CUSUM Control Charts for Censored Gamma Distributed Quality Measurements

CUSUM CONTROL CHARTS FOR CENSORED GAMMA DISTRIBUTED QUALITY MEASUREMENTS

BY

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To my parents, Mohammed and Radheya

Abstract

Many companies conduct life tests on their products to ascertain their quality. In some cases, the tests are performed under accelerated circumstances and very often under censoring. The gamma distribution is widely used as a viable model for the lifetimes. In this thesis, we develop likelihood ratio cumulative sum (CUSUM) control charts for Type I right-censored data to monitor the mean lifetime of a product. The primary focus of this research centers on monitoring changes in the scale parameter η of the gamma distribution while the shape parameter β is kept fixed. A simulation study was conducted to assess the effect of sample size and censoring rate on the chart's average run length (ARL) to detect changes in the scale parameter. We use two methods to calculate the ARL of the chart: a simulation method and a Markov chain method. In the first method, we follow Dickinson et al. (2014) by simulating data repeatedly until the chart reaches an out-of-control state then take the average, whereas in the second method we follow Lucas and Saccucci (1990) and Runger and Prabhu (1996) and use a Markov chain method along with the empirical cumulative distribution function (ECDF). Finally, we assess the off-target performance by finding the distribution and the probability of false alarms following Huh (2010)'s work. From the simulation method and the Markov chain method, we found that the CUSUM charts detect a change quickly if we increase the sample size, and increase or decrease the scale parameter. While from the probability of false alarms, the CUSUM charts perform well for small transition times τ.

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Introduction

1.1 Gamma Distribution and Its Use in the Industry

1.1.1 Features of Gamma Distribution and Properties

The gamma distribution is a well-known distribution of lifetime testing model (Escobar and Meeker, 1998). It has a probability density function (pdf) of the form

$$f(t;\beta,\eta) = \frac{t^{\beta-1}}{\Gamma(\beta)\eta^{\beta}}\, e^{-t/\eta}, \qquad t > 0; \tag{1.1}$$

and a cumulative distribution function (cdf) of the form

$$F(t;\beta,\eta) = P(T \leq t;\beta,\eta) = \frac{\gamma(t/\eta,\beta)}{\Gamma(\beta)} \tag{1.2}$$

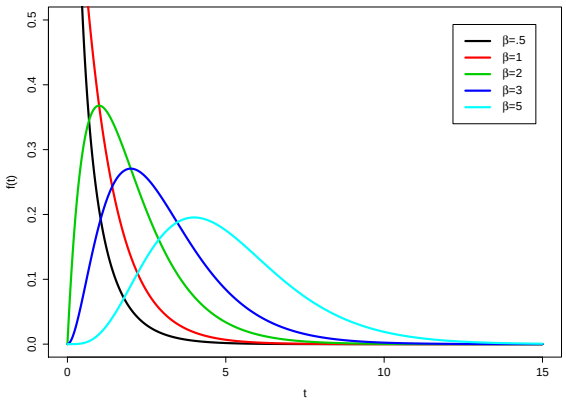


Figure 1.1: The gamma probability density function.

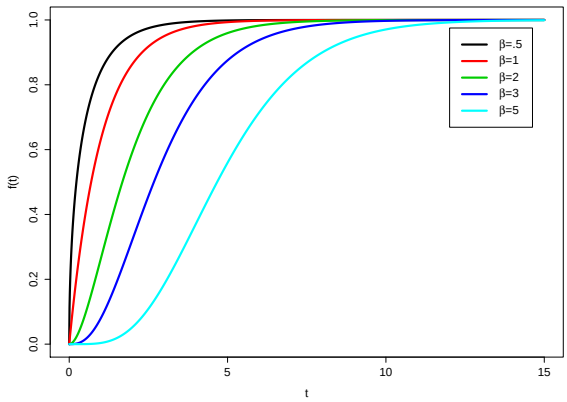


Figure 1.2: The gamma cumulative distribution function.

where $\beta > 0$ is the shape parameter, $\eta > 0$ is the scale parameter and $\gamma(z, \beta)$ is the incomplete gamma function, which can be defined by

$$\gamma(z, \beta) = \int_0^z \frac{u^{\beta-1}}{\Gamma(\beta)} e^{-u} du, \qquad u > 0. \tag{1.3}$$

The mean, variance, and raw moments respectively are given by

$$E(T) = \eta\beta, \qquad Var(T) = \eta^2\beta, \qquad E(T^m) = \frac{\eta^m \Gamma(\beta + m)}{\Gamma(\beta)} \tag{1.4}$$

There are relationships between gamma distributions and other distributions. Including, but not limited to, with the exponential distribution and chi-square distribution. When we set the shape parameter $\beta = 1$, we get an exponential distribution. Suppose that we have m independent exponential random variables with scale parameter η. The sum of those random variables will have a gamma distribution with shape parameter m and scale parameter η, which is known as the m-Erlang distribution (Hawkins and Olwell, 1998). A chi-square distribution with m degree of freedom is another special case of the gamma with parameters $\beta = \frac{m}{2}$ and

$\eta = 2$ (Dobson and Barnett, 2008).

1.1.2 Applications of Gamma Distribution

The gamma distribution has been used in different areas of research for lifetime testing and reliability. Gross and Clark (1975) used the gamma distribution in biomedical applications and survival data to model the survival time of male mice that were exposed to radiation. It was used in the environmental and climate studies to represent the monthly amount of rainfall in Africa (Husak et al., 2007). In quality control process, Soland (1968) used the gamma distribution to represent the lifetime of batteries. Also, Escobar and Meeker (1998) used it to model and analyse the ball bearing fatigue data and found the relationship between fatigue life and stress loading.

1.2 Quality Control and Control Charts

There are many ways to define quality. However, in simple terms, it means fitness for use (Montgomery, 2007). Statistical quality control (SQC) is a way that quality professionals describe statistical tools to control quality (Awasthi and Chaudhary, 2009). It involves descriptive statistics, statistical process control (SPC), and acceptance sampling. Quality control has been implemented in the manufacturing process to compare products and assess customer opinions and preferences of any products.

Most manufactured products are bound to malfunction or stop functioning altogether at some point (Dickinson et al., 2014). In other words, a certain glitch can prevent a product from functioning in a satisfactory fashion. Rates of failure are a common way to capture the reliability of a certain product. Reliability of a product can summarize and predict the

lifetime of the product during which the product is meant to work efficiently and therefore meet certain standards (Escobar and Meeker, 1998). In addition, when certain glitches or errors are discovered early, this can lead to repairs and fixtures done more economically as opposed to the cost incurred as a result of rebuilding or checking the entire system. This is beneficial for companies to reduce warranty expenses.

Usually, companies conduct tests such as life tests on their products under certain conditions (Dickinson et al., 2014). Either they place items on a test until the products stop functioning, which is called an uncensored case, or conduct the tests for a specified time limit, which results in the censored case. In the latter case not all the products under test will stop functioning. More details about censoring will be discussed in detail in Chapter 2.

Since a manufacturer is interested in maintaining and improving the performance of the process, it is important to find where and when any change occurs to the process, and that may have a negative effect on the reliability of the product. Along with lifetime test, there are tools that used to determine if a manufacturing process is in control state. One of which is a control chart also known as Shewhart charts or process-behavior charts (Montgomery, 2007). This chart is considered to be the first of such tools, which was invented by Walter Shewhart while working for Bell Labs in 1942. Shewhart's chart can effectively detect sudden changes in a process, particularly if the changes are dramatic. However, it is inefficient in detecting subtle sustained changes in the mean of a process. Another powerful and useful tool in quality control is the cumulative sum (CUSUM) control chart, which is more effective than the Shewhart chart. CUSUM's control charts were first proposed by Page (1954) and have been studied by many authors afterward. This chart is a good candidate to use in the chemical and process industries where rational subgroups are frequently of size 1. In addition, it is effective in detecting sustained changes and variations in the mean. The exponentially weighted moving

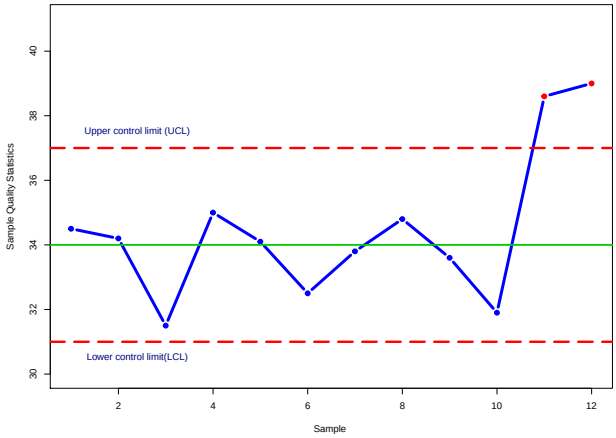


Figure 1.3: Control chart.

average (EWMA) control chart was introduced by Roberts (1959). This chart is also a good alternative to the Shewhart control chart when one is interested in detecting small shifts.

Any control chart contains a center line that represents the average value of the quality characteristic corresponding to the in-control state of the process (Montgomery, 2007). Also, there are two horizontal lines which are upper and lower control limits. They indicate the threshold, and they are chosen in a way that most of the data points fall between those limits when the process is in-control. We can assess whether a process it is in control or out-of-control by plotting a control chart of the data. If there is some points fall outside the control limits, we say the process is out-of-control. The process is then inspected to establish the cause for the out-of-control behaviour.

1.3 Thesis Objectives and Outline

The main objective of this thesis is to develop a likelihood-ratio CUSUM control chart to monitor the mean lifetime where the data follow a gamma distribution and are possibly type I right censored. The shape parameter is fixed but the scale parameter is changed. We assess the sensitivity of the chart to detect any change in the process by calculating the average run length while the probability of false alarms is used when the process is in control. For a chart to perform well, it should false alarms rarely when the process is in-control, whereas it should signal quickly when the process is out-of-control. In other words, the out-of-control ARL should be as small as possible.

The thesis is organized as follows. In Chapter 2, we describe the CUSUM chart for gamma censored data using the likelihood-ratio statistic. In Chapter 3, we discuss in detail the Markov chain method using the empirical CDF and the simulation method to calculate the average run length. Details are also provided on how we find the control limits that give the desired in-control ARL using the partition method. Then we calculate the probability mass function (pmf) of the off target run length and find the probability of false alarms in Chapter 4. Numerical illustrations are presented in Chapter 5. Finally, Chapter 6 summarizes the results along with a discussion of future possible work.

Chapter 2

Designing CUSUM Charts

2.1 Censored Data

If we wish to develop a control chart for lifetime data, we should take into account that our data are likely to be skewed or non-normally distributed and censored. Many life processes are not normally distributed, and modeld by different distributions such as the gamma distribution. Lifetime data usually take time to gather and they are expensive to produce because many products take a long time until they fail. Because of that, we use censoring which restricts the ability to observe failure times. Escobar and Meeker (1998) discuss different types of censoring such as right censoring, left censoring and interval censoring. Right censoring is the most common type, and it is classified as type I and type II right censoring. It means that there are items that do not fail by the end of the test. Type I right censoring occurs when we fixed the time of censoring C and stop testing the units that did not fail whereas in Type II censoring we predetermine the number of units that should fail and we stop testing once such number of units is completed. Left censoring happens when units may fail before the end of the study. Interval censoring means that the actual failure time for units can not be determined exactly,

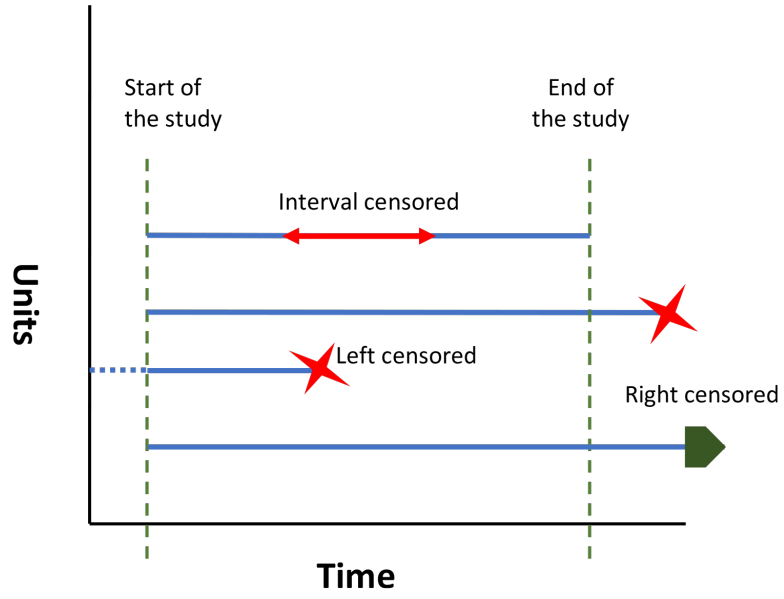


Figure 2.1: Types of censoring.

but we know it lies within an interval or between two censoring times C_1 and C_2. In this thesis we work with Type I right censoring only. Because is the most common type of censoring occurring in industry.

2.2 Likelihood Ratio Statistic

Hawkins and Olwell (1998) discussed in detail the development of CUSUM charts. They give examples of how to set the CUSUM chart for different distributions and the properties without

censoring. They design CUSUM for any distribution using the score statistic

$$z_i = \ln\left(\frac{f(t_i|\theta_1)}{f(t_i|\theta_0)}\right), \tag{2.1}$$

where $f(t_i|\theta_i)$ is the probability density function of the quality variable t, z_i is the score statistic (the log-likelihood ratio statistic) of the i^{th} observation, θ_1 is our parameter of interest when the process is out-off-control, and θ_0 the parameter of interest when the process is in-control. If we want to detect an upward shift in the process mean, the CUSUM chart will signal if the statistic $D_i^+ > h^+$, where

$$\begin{aligned} D_0^+ &= 0 \\ D_i^+ &= \max(0, D_{i-1}^+ + z_i) \end{aligned} \tag{2.2}$$

where h^+ is the upper threshold (control limit) and $i = 1, 2, ...$

On the other hand, for downward shifts, the CUSUM chart will signal if the statistic $D_i^- < h^-$, where

$$\begin{aligned} D_0^- &= 0 \\ D_i^- &= \min(0, D_{i-1}^- + z_i) \end{aligned} \tag{2.3}$$

where h^- is the lower threshold and $i = 1, 2, ...$ (Hawkins and Olwell, 1998).

Assuming that n items are independently tested at sampling period i yielding observations $\boldsymbol{t}_i = (t_{i1}, t_{i2}, ..., t_{in})$. We used the likelihood function for right censored data that Escobar and Meeker (1998) derived to calculate the score statistic z_i in equation (2.1), i.e.,

$$L(\boldsymbol{\theta}|\boldsymbol{t}_i) = \prod_{j=1}^{n} f(t_{ij}|\theta)^{\delta_{ij}}[1 - F(t_{ij}|\theta)]^{1-\delta_{ij}}, \tag{2.4}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } t_{ij} \text{ is an exact-failure time,} \\ 0 & \text{if } t_{ij} \text{ is a right-censored observation,} \end{cases}$$

where f is the pdf and F is cdf of a distribution.

Now, let's combine and apply the results from Hawkins and Olwell (1998) and Escobar and Meeker (1998) to the gamma distribution with type I right censoring to design the CUSUM chart to detect a decrease or increase in the process mean. Because of the behaviour of the gamma distribution, increasing or decreasing one or both of the parameters will result in an increase or a decrease in the mean. Suppose that a lifetime random variable T follows a gamma distribution with scale η and shape β. The score statistic z_i of the i^{th} observation to monitor a change in the scale parameter is given by:

$$\begin{aligned}
z_i &= \ln\left(\frac{L(\beta,\eta_1|\boldsymbol{t_i})}{L(\beta,\eta_0|\boldsymbol{t_i})}\right) \\
&= \ln\left[\frac{\prod_{j=1}^{n} f(t_{ij}|\beta,\eta_1)^{\delta_{ij}}[1-F(t_{ij}|\beta,\eta_1)]^{1-\delta_{ij}}}{\prod_{j=1}^{n} f(t_{ij}|\beta,\eta_0)^{\delta_{ij}}[1-F(t_{ij}|\beta,\eta_0)]^{1-\delta_{ij}}}\right] \\
&= \sum_{j=1}^{n}\delta_{ij}\ln\left[\frac{f(t_{ij}|\beta,\eta_1)}{f(t_{ij}|\beta,\eta_0)}\right] + \sum_{j=1}^{n}(\delta_{ij}-1)\ln\left[\frac{1-F(t_{ij}|\beta,\eta_1)}{1-Ff(t_{ij}|\beta,\eta_0)}\right] \\
&= \sum_{j=1}^{n}\delta_{ij}\ln\left[\frac{t_{ij}^{\beta-1}}{\Gamma(\beta)\eta_1^{\beta}}e^{-t_{ij}/\eta_1}\frac{\Gamma(\beta)\eta_0^{\beta}}{t_{ij}^{\beta-1}}\frac{1}{e^{-t_{ij}/\eta_1}}\right] + \sum_{j=1}^{n}(\delta_{ij}-1)\ln\left[\frac{1-\gamma(\beta,\frac{t_{ij}}{\eta_1})}{1-\gamma(\beta,\frac{t_{ij}}{\eta_0})}\right] \\
&= \sum_{j=1}^{n}\delta_{ij}\ln\left[\left(\frac{\eta_0}{\eta_1}\right)^{\beta}e^{\frac{-t_{ij}}{\eta_1}+\frac{t_{ij}}{\eta_0}}\right] + \sum_{j=1}^{n}(\delta_{ij}-1)\ln\left[\frac{1-\gamma(\beta,\frac{t_{ij}}{\eta_1})}{1-\gamma(\beta,\frac{t_{ij}}{\eta_0})}\right] \\
&= \sum_{j=1}^{n}\delta_{ij}\beta\ln\left(\frac{\eta_0}{\eta_1}\right) + \sum_{j=1}^{n}\delta_{ij}\left[\frac{t_{ij}}{\eta_0}-\frac{t_{ij}}{\eta_1}\right] + \sum_{j=1}^{n}(\delta_{ij}-1)\ln\left[\frac{1-\gamma(\beta,\frac{t_{ij}}{\eta_1})}{1-\gamma(\beta,\frac{t_{ij}}{\eta_0})}\right], i = 1,2,.... \quad (2.5)
\end{aligned}$$

2.3 Upper and Lower Chart Statistics

In the gamma distribution, we fix the shape parameter and monitor changes in the scale parameter. We will define the in-control values of the parameters η_0 and β, and assume that they are known. For the out-of-control situation the parameters are η_1 and β, where $\eta_1 = (1-d)\eta_0$ and $d \times 100\%$ is the percentage of the shift in η_0. For a decrease in the process mean η_0, $0 < d < 1$; whereas for an increase in η_0, $d < 0$. Using the later specification of the parameters, we can re-write the score statistic in equation (2.5) as:

$$z_i = \sum_{j=1}^{n} \delta_{ij}\beta \ln\left(\frac{\eta_0}{(1-d)\eta_0}\right) + \sum_{j=1}^{n} \delta_{ij}\left[\frac{t_{ij}}{\eta_0} - \frac{t_{ij}}{(1-d)\eta_0}\right] + \sum_{j=1}^{n}(\delta_{ij}-1)\ln\left[\frac{1-\gamma(\beta, \frac{t_{ij}}{(1-d)\eta_0})}{1-\gamma(\beta, \frac{t_{ij}}{\eta_0})}\right], \tag{2.6}$$

where $i = 1, 2, ...$ and $\gamma(\beta, \frac{t_{ij}}{\eta_0}), \gamma(\beta, \frac{t_{ij}}{(1-d)\eta_0})$ are the incomplete gamma values.

The lower chart statistic is used to detect a decrease in the process mean. It is defined as follows: for $0 < d < 1$ and $i = 1, 2, ...,$

$$\begin{aligned} C_0^- &= 0; \\ C_i^- &= \min(0, C_{i-1}^- - z_i). \end{aligned} \tag{2.7}$$

The chart signals a decrease in the mean when $C_i^- < h^-$.

Similarly, the upper chart statistic is used to detect an increase in the process mean. It is defined as follows: for $d < 0$ and $i = 1, 2, ...,$

$$\begin{aligned} C_0^+ &= 0; \\ C_i^+ &= \max(0, C_{i-1}^+ + z_i). \end{aligned} \tag{2.8}$$

The chart signals an increase in the mean when $C_i^+ > h^+$.

In the above, h^- and h^+ are the control limits that give the desired average run length (more details on how to specify them can be found in Chapter 3).

2.4 Measures of Chart Performance

There are different ways to measure and assess the performance of a control chart. According to Montgomery (2007) and Farouk and Mohamad (2012), the average run length can be used to evaluate the performance of a control chart, namely how quick it detects shifts in the process. The ARL can be defined as the average number of sampling periods until the chart signals. There are two types of average run lengths: in-control average run length and out-of-control average run length which are and denoted by ARL_0 and ARL_1, respectively. ARL_0 is the number of points plotted up to and including the first point that crosses the control limit when the process is operating in-control. While ARL_1 is the same average but when the process is operating out-of-control (more details can be found in Chapter 3). Another method to measure the performance of the chart is the probability of false alarms. In this measure, we calculate the in-control and the out-of-control run length distributions to calculate the false alarm probabilities (more details can be found in Chapter 4). When a chart performs well, ARL_0 is large and ARL_1 is relatively small. If we have large ARL_0, we reduce the false alarms whereas small ARL_1 implies the chart detects the change fast (Hawkins and Olwell, 1998).

Chapter 3

Calculation of Average Run Length

3.1 Average Run Length by Simulation

Dickinson et al. (2014) assesses the performance of the CUSUM and EWMA charts of the likelihood-ratio to detect a decrease or increase in the process mean of Weibull distribution with type I right censoring by simulation. In this section, we use this simulation approach to evaluate the performance of the likelihood-ratio based CUSUM to monitor a decrease or increase in the process mean of the gamma distribution with type I right censoring. The following subsections describe the simulation algorithm.

3.1.1 Monitoring a Decrease in the Mean of Lifetimes

The following steps are used to calculate the in-control average run length ARL_0:

(1) The desired ARL_0 is 370 which is the standard and commonly used.

(2) Initialize the in-control parameters and the out-of-control parameters:

Table 3.1: In-control and out-of-control parameters to monitor a decrease in the process mean

Process stage	Sample size n	Shift Size d	Scale parameter η	Shape parameter β
In-control	$n = 3, 5, 10$	0.15, 0.20, 0.30, 0.35	$\eta_0 = 1$	$\beta_0 = 0.5, 1, 3$
Out-of-control	$n = 3, 5, 10$	0.15, 0.20, 0.30, 0.35	$\eta_1 = (1-d)\eta_0$	$\beta_1 = 0.5, 1, 3$

(3) Generate a random variable T from a Gamma (β_0, η_0) with type I right censoring.

(a) Generate a random sample size n from Gamma (β_0, η_0).

(b) Initialize the censoring rate pc which is the probability of an observation being censored and it can be calculated from the following equation:

$$pc = P(T > C) \Rightarrow pc = 1 - P(T \leq C)$$
$$\Rightarrow pc = 1 - F(C; \beta_0, \eta_0), \tag{3.1}$$

where C is predetermined the censoring time.

(c) Predetermined the censoring time C which can be calculated using equation (3.1)

$$C = F^{-1}(1 - pc; \beta_0, \eta_0), \tag{3.2}$$

which is the cdf of Gamma (β_0, η_0) random variable. We calculate C that corresponds to $100 \times pc = 10\%, 30\%, 50\%, 80\%$.

(d) Generate $2 \times n$ matrix where all the entries of the first row are C and the second row has entries of the random sample. Take the minimum value between each rows entry to get the censored data.

(4) Initialize the control limits h^- which is the mid point of h_1^- and h_2^-, where $h_1^- > h_2^-$.

(5) Calculate the CUSUM chart statistics C_i^- from equation (2.7).

(6) Stop when $C_i^- < h^-$ and count the number of samples required.

(7) Repeat the process $50,000$ times and take the average of the counts to calculate ARL_0.

To evaluate the out-of-control ARL, we repeat the same steps above except we generate a random sample size $n = 3, 5, 10$ from Gamma (β_1, η_1).

3.1.2 Monitoring an Increase in the Mean of Lifetimes

We evaluate the upper chart statistic to detect an upward change in the process mean by calculating ARL_0 and ARL_1. We use the same steps for monitoring a decrease in the of lifetimes and we specify the following parameters:

- Initialize the in-control parameters and the out-of-control parameters

Table 3.2: In-control and out-of-control parameters to monitor an increase in the process mean.

Process stage	Sample size n	Shift Size d	Scale parameter η	Shape parameter β
In-control	$n = 3, 5, 10$	0.15, 0.20, 0.30, 0.35	$\eta_0 = 1$	$\beta_0 = 0.5, 1, 3$
Out-of-control	$n = 3, 5, 10$	0.15, 0.20, 0.30, 0.35	$\eta_1 = (1 + d)\eta_0$	$\beta_1 = 0.5, 1, 3$

- The control limit $h^+ = (h_1^+ + h_2^+)/2$, where $h^+, h_1^+$and $h_2^+ > 0$ and $h_1^+ > h_2^+$.
- Calculate the CUSUM chart statistics C_i^+ from equation (2.8) and average $50,000$ replications of the simulation to calculate ARL_0 and ARL_1.

Whether we are looking to detect a decrease or an increase in process mean, we must check if ARL_0 is close to the desired 370. If $|ARL_0 - 370| < 5$, then ARL_0 is close enough and we return the threshold. Otherwise, we repeat the simulation steps and find the threshold that gives the desired ARL_0. To find out the threshold that gives $|ARL_0 - 370| < 5$, we use the partition search method (Dickinson et al., 2014). The following steps describe how the partition method was designed:

1. Initialized h_1, h_2 that take values between 0 and 30, where $h_1 > h_2$.
2. Calculate the midpoint $h = (h_1 + h_2)/2$.
3. Obtain ARL_0 for h and check if the $ARL_0 \approx 370$. If $ARL_0 \approx 370$, stop.
4. If $ARL_0 > 370$, then take $h_2 = h$ and update the midpoint h.
5. If $ARL_0 \leq 370$, then take $h_1 = h$ and update the midpoint h.
6. Calculate the ARL_0 using the update threshold.
7. Repeat the above steps to find out the threshold h that gives $ARL_0 \approx 370$.

The results of simulations can be found in Section 3.4 in this chapter displayed in Tables 3.1 and 3.2.

3.2 Empirical Cumulative Distribution (ecdf)

In order to use the Markov chain method, we need to find the distribution of the score statistic z_t from equation (2.6). Because of the difficulties to find out a close form for its distribution, we use the empirical cdf (ecdf). The ecdf is an estimate of the cdf and can be calculated from

$$F_n(t) = \frac{\text{number of sample values} \leq t}{n}. \tag{3.3}$$

We first generate $n = 500,000$ values of the score statistics z_t and then find the ecdf of that statistic using the R function `ecdf()`. We notice that when we plot the empirical cdf, there is a discontinuity in the plot due to the censoring effect. Figures 3.1 and 3.2 illustrate the empirical distribution for the upper and the lower score statistics.

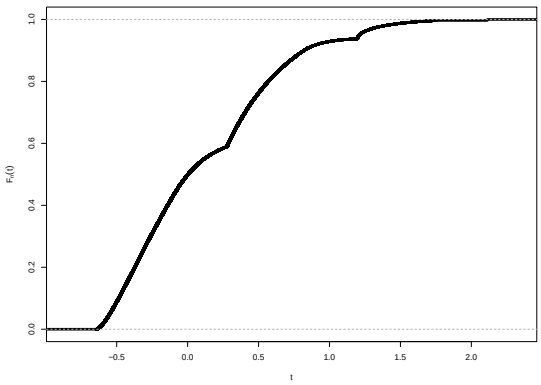


Figure 3.1: Empirical cumulative distribution function for lower CUSUM chart with $n = 3$, $\beta = 0.5$, $\eta_0 = 1$, $d = 0.35$ and $pc = 0.15$.

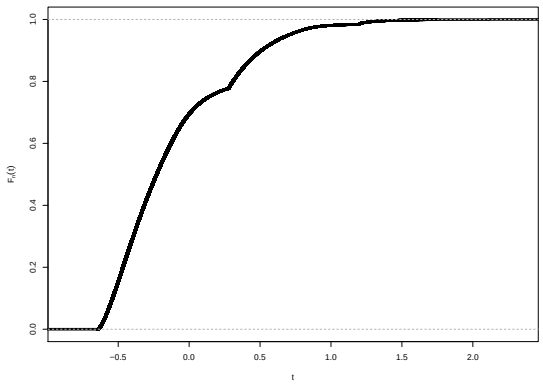


Figure 3.2: Empirical cumulative distribution function for upper CUSUM chart with $n = 3$, $\beta = 0.5$, $\eta_0 = 1$, $d = 0.35$ and $pc = 0.15$.

3.3 Average Run Length Using the Markov Chain Method with Empirical CDF

3.3.1 Markov Chain

When we used the simulation method, it took a long time to calculate the ARL and the R program failed to obtain the results. As a result, we used the Markov chain to calculate ARL. The Markov chain method is a well-known method that has been used and presented by different researchers to evaluate the run length performance of control charts. Brook and Evans (1972) calculated ARL using the Markov chain method to assess the performance of a univariate CUSUM chart; however, it is only applied to the upper CUSUM chart. Lucas and Saccucci (1990) presented this method and applied it to the EWMA chart and Runger and Prabhu (1996) applied and extended the method to multivariate EWMA charts.

In this thesis, we follow Lucas and Saccucci (1990) and Runger and Prabhu (1996) approaches to calculate and evaluate the ARL. We divide the in-control range for the upper

and the lower control limits into non-overlapping m subintervals I_i where $i = 1, 2, ..., m$ and $m = 500$. The subintervals are the transient states of the Markov chain. We choose $m = 500$ after testing many values and we got closer to 370, and after 500 states the results were stable (see Figure 3.3). Interval I_i has length $L = \frac{h}{m}$, $L > 0$, and has mid point S_i, $i = 1, 2, ..., m$. Then, we calculate the transition probability matrix $\boldsymbol{P}$ with m transient states

$$\boldsymbol{P} = \begin{bmatrix} p[1,1] & p[1,2] & \dots & p[1,m] \\ p[2,1] & p[2,2] & \dots & p[2,m] \\ \vdots & \vdots & \ddots & \vdots \\ p[m,1] & p[m,2] & \dots & p[m,m] \end{bmatrix}, \tag{3.4}$$

where $p[i,j] = P\left[C_t \in I_j \mid C_{t-i} \in I_i\right]$; C_t is calculated from equation (2.7) for the lower chart and from equation (2.8) for the upper chart.

Finally, we calculate ARL using the fallowing formula:

$$ARL \doteq \boldsymbol{s'}(\boldsymbol{I} - \boldsymbol{P})^{-1}\mathbf{1} \tag{3.5}$$

where $\boldsymbol{s}$ is the column vector of zeros with first entry equal to one, $\boldsymbol{I}$ is the identity matrix and $\mathbf{1}$ is a column vector of ones. This is a well-known formula in Markov chains, see Karlin and Taylor (1975).

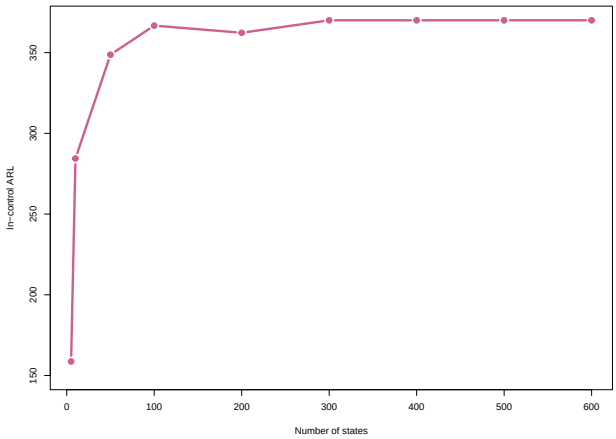


Figure 3.3: The relation between the number of states and ARL_0.

3.3.2 Monitoring a Decrease in the Mean of Lifetimes

The following steps are used to calculate and evaluate the in-control average run length ARL_0:

(1) Follow the same steps from the simulation method to initialize the in-control and out of control parameters.

(2) Obtain the empirical cdf `ecdf()` as explain in Section (3.2).

(3) Initialized the control limit h^-, $h^- < 0$ and hence the in-control range is $[h^-, 0]$.

(4) Divide the range $[h^-, 0]$ into $m = 500$ sub-intervals I_i

$$I_i = (-iL, -(i-1)L], \qquad i = 1, 2, ..., m$$

where $L = \dfrac{-h^-}{m}$, $L > 0$.

(5) Find the mid point of each interval

$$S_i = \frac{-(2i-1)L}{2}, \qquad i = 1, 2, ..., m.$$

(6) Calculate the in-control transition probability matrix $\boldsymbol{P_0}$, where

$$p[i,j] = P\left[C_t^- \in I_j \mid C_{t-i}^- \in I_i\right].$$

Since each interval has the same length $L = \dfrac{-h}{m}$, $L > 0$ and same width $\delta = \dfrac{L}{2}$ and mid point S_i, I_i can be written as

$$I_i = (S_i - \delta, S_i + \delta], \qquad i = 1, 2, ..., m.$$

Hence, $p[i,j]$ can be written as

$$\begin{aligned}
p[i,j] &= P\left[C_t^- \in (S_j - \delta, S_j + \delta] \mid C_{t-1}^- \in (S_i - \delta, S_i + \delta]\right] \\
&= P\left[S_j - \delta < C_t^- \leq S_j + \delta \mid S_i - \delta < C_{t-1}^- \leq S_i + \delta\right] \\
&\doteq P\left[S_j - \delta < C_t^- \leq S_j + \delta \mid C_{t-1}^- = S_i\right] \\
&= P\left[S_j - \delta < \min\{0, C_{t-1}^- + Z_t\} \leq S_j + \delta \mid C_{t-1}^- = S_i\right] \\
&= P\left[S_j - \delta < \min\{0, S_i + Z_t\} \leq S_j + \delta\right], i = 1, 2, ...m, j = 1, 2, ..., m, \qquad (3.6)
\end{aligned}$$

where Z_t is from (2.5).

Now, if $j \geq 2$ then equation (3.5) becomes:

$$\begin{aligned}
p[i,j] &= P\left[S_j - \delta < \min\{0, S_i + Z_t\} \leq S_j + \delta\right] \\
&= P\left[S_j - \delta < S_i + Z_t \leq S_j + \delta\right] \\
&= P\left[S_j - S_i - \delta < Z_t \leq S_j - S_i + \delta\right] \\
&= P\left(Z_t \leq S_j - S_i + \delta\right) - P\left(Z_t \leq S_j - S_i - \delta\right) \\
&\doteq \text{ecdf}\left(S_j - S_i + \delta\right) - \text{ecdf}\left(S_j - S_i - \delta\right). \qquad (3.7)
\end{aligned}$$

Now, if $j = 1$ then equation (3.5) becomes:

$$
\begin{aligned}
p[i,1] &= P\left[S_1 - \delta < \min\{0, S_i + Z_t\} \leq S_1 + \delta\right] \\
&= P\left[\frac{-L}{2} - \frac{-L}{2} < \min\{0, S_i + Z_t\} \leq \frac{-L}{2} + \frac{-L}{2}\right] \\
&= P\left[-L < \min\{0, S_i + Z_t\} \leq 0\right] \\
&= P\left[S_i + Z_t \geq -L\right] \\
&= P\left[Z_t \geq -L - S_i\right] \\
&= 1 - P\left[Z_t < -L - S_i\right] \\
&\doteq 1 - \text{ecdf}\left(-L - S_i\right). \qquad (3.8)
\end{aligned}
$$

Thus, for $i, j = 1, 2, ..., m$ the probability of going from state i to state j can be calculated as:

$$
p[i,j] = \begin{cases} \text{ecdf}\left(S_j - S_i + \delta\right) - \text{ecdf}\left(S_j - S_i - \delta\right) & \text{if } j \geq 2; \\ 1 - \text{ecdf}\left(-L - S_i\right) & \text{if } j = 1. \end{cases} \qquad (3.9)
$$

(7) Calculate ARL_0 from equation (3.4).

To calculate ARL_1 we follow the same method for calculating ARL_0 except we simulate random values of z_t from (2.5) using a Gamma (β_1, η_1) with type I right censoring to calculate the empirical distribution function.

3.3.3 Monitoring an Increase in the Mean of Lifetimes

(1) Obtain the `ecdf()`.

(2) Initialize the control limit h^+, $h^+ > 0$ and hence the in-control range is $[0, h^+]$.

(3) Divide the in-control range into $m = 500$ sub-intervals I_i

$$I_i = ((i - L), iL], \qquad i = 1, 2, ..., m,$$

where $L = \dfrac{h^+}{m}$, $L > 0$.

(4) Find the mid point of each interval

$$S_i = \frac{(2i-1)L}{2}, \qquad i = 1, 2, ..., m.$$

(5) Calculate the in-control transition probability matrix $\boldsymbol{P_0}$

where

$$p[i,j] = P\left[C_t^+ \in I_j \mid C_{t-i}^+ \in I_i\right].$$

Each interval has the same length L, same width $\delta = \dfrac{L}{2}$ and mid point S_i, thus I_i can be written as

$$I_i = (S_i - \delta, S_i + \delta], \qquad i = 1, 2, ..., m.$$

Hence, $p[i,j]$ can be written as

$$\begin{aligned}
p[i,j] &= P\left[C_t^+ \in (S_j - \delta, S_j + \delta] \mid C_{t-1}^+ \in (S_i - \delta, S_i + \delta]\right] \\
&= P\left[S_j - \delta < C_t^+ \leq S_j + \delta \mid S_i - \delta < C_{t-1}^+ \leq S_i + \delta\right] \\
&\doteq P\left[S_j - \delta < C_t^+ \leq S_j + \delta \mid C_{t-1}^+ = S_i\right] \\
&= P\left[S_j - \delta < \max\{0, C_{t-1}^+ + Z_t\} \leq S_j + \delta \mid C_{t-1}^+ = S_i\right] \\
&= P\left[S_j - \delta < \max\{0, S_i + Z_t\} \leq S_j + \delta\right], i = 1, 2, ...m, j = 1, 2, ..., m, \qquad (3.10)
\end{aligned}$$

where Z_t is from (2.5).

If $j \geq 2$, then equation (3.9) becomes:

$$\begin{aligned} p[i,j] &= P\left[S_j - \delta < \max\{0, S_i + Z_t\} \leq S_j + \delta\right] \\ &= P\left[S_j - \delta < S_i + Z_t \leq S_j + \delta\right] \\ &= P\left[S_j - S_i - \delta < Z_t \leq S_j - S_i + \delta\right] \\ &= P\left(Z_t < S_j - S_i + \delta\right) - P\left(Z_t < S_j - S_i - \delta\right) \\ &\doteq \text{ecdf}\left(S_j - S_i + \delta\right) - \text{ecdf}\left(S_j - S_i - \delta\right). \end{aligned} \tag{3.11}$$

If $j = 1$ then equation (3.9) becomes:

$$\begin{aligned} p[i,1] &= P\left[S_1 - \delta < \max\{0, S_i + Z_t\} \leq S_1 + \delta\right] \\ &= P\left[\frac{L}{2} - \frac{L}{2} < \max\{0, S_i + Z_t\} \leq \frac{L}{2} + \frac{-L}{2}\right] \\ &= P\left[0 < \max\{0, S_i + Z_t\} \leq L\right] \\ &= P\left[S_i + Z_t \leq L\right] \\ &= P\left[Z_t < L - S_i\right] \\ &\doteq \text{ecdf}\left(L - S_i\right). \end{aligned} \tag{3.12}$$

Thus, for $i, j = 1, 2, ..., m$ the probability of going from state i to state j can be computed as:

$$p[i,j] = \begin{cases} \text{ecdf}\left(S_j - S_i + \delta\right) - \text{ecdf}\left(S_j - S_i - \delta\right) & \text{if } j \geq 2, \\ \text{ecdf}\left(L - S_i\right) & \text{if } j = 1. \end{cases} \tag{3.13}$$

(6) Calculate ARL_0 from equation (3.4).

To calculate ARL_1 we follow the same method for calculating ARL_0 except we simulate a type I right censoring data from a gamma distribution using the out-of-control parameters. Then, we calculate the out-of-control transition probability matrix $\boldsymbol{P_1}$ and compute ARL_1 using equation (3.4).

Whether we are interested in monitoring a decrease or an increase in the process mean, we need to check if ARL_0 is close to the desired 370. If ARL_0 is not close to what we need, we repeat the process and search for another control limit using the partition algorithm until we get $ARL_0 \approx 370$.

The results of Markov chain method can be found in Section 3.4(Tables 3.3-3.8).

3.4 Numerical Results

In this section, we will present and summarize the results of both methods as well as the CUSUM chart performance. In both methods, we evaluate the chart's performance by having the in-control average run length ARL_0 close to 370, and we wish the chart to detect any change as soon as it occurs. In other words, we judge the performance of a CUSUM chart by out-of-control ARL_1. If the ARL_1 is small enough, we say the chart signals quickly. There are no differences in terms of accuracy and efficiency of the results between the simulation method and the Markov chain method. However, the time that the simulation took to calculate ARLs was considerably longer for the simulation method compared to the Markov chain. Each scenario in the simulation took around 28 minutes to output the result whereas less than a minute with the Markov chain.

Tables 3.1-3.5 are the numerical results for monitoring a decrease in the scale parameter η. They display ARL_0, ARL_1 values, and the control limit that was calculated for different values of sample size n, shape parameter β, censoring proportion pc, and shift size d. Tables 3.2-3.8 are the numerical results for an increase in the mean with the same cases used in monitoring a decrease in the mean. Figures 3.4 through 3.7 are based on the numbers from Tables 3.1 to 3.8, and they give a summary of the chart performance.

From the Tables and the Figures, we can see that:

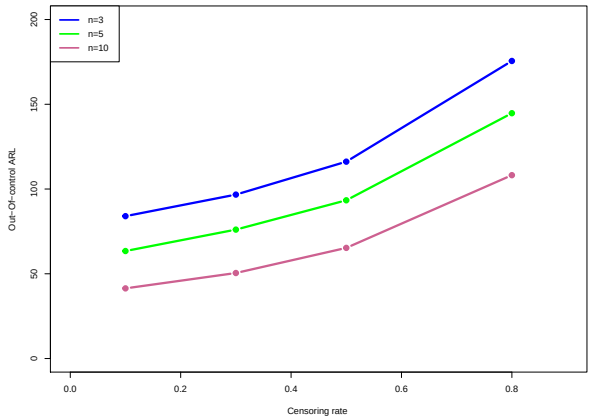
1) The shape parameter, sample size, censoring rate, and the shift size have an impact on the chart performance and ARL_1.

2) By increasing the sample size the chart signals quickly and we get a lower ARL_1. For instance, from Table 3.3 and Figure 3.4 (a), for $\beta = 0.5$, $n = 3$, $pc = 10\%$, and $d = 0.15$ the chart signals on average after 83.991 samples whereas if we use the same parameters and use $n = 10$ the chart signals after 41.389.

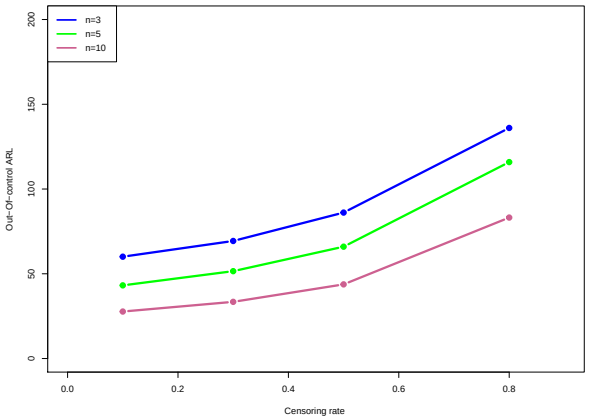
3) There is a negative correlation between censoring and the sensitivity of the chart. The chart detects the change fast if we decrease the censoring.

4) The chart signals quickly if we increase the shape parameter. For example, when we look at Figure 3.4 (d) with $\beta = 0.5$, it takes 80 samples or less on an average for the chart to signal, whereas from Figure 3.5 (d) with $\beta = 3$, it takes only 40 samples or less on an average for the chat to signal.

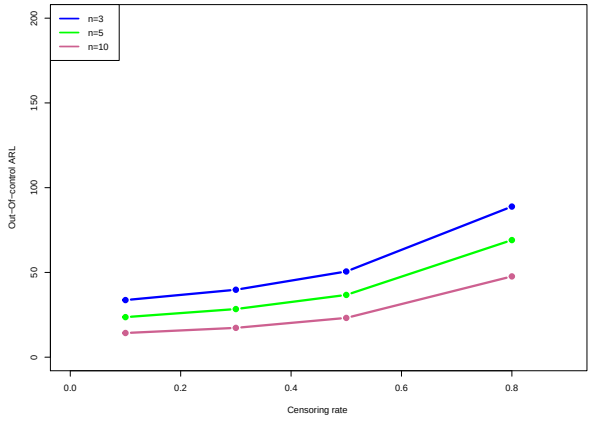
5) The scale parameter (i.e. the shift size) seems to have more impact than other parameters. With large negative or positive shift to detect, the chart signals and perform better than with small shift.



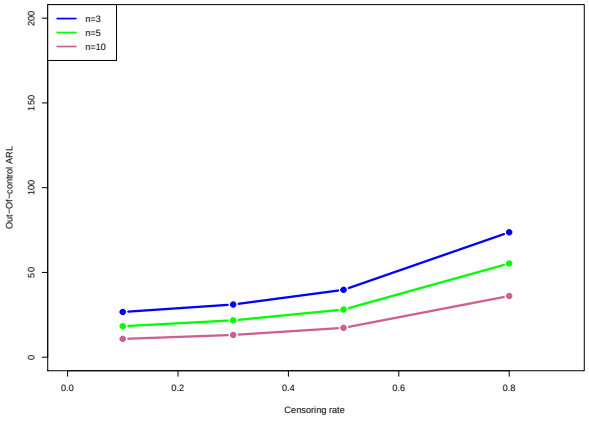
(a)



(b)

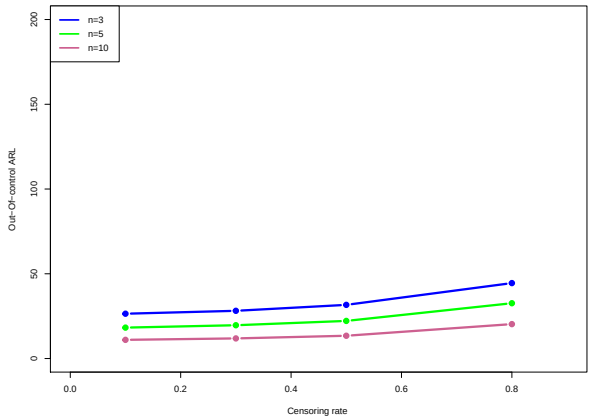


(c)

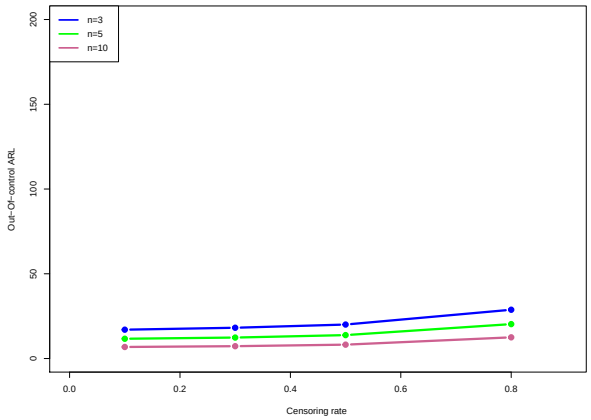


(d)

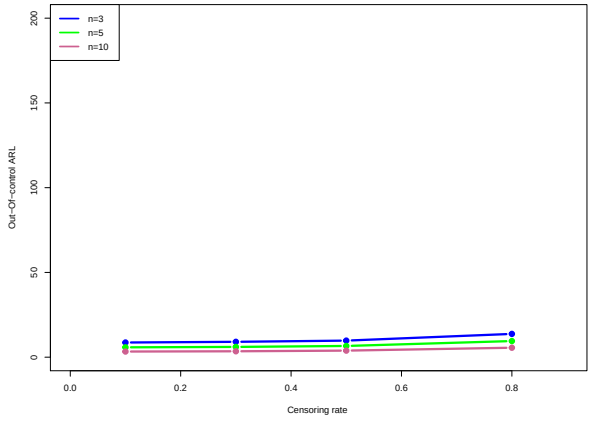
Figure 3.4: ARL_1 for lower CUSUM chart with $\beta = 0.5$, and (a) $d = 0.15$, (b) $d = 0.20$, (c) $d = 0.30$, and (d) $d = 0.35$.



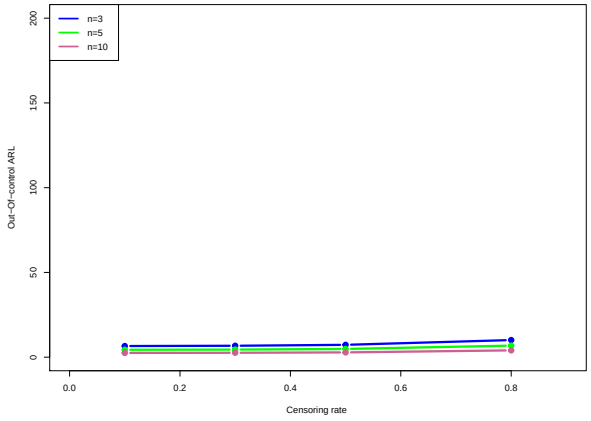
(a)



(b)

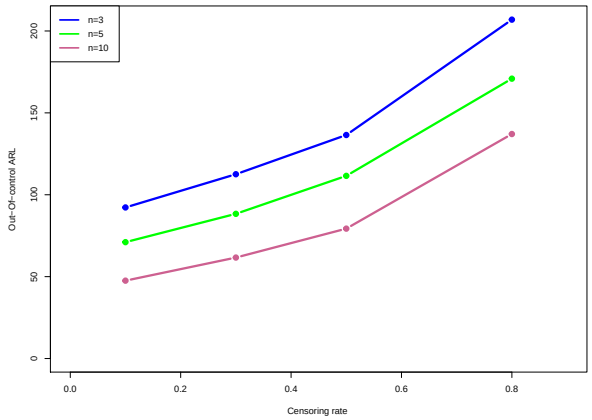


(c)

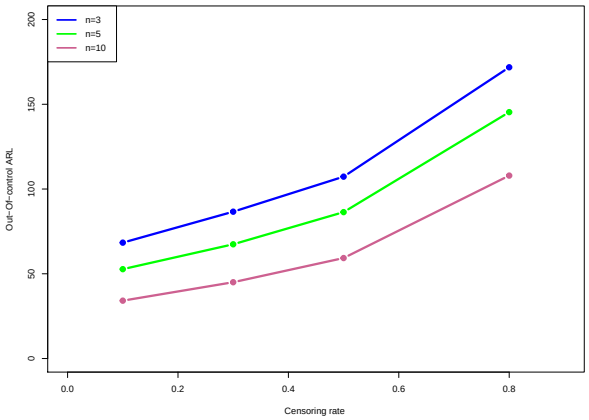


(d)

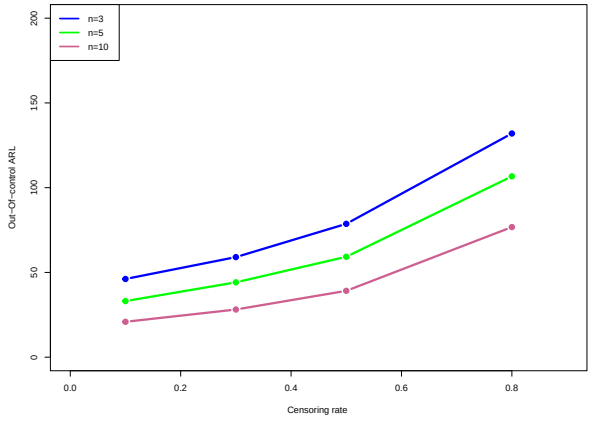
Figure 3.5: ARL_1 for the lower CUSUM chart with $\beta = 3$, and (a) $d = 0.15$, (b) $d = 0.20$, (c) $d = 0.30$, and (d) $d = 0.35$.



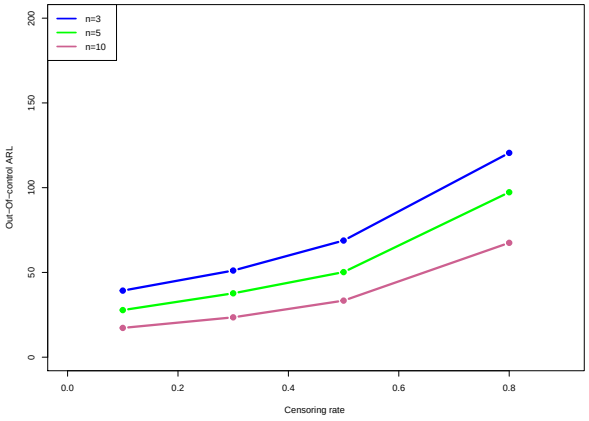
(a)



(b)

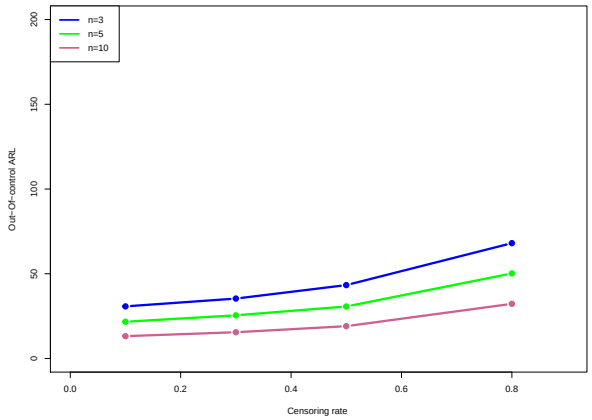


(c)

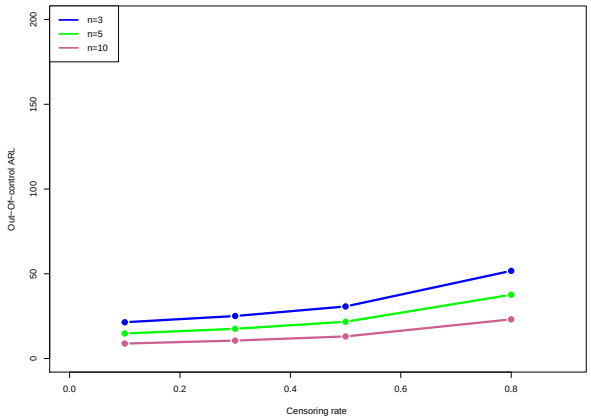


(d)

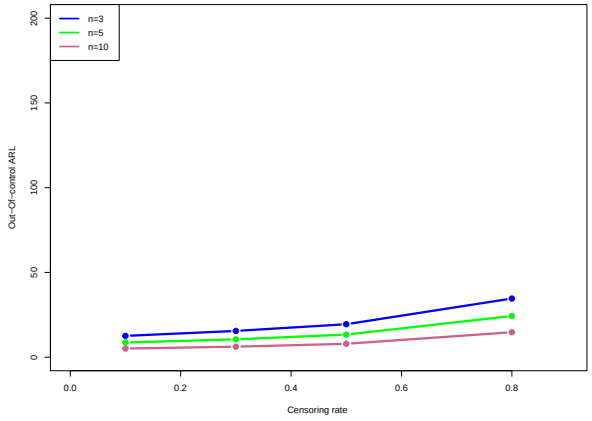
Figure 3.6: ARL_1 for upper CUSUM chart with $\beta = 0.5$, and (a) $d = 0.15$, (b) $d = 0.20$, (c) $d = 0.30$, and (d) $d = 0.35$.



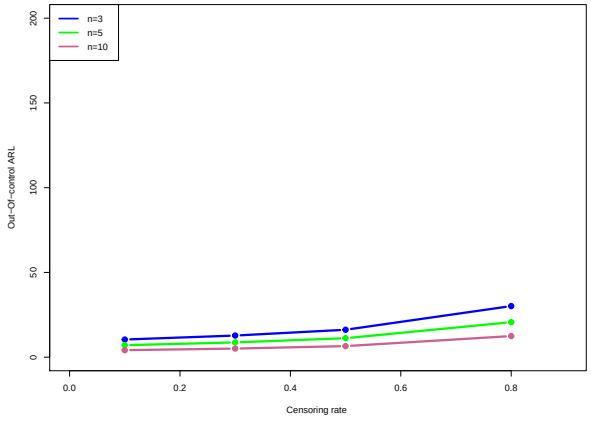
(a)



(b)



(c)



(d)

Figure 3.7: ARL_1 for upper CUSUM chart with $\beta = 3$, and (a) $d = 0.15$, (b) $d = 0.20$, (c) $d = 0.30$, and (d) $d = 0.35$.

Table 3.3: Lower CUSUM chart for a decrease in the scale parameter, in-control and out-of-control ARL and $\beta = \beta_0 = 0.5, 1, 3$ using simulation method

β_0	Censoring Proportion	Sample Size	Shift Size	Control Limit	ARL_0	ARL_1
0.5	0.10	3	0.15	-2.0785	372.718	83.116
0.5	0.10	5	0.15	-2.4196	374.944	63.178
0.5	0.10	3	0.35	-3.4631	374.021	26.660
0.5	0.10	5	0.35	-3.7531	370.087	18.270
0.5	0.50	3	0.15	-1.5978	374.348	118.000
0.5	0.50	5	0.15	-1.8819	370.302	92.748
0.5	0.50	3	0.35	-2.9026	374.630	39.626
0.5	0.50	5	0.35	-3.2184	372.823	28.142
1	0.10	3	0.15	-2.5801	372.773	54.960
1	0.10	5	0.15	-2.9199	370.684	40.002
1	0.10	3	0.35	-3.8843	371.592	15.519
1	0.10	5	0.35	-4.1698	374.759	10.594
1	0.50	3	0.15	-2.2099	372.676	72.544
1	0.50	5	0.15	-2.5224	371.602	53.702
1	0.50	3	0.35	-3.5576	371.543	20.040
1	0.50	5	0.35	-3.8289	371.834	13.814
3	0.10	3	0.15	-3.3489	374.679	26.472
3	0.10	5	0.15	-3.6514	371.095	18.374
3	0.10	3	0.35	-4.3931	373.990	6.585
3	0.10	5	0.35	-4.4808	374.524	4.383
3	0.50	3	0.15	-3.1161	374.003	31.692
3	0.50	5	0.15	-3.4185	371.367	22.158
3	0.50	3	0.35	-4.2536	373.728	7.252
3	0.50	5	0.35	-4.3675	374.087	4.858

Table 3.4: Upper CUSUM chart for an increase in the scale parameter, in-control and out-of-control ARL and $\beta = \beta_0 = 0.5, 1, 3$ using simulation method

β_0	Censoring Proportion	Sample Size	Shift Size	Control Limit	ARL_0	ARL_1
0.5	0.10	3	0.15	1.8208	370.153	91.128
0.5	0.10	5	0.15	2.1392	372.887	70.614
0.5	0.10	3	0.35	2.8151	374.483	39.157
0.5	0.10	5	0.35	3.1149	370.111	27.840
0.5	0.50	3	0.15	1.3848	373.470	138.343
0.5	0.50	5	0.15	1.6680	374.716	112.258
0.5	0.50	3	0.35	2.2717	373.963	68.998
0.5	0.50	5	0.35	2.5774	370.632	50.803
1	0.10	3	0.15	2.3242	347.982	62.533
1	0.10	5	0.15	2.6392	373.937	46.063
1	0.10	3	0.35	3.2858	372.401	23.597
1	0.10	5	0.35	3.5780	370.322	16.474
1	0.50	3	0.15	1.9739	374.998	91.010
1	0.50	5	0.15	2.2454	370.931	67.901
1	0.50	3	0.35	2.9218	372.680	38.650
1	0.50	5	0.35	3.2672	374.547	27.854
3	0.10	3	0.15	3.0758	374.867	30.823
3	0.10	5	0.15	3.3977	374.502	21.933
3	0.10	3	0.35	3.9310	370.125	10.336
3	0.10	5	0.35	4.1565	374.093	7.089
3	0.50	3	0.15	2.8189	374.152	42.648
3	0.50	5	0.15	3.1551	373.633	30.594
3	0.50	3	0.35	3.7500	370.613	16.237
3	0.50	5	0.35	4.0305	371.909	11.155

Table 3.5: Lower CUSUM chart for a decrease in the scale parameter, in-control and out-of-control ARL and $\beta = \beta_0 = 0.5$ using Markov chain method

β_0	Censoring Proportion	Sample Size	Shift Size	Control Limit	ARL_0	ARL_1
0.5	0.10	3	0.15	-2.0785	374.886	83.991
0.5	0.10	5	0.15	-2.4196	373.096	63.417
0.5	0.10	10	0.15	-2.8619	370.379	41.389
0.5	0.10	3	0.20	-2.5279	374.777	60.076
0.5	0.10	5	0.20	-2.8526	370.410	43.165
0.5	0.10	10	0.20	-3.3256	370.334	27.722
0.5	0.10	3	0.30	-3.1730	370.033	33.692
0.5	0.10	5	0.30	-3.5196	370.283	23.628
0.5	0.10	10	0.30	-3.9199	372.441	14.266
0.5	0.10	3	0.35	-3.4631	373.167	26.668
0.5	0.10	5	0.35	-3.7531	372.535	18.258
0.5	0.10	10	0.35	-4.0984	370.626	10.782
0.5	0.30	3	0.15	-1.8517	373.484	96.706
0.5	0.30	5	0.15	-2.1987	373.601	76.054
0.5	0.30	10	0.15	-2.6152	371.082	50.451
0.5	0.30	3	0.20	-2.2756	374.971	69.353
0.5	0.30	5	0.20	-2.5929	371.180	51.540
0.5	0.30	10	0.20	-3.0724	370.073	33.442
0.5	0.30	3	0.30	-2.9360	372.926	39.783
0.5	0.30	5	0.30	-3.3043	373.701	28.394
0.5	0.30	10	0.30	-3.7018	372.804	17.295
0.5	0.30	3	0.35	-3.2341	372.944	31.075
0.5	0.30	5	0.35	-3.5359	371.788	21.736
0.5	0.30	10	0.35	-3.9389	371.469	13.123
0.5	0.50	3	0.15	-1.5978	371.864	116.143
0.5	0.50	5	0.15	-1.8819	370.350	93.353
0.5	0.50	10	0.15	-2.3404	372.193	65.273
0.5	0.50	3	0.20	-1.9724	374.043	86.099
0.5	0.50	5	0.20	-2.2991	371.575	65.977
0.5	0.50	10	0.20	-2.7519	370.357	43.746
0.5	0.50	3	0.30	-2.6416	372.614	50.559
0.5	0.50	5	0.30	-2.9523	372.389	36.727
0.5	0.50	10	0.30	-3.4233	370.305	23.168
0.5	0.50	3	0.35	-2.9026	372.579	39.732
0.5	0.50	5	0.35	-3.2184	371.950	28.120
0.5	0.50	10	0.35	-3.6409	372.786	17.303
0.5	0.80	3	0.15	-1.0410	371.115	175.545
0.5	0.80	5	0.15	-1.2433	370.011	144.731
0.5	0.80	10	0.15	-1.5820	370.051	108.173
0.5	0.80	3	0.20	-1.2966	374.844	136.023
0.5	0.80	5	0.20	-1.6105	372.661	115.886
0.5	0.80	10	0.20	-2.0160	370.327	83.160
0.5	0.80	3	0.30	-1.8352	372.891	88.853
0.5	0.80	5	0.30	-2.1396	370.475	69.086
0.5	0.80	10	0.30	-2.6414	370.422	47.658
0.5	0.80	3	0.35	-2.1427	371.323	73.697
0.5	0.80	5	0.35	-2.4261	374.967	55.285
0.5	0.80	10	0.35	-2.8929	373.347	36.100

Table 3.6: Lower CUSUM chart for a decrease in the scale parameter, in-control and out-of-control ARL and $\beta = \beta_0 = 1$ using Markov chain method

β_0	Censoring Proportion	Sample Size	Shift Size	Control Limit	ARL_0	ARL_1
1	0.10	3	0.15	-2.5801	373.326	54.782
1	0.10	5	0.15	-2.9199	371.979	40.014
1	0.10	10	0.15	-3.3530	372.474	25.018
1	0.10	3	0.20	-3.0390	374.451	37.765
1	0.10	5	0.20	-3.3499	371.345	26.553
1	0.10	10	0.20	-3.7793	372.560	16.231
1	0.10	3	0.30	-3.6779	372.127	20.236
1	0.10	5	0.30	-3.9599	372.312	13.912
1	0.10	10	0.30	-4.2595	373.849	8.155
1	0.10	3	0.35	-3.8843	370.293	15.545
1	0.10	5	0.35	-4.1698	371.698	10.609
1	0.10	10	0.35	-4.3989	373.192	6.129
1	0.30	3	0.15	-2.3971	371.534	60.384
1	0.30	5	0.15	-2.7484	372.557	44.960
1	0.30	10	0.15	-3.2170	373.454	28.945
1	0.30	3	0.20	-2.8401	370.836	41.789
1	0.30	5	0.20	-3.2111	372.481	30.345
1	0.30	10	0.20	-3.6495	371.195	18.563
1	0.30	3	0.30	-3.5140	371.107	22.221
1	0.30	5	0.30	-3.8289	373.736	15.482
1	0.30	10	0.30	-4.1290	373.964	9.109
1	0.30	3	0.35	-3.7398	371.096	16.939
1	0.30	5	0.35	-4.0131	374.465	11.607
1	0.30	10	0.35	-4.2865	370.406	6.797
1	0.50	3	0.15	-2.2099	370.448	72.423
1	0.50	5	0.15	-2.5224	372.433	53.304
1	0.50	10	0.15	-2.9877	372.443	35.472
1	0.50	3	0.20	-2.6557	373.375	50.010
1	0.50	5	0.20	-2.9675	372.804	36.434
1	0.50	10	0.20	-3.4184	373.053	22.534
1	0.50	3	0.30	-3.2999	373.928	26.498
1	0.50	5	0.30	-3.5918	373.399	18.377
1	0.50	10	0.30	-3.9799	372.246	11.124
1	0.50	3	0.35	-3.5576	370.824	20.264
1	0.50	5	0.35	-3.8289	373.086	13.830
1	0.50	10	0.35	-4.1461	370.904	8.210
1	0.80	3	0.15	-1.6421	374.044	112.068
1	0.80	5	0.15	-1.9038	370.762	86.228
1	0.80	10	0.15	-2.3622	371.524	59.685
1	0.80	3	0.20	-2.0069	372.762	78.632
1	0.80	5	0.20	-2.3310	372.020	60.160
1	0.80	10	0.20	-2.7955	374.841	39.848
1	0.80	3	0.30	-2.6761	371.015	44.492
1	0.80	5	0.30	-2.9964	374.594	31.982
1	0.80	10	0.30	-3.4774	371.931	20.202
1	0.80	3	0.35	-2.9613	374.893	34.182
1	0.80	5	0.35	-3.2618	373.714	24.016
1	0.80	10	0.35	-3.7019	370.607	14.963

Table 3.7: Lower CUSUM chart for a decrease in the scale parameter, in-control and out-of-control ARL and $\beta = \beta_0 = 3$ using Markov chain method

β_0	Censoring Proportion	Sample Size	Shift Size	Control Limit	ARL_0	ARL_1
3	0.10	3	0.15	-3.3489	373.666	26.436
3	0.10	5	0.15	-3.6514	371.374	18.239
3	0.10	10	0.15	-4.0395	372.485	11.024
3	0.10	3	0.20	-3.7448	370.782	17.0132
3	0.10	5	0.20	-4.0234	372.190	11.669
3	0.10	10	0.20	-4.3001	374.536	6.817
3	0.10	3	0.30	-4.2308	373.821	8.657
3	0.10	5	0.30	-4.4099	371.385	5.811
3	0.10	10	0.30	-4.4311	374.747	3.327
3	0.10	3	0.35	-4.3931	370.085	6.575
3	0.10	5	0.35	-4.4808	372.665	4.378
3	0.10	10	0.35	-4.4209	370.602	2.511
3	0.30	3	0.15	-3.2515	372.356	28.155
3	0.30	5	0.15	-3.5693	372.593	19.645
3	0.30	10	0.15	-3.9555	372.136	11.870
3	0.30	3	0.20	-3.6806	370.221	18.160
3	0.30	5	0.20	-3.9479	374.341	12.373
3	0.30	10	0.20	-4.2355	371.141	7.283
3	0.30	3	0.30	-4.212	372.923	9.065
3	0.30	5	0.30	-4.3528	372.099	6.055
3	0.30	10	0.30	-4.4193	372.462	3.481
3	0.30	3	0.35	-4.3362	374.012	6.737
3	0.30	5	0.35	-4.4240	370.060	4.486
3	0.30	10	0.35	-4.4189	370.017	2.600
3	0.50	3	0.15	-3.1161	371.705	31.661
3	0.50	5	0.15	-3.4185	370.204	22.190
3	0.50	10	0.15	-3.8262	371.194	13.438
3	0.50	3	0.20	-3.5345	374.257	20.047
3	0.50	5	0.20	-3.8160	373.837	13.821
3	0.50	10	0.20	-4.1246	372.413	8.183
3	0.50	3	0.30	-4.080	371.879	9.775
3	0.50	5	0.30	-4.2604	372.764	6.618
3	0.50	10	0.30	-4.4070	374.829	3.870
3	0.50	3	0.35	-4.2536	371.642	7.285
3	0.50	5	0.35	-4.3675	370.196	4.868
3	0.50	10	0.35	-4.3946	374.361	2.821
3	0.80	3	0.15	-2.6548	371.055	44.505
3	0.80	5	0.15	-2.9825	370.017	32.629
3	0.80	10	0.15	-3.4139	373.139	20.303
3	0.80	3	0.20	-3.0796	371.650	28.781
3	0.80	5	0.20	-3.3901	372.117	20.305
3	0.80	10	0.20	-3.8156	373.446	12.504
3	0.80	3	0.30	-3.7070	372.767	13.737
3	0.80	5	0.30	-3.9868	371.207	9.517
3	0.80	10	0.30	-4.1962	372.332	5.578
3	0.80	3	0.35	-3.9312	374.191	10.103
3	0.80	5	0.35	-4.1484	372.499	6.824
3	0.80	10	0.35	-4.2993	373.921	4.020

Table 3.8: Upper CUSUM chart for an increase in the scale parameter, in-control and out-of-control ARL and $\beta = \beta_0 = 0.5$ using Markov chain method

β_0	Censoring Proportion	Sample Size	Shift Size	Control Limit	ARL_0	ARL_1
0.5	0.10	3	0.15	1.8208	371.516	92.249
0.5	0.10	5	0.15	2.1392	372.345	71.058
0.5	0.10	10	0.15	2.5956	374.183	47.548
0.5	0.10	3	0.20	2.1243	370.496	68.364
0.5	0.10	5	0.20	2.4790	371.284	52.723
0.5	0.10	10	0.20	2.9500	372.447	34.136
0.5	0.10	3	0.30	2.6299	373.407	46.136
0.5	0.10	5	0.30	2.9442	372.136	33.093
0.5	0.10	10	0.30	3.3995	371.103	20.880
0.5	0.10	3	0.35	2.8151	370.893	39.266
0.5	0.10	5	0.35	3.1149	371.611	27.780
0.5	0.10	10	0.35	3.5594	370.814	17.261
0.5	0.30	3	0.15	1.6241	370.295	112.534
0.5	0.30	5	0.15	1.9201	374.283	88.350
0.5	0.30	10	0.15	2.3796	371.933	61.652
0.5	0.30	3	0.20	1.9032	371.053	86.624
0.5	0.30	5	0.20	2.2587	371.305	67.398
0.5	0.30	10	0.20	2.7230	374.384	45.018
0.5	0.30	3	0.30	2.3532	372.432	59.037
0.5	0.30	5	0.30	2.7240	370.684	44.151
0.5	0.30	10	0.30	3.1791	371.744	28.116
0.5	0.30	3	0.35	2.5324	370.351	51.106
0.5	0.30	5	0.35	2.9146	372.546	37.678
0.5	0.30	10	0.35	3.3395	371.136	23.508
0.5	0.50	3	0.15	1.3848	373.934	136.484
0.5	0.50	5	0.15	1.668	372.376	111.536
0.5	0.50	10	0.15	2.0698	374.605	79.316
0.5	0.50	3	0.20	1.6333	372.245	107.291
0.5	0.50	5	0.20	1.9689	370.368	86.370
0.5	0.50	10	0.20	2.4112	372.934	59.278
0.5	0.50	3	0.30	2.1231	373.649	78.662
0.5	0.50	5	0.30	2.4278	373.679	59.226
0.5	0.50	10	0.30	2.8973	373.716	39.144
0.5	0.50	3	0.35	2.2717	370.411	68.808
0.5	0.50	5	0.35	2.5774	370.550	50.191
0.5	0.50	10	0.35	3.0741	373.875	33.367
0.5	0.80	3	0.15	0.9202	370.054	206.924
0.5	0.80	5	0.15	1.0423	374.232	170.885
0.5	0.80	10	0.15	1.4230	371.944	137.083
0.5	0.80	3	0.20	1.0882	371.679	171.825
0.5	0.80	5	0.20	1.3142	372.507	145.357
0.5	0.80	10	0.20	1.7193	372.283	107.925
0.5	0.80	3	0.30	1.4424	371.860	131.971
0.5	0.80	5	0.30	1.6913	371.350	106.657
0.5	0.80	10	0.30	2.1403	370.213	76.755
0.5	0.80	3	0.35	1.5792	370.688	120.528
0.5	0.80	5	0.35	1.8769	373.045	97.301
0.5	0.80	10	0.35	2.2959	371.040	67.463

Table 3.9: Upper CUSUM chart for an increase in the scale parameter, in-control and out-of-control ARL and $\beta = \beta_0 = 1$ using Markov chain method

β_0	Censoring Proportion	Sample Size	Shift Size	Control Limit	ARL_0	ARL_1
1	0.10	3	0.15	2.3242	373.302	63.084
1	0.10	5	0.15	2.6392	372.202	45.676
1	0.10	10	0.15	3.1031	374.698	29.423
1	0.10	3	0.20	2.6760	371.227	45.717
1	0.10	5	0.20	2.9901	370.826	32.868
1	0.10	10	0.20	3.4493	372.258	20.564
1	0.10	3	0.30	3.1267	373.536	28.388
1	0.10	5	0.30	3.4155	372.920	19.682
1	0.10	10	0.30	3.8114	370.572	12.002
1	0.10	3	0.35	3.2858	374.997	23.421
1	0.10	5	0.35	3.5780	372.409	16.437
1	0.10	10	0.35	3.9764	372.250	9.987
1	0.30	3	0.15	2.1568	374.593	74.208
1	0.30	5	0.15	2.4738	370.848	54.733
1	0.30	10	0.15	2.9293	372.683	35.547
1	0.30	3	0.20	2.4850	373.295	55.022
1	0.30	5	0.20	2.8706	374.372	41.260
1	0.30	10	0.20	3.2756	373.634	25.314
1	0.30	3	0.30	2.9394	370.850	34.979
1	0.30	5	0.30	3.2954	373.412	25.153
1	0.30	10	0.30	3.7234	371.019	17.227
1	0.30	3	0.35	3.1599	374.776	30.189
1	0.30	5	0.35	3.4520	373.758	21.086
1	0.30	10	0.35	3.8432	371.313	12.737
1	0.50	3	0.15	1.9739	370.3190	90.181
1	0.50	5	0.15	2.2454	371.495	67.085
1	0.50	10	0.15	2.7303	371.557	45.996
1	0.50	3	0.20	2.2754	370.817	68.269
1	0.50	5	0.20	2.5894	370.196	50.376
1	0.50	10	0.20	3.0904	373.618	33.043
1	0.50	3	0.30	2.7428	370.267	45.397
1	0.50	5	0.30	3.0766	372.964	32.780
1	0.50	10	0.30	3.5031	374.603	20.378
1	0.50	3	0.35	2.9218	371.942	38.615
1	0.50	5	0.35	3.2672	372.012	28.085
1	0.50	10	0.35	3.6641	372.871	17.008
1	0.80	3	0.15	1.4074	373.998	137.860
1	0.80	5	0.15	1.7031	371.797	110.338
1	0.80	10	0.15	2.0705	373.768	78.969
1	0.80	3	0.20	1.7643	370.831	114.077
1	0.80	5	0.20	1.9628	372.050	86.578
1	0.80	10	0.20	2.4621	373.767	58.857
1	0.80	3	0.30	2.1580	370.454	80.455
1	0.80	5	0.30	2.4864	374.881	60.229
1	0.80	10	0.30	2.9423	372.139	39.456
1	0.80	3	0.35	2.3677	372.792	70.757
1	0.80	5	0.35	2.6350	370.663	52.141
1	0.80	10	0.35	3.0980	372.118	33.908

Table 3.10: Upper CUSUM chart for an increase in the scale parameter, in-control and out-of-control ARL and $\beta = \beta_0 = 3$ using Markov chain method

β_0	Censoring Proportion	Sample Size	Shift Size	Control Limit	ARL_0	ARL_1
3	0.10	3	0.15	3.0758	370.013	30.766
3	0.10	5	0.15	3.3977	372.431	21.705
3	0.10	10	0.15	3.7972	371.830	13.199
3	0.10	3	0.20	3.4259	374.335	21.406
3	0.10	5	0.20	3.6901	370.116	14.785
3	0.10	10	0.20	4.0319	372.375	8.831
3	0.10	3	0.30	3.7959	370.155	12.560
3	0.10	5	0.30	4.0424	371.556	8.630
3	0.10	10	0.30	4.2734	370.886	5.099
3	0.10	3	0.35	3.931	371.348	10.414
3	0.10	5	0.35	4.1565	370.422	7.067
3	0.10	10	0.35	4.3056	373.276	4.142
3	0.30	3	0.15	2.9674	371.036	35.372
3	0.30	5	0.15	3.3287	371.348	25.500
3	0.30	10	0.15	3.7325	374.232	15.510
3	0.30	3	0.20	3.3209	374.765	25.080
3	0.30	5	0.20	3.6344	373.395	17.542
3	0.30	10	0.20	4.0085	374.942	10.599
3	0.30	3	0.30	3.7591	374.466	15.488
3	0.30	5	0.30	4.0044	373.928	10.534
3	0.30	10	0.30	4.2581	374.073	6.188
3	0.30	3	0.35	3.8880	371.186	12.760
3	0.30	5	0.35	4.1263	372.908	8.721
3	0.30	10	0.35	4.3001	371.884	5.078
3	0.50	3	0.15	2.8189	374.185	43.314
3	0.50	5	0.15	3.1551	371.869	30.710
3	0.50	10	0.15	3.6105	373.120	19.080
3	0.50	3	0.20	3.1700	373.266	30.720
3	0.50	5	0.20	3.4889	374.511	21.707
3	0.50	10	0.20	3.8779	370.267	13.040
3	0.50	3	0.30	3.6100	370.020	19.443
3	0.50	5	0.30	3.9010	370.139	13.330
3	0.50	10	0.30	4.2248	373.805	7.900
3	0.50	3	0.35	3.7500	371.010	16.179
3	0.50	5	0.35	4.0305	374.253	11.203
3	0.50	10	0.35	4.2862	372.032	6.534
3	0.80	3	0.15	2.3299	374.773	68.067
3	0.80	5	0.15	2.6991	372.077	50.211
3	0.80	10	0.15	3.1473	373.056	32.266
3	0.80	3	0.20	2.7385	372.072	51.706
3	0.80	5	0.20	3.0527	372.649	37.641
3	0.80	10	0.20	3.4855	372.229	23.109
3	0.80	3	0.30	3.1572	372.515	34.596
3	0.80	5	0.30	3.5092	370.965	24.285
3	0.80	10	0.30	3.9047	372.976	14.723
3	0.80	3	0.35	3.3763	370.241	30.147
3	0.80	5	0.35	3.6361	371.504	20.640
3	0.80	10	0.35	4.0523	371.022	12.464

Chapter 4

Assessing False Alarm Performance

4.1 Run Length Distribution

In Chapter 3, we discussed one way to evaluate the CUSUM chart performance, which was based on calculating the out-of-control ARL_1. In this chapter, we will assess the performance of a chart using the run length distribution and the probability of false alarms. The run length can be defined as the number of sampling periods that a process takes until the chart signals when exceeding the control limits for the first time. Many authors, e.g. Prabhu and Runger (1997), Page (1954), Brook and Evans (1972), Woodall (1983), Vardeman and Ray (1985), Huh (2010) and Gau (1992), use the run length distribution as a measure to assess the performance of CUSUM or EWMA charts. In this thesis, we followed Prabhu and Runger (1997) and Huh (2010) to derive and evaluate the off-target run length distribution. We can obtain the distribution of the run length by using the transition probability matrix from the Markov chain method. Prabhu and Runger (1997) found the distribution for run length for the zero-state case which means that the process starts at an out of control stage. However, Huh (2010) expanded the work to the steady-state case which means that the process starts in a control stage and a

shift to an out-of-control occurs at a later time $t = \tau$.

Now, suppose that at a time $t = \tau$ the process goes off-control from η_0 to η_1. Thus, the change in the scale parameter η will be:

$$\eta = \begin{cases} \eta_0 & t = 1, 2, ..., \tau - 1 \\ \eta_1 & t = \tau, \tau + 1, ... \end{cases}$$

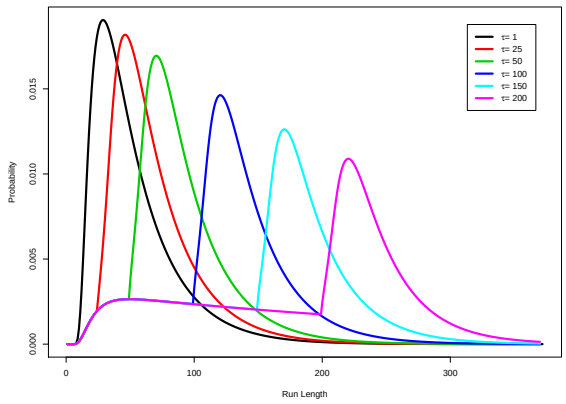
and the transition probability matrix will be

$$\boldsymbol{P} = \begin{cases} \boldsymbol{P_0} & t = 1, 2, ..., \tau - 1 \\ \boldsymbol{P_1} & t = \tau, \tau + 1, ... \end{cases}$$

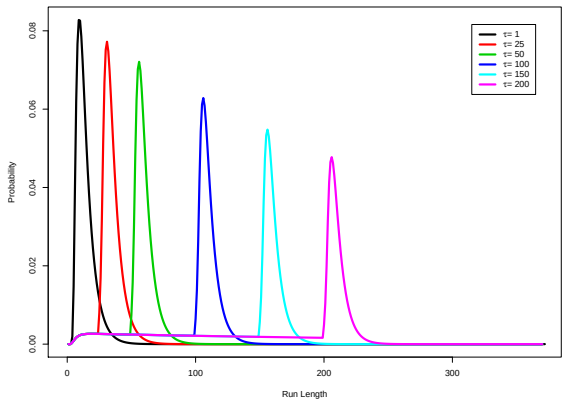
(Huh, 2010). Hence, the probability mass function (pmf) of run length N is given by:

$$f(n) = P(N = n) \begin{cases} \boldsymbol{s'P_0^{n-1}(I - P_0)1} & \text{if } n = 1, 2, ..., \tau - 1 \\ \boldsymbol{s'P_0^{\tau-1}P_1^{n-\tau}(I - P_1)1} & \text{if } \tau, \tau + 1, ... \end{cases} \tag{4.1}$$

For the complete derivation of the run length distribution, see Huh (2010). Figures 4.1-4.4 display the distribution of the run length N with $ARL_0 \approx 370$. We can see that the off-target run length probability mass function has a heavily right skewed distribution or positive skewness. Those figures also show the effect of the value τ on the run length distribution. The smaller value of τ, the higher the peak we get. Also, the distribution moved toward the right as the value of τ increase.

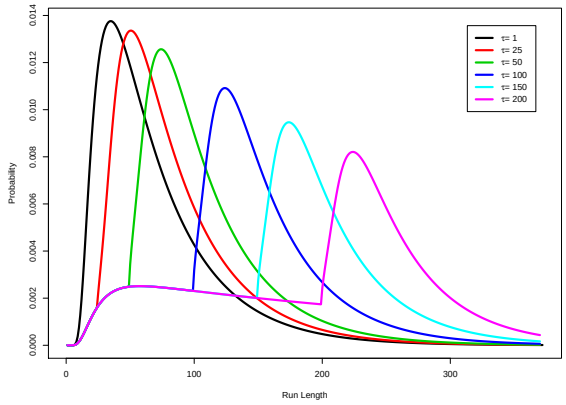


(a)

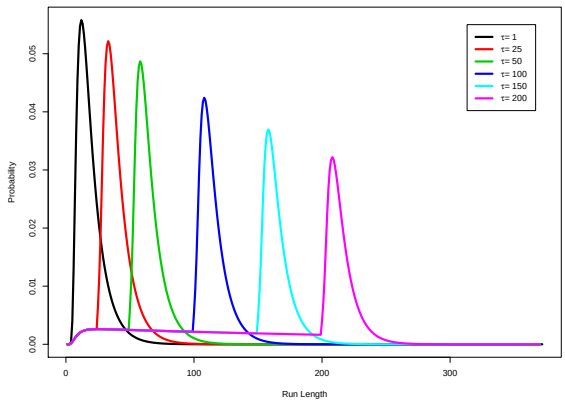


(b)

Figure 4.1: Off-target run length distribution for the lower CUSUM chart with $ARL_0 = 370$, and (a) $\beta = 0.5$ and (b) $\beta = 3$.



(a)



(b)

Figure 4.2: Off-target run length distribution for the upper CUSUM chart with $ARL_0 = 370$, and (a) $\beta = 0.5$ and (b) $\beta = 3$.

4.2 Probability of False Alarms

The main objective of using the CUSUM chart or any other control charts is to detect any change in the process quickly. However, sometimes this may lead to false alarms (Khan, 1979). The probability of a false alarm is a measure of the sensitivity of the chart to detect a change and can be defined as the probability of the chart signalling an out-of-control state when the process is in fact in-control (Huh, 2010). In other words, how likely is that the chart signals for a change when in fact, no shift or change has occurred. We want a chart to signal for false alarms infrequently because false alarms cause wasting time and disturbing the process (Hawkins and Olwell, 1998). The probability of false alarm can be calculated by using the run length distribution from the following equation:

$$\text{Probability of False Alarm} = \sum_{n=1}^{\tau-1} f(n) \tag{4.2}$$

(Huh, 2010).

4.3 Out-of-control ARL

We calculate the off-target average run length ARL_1 for a given τ with the condition that the in-control $ARL_0 = 370$ from the following:

$$ARL_1 = \boldsymbol{s'}[\boldsymbol{I} - \boldsymbol{P_0^{\tau-1}}](\boldsymbol{I} - \boldsymbol{P_0})^{-1}\mathbf{1} + \boldsymbol{s'}\boldsymbol{P_0^{\tau-1}}(\boldsymbol{I} - \boldsymbol{P_1})^{-1}\mathbf{1}, \tag{4.3}$$

where P_0 and P_1 are the transition probability matrices for in-control and out-of-control cases respectively. For the complete derivation of ARL_1 see Huh (2010).

A more meaningful measure than the ARL_1 is the effective ARL_1 which is $ARL_1 - \tau$.

The effective ARL_1 is the average number of runs required to detect a shift or change in the process mean after the change has occurred (Huh, 2010). Tables 4.1-4.2 illustrate this for the lower and the upper charts respectively, with the following parameters: $\beta_0 = 0.5, 3$, $\eta_0 = 1$, $\eta_1 = 0.8, 1.3$ and $pc = 30\%$. In those tables, we calculate the probability of false alarm, ARL_1, and the effective ARL_1 for various values of τ. We noticed that as we increase the value of the transition time τ, the percentage of false alarm increase too. For example, if $\tau = 1$ there is no false alarm whereas if $\tau = 200$ there is around 40% chance the chart would signal indicating that the process has gone out of control when in fact it has not. Also, the effective ARL_1 is less than ARL_1, and in some cases, we got a negative value of effective ARL_1. This negative sign is due to the high percentage of the false alarms. In Table 4.1, for instance, with $\tau = 200$ and $\beta_0 = 0.5$, $ARL_1 = 184.949$ and the effective $ARL_1 = -15.051$ this means that the chart takes much less runs on average to signal out-of-control when the shift appears later on ($\tau = 200$) than when it appears at the beginning ($\tau = 1$).

Table 4.1: Probability of false alarms, ARL_1 for each transition point τ with $\eta_0 = 1$, $\eta_1 = 0.8$, $pc = 0.3$ and $n = 5$ for lower CUSUM.

$\beta_0 = 0.5$			
τ	Probability of false alarms	ARL_1	Effective ARL_1
1	0.0000	51.667	50.667
25	0.0165	66.901	41.901
50	0.0786	87.345	37.345
100	0.2048	124.778	24.778
150	0.3144	157.090	7.090
200	0.4089	184.949	-15.051
$\beta_0 = 3$			
τ	Probability of false alarms	ARL_1	Effective ARL_1
1	0.0000	12.988	11.988
25	0.0459	33.369	8.369
50	0.1079	55.834	5.834
100	0.2110	96.484	-3.516
150	0.3180	132.026	-17.974
200	0.4037	163.102	-36.898

Table 4.2: Probability of false alarms, ARL_1 for each transition point τ with $\eta_0 = 1$, $\eta_1 = 1.3$, $pc = 0.3$ and $n = 5$ for upper CUSUM.

$\beta_0 = 0.5$			
τ	Probability of false alarms	ARL_1	Effective ARL_1
1	0.0000	67.891	66.981
25	0.0131	80.659	55.659
50	0.0680	100.223	50.223
100	0.1892	136.826	36.826
150	0.2962	168.710	18.710
200	0.3891	196.388	-3.612
$\beta_0 = 3$			
τ	Probability of false alarms	ARL_1	Effective ARL_1
1	0.0000	18.214	17.214
25	0.0419	37.751	12.751
50	0.1062	59.972	9.972
100	0.2224	100.043	0.043
150	0.3234	134.905	-15.095
200	0.4114	165.235	-34.765

Chapter 5

Numerical Illustrations

Because of the limited sources for quality control data that can be used to illustrate the methods, we simulated data sets that follow a gamma distribution to illustrate the methods in the thesis. Specifically, we apply the methods that have been discussed in Chapters 3 and 4 to monitor an increase and a decrease in the process mean.

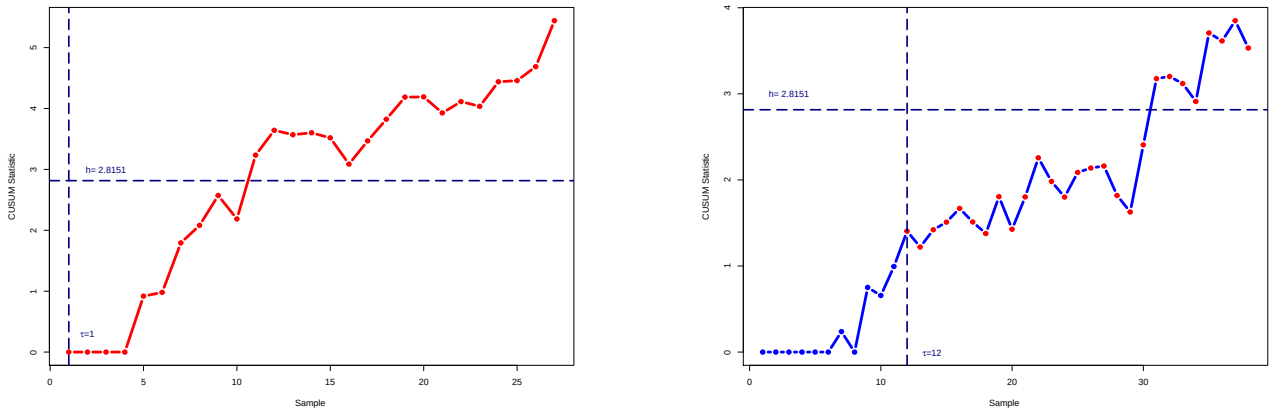
5.1 Numerical Illustrations for Upper CUSUM

First, we simulated two data sets with the following parameters $n = 3$, $\beta = 0.5$, $\eta_0 = 1$, $d = 0.35$, $pc = 0.10$, $\tau = 1, 12$. Then, we calculated ARL_0, and ARL_1 using Markov chain method and found the control limits that give $ARL_0 \approx 370$ using the partition method, the resulting control limit was $h^+ = 2.1851$. After that, we assessed the chart performance using the ARL_1 and the probability of false alarms. Finally, we plot the CUSUM chart that corresponds to each dataset. The results of this illustration are given in Table 5.1 and Figures 5.1 and 5.2. From the table, we can see that as we increase the transition time, the chart is more likely to signal for a non-real shift. In both data sets, the CUSUM signal on average on 38 to 39 samples

and necessary action needed to do an inspection. As we desire to detect a 35% upward shift in the mean, the scale parameter increased to 65%. We noticed that when the process shifts to the out-of-control stage, the chart signals for a first time after 12 samples as in Figure 5.1 and after 31 samples as in Figure 5.2.

Table 5.1: ARL results for two simulated datasets.

Data	ARL_0	ARL_1	τ	Probability of false alarm	ARL_1	Effective ARL_1
First	370.893	39.266	1	0.0000	39.182	38.182
Second	370.893	39.266	12	0.0036	45.579	33.579



(a) (b)

Figure 5.1: Upper CUSUM chart designed to detect a 35% increase in the scale parameter η. The red points represent the 26 simulated out-of-control observations and (a) $\tau = 1$ and (b) $\tau = 12$.

5.2 Numerical Illustration for Lower CUSUM

We start with simulating a data from a gamma distribution with scale parameter $\eta_0 = 1$ and shape $\beta_0 = 0.5$, sample size $n = 3$, $pc = 10\%$ and wish to detect a 35% downward in the scale. Then calculate the in-control and out-of-control ARL using Markov chain. Finally, calculate

the probability of false alarms at a transition time $\tau = 15$ and plot the CUSUM chart. From Table 5.2, the CUSUM signals on average to detect a shift on around 40 samples when the process started out-of-control at $\tau = 1$ and increased to around 49 samples at $\tau = 15$ to jump off the control limit. Figure 5.3 displays that 14 samples in control and 42 out-of-control that put under quality test. From that figure, we can see how the chart statistic decreases gradually until it goes off the control limit for the first time after 44 samples.

Table 5.2: ARL results for simulated dataset.

ARL_0	ARL_1	τ	Probability of false alarm	ARL_1	Effective ARL_1
372.579	39.732	15	0.0044	48.761	33.761

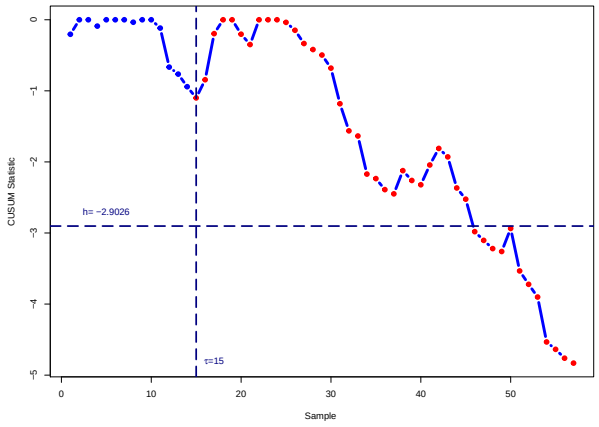


Figure 5.2: Lower CUSUM chart designed to detect a 35% decrease in the scale parameter η. The red points represent the 42 simulated out-of-control observations.

Chapter 6

Conclusions and Future Work

Cumulative sum control charts (CUSUM) were invented by Page (1954), and are well-known quality control tools to detect a small or sustained changes in a process (Hawkins and Olwell, 1998). In this thesis, we developed a likelihood-ratio CUSUM control chart to monitor a decrease or increase in the process mean for a gamma distribution with type I right censoring. We started with an introduction about the gamma distribution and quality measures followed by the design of the CUSUM chart using likelihood-ratio under censoring. Censoring is important due to the restriction of the ability to observe the failure times (Escobar and Meeker, 1998). In Chapters 3 and 4, we discussed two methods to evaluate the CUSUM chart sensitivity by calculating the out-of-control ARL_1 and the probability of false alarms. We calculated the average run length by simulation and by the Markov chain method. In the simulation method, we followed Dickinson et al. (2014); however, the main disadvantage of this method is that it is time consuming to obtain the results. In the Markov chain method, which has been used by many statistical researchers (Runger and Prabhu (1996), Brook and Evans (1972), and Lucas and Saccucci (1990)), we found the transition probability matrix and used the empirical cdf to approximate the distribution of the score statistic. We also calculated the probability of

false alarms, the run length distribution and evaluated the off-target run length at different times τ (Huh, 2010). We looked into the performance of the CUSUM chart and identified the parameters and circumstances that affect the sensitivity of the chart to detect a change in the mean life. From both evaluation methods, we found that the chart preforms well as we increase the shift size and the shape parameter and decrease the transition time. If the process is in-control, we need ARL_0 to be large enough or close to the desired average run length and the chart signals for false alarms rarely. Also, if the process is in an out-of-control stage, we want the ARL_1 small to detect any changes.

The work done provides guidance to engineers on how to set a CUSUM control chart for their process quality measurements if their distribution is adequately represented by a gamma distribution. Specifically, after the in-control parameters are established by the engineers, namely β_0, η_0 and pc, the thesis provides the required control limits necessary to detect a specified increase or decrees in the process mean with an in-control ARL of 370 runs.

In the future work, we plan to extend the methods to other distributions, including the Pareto distribution, which is a popular distribution in quality control. Recently, attention has been given to multivariate process monitoring. We would like to apply the methods to the multivariate case and with another type of censoring such as interval censoring.

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