$

$(1,\ -1,\ \ 1,\ \ 1)$

$(1,\ -1,\ \ 1,\ -1)$

$m = 3$, $(a_1,\ a_2,\ a_3,\ a_4)$

$(1,\ -1,\ -1,\ \ 1)\ \binom{-2}{-1}$

$(1,\ -1,\ -1,\ -1)\ \binom{-3}{-1}$

$m = 4$, $(a_1,\ a_2,\ a_3,\ a_4,\ a_5)$;

$(1,\ \ 1,\ \ 1,\ \ 1,\ \ 1)$

$(1,\ \ 1,\ \ 1,\ \ 1,-1)$

$(1,\ \ 1,\ \ 1,-1,\ \ 1)$

$(1,\ \ 1,\ \ 1,-1,-1)$

$(1,\ \ 1,-1,\ \ 1,-1)$

$(1,-1,\ \ 1,\ \ 1,-1)$

$(1,-1,\ \ 1,-1,-1)$

$m = 4$, $(a_1,\ a_2,\ a_3,\ a_4,\ a_5)$

$(1,\ \ 1,-1,\ \ 1,\ \ 1)\ \binom{-4}{1}$

$(1,\ \ 1,-1,-1,\ \ 1)\ \binom{-2}{-1}$

$(1,\ \ 1,-1,-1,-1)\ \binom{-3}{-1}$

$(1,-1,\ \ 1,\ \ 1,\ \ 1)\ \binom{-2}{1}$

$(1,-1,\ \ 1,-1,\ \ 1)\ \binom{-3}{1}$

$m = 5$, $(a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6)$;

$(1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1)$

$(1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,-1)$

$(1,\ \ 1,\ \ 1,\ \ 1,-1,\ \ 1)$

$(1,\ \ 1,\ \ 1,\ \ 1,-1,-1)$

$(1,\ \ 1,\ \ 1,-1,\ \ 1,-1)$

$(1,\ \ 1,-1,\ \ 1,-1,-1)$

$(1,-1,\ \ 1,\ \ 1,-1,-1)$

$(1,-1,\ \ 1,-1,-1,\ \ 1)$

$m = 5$, $(a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6)$

$(1,\ \ 1,\ \ 1,-1,\ \ 1,\ \ 1)\ \binom{-4}{1}$

$(1,\ \ 1,\ \ 1,-1,-1,\ \ 1)\ \binom{-2}{-1}$

$(1,\ \ 1,\ \ 1,-1,-1,-1)\ \binom{-3}{-1}$

$(1,\ \ 1,-1,\ \ 1,-1,\ \ 1)\ \binom{-3}{1}$

$(1,-1,\ \ 1,\ \ 1,-1,\ \ 1)\ \binom{1}{3}$

$(1,-1,\ \ 1,-1,-1,-1)\ \binom{2}{4}$

$m=6$, $(a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7)$;	$m=6$, $(a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7)$
$(1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)$	$(1,\ 1,\ 1,\ 1,-1,\ 1,\ 1)\ \binom{-4}{1}$
$(1,\ 1,\ 1,\ 1,\ 1,\ 1,-1)$	$(1,\ 1,\ 1,\ 1,-1,-1,\ 1)\ \binom{-2}{-1}$
$(1,\ 1,\ 1,\ 1,\ 1,-1,\ 1)$	$(1,\ 1,\ 1,\ 1,-1,-1,-1)\ \binom{-3}{-1}$
$(1,\ 1,\ 1,\ 1,\ 1,-1,-1)$	$(1,\ 1,\ 1,-1,\ 1,-1,\ 1)\ \binom{-3}{1}$
$(1,\ 1,\ 1,\ 1,-1,\ 1,-1)$	$(1,\ 1,-1,\ 1,-1,-1,-1)\ \binom{2}{4}$
$(1,\ 1,\ 1,-1,\ 1,-1,-1)$	$(1,-1,\ 1,\ 1,-1,-1,\ 1)\ \binom{-7}{-3}$
$(1,\ 1,-1,\ 1,-1,-1,\ 1)$	$(1,-1,\ 1,\ 1,-1,-1,-1)\ \binom{6}{2}$
	$(1,-1,\ 1,-1,-1,\ 1,\ 1)\ \binom{9}{3}$
	$(1,-1,\ 1,-1,-1,\ 1,-1)\ \binom{9}{2}$

If $m \geq 6$, sequences $\{a_j\}_{j=1}^{m+1}$ with $a_1 = 1$ which satisfy $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$ have the following forms.

$m \geq 6$	$(a_1, a_2, \ldots, a_{m-4}, a_{m-3}, a_{m-2}, a_{m-1}, a_m, a_{m+1})$
	$(1,\ 1, \ldots,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)$
	$(1,\ 1, \ldots,\ 1,\ 1,\ 1,\ 1,\ 1,-1)$
	$(1,\ 1, \ldots,\ 1,\ 1,\ 1,-1,\ 1,-1)$
	$(1,\ 1, \ldots,\ 1,\ 1,\ 1,\ 1,-1,\ 1)$
	$(1,\ 1, \ldots,\ 1,\ 1,\ 1,\ 1,-1,-1)$
	$(1,\ 1, \ldots,\ 1,\ 1,\ 1,-1,-1,-1)$
	$(1,\ 1, \ldots,\ 1,-1,\ 1,-1,-1,\ 1)$

Therefore, only the constant sequences $\pm(1, 1, \ldots, 1, 1 \ldots)$ satisfy $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$ for all $m \geq 1$. As before, it follows that $0 \in \mathcal{D}$ is necessary and sufficient for the inclusion $K(B, \mathcal{D}) \subset BK(B, \mathcal{D})$. □

Theorem 3.4.11. *Let* $A = -C_4 = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}$ *and let* $\mathcal{D} = \{d_1, d_2\}$ *with* $d_1, d_2 \in \mathbb{R}^2$. *Then*

(i) $K(B,\mathcal{D}) \subseteq BK(B,\mathcal{D})$ *if and only if* $0 \in \mathcal{D}$,

(ii) $K(B,\mathcal{D})$ *is an A-dilation scaling set if and only if* $\mathcal{D}$ *is one of the 4 digit sets* $\mathcal{D} = \{\binom{0}{0}, \binom{1}{0}\}$, $\mathcal{D} = \{\binom{0}{0}, \binom{-1}{0}\}$, $\mathcal{D} = \{\binom{0}{0}, \binom{1}{\frac{1}{2}}\}$, $\mathcal{D} = \{\binom{0}{0}, \binom{-1}{-\frac{1}{2}}\}$.

Proof. Here, $B = A^t = \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix} = -C_4^t$. Hence,

$$K_3 \bigcap \mathbb{Z}^2 = \left\{ \binom{-1}{-1}, \binom{-1}{0}, \binom{0}{-1}, \binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1} \right\}.$$

In the following, we list the sequences $\{a_j\}_{j=1}^{m+1}$ with $a_1 = 1$, where $a_j \in \{-1, 1\}$ for $j > 1$, which do and do not satisfy $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$ for each $m \in \{1, 2, 3\}$.

$m = 1$, $(a_1,\ a_2)$;
$(1,\ \ 1)$
$(1,\ -1)$

$m = 2$, $(a_1,\ a_2,\ a_3)$;
$(1,\ \ 1,\ \ 1)$
$(1,\ \ 1,\ -1)$
$(1,\ -1,\ -1)$

$m = 2$, $(a_1,\ a_2,\ a_3)$;
$(1,\ -1,\ \ 1)\ \binom{-1}{1}$

$m = 3$, $(a_1,\ a_2,\ a_3,\ a_4)$;
$(1,\ \ 1,\ \ 1,\ \ 1)$
$(1,\ \ 1,\ \ 1, -1)$
$(1,\ \ 1, -1, -1)$

$m = 3$, $(a_1,\ a_2,\ a_3,\ a_4)$
$(1,\ \ 1, -1,\ \ 1)\ \binom{-3}{0}$
$(1, -1, -1,\ \ 1)\ \binom{-3}{-2}$
$(1, -1, -1, -1)\ \binom{-3}{-1}$

If $m \geq 2$, sequences $\{a_j\}_{j=1}^{m+1}$ with $a_1 = 1$ which satisfy $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$ have the following forms.

$m \geq 2$, $(a_1, a_2, \ldots, a_{m-4}, a_{m-3}, a_{m-2}, a_{m-1}, a_m, a_{m+1})$
$(1,\ \ 1, \ldots,\ \ 1,\ \ \ 1,\ \ \ 1,\ \ \ 1,\ \ 1,\ \ \ 1)$
$(1,\ \ 1, \ldots,\ \ 1,\ \ \ 1,\ \ \ 1,\ \ \ 1,\ \ 1,\ -1)$
$(1,\ \ 1, \ldots,\ \ 1,\ \ \ 1,\ \ \ 1,\ \ \ 1,\ -1,\ -1)$

From the above listed results, only the constant sequences $\pm(1, 1, \ldots, 1, 1 \ldots)$ satisfy $K_1^{(m)} \subseteq K_2^{(m)} + K_3 \bigcap \mathbb{Z}^2$ for all $m \geq 1$. So, necessary and sufficient condition for the inclusion $K(B,\mathcal{D}) \subset BK(B,\mathcal{D})$ is $0 \in \mathcal{D}$. (ii) follows from the result in Chapter 2. $\square$

Chapter 4

Construction of Wavelet Sets Using Integral Self-Affine Multi-Tiles

The main purpose of this chapter is to construct (multi)wavelet sets using integral self-affine multi-tiles.

4.1 Properties of integral self-affine multi-tiles

Throughout this chapter, we will limit our discussion to expanding matrices $A \in M_n(\mathbb{Z})$. We let $B = A^t$ and $S = \{1, 2, \ldots, M\}$. In the following, we will always assume that $K = \bigcup_{i=1}^{M} K_i$, where the unions are essentially disjoint and $|K_i| > 0$ for each $i \in S$, and that the sets K_i, $i \in S$, satisfy (0.0.5) and (0.0.6). We call such set K a self-affine tiling set with M prototiles. In addition, if K is a $\mathbb{Z}^n$-tiling set, then the set K is called a self-affine $\mathbb{Z}^n$-tiling set (with M prototiles). Gröchenig and Hass [28] showed that a necessary condition for K to be a self-affine $\mathbb{Z}^n$-tiling set with M prototiles is that the corresponding digit set $\Gamma = \{\Gamma_{ij}, i, j = 1, \ldots, M\}$ be a standard digit set.

Lemma 4.1.1. *Let K be a self-affine tiling set with M prototiles. Given any $x \in K$, if V is a neighborhood of x, then $|V \bigcap K| > 0$.*

Proof. By assumption, for each $i \in S$, the set K_i satisfies (0.0.5). This gives

$$K_i = B^{-1}(\bigcup_{j=1}^{M} K_j + \Gamma_{ij}). \tag{4.1.1}$$

Using (4.1.1) again to each prototile K_j, we obtain

$$K_i = \bigcup_{j=1}^{M} B^{-1}\Gamma_{ij} + B^{-1}(\bigcup_{l=1}^{M} B^{-1}K_l + B^{-1}\Gamma_{jl}) = \bigcup_{j=1}^{M}\bigcup_{l=1}^{M}(B^{-1}\Gamma_{ij} + B^{-2}\Gamma_{jl} + B^{-2}K_l).$$

Iterating this procedure d times, we obtain that $x \in K_i$ if and only if for some $s \in S$

$$x \in \sum_{j=1}^{d} B^{-j}\epsilon_j + B^{-d}K_s, \tag{4.1.2}$$

where $\epsilon_j \in \Gamma_{\rho_j\rho_{j+1}}$ for $j = 1, ..., d$ and $\rho_1 = i$. Given $x \in K$, there exists $i \in S$ such that $x \in K_i$. It follows from (4.1.2) that x can be written as

$$x = \sum_{j=1}^{d} B^{-j}\epsilon_j + B^{-d}k_s$$

for some $s \in S$, $k_s \in K_s$, where $\epsilon_j \in \Gamma_{\rho_j\rho_{j+1}}$ for $j = 1, ..., d$ and $\rho_1 = i$. Let V contain a neighborhood of x satisfying $B(x, 2r) \subseteq V$, where $r = \max_{t\in K_s}\|B^{-d}t\|$. Let $x' = \sum_{j=1}^{d} B^{-j}\epsilon_j$ and $W = \{x \in \mathbb{R}^n : x = x' + B^{-d}t, t \in K_s\}$, then $W \subseteq V$. In addition, $W \subseteq K$. Hence, $W \subseteq K \bigcap V$ and $|W| = |B^{-d}K_s| = |\det B|^{-d}|K_s| > 0$, which implies that $|K \bigcap V| \geqslant |W| > 0$. Note that given any neighborhood of x, say V, the condition $B(x, 2r) \subseteq V$, where $r = \max_{t\in K_s}\|B^{-d}t\|$ can always be satisfied if we take d large enough. This proves that given any $x \in K$, if V is a neighborhood of x, then $|V \bigcap K| > 0$. □

Lemma 4.1.2. *If K is an integral self-affine $\mathbb{Z}^n$-tiling set, then K contains a neighborhood of 0 if and only if $K \bigcap \mathbb{Z}^n = \{0\}$.*

Proof. Suppose that $K \bigcap \mathbb{Z}^n = \{0\}$. This implies that $0 \in K$. If 0 is an interior point of K, then K contains a neighborhood of 0. If 0 is the boundary point of K, then there exists $j \in \mathbb{Z}^n \setminus \{0\}$ such that $0 \in K + j$ since K is a $\mathbb{Z}^n$-tiling set. This implies that

$-j \in K$, contradicting the fact that $K \bigcap \mathbb{Z}^n = \{0\}$.
Conversely, suppose that K contains a neighborhood of 0. Assume that $0 \neq \ell \in K \bigcap \mathbb{Z}^n$. Then $\ell \in K \bigcap (K+\ell)$ since $0 \in K$. Let U be a neighborhood of 0 with $U \subseteq K$. Then $U+\ell$ is a neighborhood of ℓ and $U+\ell \subseteq K+\ell$. By Lemma 4.1.1, $|(K+\ell) \bigcap K| \geq |(U+\ell) \bigcap K| > 0$ which contradicts the fact that $(K+\ell) \bigcap K = \emptyset$. Therefore, $K \bigcap \mathbb{Z}^n = \{0\}$. □

If K is an integral self-affine tiling set with M prototiles as above, there exists $\Gamma_{ij}^m \subset \mathbb{Z}^n$ satisfying

$$B^m K_i = \bigcup_{j=1}^{M} K_j + \Gamma_{ij}^m. \tag{4.1.3}$$

Indeed, it follows from (0.0.5) that we can take $\Gamma_{ij}^1 = \Gamma_{ij}$. Using (0.0.5) to each prototile K_j, we have

$$B^2 K_i = \bigcup_{j=1}^{M} BK_j + B\Gamma_{ij} = \bigcup_{j=1}^{M} \bigcup_{\ell=1}^{M} K_j + \Gamma_{\ell j} + B\Gamma_{i\ell} = \bigcup_{j=1}^{M} K_j + \bigcup_{\ell=1}^{M} \Gamma_{\ell j} + B\Gamma_{i\ell}.$$

Thus we can define

$$\Gamma_{ij}^2 = \bigcup_{\ell=1}^{M} (\Gamma_{\ell j} + B\Gamma_{i\ell}). \tag{4.1.4}$$

Inductively, we obtain

$$\Gamma_{ij}^m = \bigcup_{\ell=1}^{M} (\Gamma_{\ell j} + B\Gamma_{i\ell}^{m-1}),\ m \geq 2. \tag{4.1.5}$$

Defining

$$\mathcal{D}_j := \bigcup_{i=1}^{M} \Gamma_{ij} \text{ and } \mathcal{D}_j^m := \bigcup_{i=1}^{M} \Gamma_{ij}^m, \tag{4.1.6}$$

the corresponding self-affine multi-tile $K = \bigcup_{i=1}^{M} K_i$ satisfies

$$BK = \bigcup_{i=1}^{M} BK_i = \bigcup_{i=1}^{M} \bigcup_{j=1}^{M} K_j + \Gamma_{ij} = \bigcup_{j=1}^{M} K_j + \mathcal{D}_j, \tag{4.1.7}$$

and

$$B^m K = \bigcup_{i=1}^{M} B^m K_i = \bigcup_{i=1}^{M}\bigcup_{j=1}^{M} K_j + \Gamma_{ij}^m = \bigcup_{j=1}^{M} K_j + \mathcal{D}_j^m. \tag{4.1.8}$$

Theorem 4.1.3. *Let K be an integral self-affine $\mathbb{Z}^n$-tiling set with M prototiles and $0 \in \mathcal{D}_j$ for each $j \in S$. Then K contains a neighborhood of 0 if and only if $\bigcup_{m\geqslant 0}\bigcup_{i=1}^{M} \Gamma_{ij}^m = \mathbb{Z}^n$, where Γ_{ij}^m is defined by (4.1.3).*

Proof. Suppose that K contains a neighborhood of 0. Then $\bigcup_{m\geqslant 1} B^m K = \mathbb{R}^n$ since B is expansive (see [37]). By the definition of Γ_{ij}^m, we obtain

$$\bigcup_{m\geqslant 1} B^m K = \bigcup_{m\geqslant 1}\bigcup_{i=1}^{M} B^m K_i = \bigcup_{m\geqslant 1}\bigcup_{i=1}^{M}\bigcup_{j=1}^{M} K_j + \Gamma_{ij}^m = \bigcup_{j=1}^{M} K_j + \bigcup_{m\geqslant 1}\bigcup_{i=1}^{M} \Gamma_{ij}^m. \tag{4.1.9}$$

Thus, we have

$$\bigcup_{j=1}^{M} K_j + \bigcup_{m\geqslant 1}\bigcup_{i=1}^{M} \Gamma_{ij}^m = \mathbb{R}^n. \tag{4.1.10}$$

Since $\bigcup_{m\geqslant 1}\bigcup_{i=1}^{M} \Gamma_{ij}^m \subseteq \mathbb{Z}^n$ and $K = \bigcup_{j=1}^{M} K_j$ is a $\mathbb{Z}^n$-tiling set, it follows that $\bigcup_{m\geqslant 1}\bigcup_{i=1}^{M} \Gamma_{ij}^m = \mathbb{Z}^n$.

Conversely, suppose that $\bigcup_{m\geqslant 1}\bigcup_{i=1}^{M} \Gamma_{ij}^m = \mathbb{Z}^n$. Define $\mathcal{D}_j^m, m \geq 1$ as in (4.1.6), then using (4.1.5) and (4.1.6), we have that

$$\mathcal{D}_j^m = \bigcup_{i=1}^{M} \Gamma_{ij}^m = \bigcup_{i=1}^{M}\bigcup_{\ell=1}^{M} \Gamma_{\ell j} + B\Gamma_{i\ell}^{m-1} = \bigcup_{\ell=1}^{M} \Gamma_{\ell j} + B\mathcal{D}_\ell^{m-1}. \tag{4.1.11}$$

In particular, taking $m = 2$, (4.1.11) gives

$$\mathcal{D}_j^2 = \bigcup_{i=1}^{M} \Gamma_{ij}^2 = \bigcup_{\ell=1}^{M} \Gamma_{\ell j} + B\mathcal{D}_\ell. \tag{4.1.12}$$

By assumption, $0 \in \mathcal{D}_j$ for each $j \in S$, it follows from (4.1.12) that $\mathcal{D}_j \subseteq \mathcal{D}_j^2$. Suppose that $\mathcal{D}_j^m \subseteq \mathcal{D}_j^{m+1}$, $j \in S$, for $m \geq 1$. Then we deduce from (4.1.11) and our assumption

that

$$\mathcal{D}_j^{m+2} = \bigcup_{\ell=1}^{M} \Gamma_{\ell j} + B\mathcal{D}_\ell^{m+1} \supseteq \bigcup_{\ell=1}^{M} \Gamma_{\ell j} + B\mathcal{D}_\ell^{m} = \mathcal{D}_j^{m+1}.$$

This shows that $\mathcal{D}_j^m \subseteq \mathcal{D}_j^{m+1}$ for each $j \in S$ and for any $m \geq 1$. Since K is a $\mathbb{Z}^n$-tiling set and $\bigcup_{m\geqslant 1}\bigcup_{i=1}^{M} \Gamma_{ij}^m = \mathbb{Z}^n$, (4.1.8) and (4.1.9) show that

$$\bigcup_{m\geqslant 1} B^m K = \bigcup_{j=1}^{M} K_j + \bigcup_{m\geqslant 1} \mathcal{D}_j^m = \mathbb{R}^n. \tag{4.1.13}$$

Let V be a compact set containing a neighborhood of 0. Define

$$S_i := \Big\{k \in \mathbb{Z}^n : |(K_i + k)\bigcap V| > 0\Big\}, \quad i \in S.$$

Since $S_i \subset V - K_i$, S_i is bounded and so is $\bigcup_{s\in S_i} K_i + s$. Hence, there exists $r_i > 0$ such that

$$\bigcup_{s\in S_i} K_i + s \subseteq B(0, r_i) \text{ and } V \subseteq \bigcup_{i=1}^{M}\bigcup_{s\in S_i} K_i + s \subseteq \bigcup_{i=1}^{M} B(0, r_i).$$

Since the sets $K_i+s, s \in S_i$, are essentially disjoint, the number of elements in S_i is at most $\frac{|B(0,r_i)|}{|K_i|}$. The inclusion $\mathcal{D}_j^m \subseteq \mathcal{D}_j^{m+1}$, $j \in S$ and the fact that $\bigcup_{m\geqslant 1}\bigcup_{i=1}^{M} \Gamma_{ij}^m = \bigcup_{m\geqslant 1} \mathcal{D}_j^m = \mathbb{Z}^n$ imply that for each $j \in S$, there is $M_j \in \mathbb{N}$ such that $S_j \subseteq \mathcal{D}_j^{M_j}$. Let $M = \max_{1\leq j\leq M} M_j$. Then, we have

$$V \subseteq \bigcup_{i=1}^{M} K_i + S_i \subseteq \bigcup_{i=1}^{M} K_i + \mathcal{D}_i^M = B^M K.$$

This proves that K contains $B^{-M}V$, a neighborhood of 0. □

Theorem 4.1.4 and Corollary 4.1.5 below provide a sufficient condition for K to be a self-affine $\mathbb{Z}^n$-tiling set with M prototiles. We first introduce a metric defined on the space of subsets of $\mathbb{R}^n$. Let $H_\emptyset(\mathbb{R}^n)$ be the set containing the compact subsets of $\mathbb{R}^n$ and the empty set with the modified Hausdorff metric

$$D(X, Y) = \begin{cases} \max\{\sup_{y\in Y} d(X, y), \sup_{x\in X} d(x, Y)\}, & \text{for } X, Y \neq \emptyset, \\ \sup_{y\in Y} d(0, y) + 10, & \text{for } X = \emptyset, \end{cases}$$

where d is the Euclidean meatric on $\mathbb{R}^n$.

Theorem 4.1.4. *Let K be an integral self-affine set with M prototiles $K_1, \ldots, K_M$, and let the corresponding digit set $\Gamma = \{\Gamma_{ij}, i, j = 1, \ldots, M\}$ be a standard digit set. Then $|K|$ is a positive integer and K satisfies $\sum_{k\in\mathbb{Z}^n} \chi_K(x-k) = |K|$.*

Proof. We may choose compact sets $Z_i \subset \mathbb{R}^n$, $i = 1, \ldots, M$ with the following properties: for $i \in S$, $|Z_i| = \frac{|K_i|}{|K|}$, $Z_i \bigcap Z_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{M} Z_i = [0,1]^n$. Define $Z_i^0 = Z_i$ and $Z_i^{m+1} = B^{-1}\bigcup_{j=1}^{M} \Gamma_{ij} + Z_j^m$ for $m \geq 0$ and $i \in S$. By the proof of Theorem 2 in ([28], section 3) and using the fact that $\Gamma = \{\Gamma_{ij}, i, j = 1, \ldots, M\}$ is a standard digit set, we have, for any $m \geq 0$,

(i) $Z_i^m \bigcap Z_j^m = \emptyset$ for $i \neq j$,

(ii) $\bigcup_{i=1}^{M} Z_i^m$ is a $\mathbb{Z}^n$-tiling set.

Furthermore, as shown in [28], for each $i \in S$, $\{Z_i^m\}_{m\geq 0}$ converges in the metric D to K_i. This implies, in particular, for each $i \in S$, the existence of a fixed compact set W_i such that for any $m \geq 0$, $Z_i^m \subset W_i$. Let

$$R_i^m = \Big\{\rho = (\rho_1, \ldots, \rho_{m+1}) \in S^{m+1}, \rho_1 = i, \Gamma_{\rho_k\rho_{k+1}} \neq \emptyset \text{ for } 1 \leq k \leq m\Big\}.$$

By the definition of Z_i^m and R_i^m, we obtain $Z_i^1 = B^{-1}\bigcup_{j=1}^{M} \Gamma_{ij} + Z_j$, and

$$\begin{aligned} Z_i^2 &= B^{-1}\bigcup_{j=1}^{M} \Gamma_{ij} + Z_j^1 = B^{-1}[\bigcup_{j=1}^{M} \Gamma_{ij} + (B^{-1}\bigcup_{k=1}^{M} \Gamma_{jk} + Z_k)] \\ &= (\bigcup_{j=1}^{M}\bigcup_{k=1}^{M} B^{-1}\Gamma_{ij} + B^{-2}\Gamma_{jk} + B^{-2}Z_k) = \bigcup_{\rho\in R_i^m} B^{-2}Z_{\rho_3} + \sum_{k=1}^{2} B^{-k}\Gamma_{\rho_k\rho_{k+1}}. \end{aligned}$$

More generally,

$$Z_i^m = \bigcup_{\rho\in R_i^m} B^{-m}Z_{\rho_{m+1}} + \sum_{k=1}^{m} B^{-k}\Gamma_{\rho_k\rho_{k+1}}. \tag{4.1.14}$$

Let

$$T_\rho^m = \Big\{ y \in \mathbb{R}^n : y = \sum_{k=1}^{m} B^{-k}\epsilon_k, \epsilon_k \in \Gamma_{\rho_k\rho_{k+1}}, \rho = \{\rho_k\}_{k=1}^{m+1} \in R_i^m \Big\}.$$

Then, (4.1.14) is equivalent to the following equation for the characteristic function of Z_i^m,

$$\chi_{Z_i^m}(x) = \sum_{\rho\in R_i^m} \sum_{y\in T_\rho^m} \chi_{Z_{\rho_{m+1}}}(B^m x - B^m y). \tag{4.1.15}$$

Similarly, by the self-affine property of the sets $K_i, i \in S$, we have

$$K_i = \bigcup_{\rho\in R_i^m} B^{-m}K_{\rho_{m+1}} + \sum_{k=1}^{m} B^{-k}\Gamma_{\rho_k\rho_{k+1}} = \bigcup_{\rho\in R_i^m} \sum_{y\in T_\rho^m} B^{-m}K_{\rho_{m+1}} + y, \tag{4.1.16}$$

which can also be written as

$$\chi_{K_i}(x) = \sum_{\rho\in R_i^m} \sum_{y\in T_\rho^m} \chi_{K_{\rho_{m+1}}}(B^m x - B^m y). \tag{4.1.17}$$

Let $\psi(x)$ be any continuous and bounded function defined on $\mathbb{R}^n$. Then,

$$\begin{aligned} \chi_{Z_i^m}(x)\psi(x) &= \sum_{\rho\in R_i^m} \sum_{y\in T_\rho^m} (\psi(x)-\psi(y))\chi_{Z_{\rho_{m+1}}}(B^m x - B^m y) \\ &+ \sum_{\rho\in R_i^m} \sum_{y\in T_\rho^m} \psi(y)\chi_{Z_{\rho_{m+1}}}(B^m x - B^m y). \end{aligned}$$

Let $f_m^{(i)}(x) = \sum_{\rho\in R_i^m} \sum_{y\in T_\rho^m} (\psi(x)-\psi(y))\chi_{Z_{\rho_{m+1}}}(B^m x - B^m y)$. Using the continuity of the function of ψ, for any $\varepsilon > 0$, there exists $M > 0$ such that $|\psi(x) - \psi(y)| < \varepsilon$ if $x - y \in B^{-m}Z_{\rho_{m+1}}$, where $m > M$. This implies that, for $m > M$,

$$|f_m^{(i)}(x)| \le \varepsilon \sum_{\rho\in R_i^m} \sum_{y\in T_\rho^m} \chi_{Z_{\rho_{m+1}}}(B^m x - B^m y) = \varepsilon\chi_{Z_i^m}(x) \le \varepsilon\chi_{W_i}(x),$$

and, in particular,

$$\int_{\mathbb{R}^n} f_m^{(i)}(x)\, dx \to 0,\ m \to \infty.$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^n} \chi_{Z_i^m}(x)\psi(x)\,dx &= \int_{\mathbb{R}^n} f_m^{(i)}(x)\,dx + \sum_{\rho\in R_i^m}\sum_{y\in T_\rho^m}\int_{\mathbb{R}^n}\psi(y)\chi_{Z_{\rho_{m+1}}}(B^m x - B^m y)\,dx \\ &= \int_{\mathbb{R}^n} f_m^{(i)}(x)\,dx + \sum_{\rho\in R_i^m}\sum_{y\in T_\rho^m}\psi(y)|B^{-m}Z_{\rho_{m+1}}|. \qquad (4.1.18)\end{aligned}$$

However, by the assumption that $|Z_i| = \frac{|K_i|}{|K|}$ and (4.1.17), we obtain

$$\begin{aligned}\sum_{\rho\in R_i^m}\sum_{y\in T_\rho^m}\psi(y)|B^{-m}Z_{\rho_{m+1}}| &= \frac{1}{|K|}\sum_{\rho\in R_i^m}\sum_{y\in T_\rho^m}\psi(y)|B^{-m}K_{\rho_{m+1}}| \\ &= \frac{1}{|K|}\sum_{\rho\in R_i^m}\sum_{y\in T_\rho^m}\int_{\mathbb{R}^n}\psi(y)\chi_{K_{\rho_{m+1}}}(B^m x - B^m y)\,dx \\ &= \frac{1}{|K|}\int_{\mathbb{R}^n}\sum_{\substack{\rho\in R_i^m \\ y\in T_\rho^m}}\psi(y)\chi_{K_{\rho_{m+1}}}(B^m x - B^m y)\,dx \qquad (4.1.19)\end{aligned}$$

Let $g_m^{(i)}(x) = \sum_{\rho\in R_i^m}\sum_{y\in T_\rho^m}\psi(y)\chi_{K_{\rho_{m+1}}}(B^m x - B^m y)$. Then (4.1.17) gives that

$$|g_m^{(i)}(x)| \le c\sum_{\rho\in R_i^m}\sum_{y\in T_\rho^m}\chi_{K_{\rho_{m+1}}}(B^m x - B^m y) = c\chi_{K_i}(x).$$

On the other hand,

$$\begin{aligned}g_m^{(i)}(x) &= \sum_{\rho\in R_i^m}\sum_{y\in T_\rho^m}(\psi(y)-\psi(x))\chi_{K_{\rho_{m+1}}}(B^m x - B^m y) \\ &+ \sum_{\rho\in R_i^m}\sum_{y\in T_\rho^m}\psi(x)\chi_{K_{\rho_{m+1}}}(B^m x - B^m y) \\ &= \sum_{\rho\in R_i^m}\sum_{y\in T_\rho^m}(\psi(y)-\psi(x))\chi_{K_{\rho_{m+1}}}(B^m x - B^m y) + \psi(x)\chi_{K_i}(x).\end{aligned}$$

An argument similar to the one used to prove that $\lim_{m\to\infty}\int_{\mathbb{R}^n} f_m^{(i)}(x)\,dx = 0$ shows that

$$\int_{\mathbb{R}^n}\sum_{\rho\in R_i^m}\sum_{y\in T_\rho^m}(\psi(y)-\psi(x))\chi_{K_{\rho_{m+1}}}(B^m x - B^m y)\,dx \to 0 \text{ as } m\to\infty,$$

and thus,

$$\lim_{m\to\infty}\int_{\mathbb{R}^n} g_m^{(i)}(x)dx = \int_{\mathbb{R}^n}\psi(x)\chi_{K_i}(x)dx.$$

Combining the previous results, (4.1.18) and (4.1.19), we obtain

$$\lim_{m\to\infty}\int_{\mathbb{R}^n}\psi(x)\chi_{Z_i^m}(x)\ dx = \frac{1}{|K|}\int_{\mathbb{R}^n}\psi(x)\chi_{K_i}(x)\ dx.$$

Therefore,

$$\lim_{m\to\infty}\sum_{i=1}^{M}\int_{\mathbb{R}^n}\psi(x)\chi_{Z_i^m}(x)\ dx = \frac{1}{|K|}\sum_{i=1}^{M}\int_{\mathbb{R}^n}\psi(x)\chi_{K_i}(x)\ dx,$$

or equivalently, we have

$$\lim_{m\to\infty}\int_{\mathbb{R}^n}\psi(x)\chi_{Z^m}(x)\ dx = \frac{1}{|K|}\int_{\mathbb{R}^n}\psi(x)\chi_K(x)\ dx,\ \text{where } Z^m = \bigcup_{i=1}^{M} Z_i^m \qquad (4.1.20)$$

In particular, taking $\psi(x) = e^{-2\pi ij\cdot x}$, where $j\in\mathbb{Z}^n$, $x\in\mathbb{R}^n$, we see that $\psi(x)$ is continuous and satisfies $|\psi(x)|\leqslant 1$ in $\mathbb{R}^n$. It follows from (4.1.20) that

$$\begin{aligned}
&\frac{1}{|K|}\int_{[0,1]^n}(\sum_{\ell\in\mathbb{Z}^n}\chi_K(x-\ell))e^{-2\pi ij\cdot x}\ dx = \frac{1}{|K|}\int_{\mathbb{R}^n}\chi_K(x)e^{-2\pi ij\cdot x}\ dx\\
&= \lim_{m\to\infty}\int_{\mathbb{R}^n}\chi_{Z^m}(x)e^{-2\pi ij\cdot x}\ dx\\
&= \lim_{m\to\infty}\int_{[0,1]^n}\sum_{\ell\in\mathbb{Z}^n}\chi_{Z^m}(x-\ell)e^{-2\pi ij\cdot x}\ dx \qquad (4.1.21)
\end{aligned}$$

Since $\sum_{\ell\in\mathbb{Z}^d}\chi_{Z^m}(x-\ell) = 1$ a.e. $x\in\mathbb{R}^n$ for any $m\geq 0$, the integral in this last limit has the value $\delta_{j,0},\ j\in\mathbb{Z}^n$, for any $m\geq 0$. Thus, (4.1.21) implies that

$$\frac{1}{|K|}\int_{[0,1]^n}(\sum_{\ell\in\mathbb{Z}^n}\chi_K(x-\ell))e^{-2\pi ij\cdot x}\ dx = \delta_{j,0},\ j\in\mathbb{Z}^n,$$

or, equivalently, that $\sum_{\ell\in\mathbb{Z}^n}\chi_K(x-\ell) = |K|$ for a.e. $x\in\mathbb{R}^n$. □

By Theorem 4.1.4, we get the following corollary.

Corollary 4.1.5. *Let K be an integral self-affine set with M prototiles and suppose that the corresponding digit set $\Gamma = \{\Gamma_{ij}, i, j = 1, \dots, M\}$ is a standard digit set. If $K \bigcap \mathbb{Z}^n = \{0\}$, then K is a $\mathbb{Z}^n$-tiling set.*

Proof. Since $K \bigcap \mathbb{Z}^n = \{0\}$, then $0 \in K$ and $0 \notin K + \ell$ for $\ell \in \mathbb{Z}^n \setminus \{0\}$. This implies that $\sum_{\ell \in \mathbb{Z}^n} \chi_K(x - \ell) = \chi_K(x) = 1$ if x belongs to a sufficiently small neighborhood of 0. On the other hand, since $\sum_{\ell \in \mathbb{Z}^n} \chi_K(x - \ell) = |K|$ for a.e. $x \in \mathbb{R}^n$ by the previous result, it follows that $|K| = 1$ and thus K is a $\mathbb{Z}^n$-tiling set. □

Combining all the previous results, we obtain the following.

Proposition 4.1.6. Let K be an integral self-affine $\Lambda-$tiling set with M prototiles, where $\Lambda \subseteq \mathbb{Z}^n$, and let the corresponding digit set $\Gamma = \{\Gamma_{ij}, i, j = 1, \dots, M\}$ be a standard digit set. Assume that $0 \in \mathcal{D}_j$ for each $j \in S$, then the following conditions are equivalent.

(i) $K \bigcap \mathbb{Z}^n = \{0\}$.

(ii) $|K| = 1$ and K contains a neighborhood of 0.

(iii) $\bigcup_{m \geq 0} \bigcup_{j=1}^{M} \Gamma_{ij}^m = \mathbb{Z}^n$.

4.2 Construction of single wavelet sets using integral self-affine multi-tiles

Gabardo and Yu introduced in [24] a method for the construction of single wavelet sets using integral self-affine tiling set with one tile. The wavelet sets they constructed have the general form $Q = C^{-1} \bigcup_{\ell \in L} K + \ell$, where C is an integral matrix commuting with B and L is a finite subset of $\mathbb{Z}^n$. In this section, we will extend their results to the integral self-affine multi-tiles setting and construct wavelet sets with the form

$$Q = C^{-1} \bigcup_{i=1}^{M} \bigcup_{\ell \in L_i} K_i + \ell, \tag{4.2.1}$$

where C is an $n \times n$ integral matrix commuting with B, $K_1, \ldots, K_M$ are M-prototiles associated with a self-affine multi-tile and $L_i,\ i = 1, \ldots, M$ are finite subsets of $\mathbb{Z}^n$.

In this section, we will always assume that $K = \bigcup_{i=1}^{M} K_i$, where the K_i are essentially disjoint, is a self-affine $\mathbb{Z}^n$-tiling set with M prototiles, which satisfies (0.0.5) and (0.0.6). Then the corresponding digit set $\Gamma = \{\Gamma_{ij}, i, j = 1, \ldots, M\}$ is a standard digit set.

For the case where K is an integral self-affine $\mathbb{Z}^n$-tiling set with one tile (see Proposition 4.1 in [24]), we have the following result.

Proposition 4.2.1 ([24])**.** Let K be an integral self-affine $\mathbb{Z}^n$-tiling set with one tile and C be a nonsingular integral matrix commuting with B. If $Q := C^{-1}\bigcup_{\ell \in L}(K + \ell)$ is an A-dilation wavelet set, then L is a complete set of coset representatives for the group $\mathbb{Z}^n/C\mathbb{Z}^n$ and $|\det C| \in (q-1)\mathbb{Z}$, where $q = |\det B|$.

However, for the case where K is an integral self-affine $\mathbb{Z}^n$-tiling set with at least two prototiles, the result we obtain is different than the one above comparing to the ont-tile situation. The main difference is that the fact $|\det C| \in (q-1)\mathbb{Z}$, where $q = |\det B|$, is not necessarily true for multi-tiles, as shown in Example 4.2.9 below.

Lemma 4.2.2. *If there is a nonsingular integral matrix C commuting with B such that Q defined by (4.2.1) is an A-dilation wavelet set, then for each $i \in S$, L_i must be a complete set of coset representatives for the group $\mathbb{Z}^n/C\mathbb{Z}^n$.*

Proof. Since Q is a wavelet set, $\bigcup_{k \in \mathbb{Z}^n}(Q + k) = \mathbb{R}^n$ by Proposition 0.0.1. We have, using (4.2.1), that

$$\mathbb{R}^n = C\bigcup_{k \in \mathbb{Z}^n}(Q + k) = \bigcup_{k \in \mathbb{Z}^n}\bigcup_{i=1}^{M}\bigcup_{\ell \in L_i}(K_i + \ell + Ck).$$

The fact that K is a $\mathbb{Z}^n$-tiling set implies that $\bigcup_{z \in \mathbb{Z}^n}\bigcup_{\ell \in L_i}(\ell + Cz) = \mathbb{Z}^n$. This proves that for each $i \in S$, L_i is a complete set of coset representatives for the group $\mathbb{Z}^n/C\mathbb{Z}^n$. □

Gabardo and Yu [24] provided some necessary and sufficient conditions for the set

$Q = C^{-1}\bigcup_{\ell\in L}(K+\ell)$ to be an A-dilation wavelet set when K is an integral self-affine $\mathbb{Z}^n$-tiling set with one tile.

Theorem 4.2.3 ([24]). *Suppose that C is a nonsingular integral matrix which commutes with B and satisfies that $|\det C| = q-1$, where $q = |\det B|$. Then the set $Q = C^{-1}\bigcup_{\ell\in L}(K+\ell)$ is an A-dilation wavelet set if and only if $L = \mathcal{D}\setminus\{0\}$, L is a complete set of coset representatives for the group $\mathbb{Z}^n/C\mathbb{Z}^n$ and $K\bigcap\mathbb{Z}^n = \{0\}$.*

For the case where K is an integral self-affine $\mathbb{Z}^n$-tiling set with M prototiles, we can only give a sufficient condition for the set $Q = C^{-1}\bigcup_{i=1}^{M}\bigcup_{\ell\in L_i}(K_i+\ell)$ to be an A-dilation wavelet set. The conditions in Theorem 4.2.5 are not necessary for the set Q defined by (4.2.1) to be an A-dilation wavelet set if the corresponding set K is an integral self-affine $\mathbb{Z}^n$-tiling set with at least two prototiles, as illustrated by the following example.

Example 4.2.4. In dimension one, consider the set $K = [0,1]$ associated with the dilation $A = 2$. Then $B = A^t = 2$. Let $K_1 = [0,\frac{1}{2}]$, $K_2 = [\frac{1}{2},1]$. Then we have

$$BK_1 = [0,1] = K_1\bigcup K_2,\ BK_2 = [1,2] = (K_1+1)\bigcup(K_2+1).$$

Thus $\Gamma_{11} = \{0\}, \Gamma_{12} = \{0\}, \Gamma_{21} = \{1\}, \Gamma_{22} = \{1\}$. For $j = 1,2$, $\mathcal{D}_j := \bigcup_{i=1}^{2}\Gamma_{ij} = \{0,1\}$ is a complete set of coset representatives for the group $\mathbb{Z}/2\mathbb{Z}$ and $0\in\mathcal{D}_j$. Moreover, the set $Q := (K_1-1)\bigcup K_2 = [-1,-\frac{1}{2}]\bigcup[\frac{1}{2},1]$ is the Shannon wavelet set.

In Example 4.2.4, the set K is an integral self-affine $\mathbb{Z}$-tiling set with 2 prototiles. $q = 2$, $C = 1 = q-1$. However, $L_1 = \{-1\} = \mathcal{D}_1\setminus\{0\}$, $L_2 = \{0\} = \mathcal{D}_2\setminus\{1\}$ are both complete sets of coset representatives for $\mathbb{Z}/C\mathbb{Z}$ but $K\bigcap\mathbb{Z}^n = \{0,1\}\neq\{0\}$.

Theorem 4.2.5. *Suppose C is a nonsingular integral matrix which commutes with B and satisfies $|\det C| = q-1$, where $q = |\det B|$. If for each $i\in S$, $0\in\mathcal{D}_i$, $L_i = \mathcal{D}_i\setminus\{0\}$ is a complete set of coset representatives for the group $\mathbb{Z}^n/C\mathbb{Z}^n$, and $K\bigcap\mathbb{Z}^n = \{0\}$, then the set Q defined by (4.2.1) is an A-dilation wavelet set.*

Proof. We deduce from (4.1.7) and $L_i = \mathcal{D}_i \setminus \{0\}$ that

$$Q = C^{-1}\bigcup_{i=1}^{M}\bigcup_{\ell\in L_i} K_i + \ell = C^{-1}(BK \setminus K). \tag{4.2.2}$$

(4.2.2) implies that $|Q| = \frac{1}{|\det C|}(|\det B| - 1)|K| = 1$ since $|\det C| = |\det B| - 1$ and $|K| = 1$. By assumption, for each $i \in S$, L_i is a complete set of coset representatives for the group $\mathbb{Z}^n/C\mathbb{Z}^n$ and the set $K = \bigcup_{i=1}^{M} K_i$ is a $\mathbb{Z}^n$-tiling set, we have

$$\begin{aligned}\bigcup_{z\in\mathbb{Z}^n}(Q+z) &= C^{-1}\bigcup_{z\in\mathbb{Z}^n}\bigcup_{i=1}^{M}\bigcup_{\ell\in L_i}(K_i+\ell+Cz)\\ &= C^{-1}\bigcup_{i=1}^{M}\bigcup_{z\in\mathbb{Z}^n}(K_i+z) = \mathbb{R}^n \end{aligned}\tag{4.2.3}$$

(4.2.3) and the fact that $|Q| = 1$ show that Q is a $\mathbb{Z}^n$-tiling set. It follows from (4.2.2) that $B^mQ = C^{-1}(B^{m+1}K \setminus B^mK)$ and $\{B^mQ, m \in \mathbb{Z}\}$ is an essentially disjoint family, i.e $B^mQ \bigcap B^nQ = \emptyset$ for $m, n \in \mathbb{Z}$ and $m \neq n$. It is left to prove that $\bigcup_{m\in\mathbb{Z}} B^mQ = \mathbb{R}^n$. The condition $K \bigcap \mathbb{Z}^n = \{0\}$ implies that K contains a neighborhood of 0 by Lemma 4.1.2. Then $\bigcup_{m\in\mathbb{Z}} B^mK = \mathbb{R}^n$ since B is expansive. Moreover, $\lim_{m\to\infty}|B^{-m}K| = \lim_{m\to\infty}|\det B|^{-m} = \lim_{m\to\infty} q^{-m} = 0$ since $|K| = 1$ and $q > 1$. For any $s \in K \setminus \{0\}$, there exists $m \geq 0$ such that $s \in B^{-m}K$, but $s \notin B^{-(m+1)}K$. Thus $s \in C\bigcup_{k\geq 1} B^{-k}Q$, which implies that $K \subseteq C\bigcup_{k\geq 1} B^{-k}Q$. Since for each $j \in S$, $0 \in \mathcal{D}_j$, it follows from (4.1.7) that $K \subset BK$. Since $\bigcup_{m\in\mathbb{Z}} B^mK = \mathbb{R}^n$ and $K \subset BK$, we have thus

$$C\bigcup_{k=0}^{\infty} B^kQ = \bigcup_{k=0}^{\infty} B^{k+1}K \setminus B^kK = \mathbb{R}^n \setminus K. \tag{4.2.4}$$

The inclusion $K \subseteq C\bigcup_{k\geq 1} B^{-k}Q$ together with (4.2.4) shows that $C\bigcup_{m\in\mathbb{Z}} B^mQ = \mathbb{R}^n$ and thus that $\bigcup_{m\in\mathbb{Z}} B^mQ = \mathbb{R}^n$. □

The following two examples of MRA wavelet sets first appeared in [30]. We show below that there exists, for each one of them, a corresponding self-affine multi-tile set K which can be used to generate the wavelet sets using the method of Theorem 4.2.5.

Example 4.2.6. In dimension two, consider the set $K = conv\left\{\binom{\frac{1}{2}}{\frac{1}{2}}, \binom{\frac{1}{2}}{-\frac{1}{2}}, \binom{-\frac{1}{2}}{-\frac{1}{2}}, \binom{-\frac{1}{2}}{\frac{1}{2}}\right\}$ associated with the matrix $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, where $conv(E)$ denotes the convex hull of E. Then $B = A^t = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$. Let

$$K_1 = conv\left\{\binom{0}{0}, \binom{1/2}{0}, \binom{1/2}{1/2}, \binom{0}{1/2}\right\}, \quad K_2 = conv\left\{\binom{-1/2}{0}, \binom{-1/2}{1/2}, \binom{0}{1/2}, \binom{0}{0}\right\},$$
$$K_3 = conv\left\{\binom{0}{0}, \binom{-1/2}{0}, \binom{-1/2}{-1/2}, \binom{0}{-1/2}\right\}, K_4 = conv\left\{\binom{0}{0}, \binom{1/2}{0}, \binom{1/2}{-1/2}, \binom{0}{-1/2}\right\},$$

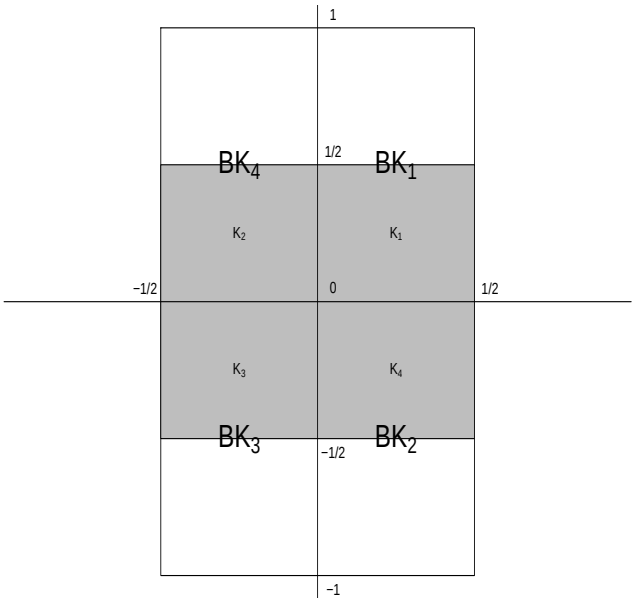


Figure 4.1: K_i and its B-dilation BK_i, i=1,2,3,4 in Example 4.2.6.

Then (see Figure 4.1), we have

$$BK_1 = K_1 \bigcup (K_4 + \tbinom{0}{1}), \quad BK_2 = (K_1 + \tbinom{0}{-1}) \bigcup K_4,$$
$$BK_3 = (K_2 + \tbinom{0}{-1}) \bigcup K_3, \quad BK_4 = K_2 \bigcup (K_3 + \tbinom{0}{1}).$$

Thus

$$\Gamma_{11} = \{\tbinom{0}{0}\}, \Gamma_{12} = \emptyset, \Gamma_{13} = \emptyset, \Gamma_{14} = \{\tbinom{0}{1}\}, \ \Gamma_{21} = \{\tbinom{0}{-1}\}, \Gamma_{22} = \emptyset, \Gamma_{23} = \emptyset, \Gamma_{24} = \{\tbinom{0}{0}\},$$
$$\Gamma_{31} = \emptyset, \Gamma_{32} = \{\tbinom{0}{-1}\}, \Gamma_{33} = \{\tbinom{0}{0}\}, \Gamma_{34} = \emptyset, \ \Gamma_{41} = \emptyset, \Gamma_{42} = \{\tbinom{0}{0}\}, \Gamma_{43} = \{\tbinom{0}{1}\}, \Gamma_{44} = \emptyset.$$

And

$$\mathcal{D}_1 = \bigcup_{i=1}^{4} \Gamma_{i1} = \Big\{\tbinom{0}{0}, \tbinom{0}{-1}\Big\}, \mathcal{D}_2 = \bigcup_{i=1}^{4} \Gamma_{i2} = \Big\{\tbinom{0}{0}, \tbinom{0}{-1}\Big\},$$
$$\mathcal{D}_3 = \bigcup_{i=1}^{4} \Gamma_{i3} = \Big\{\tbinom{0}{0}, \tbinom{0}{1}\Big\}, \ \mathcal{D}_4 = \bigcup_{i=1}^{4} \Gamma_{i4} = \Big\{\tbinom{0}{0}, \tbinom{0}{1}\Big\}.$$

Obviously, K is a $\mathbb{Z}^2$-tiling set. Note that for each $j \in \{1,2,3,4\}$, $0 \in \mathcal{D}_j$ and $\mathcal{D}_j$ is a complete set of coset representatives for the group $\mathbb{Z}^n/B\mathbb{Z}^n$. Hence, K is an integral self-affine $\mathbb{Z}^2$-tiling set with 4 prototiles which satisfies that $K \bigcap \mathbb{Z}^2 = \{0\}$. We conclude from Theorem 4.2.5 that $Q := \bigcup_{j=1}^{4} K_j + L_j$, where $L_j = \mathcal{D}_j \setminus \{0\}$, is an A-dilation wavelet set.

Example 4.2.7. In dimension two, consider the set $K = conv\Big\{\tbinom{\frac{1}{2}}{\frac{1}{2}}, \tbinom{\frac{1}{2}}{-\frac{1}{2}}, \tbinom{-\frac{1}{2}}{-\frac{1}{2}}, \tbinom{-\frac{1}{2}}{\frac{1}{2}}\Big\}$ associated with the matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Then $B = A^t = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Let

$$K_1 = conv\Big\{\tbinom{1/2}{0}, \tbinom{1/2}{1/2}, \tbinom{0}{0}\Big\}, \ K_2 = conv\Big\{\tbinom{0}{0}, \tbinom{1/2}{1/2}, \tbinom{0}{1/2}\Big\},$$
$$K_3 = conv\Big\{\tbinom{0}{0}, \tbinom{-1/2}{1/2}, \tbinom{0}{1/2}\Big\}, \ K_4 = conv\Big\{\tbinom{0}{0}, \tbinom{-1/2}{0}, \tbinom{-1/2}{1/2}\Big\},$$
$$K_5 = -K_1, \ K_6 = -K_2, \ K_7 = -K_3, K_8 = -K_4.$$

Then (see figure 4.2), we have

$$BK_1 = (K_5 + \tbinom{1}{0}) \bigcup K_8, \ BK_2 = K_1 \bigcup (K_4 + \tbinom{1}{0}), \ BK_3 = K_2 \bigcup (K_7 + \tbinom{0}{1}),$$
$$BK_4 = K_3 \bigcup (K_6 + \tbinom{0}{1}), \ BK_5 = (K_1 + \tbinom{-1}{0}) \bigcup K_4, \ BK_6 = K_5 \bigcup (K_8 + \tbinom{-1}{0}),$$
$$BK_7 = (K_3 + \tbinom{0}{-1}) \bigcup K_6, \ BK_8 = (K_2 + \tbinom{0}{-1}) \bigcup K_7.$$

Thus

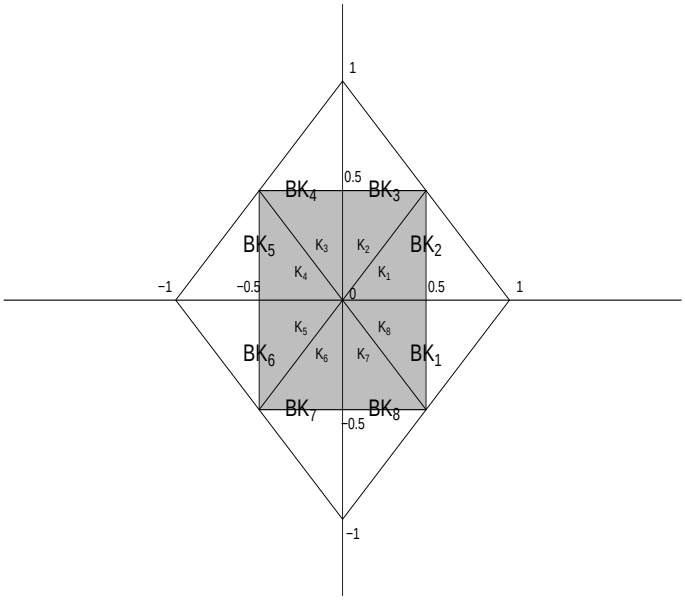


Figure 4.2: K_i and its B-dilation BK_i, i=1,2,3,4,5,6,7,8 in Example 4.2.7.

$$\Gamma_{11} = \emptyset, \Gamma_{12} = \emptyset, \Gamma_{13} = \emptyset, \Gamma_{14} = \emptyset, \Gamma_{15} = \{\binom{1}{0}\}, \Gamma_{16} = \emptyset, \Gamma_{17} = \emptyset, \Gamma_{18} = \{\binom{0}{0}\},$$
$$\Gamma_{21} = \{\binom{0}{0}\}, \Gamma_{22} = \emptyset, \Gamma_{23} = \emptyset, \Gamma_{24} = \{\binom{1}{0}\}, \Gamma_{25} = \emptyset, \Gamma_{26} = \emptyset, \Gamma_{27} = \emptyset, \Gamma_{28} = \emptyset,$$
$$\Gamma_{31} = \emptyset, \Gamma_{32} = \{\binom{0}{0}\}, \Gamma_{33} = \emptyset, \Gamma_{34} = \emptyset, \Gamma_{35} = \emptyset, \Gamma_{36} = \emptyset, \Gamma_{37} = \{\binom{0}{1}\}, \Gamma_{38} = \emptyset,$$
$$\Gamma_{41} = \emptyset, \Gamma_{42} = \emptyset, \Gamma_{43} = \{\binom{0}{0}\}, \Gamma_{44} = \emptyset, \Gamma_{45} = \emptyset, \Gamma_{46} = \{\binom{0}{1}\}, \Gamma_{47} = \emptyset, \Gamma_{48} = \emptyset,$$
$$\Gamma_{51} = \{\binom{-1}{0}\}, \Gamma_{52} = \emptyset, \Gamma_{53} = \emptyset, \Gamma_{54} = \{\binom{0}{0}\}, \Gamma_{55} = \emptyset, \Gamma_{56} = \emptyset, \Gamma_{57} = \emptyset, \Gamma_{58} = \emptyset,$$
$$\Gamma_{61} = \emptyset, \Gamma_{62} = \emptyset, \Gamma_{63} = \emptyset, \Gamma_{64} = \emptyset, \Gamma_{65} = \{\binom{0}{0}\}, \Gamma_{66} = \emptyset, \Gamma_{67} = \emptyset, \Gamma_{68} = \{\binom{-1}{0}\},$$
$$\Gamma_{71} = \emptyset, \Gamma_{72} = \emptyset, \Gamma_{73} = \{\binom{0}{-1}\}, \Gamma_{74} = \emptyset, \Gamma_{75} = \emptyset, \Gamma_{76} = \{\binom{0}{0}\}, \Gamma_{77} = \emptyset, \Gamma_{78} = \emptyset,$$
$$\Gamma_{81} = \emptyset, \Gamma_{82} = \{\binom{0}{-1}\}, \Gamma_{83} = \emptyset, \Gamma_{84} = \emptyset, \Gamma_{85} = \emptyset, \Gamma_{86} = \emptyset, \Gamma_{87} = \{\binom{0}{0}\}, \Gamma_{88} = \emptyset.$$

And

$$\mathcal{D}_1 = \bigcup_{i=1}^{8} \Gamma_{i1} = \left\{\binom{0}{0}, \binom{-1}{0}\right\}, \mathcal{D}_2 = \bigcup_{i=1}^{8} \Gamma_{i2} = \left\{\binom{0}{0}, \binom{0}{-1}\right\}, \mathcal{D}_3 = \bigcup_{i=1}^{8} \Gamma_{i3} = \left\{\binom{0}{0}, \binom{0}{-1}\right\},$$

$$\mathcal{D}_4 = \bigcup_{i=1}^{8} \Gamma_{i4} = \left\{\binom{0}{0}, \binom{1}{0}\right\}, \ \mathcal{D}_5 = \bigcup_{i=1}^{8} \Gamma_{i5} = \left\{\binom{0}{0}, \binom{1}{0}\right\}, \ \mathcal{D}_6 = \bigcup_{i=1}^{8} \Gamma_{i6} = \left\{\binom{0}{0}, \binom{0}{1}\right\},$$

$$\mathcal{D}_7 = \bigcup_{i=1}^{8} \Gamma_{i7} = \left\{\binom{0}{0}, \binom{0}{1}\right\}, \mathcal{D}_8 = \bigcup_{i=1}^{8} \Gamma_{i8} = \left\{\binom{0}{0}, \binom{-1}{0}\right\}.$$

Obviously, K is a $\mathbb{Z}^2$-tiling set. Note that for each $j \in \{1,2,3,4,5,6,7,8\}$, $0 \in \mathcal{D}_j$ and $\mathcal{D}_j$ is a complete set of coset representatives for the group $\mathbb{Z}^n/B\mathbb{Z}^n$. Hence, K is an integral self-affine $\mathbb{Z}^2$-tiling set with 8 prototiles which satisfies that $K \bigcap \mathbb{Z}^2 = \{0\}$. We conclude from Theorem 4.2.5 that $Q := \bigcup_{j=1}^{8} K_j + L_j$, where $L_j = \mathcal{D}_j \setminus \{0\}$, is an A-dilation wavelet set.

It has been shown in Chapter 3 that no single self-affine tile can yield a wavelet set associated with the matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. However, Example 4.2.7 illustrates that some A-dilation wavelet sets can be constructed by integral self-affine multi-tiles. This again demonstrates that integral self-affine multi-tiles are useful for constructing many new examples of wavelet sets. Now we give a more general result for the construction of wavelet sets using integral self-affine multi-tiles, which generalizes Theorem 4.5 in [24], with the difference that the conditions for the theorem to hold are now sufficient, but not necessary. Theorem 4.2.5 is a particular case of it obtained by taking $S_j = \{0\}$ for all $j = 1, \ldots, M$, below.

Theorem 4.2.8. *Let C be an $n \times n$ nonsingular integral matrix which commutes with B. If $K \bigcap \mathbb{Z}^n = \{0\}$ and for each $j = 1, \ldots, M$, there exists a finite set $S_j \subset \mathbb{Z}^n$ satisfying*

(i) $\{0\} \subset S_j \subset \bigcup_{i=1}^{M} \Gamma_{ij} + BS_i$,

(ii) $L_j := \bigcup_{i=1}^{M} (\Gamma_{ij} + BS_i) \setminus S_j$ *is a complete set of coset representatives for the group* $\mathbb{Z}^n/C\mathbb{Z}^n$.

Then $Q := C^{-1}\bigcup_{i=1}^{M}(K_i + L_i)$ *is an A-dilation wavelet set.*

Proof. Since, for each $j \in S$, L_j is a complete set of coset representatives for the group $\mathbb{Z}^n/C\mathbb{Z}^n$ and the set K is a $\mathbb{Z}^n$-tiling set, then we have

$$\begin{aligned}\bigcup_{k\in\mathbb{Z}^n} Q + k &= C^{-1}\bigcup_{k\in\mathbb{Z}^n}\bigcup_{j=1}^{M}(K_j + L_j + Ck) \\ &= C^{-1}\bigcup_{k\in\mathbb{Z}^n}\bigcup_{j=1}^{M}(K_j + k) = \mathbb{R}^n,\end{aligned} \tag{4.2.5}$$

and

$$|Q| = |\det C|^{-1}\sum_{j=1}^{M}|\det C||K_j| = \sum_{j=1}^{M}|K_j| = 1. \tag{4.2.6}$$

It follows from (4.2.5) and (4.2.6) that Q is a $\mathbb{Z}^n$-tiling set. Let $W = \bigcup_{i=1}^{M}(K_i + S_i)$. By the definition of Q and $L_j,\ j \in S$, we have

$$\begin{aligned}C^{-1}(BW \setminus W) &= C^{-1}[B\bigcup_{i=1}^{M}(K_i + S_i) \setminus \bigcup_{i=1}^{M}(K_i + S_i)] \\ &= C^{-1}[\bigcup_{i=1}^{M}\bigcup_{j=1}^{M}(K_j + \Gamma_{ij} + BS_i) \setminus \bigcup_{j=1}^{M}(K_j + S_j)] \\ &= C^{-1}\bigcup_{j=1}^{M}K_j + (\bigcup_{i=1}^{M}(\Gamma_{ij} + BS_i) \setminus S_j) \\ &= C^{-1}\bigcup_{j=1}^{M}(K_j + L_j) = Q.\end{aligned} \tag{4.2.7}$$

Thus $B^mQ = C^{-1}(B^{m+1}W \setminus B^mW)$ and $\{B^mQ\}_{m\in\mathbb{Z}}$ is an essentially disjoint family. Now we only need to prove that $\bigcup_{m\in\mathbb{Z}} B^mQ = \mathbb{R}^n$. The fact that $K \bigcap \mathbb{Z}^n = \{0\}$ implies that K contains a neighborhood of 0 by Lemma 4.1.2. Since for each $i \in 1, \ldots, M,\ 0 \in S_i$, $K \subset W$. Thus W contains a neighborhood of 0 and $\bigcup_{m\in\mathbb{Z}} B^mW = \mathbb{R}^n$. An argument similar to the one given in Theorem 4.2.5 shows that $\bigcup_{m\in\mathbb{Z}} B^mQ = \mathbb{R}^n$. □

Example 4.2.9. In dimension one, consider the set $K = [-\frac{1}{2}, \frac{1}{2}]$ associated with the dilation $A = 3$. Let $K_1 = [-\frac{1}{2}, 0]$, $K_2 = [0, \frac{1}{2}]$. Then $B = A^t = 3$ and we have

$$BK_1 = 3K_1 = [-\frac{3}{2}, 0] = (K_1 + \{-1, 0\})\bigcup(K_2 + \{-1\})),$$
$$BK_2 = 3K_2 = [0, \frac{3}{2}] = (K_1 + \{1\})\bigcup(K_2 + \{0, 1\}).$$

Thus $\Gamma_{11} = \{-1, 0\}, \Gamma_{12} = \{-1\}, \Gamma_{21} = \{1\}, \Gamma_{22} = \{0, 1\}$. Let

$$S_1 = \Big\{-3, -2, -1, 0, 4\Big\},\ S_2 = \Big\{-4, -3, -2, -1, 0, 1, 3, 4\Big\}.$$

Then we obtain

$$\begin{aligned} S_1 &= \Big\{-3, -2, -1, 0, 4\Big\} \\ &\subset \bigcup_{i=1}^{2}(\Gamma_{i1} + BS_i) = (\Gamma_{11} + BS_1)\bigcup(\Gamma_{21} + BS_2) \\ &= (\Big\{-1, 0\Big\} + 3\Big\{-3, -2, -1, 0, 4\Big\})\bigcup(\Big\{1\Big\} + 3\Big\{-4, -3, -2, -1, 0, 1, 3, 4\Big\}) \\ &= \Big\{-11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 4, 10, 11, 12, 13\Big\}. \end{aligned}$$

$$\begin{aligned} S_2 &= \Big\{-4, -3, -2, -1, 0, 1, 3, 4\Big\} \\ &\subset \bigcup_{i=1}^{2}(\Gamma_{i2} + BS_i) = (\Gamma_{12} + BS_1)\bigcup(\Gamma_{22} + BS_2) \\ &= (\Big\{-1\Big\} + 3\Big\{-3, -2, -1, 0, 4\Big\})\bigcup(\Big\{0, 1\Big\} + 3\Big\{-4, -3, -2, -1, 0, 1, 3, 4\Big\}) \\ &= \Big\{-12, -11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 3, 4, 9, 10, 11, 12, 13\Big\}. \end{aligned}$$

Define

$$L_1 := \bigcup_{i=1}^{2}(\Gamma_{i1} + BS_i) \setminus S_1 = \{-11, -10, -9, -8, -7, -6, -5, -4, 1, 10, 11, 12, 13\},$$

$$L_2 := \bigcup_{i=1}^{2}(\Gamma_{i2} + BS_i) \setminus S_2 = \{-12, -11, -10, -9, -8, -7, -6, -5, 9, 10, 11, 12, 13\}.$$

Then L_1 and L_2 are both complete sets of coset representatives for the group $\mathbb{Z}/13\mathbb{Z}$. It follows from Theorem 4.2.8 that the set

$$Q := 13^{-1}(3\bigcup_{i=1}^{2} K_i + S_i) \setminus (\bigcup_{i=1}^{2} K_i + S_i) = 13^{-1}\bigcup_{i=1}^{2}(K_i + L_i)$$

is a 3-dilation wavelet set.

In Example 4.2.9, the set K is an integral self-affine $\mathbb{Z}$-tiling set with 2 prototiles and $|\det C| \notin (3-1)\mathbb{Z}$ since $|\det B| = 3$ and $|\det C| = 13$. This shows that the last statement in Proposition 4.2.1 does not extend to multi-tiles.

Example 4.2.10. In dimension two, consider the set $K = conv\left\{\binom{\frac{1}{2}}{\frac{1}{2}}, \binom{\frac{1}{2}}{-\frac{1}{2}}, \binom{-\frac{1}{2}}{\frac{1}{2}}, \binom{-\frac{1}{2}}{-\frac{1}{2}}\right\}$ associated with the matrix $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. Define $K_i, i = 1, 2, 3, 4$ as in Example 4.2.6. Then K is an integral self-affine $\mathbb{Z}^2$-tiling set with 4 prototiles and $\Gamma_{ij}, i, j = 1, 2, 3, 4$ are the same to those obtained in Example 4.2.6. Let

$$S_1 = \left\{\binom{0}{0}, \binom{-1}{-1}\right\}, \quad S_2 = \left\{\binom{0}{0}, \binom{0}{-1}\right\}, \quad S_3 = \left\{\binom{0}{0}, \binom{0}{1}\right\}, \quad S_4 = \left\{\binom{0}{0}, \binom{0}{1}\right\}.$$

Then, we have

$$\left\{\binom{0}{0}\right\} \subseteq S_1 \subseteq \bigcup_{i=1}^{4} \Gamma_{i1} + BS_i = BS_1 \bigcup \binom{0}{-1} + BS_2 = \left\{\binom{0}{0}, \binom{0}{-1}, \binom{-1}{-2}, \binom{-1}{-1}\right\},$$

$$\left\{\binom{0}{0}\right\} \subseteq S_2 \subseteq \bigcup_{i=1}^{4} \Gamma_{i2} + BS_i = \binom{0}{-1} + BS_3 \bigcup BS_4 = \left\{\binom{0}{0}, \binom{0}{-1}, \binom{1}{-1}, \binom{1}{0}\right\},$$

$$\left\{\binom{0}{0}\right\} \subseteq S_3 \subseteq \bigcup_{i=1}^{4} \Gamma_{i3} + BS_i = BS_3 \bigcup \binom{0}{1} + BS_4 = \left\{\binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1}\right\},$$

$$\left\{\binom{0}{0}\right\} \subseteq S_4 \subseteq \bigcup_{i=1}^{4} \Gamma_{i4} + BS_i = \binom{0}{1} + BS_1 \bigcup BS_2 = \left\{\binom{0}{0}, \binom{0}{1}, \binom{-1}{-1}, \binom{-1}{0}\right\}.$$

Define $L_j = \bigcup_{i=1}^{4} \Gamma_{ij} + BS_i \setminus S_j,\ j = 1, 2, 3, 4$. Then

$$L_1 = \left\{\binom{0}{-1}, \binom{-1}{-2}\right\}, \quad L_2 = \left\{\binom{1}{-1}, \binom{1}{0}\right\}, \quad L_3 = \left\{\binom{1}{0}, \binom{1}{1}\right\}, \quad L_4 = \left\{\binom{-1}{-1}, \binom{-1}{0}\right\}.$$

Let $C = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$. Then $C = B$, where $B = A^t$ and obviously, it commutes with the matrix B. It is easy to check that for each $j = 1, 2, 3, 4$, L_j is a complete set of coset representatives for the group $\mathbb{Z}^2/C\mathbb{Z}^2$. It follows from Theorem 4.2.8 that $Q := C^{-1} \bigcup_{i=1}^{4} (K_i + L_i)$ is an A-dilation wavelet set.The set Q and its integer translation $Q + \binom{1}{0}, \binom{1}{-1}$ are depicted in Figure 4.3. The set Q and its B-dilations $BQ, B^{-1}Q$ are depicted in Figure 4.4.

4.3 Construction of multiwavelet sets using integral self-affine multi-tiles

This section is devoted to considering the problem of constructing multiwavelet sets using integral self-affine multi-tiles. Bownik, Rzeszotnik and Speegle ([12], section 3)

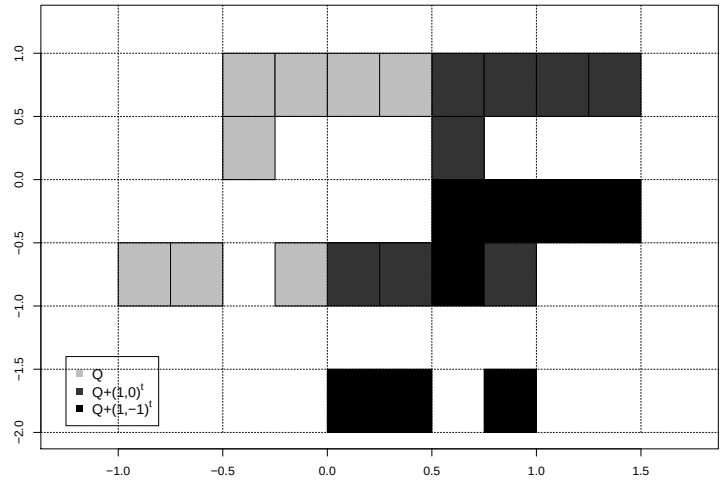


Figure 4.3: Q and its integer translations in Example 4.2.10.

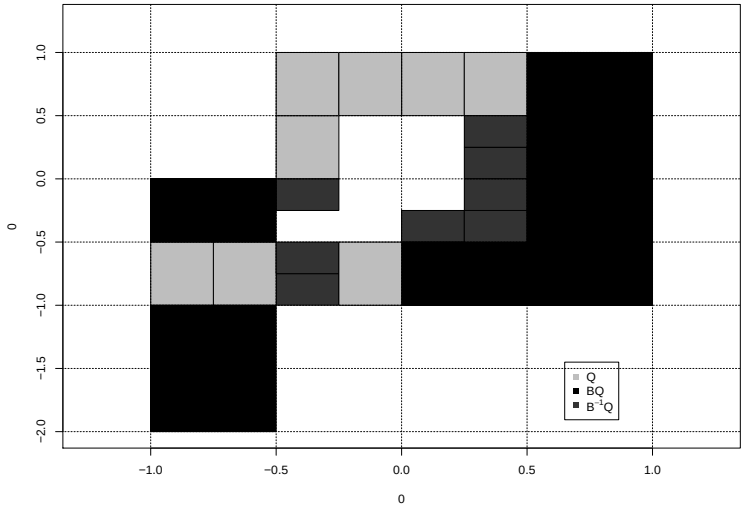


Figure 4.4: Q and its B-dilations $BQ, B^{-1}Q$ in Example 4.2.10.

showed that A-dilation generalized scaling sets and A-dilation multiwavelet sets are determined by one another. Theorem 4.3.2 below provides a sufficient condition for an integral self-affine multi-tile to be a generalized scaling set.

Theorem 4.3.1. *Let K be an integral self-affine $\mathbb{Z}^n$-tiling set with M prototiles which satisfies (0.0.5) and (0.0.6) where, for each $j \in S$, $0 \in \mathcal{D}_j := \bigcup_{i=1}^{M} \Gamma_{ij}$. In addition, let us assume that $K \bigcap \mathbb{Z}^n = \{0\}$. Then K is an A-dilation generalized scaling set of order $q-1$, where $A = B^t$ and $q = |B|$.*

Proof. Since for each $j \in S$, $0 \in \mathcal{D}_j$, (4.1.7) implies that $K \subseteq BK$. Since $K \bigcap \mathbb{Z}^n = \{0\}$, K contains a neighborhood of 0 using Proposition 4.1.6. Thus we have $\bigcup_{m\in\mathbb{Z}} B^m K = \mathbb{R}^n$ since B is expansive. The facts that $K \subseteq BK$ and $\bigcup_{m\in\mathbb{Z}} B^m K = \mathbb{R}^n$ imply that $\lim_{m\to\infty} \chi_K(B^{-m}\xi) = 1$ for a.e. $\xi \in \mathbb{R}^n$. Define $D(\xi) := \sum_{k\in\mathbb{Z}^d} \chi_K(\xi + k)$. Then $D(\xi) = 1$ for a.e. $\xi \in \mathbb{R}^n$ and $|K| = 1$ since K is a $\mathbb{Z}^n$ tiling set. This shows that $\sum_{d\in\mathcal{D}} D(\xi + B^{-1}d) = |\det B| = q$ if $\mathcal{D}$ is a set of coset representatives for the group $\mathbb{Z}^n/B\mathbb{Z}^n$. Therefore, we have $\sum_{d\in\mathcal{D}} D(\xi + B^{-1}d) = D(B\xi) + (q-1)$. By Theorem 0.0.4, K is an A-dilation generalized scaling set of order $q-1$. □

In Theorem 4.3.2, if we let $Q := \bigcup_{j=1}^{M} K_j + L_j$, where $L_j = \mathcal{D}_j \setminus \{0\}$, then $Q = BK \setminus K$ and Q is an A-dilation multiwavelet set of order $q-1$ by the definition of A-dilation generalized scaling set. If $q = 2$, then the wavelet set Q must be an A-dilation MRA wavelet set which has been extensively studied in Chapter 2 and Chapter 3. We can also construct an A-dilation multiwavelet set starting with a given A-dilation wavelet set.

Lemma 4.3.2. *Let Q be an A-dilation wavelet set. If C is an $n \times n$ nonsingular integral matrix commuting with B, where $B = A^t$, then CQ is an A-dilation multiwavelet set of order $|\det C|$.*

Proof. Let $\mathcal{D}'$ be a set of coset representatives for the group $\mathbb{Z}^n/C\mathbb{Z}^n$. By assumption that Q is an A-dilation wavelet set, then the set Q satisfies (i) and (ii) of Proposition 0.0.1. Therefore, we have, using (i) of Proposition 0.0.1, that

$$\sum_{j\in\mathbb{Z}} \chi_{CQ}(B^j\xi) = \sum_{j\in\mathbb{Z}} \chi_Q(C^{-1}B^j\xi) = \sum_{j\in\mathbb{Z}} \chi_Q(B^jC^{-1}\xi) = 1 \text{ } a.e. \text{ } \xi \in \mathbb{R}^n. \tag{4.3.1}$$

Using (ii) of Proposition 0.0.1, we have $\sum_{k\in\mathbb{Z}^n} \chi_Q(\xi + k) = 1$ for a.e. $\xi \in \mathbb{R}^n$ and thus

$$\begin{aligned} \sum_{k\in\mathbb{Z}^n} \chi_{CQ}(\xi + k) &= \sum_{t\in\mathcal{D}'} \sum_{k\in\mathbb{Z}^n} \chi_{CQ}(\xi + Ck + t) \\ &= \sum_{t\in\mathcal{D}'} \sum_{k\in\mathbb{Z}^n} \chi_Q(C^{-1}\xi + C^{-1}t + k) \\ &= |\det C| \text{ for } a.e. \text{ } \xi \in \mathbb{R}^n. \end{aligned} \tag{4.3.2}$$

By (4.3.1), (4.3.2) and using Theorem 0.0.2, we obtain that CQ is an A-dilation multi-wavelet set of order $|\det C|$. □

Similarly, we have the following result.

Proposition 4.3.3. If Q is an A-dilation multiwavelet set of order M and C is an $n \times n$ nonsingular integral matrix commuting with B, where $B = A^t$, then CQ is an A-dilation multiwavelet set of order $|\det C|M$.

The previous result raises a natural question: given an A-dilation multiwavelet set of order M, can we find a nonsingular integral matrix C with $|\det C| = M$ which commutes with the matrix B, where $B = A^t$, such that $C^{-1}Q$ is an A-dilation wavelet set. The answer turns out to be negative. The following example shows that the existence of an A-dilation multiwavelet set $Q \subset \mathbb{R}^2$ of order 3 and for which we cannot find a 2×2 nonsingular integral matrix C such that $C^{-1}Q$ is an A-dilation wavelet set.

Example 4.3.4. In $\mathbb{R}^2$, consider the unite square $K = conv\left\{\binom{-\frac{1}{2}}{\frac{1}{2}}, \binom{\frac{1}{2}}{\frac{1}{2}}, \binom{\frac{1}{2}}{-\frac{1}{2}}, \binom{-\frac{1}{2}}{-\frac{1}{2}}\right\}$ associated with the matrix $A = 2I$, where $conv(K)$ denotes the convex hull of K. Then $B = A^t = 2I$. Let

$$K_1 = conv\left\{\binom{-1/2}{0}, \binom{-1/2}{1/2}, \binom{0}{1/2}, \binom{0}{0}\right\}, \quad K_2 = conv\left\{\binom{0}{0}, \binom{1/2}{0}, \binom{1/2}{1/2}, \binom{0}{1/2}\right\},$$

$$K_3 = conv\Big\{\binom{0}{0}, \binom{1/2}{0}, \binom{1/2}{-1/2}, \binom{0}{-1/2}\Big\},\ K_4 = conv\Big\{\binom{0}{0}, \binom{-1/2}{0}, \binom{-1/2}{-1/2}, \binom{0}{-1/2}\Big\}.$$

Then (see Figure 4.5), we have

$$BK_1 = K_1 \bigcup (K_2 + \binom{-1}{0}) \bigcup (K_3 + \binom{-1}{1}) \bigcup (K_4 + \binom{0}{1}),$$

$$BK_2 = \quad (K_1 + \binom{1}{0}) \bigcup K_2 \bigcup (K_3 + \binom{0}{1}) \bigcup (K_4 + \binom{1}{1}),$$

$$BK_3 = (K_1 + \binom{1}{-1}) \bigcup (K_2 + \binom{0}{-1}) \bigcup K_3 \bigcup (K_4 + \binom{1}{0}),$$

$$BK_4 = (K_1 + \binom{0}{-1}) \bigcup (K_2 + \binom{-1}{-1}) \bigcup (K_3 + \binom{-1}{0}) \bigcup K_4.$$

Thus

$$\Gamma_{11} = \{\binom{0}{0}\}, \Gamma_{12} = \{\binom{-1}{0}\}, \Gamma_{13} = \{\binom{-1}{1}\}, \Gamma_{14} = \{\binom{0}{1}\},$$

$$\Gamma_{21} = \quad \{\binom{1}{0}\}, \Gamma_{22} = \{\binom{0}{0}\}, \Gamma_{23} = \{\binom{0}{1}\}, \Gamma_{24} = \{\binom{1}{1}\},$$

$$\Gamma_{31} = \ \{\binom{1}{-1}\}, \Gamma_{32} = \{\binom{0}{-1}\}, \Gamma_{33} = \{\binom{0}{0}\}, \Gamma_{34} = \{\binom{1}{0}\},$$

$$\Gamma_{41} = \{\binom{0}{-1}\}, \Gamma_{42} = \{\binom{-1}{-1}\}, \Gamma_{43} = \{\binom{-1}{0}\}; \Gamma_{44} = \{\binom{0}{0}\}.$$

And

$$\mathcal{D}_1 = \bigcup_{i=1}^{4} \Gamma_{i1} = \Big\{\binom{0}{0}, \binom{1}{0}, \binom{1}{-1}, \binom{0}{-1}\Big\}, \mathcal{D}_2 = \bigcup_{i=1}^{4} \Gamma_{i2} = \Big\{\binom{0}{0}, \binom{-1}{0}, \binom{-1}{-1}, \binom{0}{-1}\Big\},$$

$$\mathcal{D}_3 = \bigcup_{i=1}^{4} \Gamma_{i3} = \Big\{\binom{0}{0}, \binom{-1}{0}, \binom{-1}{1}, \binom{0}{1}\Big\},\ \mathcal{D}_4 = \bigcup_{i=1}^{4} \Gamma_{i4} = \Big\{\binom{0}{0}, \binom{1}{0}, \binom{1}{1}, \binom{0}{1}\Big\}.$$

Note that for each $j \in \{1, 2, 3, 4\}$, $0 \in \mathcal{D}_j$ and $\mathcal{D}_j$ is a complete set of coset representatives for the group $\mathbb{Z}^n / B\mathbb{Z}^n$. Moreover, K is an integral self-affine $\mathbb{Z}^2$-tiling set with 4 prototiles which satisfies that $K \bigcap \mathbb{Z}^2 = \{0\}$. We conclude from Theorem 4.3.2 that K is an A-dilation generalized scaling set of order 3. Therefore, $Q := BK \setminus K = \bigcup_{j=1}^{4} K_j + L_j$, where $L_j = \mathcal{D}_j \setminus \{0\}$, is an A-dilation multiwavelet set of order 3.

Assume that there exists a nonsingular 2×2 integral matrix C which commutes with the matrix B such that $\widetilde{Q} := C^{-1}Q = C^{-1}(BK \setminus K) = C^{-1} \bigcup_{j=1}^{4} (K_j + L_j)$ is an A-dilation

wavelet set. Then by Lemma 4.2.2, for each $j \in \{1,2,3,4\}$, $L_j = \mathcal{D}_j \setminus \{0\}$ is a complete set of coset representatives for the group $\mathbb{Z}^2/C\mathbb{Z}^2$. Hence, for each $j \in \{1,2,3,4\}$, there must exist $\ell_j \in L_j$ such that $\ell_j \in C\mathbb{Z}^2$ and the difference of any two elements in L_j is not in $C\mathbb{Z}^2$.

For $L_1 = \left\{\binom{1}{0}, \binom{1}{-1}, \binom{0}{-1}\right\}$, we have

$$\binom{1}{-1} - \binom{1}{0} = \binom{0}{-1}, \binom{1}{-1} - \binom{0}{-1} = \binom{1}{0}, \binom{1}{0} - \binom{0}{-1} = \binom{1}{1}.$$

This implies that $\binom{1}{-1} \in C\mathbb{Z}^2$ and thus $\binom{-1}{1} \in C\mathbb{Z}^2$.

For $L_2 = \left\{\binom{-1}{0}, \binom{-1}{-1}, \binom{0}{-1}\right\}$, we have

$$\binom{-1}{-1} - \binom{0}{-1} = \binom{-1}{0}, \binom{-1}{-1} - \binom{-1}{0} = \binom{0}{-1}, \binom{-1}{0} - \binom{0}{-1} = \binom{-1}{1}.$$

This shows that $\binom{-1}{-1} \in C\mathbb{Z}^2$ and thus $\binom{1}{1} \in C\mathbb{Z}^2$. Therefore, for the set L_1, we have $\binom{1}{-1} \in C\mathbb{Z}^2$ and $\binom{1}{0} - \binom{0}{-1} \in C\mathbb{Z}^2$, which contradicts the fact that L_1 is a complete set

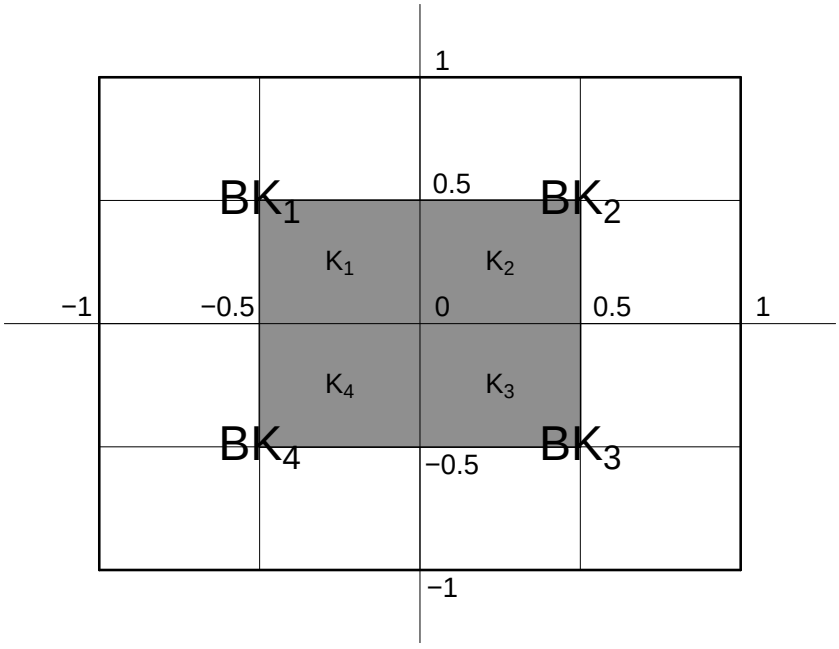


Figure 4.5: K_i and its B-dilation BK_i, i=1,2,3,4 in Example 4.3.4

of coset representatives for the group $\mathbb{Z}^2/C\mathbb{Z}^2$. Therefore, we cannot find a nonsingular matrix C commuting with B such that $\widetilde{Q} = C^{-1}Q$ is an A-dilation wavelet set.

Chapter 5

Decomposition of Integral Self-Affine Multi-Tiles

In this chapter, we consider the problem of representing a $\mathbb{Z}^n$-tiling set which is an integral self-affine multi-tile as the union of an integral self-affine collection with the minimal number of prototiles.

5.1 The representation of integral self-affine multi-tiles

Throughout this chapter, we will limit our discussion to expanding matrices $A \in M_n(\mathbb{Z})$ and let $B = A^t$. Let $S = \{1, 2, \ldots, M\}$. In the following, we will always assume that $K = \bigcup_{i=1}^{M} K_i$, where the union is essentially disjoint, that $|K_i| > 0$ for each $i \in S$, and that the sets K_i, $i \in S$, satisfy (0.0.5) and (0.0.6). Moreover, we assume here that for each $j \in S$, $\mathcal{D}_j := \bigcup_{i=1}^{M} \Gamma_{ij}$ is a complete set of coset representatives for $\mathbb{Z}^n/B\mathbb{Z}^n$. We call such set K a self-affine tiling set with M prototiles. In addition, if K is a $\mathbb{Z}^n$-tiling set, then the set K is called a self-affine $\mathbb{Z}^n$-tiling set with M prototiles and is denoted by

$K(B,\mathbb{Z}^n)$. Define $\Gamma_{ij}^m \subset \mathbb{Z}^n$ by

$$B^m K_i = \bigcup_{j=1}^{M} K_j + \Gamma_{ij}^m. \tag{5.1.1}$$

It follows from (0.0.5) that $\Gamma_{ij}^1 = \Gamma_{ij}$. Using (0.0.5) with each prototile K_j, we have

$$B^2 K_i = \bigcup_{\ell=1}^{M} BK_\ell + B\Gamma_{i\ell} = \bigcup_{\ell=1}^{M}\bigcup_{j=1}^{M} K_j + \Gamma_{\ell j} + B\Gamma_{i\ell} = \bigcup_{j=1}^{M} K_j + \bigcup_{\ell=1}^{M} \Gamma_{\ell j} + B\Gamma_{i\ell}.$$

Thus, $\Gamma_{ij}^2 = \bigcup_{\ell=1}^{M}(\Gamma_{\ell j} + B\Gamma_{i\ell})$. Inductively, we obtain

$$\Gamma_{ij}^m = \bigcup_{\ell=1}^{M}(\Gamma_{\ell j} + B\Gamma_{i\ell}^{m-1}). \tag{5.1.2}$$

Define

$$\mathcal{D}_j^m := \bigcup_{i=1}^{M} \Gamma_{ij}^m,\ m \geq 1. \tag{5.1.3}$$

Then the corresponding self-affine multi-tile $K = \bigcup_{i=1}^{M} K_i$ satisfies

$$BK = \bigcup_{i=1}^{M} BK_i = \bigcup_{i=1}^{M}\bigcup_{j=1}^{M} K_j + \Gamma_{ij} = \bigcup_{j=1}^{M} K_j + \mathcal{D}_j. \tag{5.1.4}$$

And

$$B^m K = \bigcup_{i=1}^{M} B^m K_i = \bigcup_{i=1}^{M}\bigcup_{j=1}^{M} K_j + \Gamma_{ij}^m = \bigcup_{j=1}^{M} K_j + \mathcal{D}_j^m. \tag{5.1.5}$$

Assume that K is an integral self-affine $\mathbb{Z}^n$-tiling set. Then there exists a finite integral self-affine collection $\{K_j\}_{j=1}^M$ such that $K = \bigcup_{j=1}^{M} K_j$ is an integral self-affine $\mathbb{Z}^n$-tiling set. For $m \geq 1$, let C_m be the collection of non-empty sets of the form

$$(B^{k_1}K + \ell_1)\bigcap(B^{k_2}K + \ell_2)\bigcap\cdots\bigcap(B^{k_r}K + \ell_r)\bigcap K,$$

where $1 \le k_i \le m,\ r \ge 1$. Then $C_1 \subseteq C_2 \subseteq \cdots$ and there exists $m_0 \ge 1$ such that $C_m = C_{m_0}$, for $m \ge m_0$ since each element of C_m is a finite union of sets from the finite collection $K_i,\ i = 1, \ldots, M$. By the definition of C_m, the elements of C_m are of the form

$$\bigcap_{\ell_m \in L_m} \cdots \bigcap_{\ell_1 \in L_1} (B^m K + \ell_m) \bigcap \ldots \bigcap (BK + \ell_1) \bigcap K \tag{5.1.6}$$

for some subsets $L_i \subset \mathbb{Z}^n,\ i = 1, \ldots, m$. For each $m \ge 1$, we order the sets in C_m by inclusion and pick up the minimal one in each inclusion sequence. Let C'_m be the collection of all mininal sets picked up by inclusion in C_m. Then, for general $m > 1$, the elements in C'_m can be obtained by decomposing the elements in C'_{m-1} further using (5.1.6) and picking up the minimal one in each ordered inclusion sequence.

If K is an integral self-affine $\mathbb{Z}^n$-tiling set, then there must exist a collection of sets $K_j,\ j = 1, \cdots, M$ such that $K = \bigcup_{j=1}^{M} K_j$ is an integral self-affine $\mathbb{Z}^n$-tiling set with M prototiles which satisfies (0.0.5) and (0.0.6). Define equivalent relations on the elements of the set S step by step according to the relationship among $\mathcal{D}_j^m,\ j \in S$ and $m \ge 1$ as follows. We say that i is equivalent to j at the m^{th} step if $D_i^k = \mathcal{D}_j^k$ for any $k = 1, 2, \ldots, m$. Logically, the equivalent relation defined at the m^{th} step is called the m^{th} equivalent relation, which is denoted by E^m. Furthermore, the equivalent classes obtained by the m^{th} equivalent relation is called the m^{th} equivalent classes. The equivalent classes obtained at each step give a partition of the set S and thus a corresponding partition of the set K. Let $F_{m_j},\ j = 1, \ldots, S_m$ be the m^{th} equivalent classes. Then, for each $m \ge 1$, we have $S = \bigcup_{j=1}^{S_m} F_{m_j}$, where the union is disjoint. Lemma 5.1.1 below characterizes the collection C'_m for each $m \ge 1$.

Lemma 5.1.1. *Let K be an integral self-affine $\mathbb{Z}^n$-tiling set and let $m \ge 1$. Then $E \in C'_m$ if and only if $E = \bigcup_{i \in F} K_i$, where $F \subseteq S$ is an equivalent class associated with the relation E^m and $\{K_j\}_{j=1}^M$ is an integral self-affine collection with $K = \bigcup_{j=1}^{M} K_j$. Clearly, the collection of sets in C'_m form a partition of K.*

Proof. By assumption, $K = \bigcup_{j=1}^{M} K_j$ is an integral self-affine $\mathbb{Z}^n$-tiling set with M prototiles

which satisfies (0.0.5) and (0.0.6). By the definition of C_m defined above and the self-affine property of K, any element in C_m is the union of some sets $K_i,\ i \in I \subseteq S$ and so are the elements of C'_m. Suppose that

$$\bigcap_{\ell_m \in L_m} \cdots \bigcap_{\ell_1 \in L_1} (B^m K + \ell_m) \bigcap \ldots \bigcap (BK + \ell_1) \bigcap K \in C_m.$$

Then,

$$\begin{aligned}
& \bigcap_{\ell_m \in L_m} \cdots \bigcap_{\ell_1 \in L_1} (B^m K + \ell_m) \bigcap \ldots \bigcap (BK + \ell_1) \bigcap K \\
= & \bigcap_{\ell_m \in L_m} \cdots \bigcap_{\ell_1 \in L_1} \bigcup_{j_0, j_1, \ldots, j_m = 1}^{M} (K_{j_m} + \mathcal{D}^m_{j_m} + \ell_m) \bigcap \ldots \bigcap (K_{j_1} + \mathcal{D}_{j_1} + \ell_1) \bigcap K_{j_0} \\
= & \bigcap_{\ell_m \in L_m} \cdots \bigcap_{\ell_1 \in L_1} \bigcup_{j=1}^{M} (K_j + \mathcal{D}^m_j + \ell_m) \bigcap \ldots \bigcap (K_j + \mathcal{D}_j + \ell_1) \bigcap K_j \\
= & \bigcup_{\substack{L_1 \subseteq -\mathcal{D}^1_j \\ \vdots \\ L_m \subseteq -\mathcal{D}^m_j}} K_j.
\end{aligned}$$

Since $\bigcap_{\ell_m \in L_m} \cdots \bigcap_{\ell_1 \in L_1} (B^m K + \ell_m) \bigcap \ldots \bigcap (BK + \ell_1) \bigcap K \neq \emptyset$, there must exist $j_0 \in S$ such that $K_{j_0} \subseteq \bigcup_{\substack{L_1 \subseteq -\mathcal{D}^1_j \\ \vdots \\ L_m \subseteq -\mathcal{D}^m_j}} K_j$. Moreover, if $K_{j_0} \subseteq \bigcup_{\substack{L_1 \subseteq -\mathcal{D}^1_j \\ \vdots \\ L_m \subseteq -\mathcal{D}^m_j}} K_j$, and j is equivalent to j_0 associated with E^m, then $K_j \subseteq \bigcup_{\substack{L_1 \subseteq -\mathcal{D}^1_j \\ \vdots \\ L_m \subseteq -\mathcal{D}^m_j}} K_j$. Let

$$L(j_0) = \Big\{ j \in S, j \text{ is equivalent to } j_0 \text{ at the } m^{th} \text{ step} \Big\}.$$

Then $\bigcup_{j \in L(j_0)} K_j \subseteq \bigcap_{\ell_m \in L_m} \cdots \bigcap_{\ell_1 \in L_1} (B^m K + \ell_m) \bigcap \ldots \bigcap (BK + \ell_1) \bigcap K$. In particular, taking

$L_i = -\mathcal{D}^i_{j_0}, i = 1, \ldots, m$, then

$$\begin{aligned}&\bigcap_{\ell_m \in L_m} \cdots \bigcap_{\ell_1 \in L_1} (B^m K + \ell_m) \bigcap \ldots \bigcap (BK + \ell_1) \bigcap K \\ &= \bigcup_{\substack{-\mathcal{D}^1_{j_0} \subseteq -\mathcal{D}^1_j \\ \vdots \\ -\mathcal{D}^m_{j_0} \subseteq -\mathcal{D}^m_j}} K_j = \bigcup_{\substack{-\mathcal{D}^1_{j_0} = -\mathcal{D}^1_j \\ \vdots \\ \mathcal{D}^m_{j_0} = -\mathcal{D}^m_j}} K_j = \bigcup_{j \in L(j_0)} K_j,\end{aligned}$$

which implies that $\bigcup_{j \in L(j_0)} K_j \in C_m$. This proves that $\bigcup_{j \in L(j_0)} K_j$ is the minimal one in C_m and it is in C'_m. □

Let $K = \bigcup_{i=1}^{M} K_i$ be an integral self-affine tiling set with M prototiles which satisfies (0.0.5) and (0.0.6). If $D^m_i = \mathcal{D}^m_j$ for any $m \geq 1$, we say that i is equivalent to j and denote as $i \sim j$. According to this equivalent relation, we can get a partition $\{F_j\}_{j=1}^{\ell}$, $1 < \ell \leq M$ of S, where the sets F_j, $1 \leq j \leq \ell$ are the corresponding equivalent classes.

Lemma 5.1.2. *Let $K = \bigcup_{i=1}^{M} K_i$ be an integral self-affine $\mathbb{Z}^n$-tiling set with M prototiles which satisfies (0.0.5) and (0.0.6). Let $\widetilde{K}_j = \bigcup_{i \in F_j} K_i$, $j = 1, \ldots, \ell$, where the partition $\{F_j\}_{j=1}^{\ell}$ of S is defined above. Then $\{\widetilde{K}_j\}_{j=1}^{\ell}$ is an integral self-affine collection.*

Proof. First, we will prove that $\bigcup_{i \in F_s} \Gamma_{ij}$ does not depend on j for $j \in F_t$, $t \in \{1, \ldots, \ell\}$. WLOG, we can assume that $i_1 \neq i_2 \in F_t$ for some $t \in \{1, \ldots, \ell\}$. Then for any $m \geq 1$, using (5.1.2), (5.1.3) and that $S = \bigcup_{j=1}^{\ell} F_j$, we have

$$\begin{aligned}\mathcal{D}^m_{i_1} = \mathcal{D}^m_{i_2} &\iff \bigcup_{i=1}^{M} \Gamma_{ii_1} + B\mathcal{D}^{m-1}_i = \bigcup_{i=1}^{M} \Gamma_{ii_2} + B\mathcal{D}^{m-1}_i \\ &\iff \bigcup_{s=1}^{\ell} \bigcup_{i \in F_s} \Gamma_{ii_1} + B\mathcal{D}^{m-1}_i = \bigcup_{s=1}^{\ell} \bigcup_{i \in F_s} \Gamma_{ii_2} + B\mathcal{D}^{m-1}_i\end{aligned} \tag{5.1.7}$$

Let $\phi_{si_1} = \bigcup_{i \in F_s} \Gamma_{ii_1}$, $\phi_{si_2} = \bigcup_{i \in F_s} \Gamma_{ii_2}$. Then we have

$$\mathcal{D}_{i_1} = \bigcup_{s=1}^{\ell} \phi_{si_1} = \bigcup_{s=1}^{\ell} \phi_{si_2} = \mathcal{D}_{i_2}, \tag{5.1.8}$$

which is a complete set of coset representatives for $\mathbb{Z}^n/B\mathbb{Z}^n$ by assumption. By the property of the set F_s, $\mathcal{D}_i^m$ are the same for $i \in F_s$ and any $m \geq 1$. Thus we denote $\mathcal{D}_i^m$ as $\mathcal{D}_{\tau_s}^m$ for $i \in F_s$, which is independent of i. Then equation (5.1.7) can be rewritten as the following

$$\mathcal{D}_{i_1}^m = \mathcal{D}_{i_2}^m \Longleftrightarrow \bigcup_{s=1}^{\ell} \phi_{si_1} + B\mathcal{D}_{\tau_s}^{m-1} = \bigcup_{s=1}^{\ell} \phi_{si_2} + B\mathcal{D}_{\tau_s}^{m-1}. \tag{5.1.9}$$

Suppose that, for some $s = 1, \ldots, \ell$, $\phi_{si_1} \neq \phi_{si_2}$. Then $\phi_{si_1} \setminus \phi_{si_2} \neq \emptyset$ or $\phi_{si_2} \setminus \phi_{si_1} \neq \emptyset$. Without loss of generality, we assume that there exists $x \in \phi_{si_1} \setminus \phi_{si_2}$, then $x \in \phi_{ti_2}$ for some $t \in \{1, 2, \ldots, \ell\}$ and $t \neq s$ by (5.1.8). Next we will prove that $x + B\mathcal{D}_{\tau_s}^m \bigcap y + B\mathcal{D}_{\tau_w}^m = \emptyset$ for any $y \in \phi_{wi_2}$, $w \in \{1, 2, \ldots, \ell\}$ and $y \neq x$. Otherwise, there is $y \in \phi_{wi_2}$ and $y \neq x$ so that $x + B\mathcal{D}_{\tau_s}^m \bigcap y + B\mathcal{D}_{\tau_w}^m \neq \emptyset$, which implies that $x - y \in B\mathbb{Z}^2$, contradicting the fact that x, y belong to the same complete set of coset representatives for $\mathbb{Z}^n/B\mathbb{Z}^n$ by (5.1.8). Hence, it follows from (5.1.9) that the only possibility is $x + B\mathcal{D}_{\tau_s}^{m-1} = x + B\mathcal{D}_{\tau_t}^{m-1}$, which forces that $\mathcal{D}_{\tau_s}^m = \mathcal{D}_{\tau_t}^m$ for any $m \in \mathbb{N}$, contradicting the fact that F_s and F_t belong to two different equivalent classes. Therefore, $\phi_{si_1} = \phi_{si_2}$, where $i_1,\ i_2 \in F_t$. Let $\Lambda_{st} = \bigcup_{i \in F_s} \Gamma_{ij},\ j \in F_t$. Then

$$B\widetilde{K}_s = B \bigcup_{i \in F_s} K_i = \bigcup_{i \in F_s} \bigcup_{j=1}^{M} K_j + \Gamma_{ij} = \bigcup_{t=1}^{\ell} \bigcup_{j \in F_t} K_j + \bigcup_{i \in F_s} \Gamma_{ij} = \bigcup_{t=1}^{\ell} \widetilde{K}_t + \Lambda_{st},$$

Furthermore, for $j \in F_t \subset S$, where $t \in \{1, 2, \ldots, \ell\}$, $\Lambda_t := \bigcup_{s=1}^{\ell} \Lambda_{st} = \bigcup_{s=1}^{\ell} \bigcup_{i \in F_s} \Gamma_{ij} = \mathcal{D}_j$ is a complete set of coset representatives for $\mathbb{Z}^n/B\mathbb{Z}^n$. □

As we mentioned in the introduction, the representation of an integral self-affine $\mathbb{Z}^n$-tiling set is not unique. For convenience, we call the representation of an integral self-affine tiling set $K = \bigcup_{i=1}^{M} K_i$ to be in its simplest form if the number M is minimal. Given an integral self-affine $\mathbb{Z}^n$-tiling set K, using Lemma 5.1.1, for each $m \geq 1$, C'_m gives a partition of the set K. Let $C'_m = \{K_{mi}, i = 1, \ldots, S_m\}$. Since $C_m = C_{m_0}$ for $m \geq m_0$, we have also $C'_m = C'_{m_0}$ for $m \geq m_0$. We let $S_{m_0} = N$ and $K_{m_0 i} = W_i,\ i = 1, \ldots, N$. Actually, this produce an algorithm to decompose any integral self-affine $\mathbb{Z}^n$-tiling set into disjoint

prototiles and the representation obtained here is in its simplest form. Theorem 5.1.3 below illustrates this fact.

Theorem 5.1.3. *Suppose that K is an integral self-affine $\mathbb{Z}^n$-tiling set. Then $K = \bigcup_{i=1}^{N} W_i$ is an integral self-affine $\mathbb{Z}^n$-tiling set with N prototiles, where $W_i,\ i = 1, \ldots, N$ are defined as above. Furthermore, the representation $K = \bigcup_{i=1}^{N} W_i$ is in its simplest form and the simplest form representation of K is unique.*

Proof. Since K is an integral self-affine $\mathbb{Z}^n$-tiling set, there exists a partition $\{K_i\}_{i=1}^{M}$ of K such that $K = \bigcup_{i=1}^{M} K_i$ is an integral self-affine $\mathbb{Z}^n$-tiling set with M prototiles. By the definition of W_i, the sets $W_i,\ i = 1, \ldots N$ are essentially disjoint and Lemma 5.1.1 implies that $W_i = \bigcup_{j \in T_i \subseteq S} K_j$ for each $i = 1, \ldots, N$, where the sets $T_i, i = 1, \ldots, N$ are the equivalent classes obtained by the equivalence relation "$\sim$". Thus $S = \bigcup_{i=1}^{N} T_i$. It follows from Lemma 5.1.2 that the collection $\{W_i\}_{i=1}^{N}$ is an integral self-affine collection. Therefore, $K = \bigcup_{i=1}^{N} W_i$ is an integral self-affine $\mathbb{Z}^n$-tiling set with N prototiles. We note that the sets $W_i,\ i = 1, \ldots, N$ do not depend on the sets $K_i,\ i = 1, \ldots, M$. For any representation of the set K which satisfies integral self-affine condition and have M prototiles, the set W_i is the union of some prototiles for each $i = 1, \ldots, N$. Thus, we have $N \leq M$. This proves that the representation $K = \bigcup_{i=1}^{N} W_i$ is in its simplest form and the simplest form representation of K is unique. □

Theorem 5.1.3 shows that if an integral self-affine $\mathbb{Z}^n$-tiling set $K = \bigcup_{i=1}^{M} K_i$ is not in its simplest form, then there exists a partition $\{F_j\}_{j=1}^{\ell}$ of the set S with F_{j_0} having at least two elements for at least one j_0, such that the collection $\{\bigcup_{i \in F_j} K_i\}_{j=1}^{\ell}$ is an integral self-affine collection. Combining all above results, we get the following corollary, which provides an equivalent condition for an integral self-affine $\mathbb{Z}^n$-tiling set to be in its simplest form.

Corollary 5.1.4. *Let $K = \bigcup_{i=1}^{M} K_i$ be an integral self-affine tiling set with M prototiles which satisfies (0.0.5) and (0.0.6). Then, the representation $K = \bigcup_{i=1}^{M} K_i$ is in its simplest*

form if and only if for any $i_1,\ i_2 \in S$ *with* $i_1 \neq i_2$*, there exists some* $m \geq 1$ *such that* $\mathcal{D}_{i_1}^m \neq \mathcal{D}_{i_2}^m$*, where* $\mathcal{D}_i^m,\ i \in S$ *is defined as in (5.1.3).*

Theorem 5.1.3 provides us an algorithm for decomposing an integral self-affine $\mathbb{Z}^n$-tiling set K into essentially disjoint prototiles $K_j,\ j = 1, \ldots, M$ such that the collection $\{K_j\}_{j=1}^M$ is an integral self-affine collection. Moreover, this representation is in its simplest form and the decomposition is unique in the sense that the number of prototiles is minimal by Corollary 5.1.4 and also, in the sense, that given any representation of K as a union of prototiles $K_j,\ j = 1, \ldots, M$, the elements of the minimal representation can always be written as finite unions of these sets K_j. Given an integral self-affine $\mathbb{Z}^n$-tiling set K, we compute the collection C'_m step by step untill we find an integer m_0 such that $C'_m = C'_{m_0}$ for any $m \geq m_0$. Or, alternatively, we check whether or not the collection C'_m obtained at each step is an integral self-affine collection. If it is, we stop and $K = \bigcup_{i=1}^{S_m} K_{mi}$ is an integral self-affine $\mathbb{Z}^n$-tiling set and the representation is in its simplest form.

As we mentioned before, there are many integral self-affine $\mathbb{Z}^n$-tiling sets which have different representations. However, the representation we provide here is unique by Corollary 5.1.4. Such examples will be given in section 3. Furthermore, we can also use this algorithm to determine whether or not a given measurable $\mathbb{Z}^n$-tiling set $K \subset \mathbb{R}^n$ is an integral self-affine multi-tile associated with any given $n \times n$ integral expansive matrix B. We use the above algorithm to decompose the set K. Then K is an integral self-affine $\mathbb{Z}^n$-tiling set if and only if the process stops after finitely many steps.

5.2 Examples

For some integral self-affine tiling sets with simple geometrical shape, it is easy to see how to decompose the given measurable set $K \subset \mathbb{R}^n$ into essentially disjoint pieces K_j such that $K_j,\ j \in I$, where I is a finite set, is an integral self-affine collection. However, for those with complicated geometrical shape, it is not easy to represent it as an integral self-affine collection intuitively. For such self-affine tiling sets, we can use the method introduced in section 2 to solve this problem. In this section, we will give some examples

to show how to use the algorithm given in section 2 to represent an integral self-affine $\mathbb{Z}^n$-tiling set in its simplest form.

Example 5.2.1. In dimension one, consider the set $K = [-\frac{3}{4}, \frac{1}{4}]$ associated with $B = 2$.

As we mentioned before, the set K here can be not only an integral self-affine $\mathbb{Z}$-tiling set with 4 prototiles, but an integral self-affine $\mathbb{Z}$-tiling set with 3 prototiles. In the following, we will give its representation for each case. For the first case, let

$$K_1 = [-\frac{3}{4}, -\frac{1}{2}],\ K_2 = [-\frac{1}{2}, -\frac{1}{4}],\ K_3 = [-\frac{1}{4}, 0],\ K_4 = [0, \frac{1}{4}].$$

Define $\Gamma_{ij},\ i, j = 1, 2, 3, 4$ as in (0.0.5) and $\mathcal{D}_j := \bigcup_{i=1}^{4} \Gamma_{ij}$. Then, we have

$$BK_1 = [-\frac{3}{2}, -1] = (K_2 - 1)\bigcup(K_3 - 1) \Longrightarrow \Gamma_{11} = \emptyset,\ \Gamma_{12} = \{-1\},\ \Gamma_{13} = \{-1\},\ \Gamma_{14} = \emptyset,$$
$$BK_2 = [-1, -\frac{1}{2}] = (K_4 - 1)\bigcup K_1 \Longrightarrow \Gamma_{21} = \{0\},\ \Gamma_{22} = \emptyset,\ \Gamma_{23} = \emptyset,\ \Gamma_{24} = \{-1\},$$
$$BK_3 = [-\frac{1}{2}, 0] = K_2\bigcup K_3 \Longrightarrow \Gamma_{31} = \emptyset,\ \Gamma_{32} = \{0\},\ \Gamma_{33} = \{0\},\ \Gamma_{34} = \emptyset,$$
$$BK_4 = [0, \frac{1}{2}] = (K_1 + 1)\bigcup K_4 \Longrightarrow \Gamma_{41} = \{1\},\ \Gamma_{42} = \emptyset,\ \Gamma_{43} = \emptyset,\ \Gamma_{44} = \{0\},$$

and

$$\mathcal{D}_1 = \bigcup_{i=1}^{4} \Gamma_{i1} = \Big\{0, 1\Big\},\ \mathcal{D}_2 = \bigcup_{i=1}^{4} \Gamma_{i2} = \Big\{-1, 0\Big\},$$
$$\mathcal{D}_3 = \bigcup_{i=1}^{4} \Gamma_{i3} = \Big\{-1, 0\Big\},\ \mathcal{D}_4 = \bigcup_{i=1}^{4} \Gamma_{i4} = \Big\{-1, 0\Big\}.$$

This shows that for each $j \in \{1, 2, 3, 4\}$, $\mathcal{D}_j$ is a complete set of coset representatives for the group $\mathbb{Z}/2\mathbb{Z}$ and the set K is an integral self-affine $\mathbb{Z}$-tiling set with 4 prototiles. Define $\mathcal{D}_j^m$ as in (5.1.3). It follows from (5.1.2) and (5.1.3) that $\mathcal{D}_j^m = \bigcup_{i=1}^{4} \Gamma_{ij} + 2\mathcal{D}_i^{m-1}$. Note that for $i \in \{1, 2, 3, 4\}$, $\Gamma_{i2} = \Gamma_{i3}$. Thus we get $\mathcal{D}_2^m = \mathcal{D}_3^m$ for any $m \geq 1$ by the definition of $\mathcal{D}_j^m$. On the other hand, since

$$\mathcal{D}_1^2 = \bigcup_{i=1}^{4} \Gamma_{i1} + 2\mathcal{D}_i = \Big\{-2, -1, 0, 1\Big\},\ \mathcal{D}_4^2 = \bigcup_{i=1}^{4} \Gamma_{i4} + 2\mathcal{D}_i = \Big\{-3, -2, -1, 0\Big\},$$

which implies that $\mathcal{D}_1^2 \neq \mathcal{D}_4^2$. And again, we have

$$\mathcal{D}_2^3 = \bigcup_{i=1}^{4} \Gamma_{i2} + 2\mathcal{D}_i^2 = \Big\{ -5, -4, -3, -2, -1, 0, 1, 2 \Big\},$$
$$\mathcal{D}_4^3 = \bigcup_{i=1}^{4} \Gamma_{i4} + 2\mathcal{D}_i^2 = \Big\{ -6, -5, -4, -3, -2, -1, 0, 1 \Big\},$$

which implies that $\mathcal{D}_2^3 \neq \mathcal{D}_4^3$. By Theorem 5.1.3, $K = \bigcup_{i=1}^{4} K_i$ is not in "the simplest form". By the proof in Theorem 5.1.3, we let

$$K_1' = [-\frac{3}{4}, -\frac{1}{2}],\ K_2' = [-\frac{1}{2}, 0],\ K_3' = [0, \frac{1}{4}].$$

Define $\Gamma'_{ij},\ i, j = 1, 2, 3$ to satisfy $BK_i' = \bigcup_{j=1}^{3} K_j + \Gamma'_{ij}$ and $\mathcal{D}_j' = \bigcup_{i=1}^{3} \Gamma'_{ij}$. Then, we have

$$BK_1' = [-\frac{3}{2}, -1] = (K_2' - 1) \Longrightarrow \Gamma'_{11} = \emptyset,\ \Gamma'_{12} = -1,\ \Gamma'_{13} = \emptyset,$$
$$BK_2' = [-1, 0] = K_1' \bigcup K_2' \bigcup (K_3' - 1) \Longrightarrow \Gamma'_{21} = 0,\ \Gamma'_{22} = 0,\ \Gamma'_{23} = -1,$$
$$BK_3' = [0, \frac{1}{2}] = (K_1' + 1) \bigcup K_3' \Longrightarrow \Gamma'_{31} = 1,\ \Gamma'_{32} = \emptyset,\ \Gamma'_{33} = 0.$$

Furthermore,

$$\mathcal{D}_1' = \bigcup_{i=1}^{3} \Gamma'_{i1} = \Big\{0, 1\Big\},\ \mathcal{D}_2' = \bigcup_{i=1}^{3} \Gamma'_{i2} = \Big\{ -1, 0\Big\},\ \mathcal{D}_3' = \bigcup_{i=1}^{3} \Gamma'_{i3} = \Big\{ -1, 0\Big\}.$$

This shows that for each $i \in \{1, 2, 3\}$, $\mathcal{D}_i'$ is a complete set of coset representatives for $\mathbb{Z}/2\mathbb{Z}$ and that $K = \bigcup_{i=1}^{3} K_i'$ is an integral self-affine $\mathbb{Z}$-tiling set with 3 prototiles.

Remark 5.2.2. Let $K = \bigcup_{j=1}^{M}$ be an integral self-affine $\mathbb{Z}^n$-tiling set with M prototiles which satisfy (0.0.5) and (0.0.6). If $\Gamma_{ij_1} = \Gamma_{ij_2}$ for any $i \in S$, then $j_1 \sim j_2$. But the converse is not true.

Example 5.2.3. In dimension one, consider the set $K = [-\frac{3}{4}, \frac{1}{4}]$ associated with $B = -3$.

Obviously, K is a $\mathbb{Z}$-tiling set. For this example, the set K can be represented as many different kinds of prototiles. The simplest form is that K is an integral self-affine tile, since

$$BK = [-\frac{3}{4}, \frac{9}{4}] = K \bigcup (K+1) \bigcup (K+2).$$

If we let $K_1 = [-\frac{3}{4}, -\frac{1}{4}]$ and $K_2 = [-\frac{1}{4}, \frac{1}{4}]$, then

$$BK_1 = [\frac{3}{4}, \frac{9}{4}] = (K_1 + 2) \bigcup K_2 + \{1, 2\} \Longrightarrow \Gamma_{11} = \{2\}, \Gamma_{12} = \{1, 2\},$$
$$BK_2 = [-\frac{3}{4}, \frac{3}{4}] = K_1 + \{0, 1\} \bigcup K_2 \Longrightarrow \Gamma_{21} = \{0, 1\}, \Gamma_{22} = \{0\},$$

and

$$\mathcal{D}_1 = \bigcup_{i=1}^{2} \Gamma_{i1} = \{0, 1, 2\} = \bigcup_{i=1}^{2} \Gamma_{i2} = \mathcal{D}_2.$$

This shows that $K = \bigcup_{i=1}^{2} K_i$ is an integral self-affine $\mathbb{Z}$-tiling set with 2 prototiles. Hence, K is not only an integral self-affine $\mathbb{Z}$-tiling set with one tile, but an integral self-affine $\mathbb{Z}$-tiling set with two prototiles. Corollary 5.1.4 shows that $\mathcal{D}_1^m = \mathcal{D}_2^m$ for any $m \geq 1$, i.e. $1 \sim 2$. However, $\Gamma_{i1} \neq \Gamma_{i2}$ for any $i = 1, 2$.

Example 5.2.4. In dimension two, consider the set $K = H \bigcup (-H) \bigcup K'$ associated with the matrix $B = \begin{pmatrix} -1 & 1 \\ -3 & 1 \end{pmatrix}$, where

$$H = conv\Big\{\begin{pmatrix} 1/3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/6 \\ 1/2 \end{pmatrix}\Big\} \text{ and } K' = conv\Big\{\begin{pmatrix} -1/6 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/6 \\ -1/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix}\Big\},$$

where $conv(E)$ denotes the convex hull of E.

It is easy to see that K is a $\mathbb{Z}^2$-tiling set. The sets K and BK are depicted in Figure 5.1. Clearly, we can divide K into six pieces $\{K_j\}_{j=1}^{6}$ with $K_1 = H, K_2 = E, K_3 = F, K_4 = -E, K_5 = -F, K_6 = -H$, where

$$E = conv\Big\{\begin{pmatrix} 1/3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/6 \\ 1/2 \end{pmatrix}\Big\} \text{ and } F = conv\Big\{\begin{pmatrix} -1/6 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/6 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\Big\}.$$

Moreover, we have (see Figure 5.1)

$$BK_1 = (K_1 + \begin{pmatrix} 0 \\ -1 \end{pmatrix}) \bigcup (K_2 + \begin{pmatrix} 0 \\ -1 \end{pmatrix}),\ BK_2 = (K_3 + \begin{pmatrix} 0 \\ -1 \end{pmatrix}) \bigcup K_5,\ BK_3 = K_1 \bigcup K_2,$$
$$BK_4 = K_3 \bigcup (K_5 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}),\ \ BK_5 = K_4 \bigcup K_6,\ \ BK_6 = (K_4 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \bigcup (K_6 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}),$$

which implies that $\{K_j\}_{j=1}^6$ is an integral self-affine collection. Therefore, K is an integral self-affine $\mathbb{Z}^n$-tiling set with 6 prototiles. However, this representation is not in its simplest form. We will use the algorithm introduced in section 2 to represent the set K in its simplest form. At the first step, we get a partition $C_1' = \{K_{1i}\}_{i=1}^2$ of K by computing (5.1.6) for $m = 1$ (see Figure 5.2).

$$K_{11} = (BK + \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \bigcap K = K_1 \bigcup K_2 \bigcup K_3,\ K_{12} = (BK + \begin{pmatrix} 0 \\ -1 \end{pmatrix}) \bigcap K = K_4 \bigcup K_5 \bigcup K_6.$$

It is easy to check that $\{K_{1i}\}_{i=1}^2$ is not an integral self-affine collection. Thus, we need to decompose $K_{1i},\ i = 1, 2$ further using (5.1.6) (see Figure 5.3) and we have

$$K_{21} = (B^2K + \begin{pmatrix} 1 \\ 1 \end{pmatrix}) \bigcap K_{11} = K_1 \bigcup K_2,\ K_{22} = (B^2K + \begin{pmatrix} -1 \\ -1 \end{pmatrix}) \bigcap K_{11} = K_3,$$
$$K_{23} = (B^2K + \begin{pmatrix} -1 \\ -1 \end{pmatrix}) \bigcap K_{12} = K_4 \bigcup K_6,\ K_{24} = (B^2K + \begin{pmatrix} 1 \\ 1 \end{pmatrix}) \bigcap K_{12} = K_5.$$

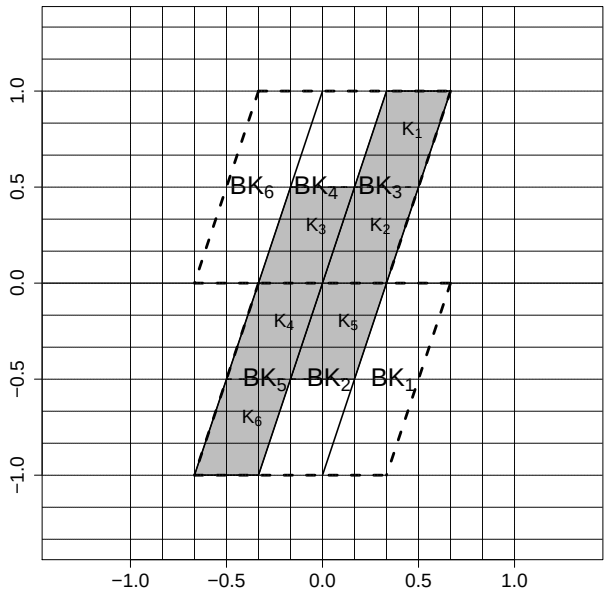
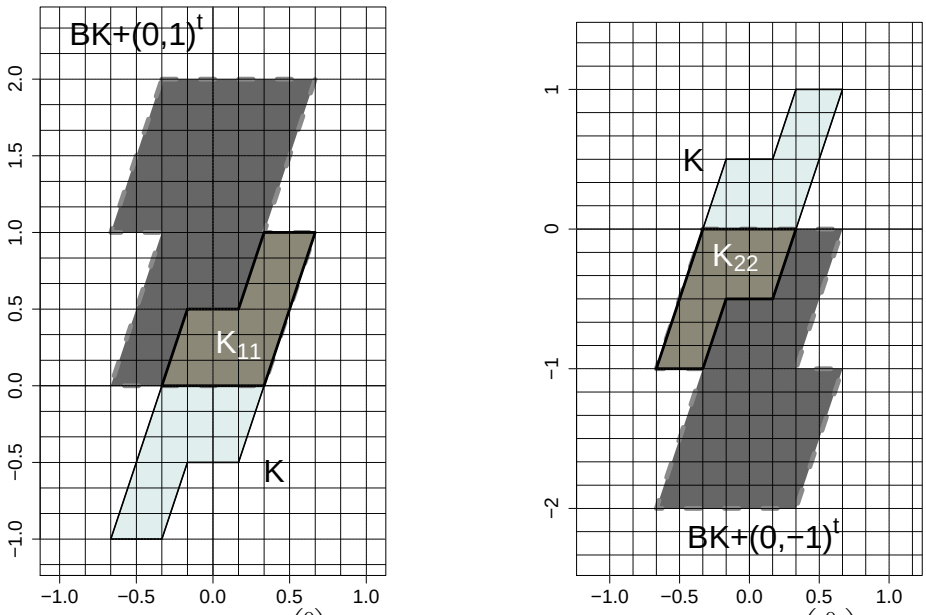


Figure 5.1: K_i and its B-dilation BK_i, i=1,2,3,4 in Example 5.2.4



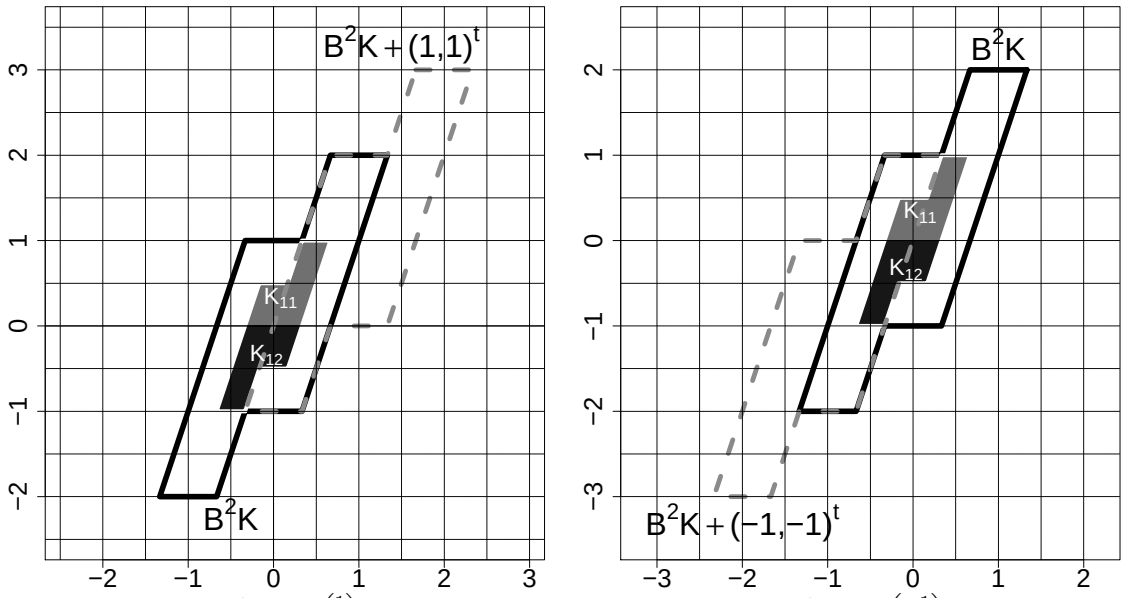
(a).$K \bigcap BK + \binom{0}{1}$ (b).$K \bigcap BK + \binom{0}{-1}$

Figure 5.2: The intersection of K and integer translations of BK in Example 5.2.4

Furthermore, $\{K_{2i}\}_{i=1}^{4}$ is an integral self-affine collection since

$$BK_{21} = (K_{21} + \binom{0}{-1}) \bigcup (K_{21} + \binom{0}{-1}) \bigcup K_{24}, \ BK_{22} = K_{21},$$
$$BK_{23} = (K_{23} + \binom{0}{1}) \bigcup K_{22} \bigcup (K_{24} + \binom{0}{1}), \quad BK_{24} = K_{23}.$$

Therefore, $K = \bigcup_{i=1}^{4} K_{2i}$ is an integral self-affine $\mathbb{Z}^2$-tiling set with 4 prototiles and this



(a). $K_{1i} \bigcap B^2K + \binom{1}{1}, \ i = 1, 2$ (b). $K_{1i} \bigcap B^2K + \binom{-1}{-1}, \ i = 1, 2$

Figure 5.3: The intersection of $K_{1i}, \ i = 1, 2$ and integer translations of B^2K in Example 5.2.4

representation is in its simplest form.

It has been shown in [28] that the theory of interal self-affine multi-tiles is closely related to the theory of wavelet. We also considered in Chapter 4 the problem of constructing wavelet sets using integral self-affine multi-tiles and gave a sufficient condition for an integral self-affine $\mathbb{Z}^n$ multi-tiling set to be a scaling set. The example below shows that some wavelet sets cannot be constructed using integral self-affine $\mathbb{Z}^n$-tiling sets with multi-prototiles as was done in Chapter 4.

Example 5.2.5. In dimension one, consider the set $K = [-a, 1-a]$ where $0 < a < 1$ associated with $B = 2$. Then K is an integral self-affine $\mathbb{Z}^n$ tiling set with several tiles if and only if $a \in \mathbb{Q}$.

Proof. It has been proved in Chapter 3 that K is an A-dilation MRA scaling set and the set $Q := BK \setminus K$ is an A-dilation MRA wavelet set, where $A = B^t = 2$. Obviously, K is a $\mathbb{Z}$-tiling set. We will divide the proof into two cases to prove our claim.
Case 1: $a \in \mathbb{Q}$. Then $a = \frac{p}{q}$, for some p, q $\in \mathbb{N}$ and $(p, q) = 1$, $p < q$ since $0 < a < 1$. In this case, $K = [-\frac{p}{q}, 1 - \frac{p}{q}] = [-\frac{p}{q}, \frac{q-p}{q}]$. Let

$$K_1 = [-\frac{p}{q}, -\frac{p-1}{q}], \ldots, K_i = [-\frac{p-i+1}{q}, -\frac{p-i}{q}], \ldots,$$
$$K_p = [-\frac{1}{q}, 0], \ldots K_q = [\frac{q-p-1}{q}, \frac{q-p}{q}].$$

Then $K = \bigcup_{i=1}^{q} K_i$, where the union is essentially disjoint. For $i \leq p$,

$$\begin{aligned} BK_i &= 2[-\tfrac{p-i+1}{q}, -\tfrac{p-i}{q}] = [-\tfrac{2p-2i+2}{q}, -\tfrac{2p-2i}{q}] \\ &= [-\tfrac{2(p-i+1)}{q}, -\tfrac{2p-2i+1}{q}] \bigcup [-\tfrac{2p-2i+1}{q}, -\tfrac{2(p-i)}{q}]. \end{aligned}$$

Note that

$$-\frac{2(p-i+1)}{q} \in \Big\{ -\frac{p}{q}, -\frac{p-1}{q}, ..., -\frac{1}{q}, 0, \frac{1}{q}, ..., \frac{q-p-1}{q} \Big\} + \mathbb{Z},$$

and

$$-\frac{2(p-i)}{q} \in \Big\{ -\frac{p-1}{q}, -\frac{p-2}{q}, ..., 0, \frac{1}{q}, ..., \frac{q-p}{q} \Big\} + \mathbb{Z}.$$

Then we have $BK_i = (K_{j_1} + \ell_1) \bigcup (K_{j_2} + \ell_2)$ for some $\ell_1,\ \ell_2 \in \mathbb{Z},$ and $j_1,\ j_2 \in \{1, 2, ..., q\}$. For $i > p$, $K_i = [\frac{i-p-1}{q}, \frac{i-p}{q}]$ and $BK_i = 2[\frac{i-p-1}{q}, \frac{i-p}{q}]$. Similarly as above, we can also show that $BK_i = (K_{j_1'} + \ell_1') \bigcup (K_{j_2'} + \ell_2')$ for some $\ell_1',\ \ell_2' \in \mathbb{Z}$ and $j_1',\ j_2' \in \{1, 2, ..., q\}$. This proves that the collection $\{K_i\}_{i=1}^{q}$ is an integral self-affine collection. Assume that

$$BK_i = \bigcup_{j=1}^{q} K_j + \Gamma_{ij}, \text{ where } \Gamma_{\text{ij}} \subseteq \mathbb{Z}.$$

Then $BK = \bigcup_{j=1}^{q} K_j + \mathcal{D}_j$, where $\mathcal{D}_j := \bigcup_{i=1}^{q} \Gamma_{ij}$. Since K is a $\mathbb{Z}$-tiling set,

$$\mathbb{R} = \bigcup_{\ell\in\mathbb{Z}} BK + B\ell = \bigcup_{\ell\in\mathbb{Z}}\bigcup_{j=1}^{q} K_j + \mathcal{D}_j + B\ell = \bigcup_{j=1}^{q} K_j + \bigcup_{\ell\in\mathbb{Z}} \mathcal{D}_j + B\ell,$$

which implies that $\bigcup_{\ell\in\mathbb{Z}} \mathcal{D}_j + B\ell = \mathbb{Z}$ for each $j \in \{1, 2, ..., q\}$. Therefore, for each $j \in \{1, 2, ..., q\}$, $\mathcal{D}_j$ is a complete set of coset representatives for the group $\mathbb{Z}/2\mathbb{Z}$.

Case 2: $a \notin \mathbb{Q}$. Assume that $K = [-a, 1-a]$ is an integral self-affine $\mathbb{Z}$-tiling set with M prototiles. Then we can use the method given in section 2 to decompose K as essentially disjoint union of prototiles. First, we compute C_1' using (5.1.6) and get a partition $\{K_{1i}\}_{i=1}^{2}$ of K which are defined by

$$K_{11} := (BK + 1) \bigcap K = [1-2a, 3-2a] \bigcap [-a, 1-a] = [1-2a, 1-a],$$

$$K_{12} := (BK - 1) \bigcap K = [-1-2a, 1-2a] \bigcap [-a, 1-a] = [-a, 1-2a].$$

Then the endpoints of $K_{1i},\ i = 1, 2$ are $-a,\ 1-2a,\ 1-a$. Let $E_1 = \{-a,\ 1-2a,\ 1-a\}$. We note that $2(1-2a) = 2-4a \neq x + \ell$ for any $x \in E_1$ and $\ell \in \mathbb{Z}$, which implies that $K_{1i},\ i = 1, 2$ is not an integral self-affine collection. Thus we need to compute C_2' and get a new partition $\{K_{2i}\}_{i=1}^{M_2}$ of the set K. The endpoints of $K_{2i},\ i = 1, 2, \ldots, M_2$ belong to the set $E_2 := \{-a, k-4a, 1-2a, 1-a,\ k \in \mathbb{Z}\}$. Since $2(k-4a) = 2k-8a \neq x_2 + \ell$ for any $x_2 \in E_2$ and $\ell \in \mathbb{Z}$, the collection $\{K_{2i}\}_{i=1}^{M_2}$ is not an integral self-affine collection. Thus we have to proceed further steps again. Generally, at the j^{th} step, we get a partition $\{K_{ji}\}_{i=1}^{M_j}$ of K and the endpoints of K_{ji} should be in the set $E_j := \{-a, k_2 - 2^2a, k_3 - 2^3a, ..., k_j - 2^ja, 1-2a, 1-a,\ k_2, \ldots, k_j \in \mathbb{Z}\}$. At the j^{th} step, there must exist some $k_j \in \mathbb{Z}$ such that $k_j - 2^ja$ is one of endpoints of the sets $K_{ji},\ i = 1, 2, \ldots, M_j$. However, $2(k_j - 2^ja) = 2k_j - 2^{j+1}a \neq x + \ell$ for any $x \in E_j$ and $\ell \in \mathbb{Z}$. Therefore, $\{K_{ji}\}_{i=1}^{M_j}$ is not an integral self-affine collection. Hence, if $a \notin \mathbb{Q}$, this process will go on infinitely and for any $j \in \mathbb{N}$, $\{K_{ji}\}_{i=1}^{M_j}$ is not an integral self-affine collection. This proves that $K = [-a, 1-a]$ is not an integral self-affine multi-tile if $a \notin \mathbb{Q}$. □

Chapter 6

The Measure of Self-Affine Sets

If U is a non-empty subset of $\mathbb{R}^n$, we denote the diameter of U as $\mathrm{diam}(U)$, which is defined by $\mathrm{diam}(U) = \sup\{\|x - y\| : x, y \in U\}$. Throughout this chapter, we assume that $B \in M_n(\mathbb{R})$ is an expanding matrix and that $\mathcal{D} = \{d_1, d_2, \ldots, d_m\} \subseteq \mathbb{R}^n$ is a set of m distinct vectors, which is called a digit set. Under this assumption, the self-affine set $K(B, \mathcal{D})$ is the unique compact set K satisfying the set-valued equation

$$BK = \bigcup_{i=1}^{m} (K + d_i).$$

Let $f_i(x) = B^{-1}(x + d_i),\ 1 \leq i \leq m$ and $x \in \mathbb{R}^n$. The family of mappings $\{f_i(x)\}_{i=1}^m$ is called a self-affine iterated function system (IFS) and we have $K = K(B, \mathcal{D}) = \bigcup_{i=1}^{m} f_i(K)$. We say that the IFS $\{f_i\}_{i=1}^m$ satisfies the *open set condition* if there exists a non-empty bounded open set V such that

$$\bigcup_{i=1}^{m} f_i(V) \subset V \text{ and } f_i(V) \bigcap f_j(V) = \emptyset \text{ for } i \neq j.$$

In particular, the set $K(B, \mathcal{D})$ is called a *self-similar set* if the matrix $B = \rho R$, where $\rho > 1$ and R is an orthonormal matrix. In this case, the matrix B is called a *similarity* with scaling factor $\rho > 1$. A set $A \subseteq \mathbb{R}^n$ is called *uniformly discrete* if there exists $\delta > 0$ such that $x, y \in A$ implies that $\|x - y\| > \delta$.

Without loss of generality, we will suppose that $0 \in \mathcal{D}$ in this chapter. Define

$$\mathcal{D}_s := \Big\{ \sum_{j=0}^{s-1} B^j \ell_j : \ell_j \in \mathcal{D}, j \geqslant 0 \Big\} \text{ for } s \geqslant 1 \text{ and } \mathcal{D}_\infty := \bigcup_{s=1}^{\infty} \mathcal{D}_s.$$

Then it is easy for us to prove that $\mathcal{D}_s \subset \mathcal{D}_{s+1}$ for any $s \geq 1$ since $0 \in \mathcal{D}$. Many researchers have considered the problem of characterizing the open set condition of IFS $\{f_i\}_{i=1}^m$. Bandt, Hung and Rao [6] characterized the open set condition for self-similar sets and He and Lau [32] provided an equivalent condition for an IFS $\{f_i\}_{i=1}^m$ to satisfy the open set condition for the case where K is a self-affine set.

Theorem 6.0.6 ([32]). *IFS $\{f_i\}_{i=1}^m$ satisfies the open set condition if and only if $\mathcal{D}_\infty$ is a uniformly discrete set and m^k expansions in $\mathcal{D}_k$ are distinct for all $k \geq 1$.*

Our main goal in this chapter is to prove that the Lebesgue measure of K or the Hausdorff measure $\mathcal{H}^s(K)$, where s is the similarity dimension of the IFS is equal to the inverse of the corresponding upper density of the measure μ, where μ is defined by (0.0.8).

6.1 The Lebesgue measure of self-affine sets

In this section, we will assume that $m = |\det B|$. Under this assumption, Lagarias and Wang [37] gave conditions characterizing when $K(B, \mathcal{D})$ is a self-affine tile.

Theorem 6.1.1 ([37]). *The following four conditions are equivalent.*

(i) *$K(B, \mathcal{D})$ has positive Lebesgue measure.*

(ii) *$K(B, \mathcal{D})$ has nonempty interior.*

(iii) *$K(B, \mathcal{D})$ is the closure of its interior K°, and its boundary $\partial K := K - K^\circ$ has Lebesgue measure zero.*

(iv) *For each $k \geq 1$, all m^k expansions in $\mathcal{D}_k$ are distinct, and $\mathcal{D}_\infty$ is a uniformly discrete set.*

Let $\Lambda \subset \mathbb{R}^n$ be a discrete subset, a measurable set $K \subset \mathbb{R}^n$ is said to Λ-tile $\mathbb{R}^n$, if it satisfies (0.0.7) or equivalently,

$$\sum_{\ell \in \Lambda} \chi_K(x + \ell) = 1 \text{ for } a.e.\ x \in \mathbb{R}^n. \tag{6.1.1}$$

The tiling property (0.0.7) of a measurable set $K \subset \mathbb{R}^n$ can be characterized in terms of Beurling density of Λ. Lemma 6.1.8 below illustrates this fact. In the following, we introduce the definitions of upper and lower Beurling density of a measure μ and a discrete subset $\Lambda \subset \mathbb{R}^n$ respectively.

Definition 6.1.2. Let μ be a Borel measure in $\mathbb{R}^n$. The upper Beurling density of the measure μ with respect to $r > 0$ is defined by

$$\mathcal{D}_r^+(\mu) = \limsup_{N \to \infty} \sup_{z \in \mathbb{R}^n} \frac{\mu(I_N(z))}{N^r}, \tag{6.1.2}$$

and the lower Beurling density of the measure μ with respect to $r > 0$ is defined by

$$\mathcal{D}_r^-(\mu) = \liminf_{N \to \infty} \inf_{z \in \mathbb{R}^n} \frac{\mu(I_N(z))}{N^r}, \tag{6.1.3}$$

where $I_N(z) = \Big\{ y = (y_1, \ldots, y_n) \in \mathbb{R}^n, |y_i - z_i| \leq N, i = 1, \ldots, n \Big\}$. If $\mathcal{D}_r^+(\mu) = \mathcal{D}_r^-(\mu)$, we say that the Beurling density of the measure μ with respect to r exists and we denote it as $\mathcal{D}_r(\mu)$.

Definition 6.1.3 ([13])**.** Let Λ be a discrete subset of $\mathbb{R}^n$. For $r > 0$, the upper Beurling density of Λ corresponding to r (or r-upper Beurling density) is defined by

$$D_r^+(\Lambda) := \limsup_{N \to \infty} \sup_{z \in \mathbb{R}^n} \frac{\#(\Lambda \bigcap I_N(z)}{N^r}, \tag{6.1.4}$$

and the lower Beurling density of Λ with respect to r is defined by

$$D_r^-(\Lambda) := \liminf_{N \to \infty} \inf_{z \in \mathbb{R}^n} \frac{\#(\Lambda \bigcap I_N(z)}{N^r}. \tag{6.1.5}$$

If $D_r^+(\Lambda) = D_r^-(\Lambda)$, then we say that Λ has uniform Beurling density with respect to $r > 0$ and we denote this density by $D_r(\Lambda)$. If we consider the measure $\sigma = \sum_{\lambda \in \Lambda} \delta_\lambda$, it is clear that $D_r^+(\sigma) = D_r^+(\Lambda)$ and $D_r^-(\sigma) = D_r^-(\Lambda)$.

Gabardo studied the notion of Beurling density in [23] and he provided some properties of Beurling density which are related to tilings. In the following, we will list some of his results for later use. For now, we cite the notion of the convolution of $f * \mu$ in [23], where $f \in L^1(\mathbb{R}^n)$ with $f \geq 0$ and μ is a positive Borel measure on $\mathbb{R}^n$. Let $C_c^+(\mathbb{R}^n)$ denote the space of continuous functions φ with compact support on $\mathbb{R}^n$ and with $\varphi \geq 0$.

Definition 6.1.4. Let $f \in L^1(\mathbb{R}^n)$ with $f \geq 0$ and let μ be a positive Borel measure on $\mathbb{R}^n$. The convolution $f * \mu$ is defined by

$$\int_{\mathbb{R}^n} \varphi(t) d(f * \mu)(t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi(x+y) f(y) dy d\mu(x),$$

where $\varphi \in C_c^+(\mathbb{R}^n)$.

Theorem 6.1.5 ([23])**.** *Let $f \in L^1(\mathbb{R}^n)$ with $f \geq 0$ and let μ be a positive Borel measure on $\mathbb{R}^n$. If there exists a constant $C > 0$ such that $f * \mu \leq C$ a.e. on $\mathbb{R}^n$, then $\int_{\mathbb{R}^n} f(x)dx \; D_n^+(\mu) \leq C$. If there exists a constant $C > 0$ such that $f * \mu \geq C$ a.e. on $\mathbb{R}^n$, then $\int_{\mathbb{R}^n} f(x)dx \; D_n^-(\mu) \geq C$.*

Theorem 6.1.6 ([23])**.** *Let μ be a positive Borel measure on $\mathbb{R}^n$. Then*

$$D_n^+(\mu) = \inf_{\substack{f \geq 0 \\ \int f = 1}} \|\mu * f\|_\infty.$$

Lemma 6.1.7. *Let $\Lambda \subset \mathbb{R}^n$ be a discrete set and let $C \in M_n(\mathbb{R}^n)$ be an invertible matrix. Then $|\det C| D_n^+(C\Lambda) = D_n^+(\Lambda)$.*

Proof. Define $\mu = \sum_{\lambda \in \Lambda} \delta_\lambda$ and $\widetilde{\mu} = |\det C| \sum_{\lambda \in \Lambda} \delta_{C\lambda}$. Then, we have

$$\int_{\mathbb{R}^n} f(x) \; d\mu(C^{-1}x) = |\det C| \int_{\mathbb{R}^n} f(Cx) \; d\mu(x).$$

Using the definition of $\widetilde{\mu}$ and the previous equality, we obtain

$$\widetilde{\mu} * f = |\det C| \int_{\mathbb{R}^n} f(C(x-y)) \; d\mu(y) = \mu * h,$$

where $h(x) = |\det C| f(Cx)$. Particularly, if $f \geq 0$ and $\int_{\mathbb{R}^n} f(x) \; dx = 1$, then $h(x) \geq 0$ and $\int_{\mathbb{R}^n} h(x) \; dx = 1$, and vice versa. Thus, $\inf\limits_{\substack{f \geq 0 \\ \int f = 1}} \|\mu * f\|_\infty = \inf\limits_{\substack{f \geq 0 \\ \int f = 1}} \|\widetilde{\mu} * f\|_\infty$. It follows from Theorem 6.1.6 that $D_n^+(\mu) = D_n^+(\widetilde{\mu})$. Then, clearly, $|\det C| D_n^+(C\Lambda) = D_n^+(\Lambda)$. □

Lemma 6.1.8. *Let Λ be a discrete subset of $\mathbb{R}^n$. If a measurable subset $K \subset \mathbb{R}^n$ Λ-tiles $\mathbb{R}^n$, then the uniform n-Beurling density $D_n(\Lambda)$ of Λ exists and $|K|D_n(\Lambda) = 1$.*

Proof. Let δ_ℓ be a dirac measure at ℓ. Then, we have

$$\sum_{\ell\in\Lambda} \chi_{K+\ell}(x) = \sum_{\ell\in\Lambda} \chi_K * \delta_\ell = \chi_K * \sum_{\ell\in\Lambda} \delta_\ell. \tag{6.1.6}$$

Let $\mu := \sum_{\ell\in\Lambda} \delta_\ell$, then μ defines a Borel measure and $\mu \geq 0$. Using our assumption that the set K Λ-tiles $\mathbb{R}^n$, we obtain that $\sum_{\ell\in\Lambda} \chi_{K+\ell}(x) = 1$ a.e. $x \in \mathbb{R}^n$ and $\chi_K \in L^1(\mathbb{R}^n)$. Thus, it follows from (6.1.6) that $\chi_K * \mu = 1$. Theorem 6.1.5 implies that

$$\int_{\mathbb{R}^n} \chi_K(x)dx \; D_n^+(\mu) = \int_{\mathbb{R}^n} \chi_K(x)dx \; D_n^-(\mu) = 1,$$

which yields that $|K|D_n(\mu) = 1$. This shows that $D_n(\Lambda)$ exists and $|K|D_n(\Lambda) = 1$ by the definition of μ. □

A positive Borel measure μ on $\mathbb{R}^n$ is called translation-bounded if, for every compact set $K \subset \mathbb{R}^n$, there exists a constant $C_\mu(K) \geq 0$ such that $\mu(K + z) \leq C_\mu(K)$, $z \in \mathbb{R}^n$. Gabardo [23] provided an equivalent condition for the measure μ, which is defined by (0.0.8), to be translation-bounded.

Lemma 6.1.9. *The measure μ defined in (0.0.8) is translation-bounded if and only if $0 < D_n^+(\mu) < \infty$.*

Theorem 6.1.10. *Let $B \in M_n(\mathbb{R})$ be an expansive matrix with $|\det B| = m \in \mathbb{Z}$ and let $\mathcal{D}$ be a finite subset of $\mathbb{R}^n$ with card$(\mathcal{D}) = m$. Then, $|K(B, \mathcal{D})| = (D_n^+(\mu))^{-1}$, where μ is defined by (0.0.8), with the convention that $|K(B, \mathcal{D})| = 0$ if $D_n^+(\mu) = \infty$.*

Proof. We will consider two cases: $|K(B, \mathcal{D})| > 0$ and $|K(B, \mathcal{D})| = 0$. Assume that $|K(B, \mathcal{D})| > 0$. Then the sets $K+\ell, \ell \in \mathcal{D}_\infty$ are essentially disjoint since $B^k K = \bigcup_{\ell\in\mathcal{D}_k} K+\ell$ has Lebesgue measure $m^k|K|$ and $\mu = \sum_{\lambda\in\mathcal{D}_\infty} \delta_\lambda$ is translation bounded by Theorem 6.1.1

and Lemma 6.1.9. Hence, we have

$$\begin{aligned}\mu * \chi_K(x) &= \sum_{\lambda\in\mathcal{D}_\infty}\int_{\mathbb{R}^n}\chi_K(y)\delta_\lambda(x-y)dy = \sum_{\lambda\in\mathcal{D}_\infty}\int_{\mathbb{R}^n}\chi_K(y)\delta(y-(x-\lambda))dy \\ &= \sum_{\lambda\in\mathcal{D}_\infty}\chi_K(x-\lambda) = \sum_{\lambda\in\mathcal{D}_\infty}\chi_{K+\lambda}(x) = \chi_{\bigcup\limits_{\ell\in\mathcal{D}_\infty}(K+\ell)}(x) \le 1. \qquad (6.1.7)\end{aligned}$$

It follows from (6.1.7) and Theorem 6.1.5 that $\int_{\mathbb{R}^n}\chi_K(d)dxD_n^+(\mu) \le 1$, i.e. $D_n^+(\mu) \le \frac{1}{|K|}$. To prove the converse inequality, use the identity $D_n^+(\mu) = \inf\limits_{\substack{f\ge 0\\ \int f=1}}\|\mu * f\|_\infty$ from Theorem 6.1.6. Thus for any fixed $\varepsilon > 0$, there exists $f \ge 0$ with $\int f = 1$ such that $\|\mu * f\|_\infty \le D_n^+(\mu) + \varepsilon$, which implies that $\mu * f(x) \le D_n^+(\mu) + \varepsilon$ a.e. $x \in \mathbb{R}^n$. Then, using the definition of convolution, we have

$$\mu * f * \frac{\chi_K}{|K|} \le D_n^+(\mu) + \varepsilon. \qquad (6.1.8)$$

On the other hand,

$$\begin{aligned}\mu * f * \tfrac{\chi_K}{|K|}(x) &= \mu * \tfrac{\chi_K}{|K|} * f(x) \\ &= \tfrac{1}{|K|}\int_{\mathbb{R}^n}\mu * \chi_K(x-y)f(y)dy \\ &= \tfrac{1}{|K|}\Big(\int_{B(0,N)}\mu * \chi_K(x-y)f(y)dy + \int_{\mathbb{R}^n\setminus B(0,N)}\mu * \chi_K(x-y)f(y)dy\Big).\end{aligned} \qquad (6.1.9)$$

Since $\mu * \chi_K \le 1$, $f \ge 0$ and $\int_{\mathbb{R}^n} f(y)dy = 1$, then, given $\varepsilon > 0$, using the Dominated Convergence Theorem, we have

$$\int_{\mathbb{R}^n\setminus B(0,N)}\mu * \chi_K(x-y)f(y)\,dy < \varepsilon \text{ if } N \ge N_0. \qquad (6.1.10)$$

It follows from Theorem 6.1.1 that K must contain a non-empty interior since $|K| > 0$. Thus, for any given $N > 0$, $\bigcup\limits_{m\ge 0} B^m K$ must contain a ball $B(x_N, 2N)$ for some $x_N \in \mathbb{R}^n$. It follows from (6.1.7) that

$$\mu * \chi_K(x-y) = \chi_{\bigcup\limits_{\ell\in\mathcal{D}_\infty}(K+\ell)}(x-y) = \chi_{\bigcup\limits_{m\ge 0}B^m K}(x-y).$$

Note that for $y \in B(0,N)$, $\chi_{\bigcup\limits_{m\ge 0}B^m K}(x-y)$ has the value 1 in the ball $B(x_N, N)$. Therefore, we have

$$\int_{B(0,N)}\mu * \chi_K(x-y)f(y)dy = \int_{B(0,N)}f(y)dy \to 1 \text{ as } N\to\infty,\ x \in B(x_N, N). \qquad (6.1.11)$$

We deduce from (6.1.9), (6.1.10) and (6.1.11) that $\|\mu * f * \frac{\chi_K}{|K|}\|_\infty = \frac{1}{|K|}$. Therefore, using (6.1.8), $\frac{1}{|K|} - \varepsilon \leq D_n^+(\mu) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\frac{1}{|K|} \leq D_n^+(\mu)$. This proves our claim if $|K(B, \mathcal{D})| > 0$.

Assume that $|K(B, \mathcal{D})| = 0$. Then either m^k expansions in $\mathcal{D}_k$ are not distinct for some k or the set $\mathcal{D}_\infty$ is not a uniformly discrete set.

Let us assume first that the m^k expansions in $\mathcal{D}_k$ are not distinct for a given k. Let $a \in \mathcal{D}_k$ which can be represented in at least two different forms, i.e.

$$a = \sum_{j=0}^{k-1} B^j d_j = \sum_{j=0}^{k-1} B^j d'_j, \ d_j, d'_j \in \mathcal{D}.$$

Then the element $a + B^k a \in \mathcal{D}_{2k}$ has at least four distinct representations and more generally, $\sum\limits_{j=0}^{M-1} B^{kj} a$ has at least 2^M distinct expansions in $\mathcal{D}_{Mk}$. It follows that if $z_M = \sum\limits_{j=0}^{M-1} B^{kj} a$, then $\mu(\{z_M\}) \geq 2^M$. Hence, $\sup\limits_{z \in \mathbb{R}^n} \frac{\mu(I_N(z))}{N^n} = \infty$ and, in particular,

$$\mathcal{D}_n^+(\mu) = \limsup_{N \to \infty} \sup_{z \in \mathbb{R}^n} \frac{\mu(I_N(z))}{N^n} = \infty.$$

Secondly, we assume that $\mathcal{D}_\infty$ is not a uniformly discrete set. Then given any $M \geq 1$, we can find $x, y \in \mathcal{D}_k \subset \mathcal{D}_\infty$ for some $k \geq 1$ with $\|x - y\| \leq (\sum\limits_{j=0}^{M-1} \|B\|^{kj})^{-1}$. Let

$$F = \Big\{ \sum_{j=0}^{M-1} B^{kj} \varepsilon_j, \ \varepsilon_j \in \{x, y\} \Big\} \text{ and } b = \sum_{j=0}^{M-1} B^{kj} x.$$

Then $F \subset \mathcal{D}_{Mk}$, $b \in \mathcal{D}_{Mk}$ and for any $z \in F$, we have

$$\|z - b\| = \|\sum_{j=0}^{M-1} B^{kj}(\varepsilon_j - x)\| \leq 1.$$

In the definition of $D_n^+(\mu)$, we can take the center of $I_N(z)$ as $z = \sum\limits_{j=0}^{M-1} B^{kj} x$. Then, we have $\mu(I_N(z)) \geq 2^M$ if $N \geq 1$. Therefore, $\sup\limits_{z \in \mathbb{R}^n} \frac{I_N(z)}{N^n} = \infty$ and $D_n^+(\mu) = \infty$ as before. □

By the proof of Theorem 6.1.10 and combining with Theorem 6.0.6 and Lemma 6.1.9, we can deduce the following result.

Theorem 6.1.11. *If the measure μ is defined as in (0.0.8), then the following conditions are equivalent.*

(i) *The IFS $\{f_i\}_{i=1}^m$ satisfies the open set condition.*

(ii) *The m^k expansions in $\mathcal{D}_k$ are distinct for all $k \geq 1$ and $\mathcal{D}_\infty$ is a uniformly discrete set.*

(iii) $|K(B, \mathcal{D})| > 0$

(iv) $0 < D_n^+(\mu) < \infty$.

(v) *μ is translation-bounded.*

Proof. It follows from Theorem 6.0.6 that $(i) \Leftrightarrow (ii)$. Theorem 6.1.1 implies that $(ii) \Leftrightarrow (iii)$. $(iii) \Leftrightarrow (iv)$ is obtained from Theorem 6.1.10. Lemma 6.1.9 shows that $(iv) \Leftrightarrow (v)$. □

The Beurling density of μ can also be used to check whether or not a self-affine tile contains a neighborhood of 0. Theorem 6.1.12 below provides an equivalent condition for a self-affine tile to contain a neighborhood of 0.

Theorem 6.1.12. *Let $K(B, \mathcal{D})$ be a self-affine tile. Then $K(B, \mathcal{D})$ contains a neighborhood of 0 if and only if $D_n^+(\mathcal{D}_\infty) = D_n^-(\mathcal{D}_\infty)$. Furthermore, $D_n^-(\mathcal{D}_\infty) = 0$ if $K(B, \mathcal{D})$ does not contain a neighborhood of 0.*

Proof. We have $|K| > 0$ since $K = K(B, \mathcal{D})$ is a self-affine tile. Then the sets $K + \ell,\ \ell \in \mathcal{D}_\infty$ are essentially disjoint by the proof in Theorem 6.1.10. Suppose that $K(B, \mathcal{D})$ contains a neighborhood of 0. Then $\bigcup_{k\in\mathbb{Z}} B^k K = \bigcup_{\ell\in\mathcal{D}_\infty} K + \ell = \mathbb{R}^n$ since B is expansive. Thus, K is a $\mathcal{D}_\infty$-tiling set. It follows from Lemma 6.1.8 that $D_n^+(\mathcal{D}_\infty) = D_n^-(\mathcal{D}_\infty) = \frac{1}{|K|}$.
Conversely, suppose that $D_n^+(\mathcal{D}_\infty) = D_n^-(\mathcal{D}_\infty)$. Assume that K does not contain a neighborhood of 0. Then $(K+\mathcal{D}_\infty)^c$ is a nonempty open set in $\mathbb{R}^n$ since K is not a $\mathcal{D}_\infty$-tiling set and $\mathcal{D}_\infty$ is a uniformly discrete set. Thus we can find a ball $D(a, r) = \Big\{x \in \mathbb{R}^n, \|x - a\| <$

$r\Big\}$ contained in $(K+\mathcal{D}_\infty)^c$. On the other hand, since, $B(K+\mathcal{D}_\infty)=K+\mathcal{D}_\infty$, we have $B^mD(a,r)\subset(K+\mathcal{D}_\infty)^c$ for any $m\geq 0$. Since B is expansive, $B^mD(a,r)$ contains a cube $I_{N_m}(B^ma)$ with $\lim\limits_{m\to\infty}N_m=\infty$. Hence, we have

$$D_n^-(\mathcal{D}_\infty)=\liminf_{N\to\infty}\inf_{z\in\mathbb{R}^n}\frac{\#(\mathcal{D}_\infty\bigcap(I_N(z))}{N^n}\leq\lim_{m\to\infty}\frac{\#(\mathcal{D}_\infty\bigcap I_{N_m}(B^ma))}{N_m^n}=0.$$

However, using Theorem 6.1.10, we have $D_n^+(\mathcal{D}_\infty)=\frac{1}{|K|}>0$. This contradicts the fact that $D_n^+(\mathcal{D}_\infty)=D_n^-(\mathcal{D}_\infty)$. □

As we mentioned before, if $m>|\det B|$, then the pieces $K+d, d\in\mathcal{D}$ overlap if $|K|>0$, which makes the computation of the Lebesgue measure of K more difficult. We can not obtain the same result to that in Theorem 6.1.10. The following example illustrates this fact.

Example 6.1.13. In dimension one, consider the set K associated with the dilation $\frac{3}{2}$ and the digit set $\mathcal{D}=\{0,1\}$, i.e. the set K satisfies that $\frac{3}{2}K=K\bigcup(K+1)$. Then $K=[0,2]$ and $|K|=2$. By the definition of $\mathcal{D}_s$, we have

$$\mathcal{D}_s=\Big\{\sum_{j=0}^{s-1}B^j\ell_j,\ell_j\in\mathcal{D}\Big\}=\Big\{\sum_{j=0}^{s-1}(\frac{3}{2})^j\ell_j,\ell_j\in\{0,1\}\Big\}.$$

The number of elements in $\mathcal{D}_s$ is 2^s and the maximum element in $\mathcal{D}_s$ is $\sum\limits_{j=0}^{s-1}(\frac{3}{2})^j=2[(\frac{3}{2})^s-1]$. Then, using the definition of $D_n^+(\mu)$, where μ is defined by (0.0.8), we have

$$D_n^+(\mu)\geq\lim_{s\to\infty}\frac{2^s}{2[(\frac{3}{2})^s-1]}=\infty.$$

This shows that $|K|\neq(D_n^+(\mu))^{-1}$.

6.2 The Hausdorff measure of self-affine sets

In section 2, we considered the problem of computing the Lebesgue measure of self-affine set $K(B,\mathcal{D})$ with $m=|\det B|$. In this section, we will assume that $m<|\det B|$.

Then $|K(B,\mathcal{D})| = 0$ in this case and we will consider the problem of computing the s-Hausdorff measure of K for a suitable s. Firstly, let us introduce the definition of s-dimensional Hausdorff measure of a subset $E \subset \mathbb{R}^n$. Let E be a subset of $\mathbb{R}^n$ and let s be a non-negative number. For $\delta > 0$, define

$$\mathcal{H}^s_\delta(E) = \inf\Big\{ \sum_{i=1}^{\infty} [\mathrm{diam}(U_i)]^s : E \subseteq \bigcup_{i=1}^{\infty} U_i, \mathrm{diam}(U_i) < \delta \Big\}.$$

s-dimensional Hausdorff measure of a subset $E \subset \mathbb{R}^n$ is defined by

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E).$$

Under this definiton, n-dimensional Hausdorff measure of $\mathbb{R}^n$ is related to the usual Lebesgue measure if n is a positive integer. Clearly, the definitions of Lebesgue measure and $\mathcal{H}^1$ on $\mathbb{R}$ coincide. For $n > 1$, Falconer in [19] showed that they differ only by a constant multiple.

Theorem 6.2.1. *If $E \subset \mathbb{R}^n$, then $|E| = c_n \mathcal{H}^n(E)$, where $c_n = \pi^{\frac{1}{2}n}/2^n(\frac{1}{2}n)!$. In particular, $c_1 = 1$ and $c_2 = \frac{\pi}{4}$.*

In the following, we will limit our discussion to self-similar sets, i.e. we will assume that B is a similarity with scaling factor $\rho > 1$, i.e. $B = \rho R$, where R is an $n \times n$ orthogonal matrix. Define $f_i(x) = B^{-1}(x + d_i),\ 1 \le i \le m$. Then, under our assumption, we have

$$\|f_i(x) - f_i(y)\| = \rho^{-1}\|x - y\|. \tag{6.2.1}$$

We say that $f_i,\ 1 \le i \le m$ are similarities if they satisfy (6.2.1) for each $1 \le i \le m$. It is obvious that $f_i,\ 1 \le i \le m$ are contractions since $0 < \rho^{-1} < 1$ and therefore, by Hutchinson's theorem [35], there is a unique Borel probability measure σ supported on the set $K(B,\mathcal{D})$ satisfying

$$\int f \, d\sigma = \frac{1}{m} \sum_{i=1}^{m} \int f \circ f_i \, d\sigma, \tag{6.2.2}$$

for all compactly supported continuous function f on $\mathbb{R}^n$. The number $s = \log_\rho^m$, which satisfies that $\sum_{i=1}^{m}(\rho^{-1})^s = 1$, is called the similarity dimension of the set $K(B, \mathcal{D})$. Falconer showed in [18] that the Hausdorff dimension of a self-similar set is equal to its similarity dimension and its corresponding Hausdorff measure is positive if IFS $\{f_i\}_{i=1}^m$ satisfies the open set condition. We list this result below as a theorem.

Theorem 6.2.2 ([18]). *Suppose that the open set condition holds for the similarities f_i, $1 \le i \le m$ on $\mathbb{R}^n$ with ratios $\rho > 1$. Then the Hausdorff dimension of $K(B, \mathcal{D})$ is given by the formula $s := \dim_H K = \log_\rho^m$. Moreover, for this value of s, the corresponding Hausdorff measure of $K(B, \mathcal{D})$ is positive and finite, i.e. $0 < \mathcal{H}^s(K) < \infty$.*

Our main goal in this section is to extend the results of section 2 concerning the Lebesgue measure of self-affine set K. In this section, the Lebesgue measure will be replaced by the s-Hausdorff measure and the Beurling density by an analogous notion of "s-density". Let μ be a measure, the upper density of the measure μ with associated convex sets with respect to the parameter $s > 0$ is defined by

$$\mathcal{E}_s^+(\mu) = \limsup_{r\to\infty} \sup_{\mathrm{diam}(U)\ge r>0} \frac{\mu(U)}{[\mathrm{diam}(U)]^s},$$

where the supremum is over all convex sets U with $\mathrm{diam}(U) \ge r > 0$. We will also need to define the convolution of two measures. If μ is a Borel measure and σ is a Borel probability measure, the convolution $\mu * \sigma$ is a new measure defined by

$$\int_{\mathbb{R}^n} \phi(z) d(\mu * \sigma)(z) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi(x+y) d\mu(x) d\sigma(y),$$

for any compactly supported continuous function ϕ on $\mathbb{R}^n$. If E is a bounded Borel set, we can define $(\mu * \sigma)(E)$ by replacing ϕ by χ_E, the characteristice function of E, in the previous formula.

Lemma 6.2.3. *Let μ be a measure and σ be a Borel probability measure. Then $\mathcal{E}_s^+(\mu*\sigma) = \mathcal{E}_s^+(\mu)$.*

Proof. By the definition of $\mathcal{E}_s^+(\mu)$, we get

$$\begin{aligned}\mathcal{E}_s^+(\mu*\sigma) &= \limsup_{r\to\infty}\sup_{\mathrm{diam}(U)\geq r>0}\frac{\mu*\sigma(U)}{[\mathrm{diam}(U)]^s}\\ &= \limsup_{r\to\infty}\sup_{\mathrm{diam}(U)\geq r>0}\frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\chi_U(x+y)d\mu(x)d\sigma(y)}{[\mathrm{diam}(U)]^s}\\ &= \limsup_{r\to\infty}\sup_{\mathrm{diam}(U)\geq r>0}\frac{\int_{\mathbb{R}^n}\mu(U-y)d\sigma(y)}{[\mathrm{diam}(U)]^s}, \qquad (6.2.3)\end{aligned}$$

where the supremum is over all convex sets U with $\mathrm{diam}(U)\geq r>0$. Since σ is a Borel probability measure, using (6.2.3), we have

$$\limsup_{r\to\infty}\sup_{\mathrm{diam}(U)\geq r>0}\frac{\int_{\mathbb{R}^n}\mu(U-y)d\sigma(y)}{[\mathrm{diam}(U)]^s}\leq\limsup_{r\to\infty}\sup_{\mathrm{diam}(U)\geq r>0}\frac{\mu(U)}{[\mathrm{diam}(U)]^s},$$

which implies that $\mathcal{E}_s^+(\mu*\sigma)\leq\mathcal{E}_s^+(\mu)$. For the converse inequality, we can assume that $\mathcal{E}_s^+(\mu*\sigma)<\infty$. Let V be the convex hull of the sets U and $U+y,\ y\in D(0,R)$ for some fixed $R>0$. Then $U\subseteq V\bigcap(V-y)$ and $\mathrm{diam}(V)\leq\mathrm{diam}(U)+R$. Furthermore, we have

$$\frac{\mu(U)}{[\mathrm{diam}(U)]^s}\leq\frac{\mu(V-y)}{[\mathrm{diam}(U)]^s}. \qquad (6.2.4)$$

It follows from (6.2.4) that, for fixed $R>0$,

$$\begin{aligned}\frac{\int_{D(0,R)}\mu(U)d\sigma(y)}{[\mathrm{diam}(U)]^s} &\leq \frac{\int_{D(0,R)}\mu(V-y)d\sigma(y)}{[\mathrm{diam}(V)]^s}\cdot\frac{[\mathrm{diam}(V)]^s}{[\mathrm{diam}(U)]^s}\\ &\leq \frac{\int_{D(0,R)}\mu(V-y)d\sigma(y)}{[\mathrm{diam}(V)]^s}\cdot\frac{(\mathrm{diam}(U)+R)^s}{[\mathrm{diam}(U)]^s}.\end{aligned}$$

Thus we have

$$\begin{aligned}\limsup_{r\to\infty}\sup_{\mathrm{diam}(U)\geq r>0}\frac{\int_{D(0,R)}\mu(U)d\sigma(y)}{[\mathrm{diam}(U)]^s} &\leq \limsup_{r\to\infty}\sup_{\mathrm{diam}(V)\geq r>0}\frac{\int_{D(0,R)}\mu(V-y)d\sigma(y)}{[\mathrm{diam}(V)]^s}\\ &\leq \limsup_{r\to\infty}\sup_{\mathrm{diam}(V)\geq r>0}\frac{\int_{\mathbb{R}^n}\mu(V-y)d\sigma(y)}{[\mathrm{diam}(V)]^s}.\end{aligned}$$

Letting $R\to\infty$, we obtain that $\mathcal{E}_s^+(\mu)\leq\mathcal{E}_s^+(\mu*\sigma)$. □

Lemma 6.2.4. *Let σ be the Borel probability measure supported on the set $K(B,\mathcal{D})$ which satisfies (6.2.2). Define $\mu_N=\sum_{d_0,\ldots,d_{N-1}\in\mathcal{D}}\delta_{d_0+Bd_1+\cdots+B^{N-1}d_{N-1}}$. Then for any Borel measurable set $W\subset\mathbb{R}^n$, we have $\sigma(B^{-N}W)=\frac{1}{m^N}\mu_N*\sigma(W)$.*

Proof. It follows from (6.2.2) that, for any compactly supported continuous function f on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} f(x)d\sigma(x) = \frac{1}{m}\sum_{i=1}^{m}\int_{\mathbb{R}^n} f(B^{-1}(x+d_i))d\sigma(x). \tag{6.2.5}$$

Moreover, using (6.2.5),

$$\begin{aligned}\textstyle\int_{\mathbb{R}^n} f(x)d\sigma(B^{-1}x) &= |\det B|\textstyle\int_{\mathbb{R}^n} f(Bx)d\sigma(x) = \frac{|\det B|}{m}\sum_{i=1}^{m}\int_{\mathbb{R}^n} f(x+d_i)d\sigma(x)\\ &= \textstyle\frac{|\det B|}{m}\sum_{i=1}^{m}\int_{\mathbb{R}^n} f(x)d\sigma * \delta_{d_i}(x).\end{aligned}$$

This gives that $\frac{\sigma(B^{-1}x)}{|\det B|} = \frac{1}{m}\sum_{i=1}^{m}\sigma * \delta_{d_i}(x) = \frac{1}{m}\sigma * \mu_1(x)$. Suppose that the equality $\frac{\sigma(B^{-N}x)}{|\det B|^N} = \frac{1}{m^N}\sigma * \mu_N(x)$ holds for any $N > 1$. Then, by assumption and (6.2.5), we have

$$\begin{aligned}\textstyle\int_{\mathbb{R}^n} f(x)d\sigma(B^{-(N+1)}x) &= |\det B|^{N+1}\textstyle\int_{\mathbb{R}^n} f(B^{N+1}x)d\sigma(x)\\ &= \textstyle\frac{|\det B|^{N+1}}{m}\sum_{i=1}^{m}\int_{\mathbb{R}^n} f(B^N(x+d_i))d\sigma(x)\\ &= \textstyle\frac{|\det B|}{m}\sum_{i=1}^{m}\int_{\mathbb{R}^n} f(x+B^Nd_i)d\sigma(B^{-N}x)\\ &= \textstyle\frac{|\det B|^{N+1}}{m^{N+1}}\sum_{i=1}^{m}\int_{\mathbb{R}^n} f(x+B^Nd_i)d\sigma * \mu_N(x)\\ &= \textstyle\frac{|\det B|^{N+1}}{m^{N+1}}\sum_{i=1}^{m}\int_{\mathbb{R}^n} f(x)d\sigma * \mu_N * \delta_{B^Nd_i}(x)\\ &= \textstyle\frac{|\det B|^{N+1}}{m^{N+1}}\int_{\mathbb{R}^n} f(x)d\sigma * \mu_{N+1}(x).\end{aligned}$$

The last equality is obtained from the fact that $\sum_{i=1}^{m}\mu_N * \delta_{B^Nd_i}(x) = \mu_{N+1}(x)$ by the definition of the convolution of two measures and μ_N. This proves that for any $N > 1$,

$$\frac{\sigma(B^{-N}x)}{|\det B|^N} = \frac{1}{m^N}\sigma * \mu_N(x). \tag{6.2.6}$$

Then for any Borel measurable set $W \subset \mathbb{R}^n$, using (6.2.6), we have

$$\int_{\mathbb{R}^n}\chi_W(x)d\sigma(B^{-N}x) = |\det B|^N\int_{\mathbb{R}^n}\chi_W(B^Nx)d\sigma(x) = \frac{|\det B|^N}{m^N}\int_{\mathbb{R}^n}\chi_W(x)d\sigma * \mu_N(x),$$

which implies that $\sigma(B^{-N}W) = \frac{1}{m^N}\sigma * \mu_N(W)$. □

In the following, we assume that the IFS $\{f_i\}_{i=1}^m$ satisfies the open set condition. Then $dim_H K = \log_\rho^m$, which we denote s, and the corresponding Hausdorff measure $0 < \mathcal{H}^s(K) < \infty$ by Theorem 6.2.2. It has been shown in [35] that σ in (6.2.2) is a multiple of the restriction of the s-Hausdorff measure $\mathcal{H}^s$ to the set K, i.e.

$$\sigma = (\mathcal{H}^s(K))^{-1}\mathcal{H}^s \restriction K, \tag{6.2.7}$$

A subset $E \subset \mathbb{R}^n$ is called an s-set ($0 \le s \le n$) if E is $\mathcal{H}^s$-measurable and $0 < \mathcal{H}^s(E) < \infty$. We will need the definition of upper convex density of an s-set at x, defined in [19] as

$$D_c^s(E,x) = \overline{\lim_{r\to 0}} \sup_{0<\mathrm{diam}(U)\le r} \frac{\mathcal{H}^s(E\bigcap U)}{[\mathrm{diam}(U)]^s},$$

where the supremum is over all convex sets U with $x \in U$ and $0 < \mathrm{diam}(U) \le r$. Furthermore, we have the following theorem.

Theorem 6.2.5 ([19]). *If E is an s-set in $\mathbb{R}^n$, then $D_c^s(E,x) = 1$ at $\mathcal{H}^s$-almost all $x \in E$ and $D_c^s(E,x) = 0$ at $\mathcal{H}^s$-almost all $x \in E^c$.*

Combining the formula (6.2.7) and Theorem 6.2.5, we can get the following corollary.

Corollary 6.2.6. *Let K be a self-similar set and contractions f_i, $1 \le i \le m$ satisfy the open set condition. Then $\overline{\lim_{r\to 0}} \sup_{0<diam(U)\le r} \frac{\sigma(U)}{[diam(U)]^s} = (\mathcal{H}^s(K))^{-1}$, where s is the Hausdorff dimension of the set K, σ is defined by (6.2.2) and the supremum is over all convex sets U with $U\bigcap K \neq \emptyset$ and $0 < diam(U) \le r$.*

Proof. By assumption, K is an s-set. It follows from (6.2.7) that

$$\begin{aligned}\overline{\lim_{r\to 0}} \sup_{0<\mathrm{diam}(U)\le r} \frac{\sigma(U)}{[\mathrm{diam}(U)]^s} &= (\mathcal{H}^s(K))^{-1}\overline{\lim_{r\to 0}} \sup_{0<\mathrm{diam}(U)\le r} \frac{\mathcal{H}^s(K\bigcap U)}{[\mathrm{diam}(U)]^s} \\ &= (\mathcal{H}^s(K))^{-1} \sup_{x\in K} D_c^s(K,x). \end{aligned} \tag{6.2.8}$$

Since for any $x \in K\bigcap U$, Theorem 6.2.5 implies that $D_c^s(K,x) = 1$, we deduce from (6.2.8) that

$$\overline{\lim_{r\to 0}} \sup_{0<\mathrm{diam}(U)\le r} \frac{\sigma(U)}{[\mathrm{diam}(U)]^s} = (\mathcal{H}^s(K))^{-1}.$$

□

Actually, under certain conditions, the upper convex density of an s-set E at x can also be defined as

$$D_c^s(E,x) = \sup_{r>0} \sup_{0<\mathrm{diam}(U)\le r} \frac{\mathcal{H}^s(E\bigcap U)}{[\mathrm{diam}(U)]^s},$$

where the supremum is over all convex sets U with $x \in U$ and $0 < \mathrm{diam}(U) \le r$. The following lemma clarifies this fact.

Lemma 6.2.7. *Let K be a self-similar set associated with an $n\times n$ integral expansive similarity B with scaling factor $\rho > 1$. Then* $\overline{\lim_{r\to 0}} \sup_{0<diam(U)\le r} \frac{\mathcal{H}^s(K\bigcap U)}{[diam(U)]^s} = \sup_{r>0} \sup_{0<diam(U)\ge\le r} \frac{\mathcal{H}^s(K\bigcap U)}{[diam(U)]^s}$, *where the supremum is over all convex sets U with $U\bigcap K \neq \emptyset$ and $0 < diam(U) \le r$.*

Proof. Obviously, $\overline{\lim_{r\to 0}} \sup_{0<\mathrm{diam}(U)\le r} \frac{\mathcal{H}^s(K\bigcap U)}{[\mathrm{diam}(U)]^s} \le \sup_{r>0} \sup_{0<\mathrm{diam}(U)\le r} \frac{\mathcal{H}^s(K\bigcap U)}{[\mathrm{diam}(U)]^s}$. Conversely, since B is a similarity with scaling factor $\rho > 1$, $\mathcal{H}^s(B^{-1}K) = \rho^{-s}\mathcal{H}^s(K)$. Using the fact that $K \subset BK$, we have

$$\begin{aligned}
\sup_{r>0} \sup_{0<\mathrm{diam}(U)\le r} \frac{\mathcal{H}^s(K\bigcap U)}{[\mathrm{diam}(U)]^s} &= \sup_{r>0} \sup_{0<\mathrm{diam}(U)\le r} \frac{\mathcal{H}^s(B^{-1}(K\bigcap U))}{[\mathrm{diam}(B^{-1}U)]^s} \\
&\le \sup_{r>0} \sup_{0<\mathrm{diam}(U)\le r} \frac{\mathcal{H}^s(K\bigcap B^{-1}U)}{[\mathrm{diam}(B^{-1}U)]^s} \\
&\le \sup_{r>0} \sup_{0<\mathrm{diam}(U)\le r} \frac{\mathcal{H}^s(K\bigcap B^{-m}U)}{[\mathrm{diam}(B^{-m}U)]^s}, \text{ for any m} \ge 1 \\
&\le \limsup_{r\to 0} \sup_{0<\mathrm{diam}(U)\le r} \frac{\mathcal{H}^s(K\bigcap U)}{[\mathrm{diam}(U)]^s}.
\end{aligned}$$

This proves our claim. □

If the similarities $f_i,\ 1 \le i \le m$ satisfy the open set condition and σ is defined by (6.2.2), then $\overline{\lim_{r\to 0}} \sup_{0<\mathrm{diam}(U)\le r} \frac{\sigma(U)}{[\mathrm{diam}(U)]^s} = \sup_{r>0} \sup_{0<\mathrm{diam}(U)\ge\le r} \frac{\sigma(U)}{[\mathrm{diam}(U)]^s}$, where the supremum is over all convex sets U with $U\bigcap K \neq \emptyset$ and $0 < \mathrm{diam}(U) \le r$, by Lemma 6.2.7. Schief [47] proved that similarities $f_i,\ 1 \le i \le m$ satisfy the open set condition if and only if $\mathcal{H}^s(K) > 0$, where $s = \log_\rho^m$ is the similarity dimension, in the Euclidean spaces. However, Schief [48] showed that the open set condition no longer implies the positivity of the Hausdorff measure of the set K associated with the similarity dimension $s = \log_\rho^m$ in the general complete metric spaces. Combining the result provided by Schief and Theorem 6.1.11, we have the following representation for the Hausdorff measure of self-similar sets.

Theorem 6.2.8. *Let K be a self-similar set and $s := \log_\rho^m \leq n$ be the similarity dimension of K. Then $\mathcal{H}^s(K) = (\mathcal{E}_s^+(\mu))^{-1}$, where μ is defined by (0.0.8).*

Proof. Firstly, we assume that $\mathcal{H}^s(K) > 0$, thus that the open set condition holds (by [47]). By Corollary 6.2.6, it is enough to prove that

$$\limsup_{r\to 0} \sup_{0<\mathrm{diam}(U)\leq r} \frac{\sigma(U)}{[\mathrm{diam}(U)]^s} = \mathcal{E}_s^+(\mu),$$

where the supremum is over all convex sets U with $U \bigcap K \neq \emptyset$ and $0 < \mathrm{diam}(U) \leq r$. It follows from Lemma 6.2.7 that $\limsup_{r\to 0} \sup_{0<\mathrm{diam}(U)\leq r} \frac{\sigma(U)}{[\mathrm{diam}(U)]^s} = \sup_{r>0} \sup_{0<\mathrm{diam}(U)\leq r} \frac{\sigma(U)}{[\mathrm{diam}(U)]^s} < \infty$ and both quantities are thus finite by Corollary 6.2.6. Then, for any given $\varepsilon > 0$, there exists a convex set U_0 with $U_0 \bigcap K \neq \emptyset$ such that

$$\frac{\sigma(U_0)}{[\mathrm{diam}(U_0)]^s} \geq \sup_{r>0} \sup_{0<\mathrm{diam}(U)\leq r} \frac{\sigma(U)}{[\mathrm{diam}(U)]^s} - \varepsilon. \tag{6.2.9}$$

Define $\mu_N = \sum_{d_0,\ldots,d_{N-1}\in\mathcal{D}} \delta_{d_0+Bd_1+\cdots+B^{N-1}d_{N-1}}$. On the other hand, using Lemma 6.2.4 and Lemma 6.2.3, we have

$$\begin{aligned} \frac{\sigma(U_0)}{[\mathrm{diam}(U_0)]^s} = \frac{\sigma * \mu_N(B^N U_0)}{[\mathrm{diam}(B^N U_0)]^s} &\leq \limsup_{r\to\infty} \sup_{\mathrm{diam}(U)\geq r>0} \frac{\sigma * \mu(U)}{[\mathrm{diam}(U)]^s} \\ &= \mathcal{E}_s^+(\sigma * \mu) = \mathcal{E}_s^+(\mu). \end{aligned} \tag{6.2.10}$$

It follows from (6.2.9) and (6.2.10) that $\limsup_{r\to 0} \sup_{0<\mathrm{diam}(U)\leq r} \frac{\sigma(U)}{[\mathrm{diam}(U)]^s} \leq \mathcal{E}_s^+(\mu)$.
Conversely, it follows from Lemma 6.2.3 that $\mathcal{E}_s^+(\mu) = \mathcal{E}_s^+(\mu * \sigma)$. For any given convex set U, using Lemma 6.2.4, we have,

$$\begin{aligned} \frac{\sigma * \mu(U)}{[\mathrm{diam}(U)]^s} &= \lim_{N\to\infty} \frac{\sigma * \mu_N(U)}{[\mathrm{diam}(U)]^s} = \lim_{N\to\infty} \frac{\sigma(B^{-N}U)}{[\mathrm{diam}(B^{-N}U)]^s} \\ &\leq \limsup_{r\to 0} \sup_{0<\mathrm{diam}(U)\leq r} \frac{\sigma(U)}{[\mathrm{diam}(U)]^s}. \end{aligned} \tag{6.2.11}$$

By the definition of $\mathcal{E}_s^+(\mu * \sigma)$, (6.2.11) implies that $\mathcal{E}_s^+(\mu) \leq \limsup_{r\to 0} \sup_{0<\mathrm{diam}(U)\leq r} \frac{\sigma(U)}{[\mathrm{diam}(U)]^s}$.
If $\mathcal{H}^s(K) = 0$, then the contractions $f_i,\ 1 \leq i \leq m$ do not satisfy the open set condition

by Schief's result [47]. Thus by Theorem 6.1.11, we have $D_n^+(\mu) = \infty$. Since $s \leq n$ by our assumption, we obtain

$$\begin{aligned} D_n^+(\mu) &= \limsup_{N\to\infty} \sup_{z\in\mathbb{R}^n} \frac{\mu(I_N(z))}{N^n} \leq \limsup_{N\to\infty} \sup_{z\in\mathbb{R}^n} \frac{\mu(I_N(z))}{N^s} = \limsup_{N\to\infty} \sup_{z\in\mathbb{R}^n} \frac{\mu(I_N(z))}{[\frac{\mathrm{diam}(I_N(z))}{\sqrt{n}}]^s} \\ &\leq \sqrt{n}^s \limsup_{r\to\infty} \sup_{\mathrm{diam}(U)\geq r>0} \frac{\mu(U)}{[\mathrm{diam}(U)]^s} = \sqrt{n}^s \mathcal{E}_s^+(\mu), \end{aligned}$$

which implies that $\mathcal{E}_s^+(\mu) = \infty$. Therefore, we have $\mathcal{H}^s(K) = (\mathcal{E}_s^+(\mu))^{-1}$. □

Remark 6.2.9. Let K be a self-similar set and $s := \log_\rho^m \leq n$ be the similarity dimension of K. It follows from the proof in Theorem 6.2.8 that $D_n^+(\mu) = \infty$ implies that $\mathcal{E}_s^+(\mu) = \infty$. Conversely, it is also true, i.e. if $\mathcal{E}_s^+(\mu) = \infty$, then $D_n^+(\mu) = \infty$, where μ is defined by (0.0.8).

Proof. By Theorem 6.2.8, $\mathcal{E}_s^+(\mu) = \infty$ implies that $\mathcal{H}^s(K) = 0$. Then the similarities $f_i,\ 1 \leq i \leq m$ do not satisfy the open set condition by Schief's result [47], which is equivalent to $D_n^+(\mu) = \infty$ by Theorem 6.1.11. □

In the following, we will give an example to show how to use Theorem 6.2.8 to compute the s-Hausdorff measure $\mathcal{H}^s(K)$, where s is the similarity dimension of the self-affine set K.

Example 6.2.10. Consider the set K which satisfies that $NK = K \bigcup (K + d)$, where $d \in \mathbb{R}$.

For this set K, the corresponding digit set $\mathcal{D} = \{0, d\}$ and the similarity dimension $s = \log_N^2$. In particular, take a sequence of convex sets $U_m = [0, \sum_{j=0}^{m-1} N^j \cdot d] = [0, \frac{N^m-1}{N-1}d]$. By the definition of $\mathcal{E}_s^+(\mu)$, where μ is defined by (0.0.8), we have

$$\begin{aligned} \mathcal{E}_s^+(\mu) &= \limsup_{r\to\infty} \sup_{\mathrm{diam}(U)\geq r>0} \frac{\mu(U)}{[\mathrm{diam}(U)]^s} \\ &\geq \lim_{m\to\infty} \frac{\mu(U_m)}{[\mathrm{diam}(U_m)]^s} = \lim_{m\to\infty} \frac{2^m}{(\frac{N^m-1}{N-1}d)^s} = (\frac{N-1}{d})^s \end{aligned} \tag{6.2.12}$$

Since any convex set is an interval in dimension one, we can let $U = [a, b],\ a, b \in \mathbb{R}$ and assume that $\ell = \sum_{j=0}^{m_1} N^j r_j \in [a, b]$ and $s = \sum_{j=0}^{m_1} N^j d_j$ with $r_j, d_j \in \mathcal{D}$ are the largest and smallest elements of $\mathcal{D}_\infty$ which belong to $[a, b]$ with $r_{m_1} \neq 0$. Then we have $a \leq \sum_{j=0}^{m_1} N^j d_j \leq \sum_{j=0}^{m_1} N^j r_j \leq b$. To obtain an upper-bound for $\mathcal{E}_s^+(\mu)$, we can assume, without loss of generality, that $[a, b] = [\sum_{j=0}^{m_1} N^j d_j, \sum_{j=0}^{m_1} N^j r_j]$. Furthermore, we observe that $\mu([0, b-a]) \geq \mu([a, b])$, i.e. for the same lenth convex sets, the number of elements of $\mathcal{D}_\infty$ in the convex set with 0 being one of boundary point is the maximal. Hence, we can assume that $a = 0$ and $b = \sum_{j=0}^{m} N^j r_j$ with $r_m = d$ and $r_j \in \mathcal{D}$ for $j = 1, \ldots, m-1$. Suppose that $\sum_{j=0}^{m} N^j s_j \in [0, \sum_{j=0}^{m} N^j r_j]$. Then the total number of possibilities for $\sum_{j=0}^{m} N^j s_j \in [0, \sum_{j=0}^{m} N^j r_j]$ is $\sum_{j=0}^{m} 2^j \frac{r_j}{d} + 1$, which implies that

$$\mu([0, b]) = \mu([0, \sum_{j=0}^{m} N^j r_j]) = \sum_{j=0}^{m} 2^j \frac{r_j}{d} + 1.$$

On the other hand, if $\sum_{j=0}^{m-1} N^j \cdot d < b \leq \sum_{j=0}^{m} N^j \cdot d$, then $b = \sum_{j=0}^{m} N^j r_j$ with $r_m = d$ and $r_j \in \mathcal{D}$ for $0 \leq j < m$ and we have

$$\begin{aligned} \frac{\mu([0, b])}{b^s} \leq \frac{\mu([0, \sum_{j=0}^{m} N^j \cdot d])}{(\sum_{j=0}^{m} N^j \cdot d)^s} &\iff \frac{\sum_{j=0}^{m} 2^j \frac{r_j}{d} + 1}{(\sum_{j=0}^{m} N^j r_j)^s} \leq \frac{2^{m+1}}{(\sum_{j=0}^{m} N^j \cdot d)^s} \\ &\iff \frac{(\sum_{j=0}^{m} N^j \cdot d)^s}{(\sum_{j=0}^{m} N^j r_j)^s} \leq \frac{2^{m+1}}{\sum_{j=0}^{m} 2^j \frac{r_j}{d} + 1}. \end{aligned} \tag{6.2.13}$$

Next, we will prove (6.2.13). Since $\sum_{j=0}^{m} N^j r_j \le \sum_{j=0}^{m} N^j \cdot d$ and $0 < s < 1$,

$$\frac{(\sum_{j=0}^{m} N^j \cdot d)^s}{(\sum_{j=0}^{m} N^j r_j)^s} < \frac{\sum_{j=0}^{m} N^j \cdot d}{\sum_{j=0}^{m} N^j r_j}$$

and thus, $\frac{\sum_{j=0}^{m} N^j \cdot d}{\sum_{j=0}^{m} N^j r_j} \le \frac{2^{m+1}}{\sum_{j=0}^{m} 2^j \frac{r_j}{d}+1}$ implies (6.2.13). Moreover,

$$\begin{aligned}
\frac{\sum_{j=0}^{m} N^j \cdot d}{\sum_{j=0}^{m} N^j r_j} \quad &\le \quad \frac{2^{m+1}}{\sum_{j=0}^{m} 2^j \frac{r_j}{d} + 1} \\
&\iff \quad \frac{\frac{N^{m+1}-1}{N-1} d}{\sum_{j=0}^{m} N^j r_j} \le \frac{2^{m+1}}{\sum_{j=0}^{m} 2^j \frac{r_j}{d} + 1} \\
&\iff \quad \sum_{j=0}^{m} (2^j \frac{r_j}{d} + 1)(\frac{N^{m+1}-1}{N-1} d) \le \sum_{j=0}^{m} 2^{m+1} N^j r_j \\
&\iff \quad \sum_{j=0}^{m-1} (2^j \frac{N^{m+1}-1}{N-1} - 2^{m+1} N^j) r_j \le \frac{N^m 2^m (N-2) + 2^m - (N^{m+1}-1)}{N-1} d,
\end{aligned}$$

the last inequality is obtained using $r_m = d$. Since $2^j \frac{N^{m+1}-1}{N-1} - 2^{m+1} N^j \ge 0$ for any $0 \le j \le m-1$ and

$$\sum_{j=0}^{m-1} (2^j \frac{N^{m+1}-1}{N-1} - 2^{m+1} N^j) d = \frac{N^m 2^m (N-2) + 2^m - (N^{m+1}-1)}{N-1} d,$$

then $\frac{\sum_{j=0}^{m} N^j \cdot d}{\sum_{j=0}^{m} N^j r_j} \le \frac{2^{m+1}}{\sum_{j=0}^{m} 2^j \frac{r_j}{d}+1}$ holds. Hence, (6.2.13) holds. Then by the definition of $\mathcal{E}_s^+(\mu)$ and above argument, we have

$$\mathcal{E}_s^+(\mu) \le \lim_{m\to\infty} \frac{\mu[0, d\sum_{j=0}^{m} N^j]}{(d\sum_{j=0}^{m} N^j)^s} = \lim_{m\to\infty} \frac{2^{m+1}}{(\frac{N^{m+1}-1}{N-1} d)^{\log_N^2}} = (\frac{N-1}{d})^{\log_N^2}. \qquad (6.2.14)$$

It follows from (6.2.12) and (6.2.14) that $\mathcal{E}_s^+(\mu) = (\frac{N-1}{d})^{\log_N^2}$. Thus, $\mathcal{H}^s(K) = (\frac{N-1}{d})^{-\log_N^2}$ by Theorem 6.2.8.

Using the result of Example 6.2.10, it is easy to compute the Hausdorff measure of the middle Cantor set K satisfying $3K = K \bigcup (K+2)$ associated with its Hausdorff dimension $s = \log_3^2$. That is,

$$\mathcal{H}^s(K) = (\frac{N-1}{d})^{-\log_N^2} = (\frac{3-1}{2})^{-\log_3^2} = 1.$$

Conclusion

To conclude this thesis, we would like to make some further comments on the results we have obtained in our three main topics: self-affine scaling sets, integral self-affine multi-tiles, and self-affine sets.

Self-Affine Scaling sets

Gröchenig and Madych [29] firstly revealed an interesting connection between the theory of compactly supported wavelet bases and the theory of integral self-affine tiles. After that, many researchers (for example, see [39, 38, 37, 40]) contributed their work to the study of self-affine tiles. In particular, the study of the structure and tiling properties of self-affine tiles since these problems are closely connected with the construction of orthonormal wavelet basis in $\mathbb{R}^n$. Furthermore, Gabardo and Yu [24] considered the problem of constructing wavelet sets using integral self-affine tiles. However, they only considered the connection between wavelet theory and the theory of integral self-affine tiles in their work. To our knowledge, no one ever studied the connection between wavelet theory and non-integral self-affine tiles. Our work in this thesis supplied this gap. We considered the relationship between the theory of self-affine tiles including non-integral self-affine tiles and the theory of wavelets. The first three chapters in this thesis are devoted to studying A-dilation self-affine (generalized) scaling sets. Our limitation is that we always assume that the dilation A is an $n \times n$ integral expansive matrix with determinant equal to 2 or -2, i.e. we always assume that the A-dilation (generalized) scaling set K satisfies a self-affine equation $BK = (K + d_1) \bigcup (K + d_2)$ with $d_1, d_2 \in \mathbb{R}^n$

which are not necessarily in $\mathbb{Z}^n$, where $B = A^t$.

In Chapter 1, we characterized the properties of the dimension function of all A-dilation self-affine generalized scaling sets in $\mathbb{R}^n$. We proved that the dimension function of any A-dilation self-affine generalized scaling set K is a constant and is equal to the Lebesgue measure of the set K in $\mathbb{R}$ and $\mathbb{R}^2$ respectively. This result shows that all A-dilation self-affine scaling sets must be A-dilation MRA scaling sets in dimensions one and two. However, in dimension $n \geq 3$, we can only prove that the dimension function of a self-affine generalized scaling set is bounded by twice its Lebesgue measure. It is a pity that we can even not give an example to show that the dimension function of a self-affine generalized scaling set is not a constant in dimension $n \geq 3$.

In Chapter 2 and Chapter 3 of this thesis, we gave a complete characterization of all one and two dimensional self-affine scaling sets with a set of associated digits of the form $\{d_1, d_2\} \subset \mathbb{R}^n$, for an $n \times n$ integral expansive matrix A with $|\det A| = 2$, where $n = 1, 2$. Given any matrix $A \in M_n^{(2)}(\mathbb{Z})$, where $n = 1, 2$, the corresponding self-affine scaling sets can be computed using our results in Chapter 2 and Chapter 3. Furthermore, we determined which digit set $\{d_1, d_2\} \subset \mathbb{R}^n$ can make K a $\mathbb{Z}^n$-tiling set with $n = 1, 2$. During the process, we provided an interesting self-affine tile K which can tile $\mathbb{R}^2$ using two different tilings (one is $\mathbb{Z}^2$-tiling, the other is non-integral tiling).

Integral Self-Affine Multi-Tiles

The study of self-affine multi-tiles is not well developed because of their complicated structure. Gröchenig and Hass in [28] first provided a detailed description on integral self-affine multi-tiles. They extended the work in [29] to establish the relationship between the theory of multiwavelets and the theory of integral self-affine multi-tiles. In Chapter 4, we extended Gabardo and Yu's work in [24] to the integral self-affine multi-tiles setting. We characterized some analytic properties of integral self-affine multi-tiles under certain conditions. We also considered the problem of constructing (multi)wavelet sets using integral self-affine multi-tiles. Comparing to the case of integral self-affine tiles, integral self-affine multi-tiles can provide a nice framework in which the known examples of wavelet

sets appear naturally and also constitute a new tool with the help of which new examples can be created, but it is more restrictive to construct wavelet sets using integral self-affine multi-tiles.

In view of the fact that the representation of some integral self-affine multi-tile is not unique, we provide an algorithm to decompose an integral self-affine multi-tile K which is a $\mathbb{Z}^n$-tiling set into essentially disjoint pieces K_j which satisfy $K = \bigcup K_j$ such that the collection of sets K_j is an integral self-affine collection associated with some given expansive matrix $B \in M_n(\mathbb{Z})$ and the number of pieces K_j is minimal. Using this algorithm, we can determine whether or not a given measurable $\mathbb{Z}^n$-tiling set $K \subset \mathbb{R}^n$ is an integral self-affine multi-tile associated with any given $n \times n$ integral expansive matrix B. Furthermore, the minimal decomposition we provide is unique. However, our algorithm can only be used to $\mathbb{Z}^n$-tiling sets. For those which are not $\mathbb{Z}^n$-tiling sets, our method is not applicable.

The study of non-integral self-affine multi-tiles has not been touched before, even for the dilation 2 in dimension one. This will be our next study direction.

Self-Affine Sets

Hutchinson [35] proved that a self-affine set $K(B, \mathcal{D})$ is uniquely determined by the expanding dilation matrix B and digit set $\mathcal{D}$. For the case where $\text{card}(\mathcal{D}) = |\det B|$, Lagarias and Wang [37] provided some equivalent conditions for the Lebesgue measure of $K(B, \mathcal{D})$ to be positive. In the particular case where B is an integral matrix and the digit set $\mathcal{D} \subset \mathbb{Z}^n$ is a complete set of coset representatives for $\mathbb{Z}^n/B\mathbb{Z}^n$, Lagarias and Wang [38] showed that the Lebesgue measure of $K(B, \mathcal{D})$ is a positive integer. Furthermore, Gabardo and Yu [25] gave an algorithm to evaluate the Lebesgue measure of any such self-affine set. However, the problem of how to compute the exact Lebesgue measure of self-affine set $K(B, \mathcal{D})$ for the more general case where $\mathcal{D} \subset \mathbb{R}^n$ is still open. In Chapter 6, we considered this problem and we related the Lebesgue measure of $K(B, \mathcal{D})$ to the upper Beurling density of the associated measure $\mu = \lim_{s\to\infty} \sum_{\ell_0,\ldots,\ell_{s-1}\in\mathcal{D}} \delta_{\ell_0+B\ell_1+\cdots+B^{s-1}\ell_{s-1}}$. If, on

the other hand, $\mathrm{card}(\mathcal{D}) < |\det B|$, then, obviously, the Lebesgue measure of $K(B, \mathcal{D})$ is zero. We restricted our discussion to the expanding dilation B to be a similarity matrix with scaling factor bigger than one and we related the Hausdorff measure $\mathcal{H}^s(K)$, where B is a similarity matrix and s is the similarity dimension of K, to a corresponding notion of upper density for the measure μ. For the general expanding matrix $B \in M_n(\mathbb{R})$, we still did not know how to characterize the Hausdorff measure $\mathcal{H}^s(K)$ associated with its Hausdorff dimension s. This will be our future work. Moreover, for the case where $\mathrm{card}(\mathcal{D}) > |\det B|$, the situation becomes more complicated. This also deserves further study.

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