

The van der Waerden Simplicial Complex and the Lefschetz Properties

by

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Abstract

In this thesis, we examine the van der Waerden simplicial complexes, defined on two parameters n and k , and the Artinian rings we can form by modding out the squares of the variables from the Stanley-Reisner ring of the van der Waerden complex. We begin by providing a thorough background on all the necessary background, from abstract algebra, graph theory, and linear algebra. The main question investigated in this thesis is when the Artinian rings constructed from the van der Waerden complexes have either the Weak or Strong Lefschetz property. We examine for what values of n and k do these Artinian rings have the Weak, and in some instances Strong, Lefschetz Property. We focus on the smallest possible values of k , namely $k = 1$ and $k = 2$; and the largest possible value k can take, which is $n - 1$. We also focus on the case where $k = 3$, as the first instance of the failure of the Weak Lefschetz Property occurs here.

We also investigate in what degrees the Artinian ring always has the Weak Lefschetz Property. We then give a characterization of when these simplicial complexes are pseudo-manifolds, which provides some further insight on what degrees the associated Artinian ring has the Weak Lefschetz Property. We conclude by providing some conjectures on both Lefschetz Properties, as well as further areas of possible future research. A Macaulay2 package on the van der Waerden simplicial complexes is also provided.

Key Words: Simplicial complexes, van der Waerden simplicial complexes, Artinian rings, Graded rings, Weak Lefschetz Property, Strong Lefschetz Property, Pseudo-manifolds

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Dedication

I dedicate this thesis to my family; their endless support and love is what shaped me into the person I am today. If I can see from great heights, its because I've been sitting on their shoulders my whole life.

"I loved you completely. And you loved me the same. That's all. The rest is confetti."

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CHAPTER 1

Introduction

In this thesis, we study the Weak Lefschetz Property of monomial algebras constructed from the family of simplicial complexes called the the van der Waerden simplicial complexes. These simplicial complexes, denoted $\text{vdw}(n, k)$, are defined on two non-zero natural numbers, $n > k$. The facets that generate these simplicial complexes are all valid arithmetic progressions of the form $\{x_i, x_{i+d}, \dots, x_{i+kd}\}$ where $0 < i < i + kd \leq n$. Each facet of $\text{vdw}(n, k)$ has cardinality $k + 1$.

EXAMPLE 1.1. Consider $\text{vdw}(10, 3)$. The facets of $\text{vdw}(10, 3)$ are all arithmetic progressions of size 3 on 10 vertices. By considering all such valid progressions, we have the following arithmetic progressions that are the facets that generate this simplicial complex:

- Progressions of size 1: $\{x_1, x_2, x_3, x_4\}$, $\{x_2, x_3, x_4, x_5\}$, $\{x_3, x_4, x_5, x_6\}$,
 $\{x_4, x_5, x_6, x_7\}$, $\{x_5, x_6, x_7, x_8\}$, $\{x_6, x_7, x_8, x_9\}$, $\{x_7, x_8, x_9, x_{10}\}$,
- Progressions of size 2: $\{x_1, x_3, x_5, x_7\}$, $\{x_2, x_4, x_6, x_8\}$, $\{x_3, x_5, x_7, x_9\}$,
 $\{x_4, x_6, x_8, x_{10}\}$,
- Progressions of size 3: $\{x_1, x_4, x_7, x_{10}\}$,

These simplicial complexes were introduced in 2016 by Ehrenborg, Govindaiah, Park, and Readdy in [13], and were inspired by the van der Waerden's theorem in graph theory. van der Waerden's theorem states that given positive integers r and k , there exists some positive integer N such that if the set $\{1, 2, \dots, N\}$ is coloured with r different colours, then there exists an arithmetic progression of length k among $\{1, 2, \dots, N\}$ that all have the same colour. The smallest N for which this is true is denoted by $W(r, k)$. More about van der Waerden's theorem can be found in [17] and [44].

One can study simplicial complexes through commutative algebra by using Stanley-Reisner theory. Given a simplicial complex Δ on the vertex set $\{x_1, x_2, \dots, x_n\}$, the Stanley-Reisner ring of Δ is the quotient ring $K[x_1, x_2, \dots, x_n]/I_\Delta$, where I_Δ is the Stanley-Reisner ideal of Δ , and is generated by the non-faces of Δ .

Besides Stanley-Reisner rings, we are also interested in Artinian rings. Rings that do not have any infinite sequence of descending ideals (by containment) are referred to as Artinian rings. Equivalently, these are rings whose Krull dimension, or maximal length of chains of prime ideals, is 0. A quotient ring $K[x_1, x_2, \dots, x_n]/I$ by a homogeneous ideal is Artinian if and only if its radical, that is

$$\sqrt{I} = \{r \in R \mid \text{there exists } n \in \mathbb{Z} \text{ such that } r^n \in I\}$$

is equal to the homogeneous maximal ideal (x_1, x_2, \dots, x_n) . While the Stanley-Reisner ring of a simplicial complex is not necessarily an Artinian Ring, we can make it so by adjoining powers of our variables.

In this thesis, we focus on Artinian rings created by taking the Stanley-Reisner ring of our van der Waerden simplicial complexes, but also adjoining to the ideal the squares of all of our variables. For $\Delta = \text{vdw}(n, k)$, we construct

$$R/(I_\Delta, x_1^2, x_2^2, \dots, x_n^2), \text{ and denote it by } A(\text{vdw}(n, k)).$$

Since $A(\text{vdw}(n, k))$ is a quotient of a polynomial ring, it is also naturally graded, by their degrees. So our constructions of interest are graded Artinian rings.

Graded Artinian rings have many properties of interest, one of them being the Lefschetz Properties. A graded Artinian ring A has the **Weak Lefschetz Property (WLP)** if the homomorphism created by multiplication by ℓ , a general linear form, has maximal rank in consecutive degrees. That is, the map

$$\times \ell : A_i \rightarrow A_{i+1}$$

is either injective or surjective, where A_i is the graded component of A consisting of all the elements of degree i . We will say A has the Weak Lefschetz Property in degree d if the map $\times \ell : A_d \rightarrow A_{d+1}$ has maximal rank. One can also examine whether or not a graded Artinian ring has the Strong Lefschetz Property (*SLP*). A graded Artinian ring A has the **Strong Lefschetz Property** if the homomorphism created by multiplication by ℓ^d , some power of a general linear form, has maximal rank in consecutive degrees. That is, the map

$$\times \ell^d : A_i \rightarrow A_{i+d}$$

is either injective or surjective all values of $i \geq 0$ and $d \geq 0$.

The Lefschetz properties are a widely studied phenomena in the area of commutative combinatorial algebra. In his 1980 paper [43], Stanley first examined the Lefschetz properties, which he referred to as the Hard Lefschetz Theorem, in a commutative algebra setting. He helped popularize the use of the Hard Lefschetz Theorem, a result about the singular co-homological ring, to its uses in combinatorics. In this paper [43], Stanley proves that quotient rings of the form

$$K[x_1, x_2, \dots, x_n]/(x_1^2, x_2^2, \dots, x_n^2),$$

always has both Lefschetz Properties, a result we will examine further. In [33] and [38], Migliore and Nagel, and Phuong and Tran study Stanley's classic contributions to the study of the Lefschetz Properties, and examine contributions that have been made to the field over time, as well as open questions that remain to be solved. In [33], Harima, Migliore, Nagel, and Watanabe study both the Weak and Strong Lefschetz Properties for Artinian K -algebras, giving a complete characterization of the possible Hilbert Functions that can occur for such algebras. More recently, the Lefschetz Properties have been

studied in various settings such as different monomial ideals, Gorenstein algebras, and Rees algebras, in [1], [2], [5], [25], [26], and [28].

The Lefschetz properties also appear in [32], where Migliore, Miró-Roig, and Nagel study the Lefschetz Properties of monomial ideals and almost complete intersections. They prove that showing multiplication by the standard linear form $\times(x_1 + x_2 + \cdots + x_n)$ has maximal rank is a sufficient and necessary condition to show that $\times\ell$ has maximal rank, where ℓ is some arbitrary general linear form (please see Theorem 2.42 for more details). They also examine how the characteristic of the field in question impacts whether or not the constructed Artinian ring has the Lefschetz Properties. Finally, we mention the work of Dao-Nair [11], who examine Artinian rings that are quotient by monomial ideals that contain the squares of variables, the exact context we examine as well. They take a more graph-theoretical and topological view on the problem, examining the 1-skeleton of the simplicial complex in question and testing the 1-skeleton for criteria regarding bipartiteness.

The van der Waerden simplicial complexes on the other hand are a fairly new construction. They have been studied in [27], where sufficient and necessary conditions for when these simplicial complexes are shellable were given. They were also more recently studied in [23], where further results on when these simplicial complexes were vertex-decomposable or Cohen-Macaulay were given.

We now state the central questions of this thesis, and provide a breakdown of which aspects are examined in the different chapters of this thesis:

QUESTION 1.2. *For which values of n and k does the graded Artinian ring $A(\mathbf{vdw}(n, k))$ have the Weak Lefschetz Property?*

While we do not give a complete answer, we are able to answer the question for small values of k , as well as large values of k . We also give conjectures for the general case.

In Chapter 3, we examine several values of n and k such that $A(\mathbf{vdw}(n, k))$ does have the Weak Lefschetz Property. We begin with the smallest possible values of k , namely $k = 1$ and $k = 2$. For $k = 1$, we deduce the following result:

THEOREM 1.3 (Theorem 3.4). *$A(\mathbf{vdw}(n, 1))$ has the Weak Lefschetz Property for all values of $n \geq 2$.*

We provide two separate proofs that $A(\mathbf{vdw}(n, 1))$ always has the Weak Lefschetz Property. The first proof uses a result of Dao-Nair in [11], which examines the bipartite components of the 1-skeleton of simplicial complexes to deduce whether or not the corresponding square-free Artinian ring has the Weak Lefschetz Property. We examine the $n \geq 3$ and the $n = 2$ cases separately, as they fall under the different cases listed in the theorem. In either case, we prove that since the 1-skeleton of $\mathbf{vdw}(n, 1)$ is the complete graph on n vertices, K_n , the conditions listed by the theorem always hold. So for all of the non-trivial components of our Artinian ring, multiplication by ℓ does have maximal rank, so the ring has the Weak Lefschetz Property.

We also provide a second proof for Theorem 3.4 using the matrix representation of map for the non-trivial degree cases, and showing that it must always have maximal rank. Examples of such matrices for smaller values of n are also provided.

For $k = 2$, we also deduce the following result:

THEOREM 1.4 (Theorem 3.6). *$A(\mathbf{vdw}(n, 2))$ has the Weak Lefschetz Property for all values of $n \geq 3$.*

We prove Theorem 3.6 in degree 1 by examining the facet structure of $A(\mathbf{vdw}(n, 2))$, and showing that for all values of n , $\dim A_2 \geq \dim A_3$. Then the result can be deduced from the work of Dao-Nair. For the other non-trivial degree 2, we provide an induction proof using the matrix representation of the linear map.

After examining the two smallest values of k , we study the largest valid value k can take for $A(\mathbf{vdw}(n, k))$, which is $k = n - 1$. We get the following result for $k = n - 1$:

THEOREM 1.5 (Theorem 3.11). *Let K be an infinite field of characteristic 0. Then $A(\mathbf{vdw}(n, n - 1))$ has the Weak Lefschetz Property for all values of $n \geq 2$.*

This case requires the study of complete intersections, which are quotient rings of the form $S = R/I$ where I is generated by regular sequences. Regular sequences are sequences of the form $\{F_1, F_2, \dots, F_p\}$, where $F_1 + I$ is not a zero divisor of S , and $F_i + I$ is not a zero divisor on $R/(I, F_1, \dots, F_{i-1})$, for $i = 2, \dots, p$. We will see that $\mathbf{vdw}(n, n - 1)$ is generated by one facet only, so the resulting Artinian ring is actually a complete intersection. We then prove the desired result Theorem 3.11 using a classic theorem of Stanley in [43]. Note that this statement requires K to be an infinite field of characteristic 0, an assumption we make throughout this thesis.

We conclude Chapter 3 with the following result on the smallest possible degree one can examine when considering the Weak Lefschetz Property:

THEOREM 1.6 (Theorem 3.12). *$A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property in degree 1 for all values of $n > k \geq 1$.*

This result allows us to deduce that in degree 1, all $A(\mathbf{vdw}(n, k))$ have the Weak Lefschetz Property, regardless of what values n and k take. This result is also proved using the result from Dao-Nair.

In Chapter 4, we carefully study the first instance where $A(\mathbf{vdw}(n, k))$ fails to have the Weak Lefschetz Property, which occurs when $n = 7$ and $k = 3$. We give the following complete characterization of when $A(\mathbf{vdw}(n, 3))$ has the Weak Lefschetz Property:

THEOREM 1.7 (Theorem 4.19). *$A(\mathbf{vdw}(n, 3))$ has the Weak Lefschetz Property if and only if $n = 4, 5$, or 6 . For $n \geq 7$, $A(\mathbf{vdw}(n, 3))$ fails to have the Weak Lefschetz Property in degree 2.*

We prove that $A(\mathbf{vdw}(n, 3))$ has the Weak Lefschetz Property for $n = 4, 5, 6$ using the matrix representations of the map in non-trivial degrees. We then study the case of

$A(\mathbf{vdw}(7, 3))$, the smallest value of n and k such that $A(\mathbf{vdw}(n, 3))$ does not have the Weak Lefschetz Property, by examining the matrix representation of the map in degree 2, where the failure occurs. The transpose map and Rank-Nullity Theorem are used extensively to obtain the desired conclusion. The $n = 7$ case is treated differently also for due to the reason that it is the only instance of $A(\mathbf{vdw}(n, 3))$ where $\dim_K A_2 > \dim_K A_3$. For values of $n \geq 8$, we prove that $\dim_K A_2 \leq \dim_K A_3$, so the multiplication by ℓ in degree two cannot be surjective. We then use the facet structure of $\mathbf{vdw}(n, 3)$ to find an element in the kernel of this map, thus proving that it cannot have maximal rank.

Certain simplicial complexes are also pseudo-manifolds. A pure d -dimensional simplicial complex is a pseudo-manifold if following conditions hold:

- (1) all $d - 1$ dimensional faces of the simplicial complex are contained in at most two facets; and
- (2) given any two facets, F_a, F_b of the simplicial complex, there exists a sequence of facets $F_a = G_0, G_1, G_2, \dots, G_n = F_b$ such that the dimension of $G_i \cap G_{i+1}$ is $d - 1$ for all $0 \leq i \leq n - 1$.

A d -dimensional pseudo-manifold has a boundary if it has at least one $d - 1$ dimensional face that belongs to exactly and only one facet.

Pseudo-manifolds are of interest to us because of a result from Dao-Nair [11] that relates the presence of the Weak Lefschetz Property of

$$\frac{K[x_1, x_2, \dots, x_n]}{I_\Delta + \langle x_1^2, x_2^2, \dots, x_n^2 \rangle}$$

to the pseudo-manifold property of Δ .

The work of Dao-Nair raises the natural question:

QUESTION 1.8. *For what values of n and k is $A(\mathbf{vdw}(n, k))$ a pseudo-manifold? And for the values of n, k such that $A(\mathbf{vdw}(n, k))$ is a pseudo-manifold, which of these pseudo-manifolds have boundaries?*

In Chapter 5, we examine the facet structure of $A(\mathbf{vdw}(n, k))$, and do a case-by-case analysis for smaller values of n and k to deduce the following answer to Question 1.8:

COROLLARY 1.9. *(Corollary 5.6). $\mathbf{vdw}(n, k)$ is a pseudo-manifold if and only if:*

- (1) $k = 1$, and $n = 2$ or 3 , or
- (2) $k = 2$, and $n = 2, 3, 4, 5$, or $n = 6$, or
- (3) $k \geq 3$, and $\frac{n}{2} \leq k < n$.

COROLLARY 1.10. *(Corollary 5.8). $\mathbf{vdw}(n, k)$ is a pseudo-manifold with boundary if and only if:*

- $k = 1$, and $n = 2$, or
- $k = 2$, and $n = 2, 3, 4, 5$, or $n = 6$, or
- $k \geq 3$ and $\frac{n}{2} \leq k < n$.

Both corollaries combined with Theorem 5.9 gives us some partial information for when $A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property in degree k :

COROLLARY 1.11 (Corollary 5.10). *Let $k \geq 3$. Then $A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property in degree k if $\frac{n}{2} \leq k < n$.*

In Chapter 6 we briefly discuss some results about when $A(\mathbf{vdw}(n, k))$ has the Strong Lefschetz Property. Namely we consider two cases, when $k = 1$ and when $k = n - 1$. We then state some conjectures that were formed during the writing of this thesis for both the Strong and Weak Lefschetz Property, as well as possible areas of future research. For example, we conjecture that if $A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property, then so does $A(\mathbf{vdw}(m, k))$ for all values of $m < n$. However, this appears to be difficult to prove, and we have counterexamples showing this is untrue for the Strong Lefschetz Property, which limits some of the tools we are allowed to use to prove this statement. We also conjecture that for all even values of k , $A(\mathbf{vdw}(m, k))$ always has the Weak Lefschetz Property. On the other hand, this is not true for the odd values of k . For all odd values of k , there appears to exist some m such that $A(\mathbf{vdw}(m, k))$ does not have the Weak Lefschetz Property. We conclude with future areas of research, which include studying Artinian rings where we do not mod out by the squares of the variables, but by different powers of the variables. While this increases the complexity of the rings, it may shed more insight on their properties.

Finally, in Chapter 7 we present tables that clearly indicate for what values of n and k does $A(\mathbf{vdw}(n, k))$ have the Weak Lefschetz Property, the Strong Lefschetz Property, or is a pseudo-manifold. The information in these tables was partially generated using Macaulay2, a computer algebra system [18]. We also present the Macaulay2 code that was used to figure out when $A(\mathbf{vdw}(n, k))$ has the Weak or Strong Lefschetz Properties, as well as a package written for the Fields course “Computational Commutative Algebra and Computational Algebraic Geometry” course with Dr. Stillman to do computations regarding $A(\mathbf{vdw}(n, k))$. Some of the code provided in this section and used in this thesis was written by Dr. Lisa Nicklasson and Becky Hooper.

CHAPTER 2

Background

In this chapter, we go over the fundamental concepts required to examine the central question of this thesis: for which values of n and k does $\text{vdw}(n, k)$ have the Weak or Strong Lefschetz properties. Chapter 2.1 recalls the background on simplicial complexes and their properties, as well as essential related constructions. Chapter 2.2 reviews Artinian rings and their properties, whereas Chapter 2.3 introduces the Lefschetz properties. Chapter 2.4 introduces the van der Waerden simplicial complexes, with several examples and known results. Finally, Chapters 2.5 and 2.6 provide some required background on graph theory and linear algebra, respectively. The main sources for Chapter 2.1 are [12], [14], [16], [35], and [45]; and the main sources for Chapter 2.2 are [10], [37], [47], and [48]. The main sources for Chapters 2.3 and 2.4 are [8], [9], [20], [31], [34], [40], and [41] for the Lefschetz Properties; and [23], [27] for the van der Waerden simplicial complexes. Finally, the main sources for Chapters 2.5 and 2.6 are [4], [6], and [42]. Experts may wish to move directly to the next chapter.

1. Simplicial Complexes

In this first section of this chapter, we define simplicial complexes, and examine some of their properties.

DEFINITION 2.1. A **simplicial complex** on a finite set of vertices $X = \{x_1, x_2, \dots, x_n\}$, denoted by Δ , is a collection of subsets of X (so Δ by definition is a subset of the power set of X) such that the following properties hold:

- $\{x_i\} \in \Delta$ for all $x_i \in X$, i.e., all sets of singletons are in Δ , and
- if $S \in \Delta$ and $A \subseteq S$, then $A \in \Delta$, i.e., Δ is closed under subsets.

Note that we often omit the empty set when listing out the elements of Δ . We also note that the first defining trait is omitted by some authors when defining simplicial complexes, but for our purpose we include it.

We call elements of Δ **faces**. The faces that are maximal with respect to containment are called the **facets** of Δ . If F_1, F_2, \dots, F_s are the facets of a Δ , then we write $\Delta = \langle F_1, F_2, \dots, F_s \rangle$, and say Δ is generated by F_1, F_2, \dots, F_s .

The **dimension** of a particular face $F \in \Delta$ is its cardinality minus 1. Accordingly, the dimension of the empty set is -1 . The **dimension of Δ** is the maximal dimension of all of its faces.

DEFINITION 2.2. For a simplicial complex Δ , we denote by f_i the number of i -dimensional faces of Δ . Then the **f-vector** of a d -dimensional Δ is defined as the d -tuple $(f_0, f_1, \dots, f_{d-1})$.

DEFINITION 2.3. A simplicial complex is called **pure** if all its facets have the same dimension.

The following is a technical definition describing certain faces of pure simplicial complexes.

DEFINITION 2.4. The **ridge** of a pure, d -dimensional simplicial complex Δ is any face of dimension $d - 1$.

Though this definition is used in some literature, we will simply call the ridges of a d dimensional pure simplicial complex as $d - 1$ dimensional faces.

EXAMPLE 2.5. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Then

$$\Delta = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_6\}, \{x_1, x_3, x_5\}, \{x_2, x_4, x_6\}, \{x_1, x_3\}, \{x_1, x_5\}, \{x_3, x_5\}, \{x_2, x_4\}, \{x_2, x_6\}, \{x_4, x_6\}\}$$

is a valid simplicial complex. Its f-vector is $(6, 6, 2)$, so the dimension of Δ is 2.

EXAMPLE 2.6. Simplicial complexes also arise in graph theory. For more on graph theory, see later in this chapter. For example, we denote by K_n the **complete graph** on n vertices, which is the graph where each of the n vertices are connected to every other vertex by an edge. When a complete graph is a subgraph of a bigger graph, it is also referred to as a **clique**. Given a finite, simple graph G on the set of vertices $V_G = \{x_1, x_2, \dots, x_n\}$, the **induced graph** of G on $S \subseteq V_G$ is the subgraph of G consisting of all the vertices in S and the edges of G associated to these vertices. This subgraph is denoted G_S . We are then able to construct the **clique complex** of G :

$$\Delta(G) = \{S \subseteq V_G \mid G_S \text{ is a clique}\}$$

The set $\Delta(G)$ is a simplicial complex.

Now that we have the core definition of simplicial complexes, we can examine key properties and some related constructions. We begin by constructing the entities needed to define three main classes of simplicial complexes: shellable simplicial complexes, vertex decomposable simplicial complexes, and Cohen Macaulay simplicial complexes.

DEFINITION 2.7. A pure simplicial complex is called **shellable** if we can order its facets $F = \{F_1, F_2, \dots, F_n\}$ such that for all $1 \leq j < i \leq n$, there exists a $v \in F_i \setminus F_j$ and $k \in \{1, \dots, i - 1\}$ such that $F_i \setminus F_k = \{v\}$.

More intuitively, we call a simplicial complex shellable if its facets can be joined together in a nice, geometric way. In particular, if the facets can be ordered and then adjoined together in such a way that the connections, or the intersections, occur along a face whose dimension is $(\dim \Delta) - 1$, then the simplicial complex is shellable.

DEFINITION 2.8. Given a simplicial complex Δ on x_1, \dots, x_n , the **deletion** of vertex x_i is defined to be:

$$\text{del}_\Delta(x_i) = \{F \in \Delta \mid x_i \notin F\}.$$

A vertex x_i is a **shedding vertex** if

$$\{F_i \mid F_i \text{ is a facet of } \text{del}_\Delta(x_i)\} \subseteq \{F_i \mid F_i \text{ is a facet of } \Delta\}.$$

DEFINITION 2.9. Given a simplicial complex Δ on x_1, \dots, x_n , the **link** of vertex x_i is defined to be:

$$\text{link}_\Delta(x_i) = \{F \in \Delta \mid x_i \notin F \text{ and } F \cup \{x_i\} \in \Delta\}.$$

DEFINITION 2.10. A pure simplicial complex Δ is **vertex decomposable** if either of the following hold:

- $\Delta = \{\}$, or $\Delta = \langle F \rangle$ for some $F \in \Delta$, or
- $\text{link}_\Delta(x_i)$, and $\text{del}_\Delta(x_i)$ are vertex decomposable for some shedding vertex x .

Note that vertex decomposability is defined recursively.

We now introduce some relevant abstract algebra. The ring we are working in is $R = K[x_1, x_2, \dots, x_n]$. Throughout this paper, we assume K is infinite and has characteristic 0.

DEFINITION 2.11. The **Stanley Reisner ideal** of a simplicial complex Δ , denoted I_Δ , is the ideal generated by the non-faces of Δ . More formally,

$$I_\Delta = \langle x_{i_1} \cdot x_{i_2} \cdots x_{i_t} \mid \{x_{i_1}, x_{i_2}, \dots, x_{i_t}\} \notin \Delta \rangle.$$

Note that by construction, this ideal is a square-free monomial ideal.

DEFINITION 2.12. The **Stanley-Reisner ring** of a simplicial complex Δ on $X = \{x_1, x_2, \dots, x_n\}$, is the quotient ring $K[x_1, x_2, \dots, x_n]/I_\Delta = R/I_\Delta$.

The Stanley Reisner ring is an important concept in combinatorial commutative algebra due to the one-to-one correspondence between algebraic objects such as square free monomial ideals and the combinatorial objects associated to them, such as simplicial complexes. Before we examine some more properties of Stanley Reisner rings and similar constructions, we review some required algebra and geometry.

DEFINITION 2.13. If F_1, F_2, \dots, F_p are elements in R , they form a **regular sequence** on $S = R/I$ if

- $F_1 + I$ is not a zero divisor of S (this is also known as F_1 being regular), and
- $F_i + I$ is not a zero divisor on $J = R/(I, F_1, \dots, F_{i-1})$, for $i = 2, \dots, p$.

DEFINITION 2.14. The **depth** of $S = R/I$ is the maximal length of all regular sequences $F_1, \dots, F_p \in (x_1, \dots, x_n) \subseteq R$.

DEFINITION 2.15. The **Krull dimension** of a ring R is the maximal length of all chains of prime ideals in R .

DEFINITION 2.16. A pure simplicial complex Δ is called **Cohen Macaulay** if the depth of R/I_Δ is equal to the dimension of R/I_Δ .

We have now defined the three main classes of simplicial complexes we will examine in this thesis. We will see in later sections that for the specific simplicial complexes studied in this thesis, the presence of either of the three properties implies that all three hold. In a general setting, we have the following result connecting the three classes of simplicial complexes we have just discussed:

THEOREM 2.17. *Let Δ be a pure simplicial complex. Then the following implications hold:*

$$\text{vertex decomposable} \rightarrow \text{shellable} \rightarrow \text{Cohen Macaulay}.$$

2. Graded and Artinian Rings

We now examine Artinian rings, and their main properties. While a Stanley-Reisner ring R/I_Δ is never Artinian except in the case that $\Delta = \{\}$, we show how to associate to a simplicial complex Δ an Artinian Ring.

DEFINITION 2.18. A commutative ring is a **Noetherian ring** if every increasing sequence of ideals $I_1 \subseteq I_2 \subseteq \dots$ has a maximal element.

Note that we make an assumption that all of our rings are commutative.

DEFINITION 2.19. A ring is **local** if it has a unique maximal left (or right) ideal.

REMARK 2.20. *An equivalent definition states a ring is Noetherian if every ideal is finitely generated.*

DEFINITION 2.21. A ring R is said to be **graded** if it is a direct sum of abelian groups, $R = \bigoplus_{i \in G} R_i$, such that $R_i R_j \subseteq R_{i+j}$ for all $i, j \in G$.

DEFINITION 2.22. An element $f \in R$ is **homogeneous** if $f \in R_i$ for some i . An ideal I of a graded ring R is a **homogeneous ideal** if it is generated by homogeneous elements.

A polynomial ring, $R = K[x_1, x_2, \dots, x_n]$ is naturally graded. We can consider each graded component to be the vector space spanned by the monomials of degree i . So

$$R_d = \{f \in R \mid \text{every term of } f \text{ has degree } d\}.$$

Also note that by construction, the Stanley-Reisner ideal I_Δ is generated by monomials, making the ideal I_Δ a homogeneous ideal.

We now define a key concept:

DEFINITION 2.23. The **radical** of an ideal I in R , denoted \sqrt{I} is:

$$\sqrt{I} = \{r \in R \mid r^n \in I \text{ for some positive integer } n\}$$

DEFINITION 2.24. An **Artinian ring** is a ring whose Krull dimension is 0.

Note that this is one of many equivalent definitions of Artinian rings, though its the one that best fits our context. Another way to think of Artinian rings is as graded rings where there is a finite number of non-zero graded components.

LEMMA 2.25. *A ring $R=K[x_1, \dots, x_n]/I$ is Artinian if and only if $\sqrt{I} = (x_1, x_2, \dots, x_n)$, where the ideal on the right is the maximal homogeneous ideal.*

In our case, when considering Stanley-Reisner Ideals I_Δ , we see that its radical is equal to itself and not the maximal homogeneous ideal, making the Stanley Reisner ring of our simplicial complex not Artinian. However, we can adjoin to our ideal powers of our variables. Each power can be different provided that they all are greater than 2, as powers of 1 change I_Δ .

DEFINITION 2.26. Given a simplicial complex Δ , the ring $R/(I_\Delta, x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n})$, is a Artinian ring and is denoted by

$$A = A(\Delta, a_1, a_2, \dots, a_n).$$

For a fixed simplicial complex Δ ,

$$A(\Delta, a_1, a_2, \dots, a_n) \cong K[x_1, \dots, x_n]/(I_\Delta, x_1^{a_1}, \dots, x_n^{a_n}).$$

This is a graded Artinian ring, so there exists some e such that

$$A = A_0 \oplus A_1 \oplus \dots \oplus A_e,$$

where A_i consists of all the elements of degree i of A . Note that $A_{e+1} = A_{e+2} = \dots = 0$ because $(I_\Delta, x_1^{a_1}, \dots, x_n^{a_n})_{e+1} = (x_1, \dots, x_n)_{e+1}$ for some $e \gg 0$.

When we examine the Artinian rings related to the van der Waerden simplicial complexes in the following sections will consider the case where $a_i = 2$ for all i .

When $a_1 = \dots a_n = 2$, then a basis for A_i is

$$\{x_{i_1} \cdots x_{i_j} + J \mid \{x_{i_1}, \dots, x_{i_j}\} \text{ is a face of dim } i \text{ of } \Delta\},$$

where $J = I_\Delta + (x_1^2, \dots, x_n^2)$.

EXAMPLE 2.27. Let $\Delta = \{x_1, x_2, x_3, x_1x_2, x_2x_3\}$. Then

$$A(\Delta, 2, 2, 2) \cong K[x_1, x_2, x_3]/(I_\Delta, x_1^2, x_2^2, x_3^2) = K[x_1, x_2, x_3]/(x_1x_3, x_1^2, x_2^2, x_3^2).$$

Then we get the following basis for the graded components:

$$\begin{aligned} A_0 &= \langle 1 \rangle, \\ A_1 &= \langle x_1, x_2, x_3 \rangle, \\ A_2 &= \langle x_1x_2, x_2x_3 \rangle, \\ A_i &= 0 \text{ for all } i \geq 3. \end{aligned}$$

Although the notion of socle is not essential in our study, the socle vector of the van der Waerden simplicial complexes are nicely defined and easy to compute. We thus introduce the notion and related constructions here.

DEFINITION 2.28. The **socle** of an Artinian ring A , denoted $\text{soc}(A)$, is a homogeneous ideal that is the annihilator of A 's maximal homogeneous ideal (which we have previously defined as (x_1, x_2, \dots, x_n)). More formally (keeping in mind that we are working with elements in quotient rings),

$$\text{soc}(A) = \{a \in A \mid ax_i = 0 \text{ for all } i\}.$$

We can also define the socle vector of A , denoted $s(A)$, as the following:

$$s(A) = (s_0, s_1, \dots, s_k)$$

where each component s_i is the $\dim_K \text{soc}(A)_i$. The socle vector essentially tells us how many linearly independent socle elements exist in each degree

DEFINITION 2.29. A **top socle** is any monomial in the Artinian ring with highest degree.

DEFINITION 2.30. The **top socle degree** is the degree of any top socle monomial.

DEFINITION 2.31. The **type** of a socle vector $s(A) = (s_0, s_1, \dots, s_k)$ is $\sum_{i=0}^k s_i$.

We can define two new classes of algebras using the socle vector:

DEFINITION 2.32. If the socle vector of a graded algebra is $s(A) = (0, 0, 0, \dots, s_k = t)$, then the ring is said to be **level** of type t . If a level algebra is of type 1, it is said to be **Gorenstein**.

A result by Van Tuyl and Zanello in [46] significantly reduces the complexity of computing the socle vector for the Artinian Ring $A(\Delta, a_1, a_2, \dots, a_n)$. This result changes the computation of the socle vector from a problem dealing with the annihilators of ideals to one of counting over facets. The result is stated below (without proof):

THEOREM 2.33. *Let Δ be a simplicial complex and $F(\Delta)$ be the set of its facets. Then the Artinian ring $R/(I_\Delta, x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n})$ with $a_i \geq 2$ for all i has the following socle vector:*

$$s(A) = (s_0, s_1, \dots, s_k), \text{ where} \\ s_j = \# \{F \in F(\Delta) \mid \sum_{x_i \in F} (a_i - 1) = j\}.$$

For the Artinian ring in focus for this thesis, we get the following corollary:

COROLLARY 2.34. *If $A = A(\Delta, 2, 2, \dots, 2)$, the statement for s_j becomes:*

$$\begin{aligned} s_j &= \# \{F \in F(\Delta) \mid \sum_{x_i \in F} (a_i - 1) = j\} \\ &= \# \{F \in F(\Delta) \mid \sum_{x_i \in F} (2 - 1) = j\} \\ &= \# \{F \in F(\Delta) \mid \sum_{x_i \in F} 1 = j\} \\ &= \# \{F \in F(\Delta) \text{ with } j \text{ elements}\}. \end{aligned}$$

So if our simplicial complex is pure (and all facets are of the same size), then the socle vector of its corresponding Artinian ring has all zeros except for one particular s_j . We will return to this idea once we have defined the van der Waerden simplicial complexes.

Another key characteristic of graded rings is their Hilbert function, which we now define.

DEFINITION 2.35. The **Hilbert function** of a graded ring $R = \bigoplus_{i \in \mathbb{N}} R_i$ is the function that maps a natural number to i to $\dim_K R_i$.

DEFINITION 2.36. The **Hilbert Series** of a simplicial complex is the power series in the variable t where the coefficient of t^a is the number a was mapped to by the Hilbert function, i.e the coefficient is $\dim_K R_a$.

When we are working with simplicial complexes, the definition of the Hilbert Series simplifies using f -vectors using a result from [45] as follows:

THEOREM 2.37. For a simplicial complex Δ with f -vector $(f_0, f_1, \dots, f_{d-1})$, if we define $f_{-1} = 1$, then the Hilbert Series is of the form

$$HS(R/I_\Delta, t) = \sum_{i=-1}^{d-1} \frac{f_i t^{i+1}}{(1-t)^{i+1}}.$$

3. Lefschetz Properties

The crux of this project will be examining which van der Waerden simplicial complexes have the Weak Lefschetz Property, and which ones have the Strong Lefschetz Property. We define the properties of interest here, but the set-up remains the same as the last section. We have an Artinian ring $A = R/(I_\Delta, x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n})$ with $a_i \geq 2$ for all i , where R is the polynomial ring on x_1, \dots, x_n over a field of characteristic 0. Let ℓ be a general linear form.

DEFINITION 2.38. The graded Artinian ring A has the **Weak Lefschetz Property (WLP)** if the homomorphism created by multiplication by ℓ :

$$\times \ell : A_i \rightarrow A_{i+1}$$

is either injective or surjective (i.e., has maximal rank) for all i .

DEFINITION 2.39. The graded Artinian ring A has the **Strong Lefschetz Property (SLP)** if the homomorphism created by multiplication by ℓ^d :

$$\times \ell^d : A_i \rightarrow A_{i+d}$$

is either injective or surjective (i.e., has maximal rank) for all values of $i \geq 0$ and $d \geq 0$.

Note that this implies if $\dim_k A_i > \dim_k A_{i+1}$, then the map $\times \ell$ cannot be injective, so for it to have maximal rank it must be surjective. Similarly, if $\dim_k A_i < \dim_k A_{i+1}$, then the map $\times \ell$ cannot be surjective so for it to have maximal rank it must be injective.

DEFINITION 2.40. The **Non-Lefschetz Locus** of an Artinian Ring $A(\Delta)$, denoted $\mathcal{L}_{\mathcal{R}/\mathcal{I}}$, is the set of all linear forms ℓ such that multiplication by ℓ does not have maximal rank.

This definition is also known as the Non-Weak Lefschetz Locus. The Non-Strong Lefschetz Locus is defined analogously.

In [21], a complete characterization of what Hilbert Functions are possible for K -algebras with the Weak Lefschetz Property is given. We state one important instance below, which is also studied closely in [36]:

THEOREM 2.41. *If the Artinian algebra R/I has the Weak Lefschetz Property (or even the Strong Lefschetz Property), then its Hilbert Function must be unimodal (increases or stays constant up to a certain maximal point, and then decreases or stays constant).*

This project also uses the following result from [32] for computations:

THEOREM 2.42. *An Artinian ring defined on $R = K[x_1, \dots, x_n]/I$, for a monomial ideal I and infinite field K , has the Weak Lefschetz Property if and only if multiplication by the standard linear form*

$$\ell = x_1 + x_2 + \cdots + x_n$$

has maximal rank.

This theorem greatly simplifies detecting when a ring has the Weak Lefschetz Property, as generally one would need to prove that multiplication by an arbitrary general linear form has maximal rank, but this lemma reduces the problem to checking if multiplication by one, standard linear form is enough. Due to the essential nature of this result, we present a proof for this theorem from [32].

PROOF. We define $A = \frac{K[x_1, \dots, x_n]}{I}$. If A has the Weak Lefschetz Property, then multiplication by ℓ , a general linear form, has maximal rank by definition. Thus one direction of the proof is straightforward.

For the opposite direction, let M be the set of minimal generators of I . We know such a set exists because all monomial ideals have a unique minimal set of generators [22].

Let $L = a_1x_1 + a_2x_2 + \cdots + a_nx_s$ be a general linear form with $a_i \in K$. We begin by assuming that a_i is non-zero for all i .

Thus, we can rescale all of our coefficients so that the final one, a_s , is equal to 1. We can then create a dependency relation where x_s is written as a linear combination of the other $a_i x_i$, for $1 \leq i \leq s-1$.

We now define a new polynomial ring $S = K[x_1, \dots, x_{s-1}]$. In S , we construct an ideal J generated by the generators of M altered so that x_s is rewritten using the other variables and the a_i 's based on the dependency relation constructed above. Note that this construction now implies $A/LA \cong S/J$. The minimal generators of J have the form

$x_1^{j_1} \dots x_{s-1}^{j_{s-1}} (a_1 x_1 + a_2 x_2 + \dots + a_{s-1} x_{s-1})^{j_s}$. Since all of the a_i were chosen to be non-zero, altering these generators to the form $a_1 x_1^{j_1} \dots (a_{s-1} x_{s-1})^{j_{s-1}} (-a_1 x_1 - a_2 x_2 + \dots - a_{s-1} x_{s-1})^{j_s}$ does not alter the idea J . Again by construction, our ring S is isomorphic to the ring $K[y_1, y_2, \dots, y_{s-1}]$ by the isomorphism that maps each y_i to $a_i x_i$. A/LA and $A/(x_1 + x_2 + \dots + x_s)$ both have the same Hilbert function. But by the remarks regarding the Hilbert function of algebras with the Weak Lefschetz Property above, the Hilbert function of A/LA indicates whether L is a Lefschetz element of A , so $x_1 + x_2 + \dots + x_n$ is a Lefschetz element in both rings, completing the proof. \square

Note that K has to be an infinite field for this result to hold. This is an assumption we will make throughout this thesis: our field K is infinite and of characteristic 0. Note that for the computations shown in the next section, the Weak Lefschetz Property of the the van der Waerden simplicial complexes were computed using Theorem 2.42 on Macaulay2.

LEMMA 2.43. *Multiplication by the standard linear form $\times \ell = \times(x_1 + x_2 + \dots + x_n) : A_i \rightarrow A_{i+1}$ is a linear transformation of vector spaces.*

PROOF. Let a_{i1}, a_{i2} be two arbitrary elements in A_i , and r_1, r_2 be two arbitrary constants in K . Then $(r_1 a_{i1} + r_2 a_{i2}) \times \ell = (r_1 a_{i1} + r_2 a_{i2}) \times (x_1 + x_2 + \dots + x_n) = r_1 a_{i1} \times (x_1 + x_2 + \dots + x_n) + r_2 a_{i2} \times (x_1 + x_2 + \dots + x_n) = r_1 (a_{i1} \times \ell) + r_2 (a_{i2} \times \ell)$, as wanted. \square

We worked out the details shown above so that we can use the matrix representation of this map to deduce its properties.

We now state some key properties of the Lefschetz Properties. While we do not use all of them in this paper, they provide some insight into why these properties are studied carefully.

THEOREM 2.44. [48] *Taking the tensor product of two algebras that have the Strong Lefschetz Property also has the Strong Lefschetz Property. More formally, if (R_1, ℓ_1) and (R_2, ℓ_2) have the Strong Lefschetz Property, where ℓ_i are general linear forms, then*

$$(R_1 \otimes R_2, \ell_1 \otimes 1 + 1 \otimes \ell_2)$$

also has the Strong Lefschetz Property.

This result enables us to construct algebras with the Strong Lefschetz Property from algebras that are known to have the Strong Lefschetz Property.

We now state a theorem from [33] that gives conditions on when the map $\times \ell$ has maximal rank in certain degrees.

THEOREM 2.45. *Consider a graded Artinian ring $A = R/I$ with graded components A_i , and the map $\times \ell : A_i \rightarrow A_{i+1}$ for some linear form ℓ . Denote by $\times \ell_d$ the homomorphism obtained by $\times \ell : A_d \rightarrow A_{d+1}$ for some $d \geq 0$.*

- *If $\times \ell_d$ is surjective for some \tilde{d} , then $\times \ell_d$ is surjective for all $d \geq \tilde{d}$.*

- If $\times \ell_d$ is injective for some \tilde{d} and A is level, then $\times \ell_d$ is injective for all $d \leq \tilde{d}$.
- If $\dim_K A_d = \dim_K A_{d+1}$ for some d and A is level, then A has the Weak Lefschetz Property if and only if $\times \ell_d$ is injective (or equivalently if $\times \ell_d$ is an isomorphism).

4. The van der Waerden Simplicial Complex

We are now ready to introduce the main object of this project, the van der Waerden simplicial complexes. We also examine several constructions and properties.

DEFINITION 2.46. The **van der Waerden simplicial complexes** is a family of simplicial complexes defined on two parameters n (the number of vertices) and k (the dimension). For $n > k$, the van der Waerden simplicial complex over (n, k) , denoted $\text{vdw}(n, k)$, is generated by facets that correspond to all valid arithmetic progressions $\{x_i, x_{i+d}, \dots, x_{i+kd}\}$ where $0 < i < i + kd \leq n$.

One might find it easier to interpret the value of k as the number of "jumps", so the facets of $\text{vdw}(n, k)$ are of size $k + 1$.

Note that we consider $\{x_i, x_{i+d}, \dots, x_{i+kd}\}$ instead of $\{i, i + d, \dots, i + kd\}$, as we are working in the polynomial ring $R = K[x_1, x_2, \dots, x_n]$.

EXAMPLE 2.47. Consider $n = 5, k = 3$, and $\text{vdw}(5, 3)$. Facets derived from arithmetic progressions with $d = 1$ are $F_1 = \{x_1, x_2, x_3, x_4\}$, and $F_2 = \{x_2, x_3, x_4, x_5\}$. There can be no facets derived from arithmetic progressions with $d = 2$, as even starting with the smallest index 1 results in the facet $= \{x_1, x_3, x_5, x_7\}$, and $7 \geq 5$. Thus $\text{vdw}(5, 3)$ is the simplicial complex generated by facets F_1 and F_2 .

Although the family of van der Waerden simplicial complexes is algebraic in nature, the inspiration and construction came from different fields, such as number theory and graph theory. The construction is strongly related to van der Waerden's Theorem in graph theory (particularly Ramsey Theory), which can be found in [19] and [44] and is stated below for the reader's interest:

THEOREM 2.48. *Let r and k be positive integers. Then there exists some positive integer N such that if the set $\{1, 2, \dots, N\}$ is coloured (with each integer coloured with one colour out of r **different** options), then the arithmetic progression of elements of the same colour has at least k integer elements. We denote by $W(r, k)$ the smallest N for which this is true.*

We now define some useful characteristics of the van der Waerden simplicial complexes.

DEFINITION 2.49. Given a facet of $\text{vdw}(n, k)$, say $F = \{x_i, x_{i+d}, \dots, x_{i+kd}\}$, we define d to be the **jump factor** of F , and denote it $\text{jump}(F)$.

EXAMPLE 2.50. For $\text{vdw}(7, 2)$, the facet $\{x_1, x_3, x_5\}$, has jump factor 2.

DEFINITION 2.51. For a valid facet of $\text{vdw}(n, k)$, vertices of the form x_a for odd a are **odd entries**, and vertices of the form x_b for even b are **even entries**.

EXAMPLE 2.52. For $\text{vdw}(7, 2)$, the facet $\{x_1, x_2, x_3\}$, the vertices x_1 and x_3 are odd entries, whereas x_2 is an even entry vertex.

Let us now consider a couple of examples of van der Waerden simplicial complexes:

EXAMPLE 2.53. Consider $\text{vdw}(7, 2)$. The facets of this simplicial complex should have $2+1 = 3$ elements. Since the n and k values are relatively small, we can compute the generators of $\text{vdw}(7, 2)$ by considering all possible arithmetic progressions for all different valid jump sizes:

- Progressions of size 1: $\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_3, x_4, x_5\}, \{x_4, x_5, x_6\}, \{x_5, x_6, x_7\}$
- Progressions of size 2: $\{x_1, x_3, x_5\}, \{x_2, x_4, x_6\}, \{x_3, x_5, x_7\}$
- Progressions of size 3: $\{x_1, x_4, x_7\}$.

At this point, we can no longer have any more arithmetic progressions of size 1, 2, or 3, as the index of the last x_j in each set would exceed $n = 7$. Furthermore, we can not have any arithmetic progressions of size 4 (or higher) for the same reason. Thus this list of 9 facets exhaust the list of all generators for $\text{vdw}(7, 2)$.

EXAMPLE 2.54. Consider $\text{vdw}(8, 3)$: The facets of this simplicial complex have $3+1 = 4$ elements. Since the n and k values are relatively small, we can compute the facets of $\text{vdw}(8, 3)$ by considering all possible arithmetic progressions for all different valid jump sizes:

- Progressions of size 1: $\{x_1, x_2, x_3, x_4\}, \{x_2, x_3, x_4, x_5\}, \{x_3, x_4, x_5, x_6\}, \{x_4, x_5, x_6, x_7\}, \{x_5, x_6, x_7, x_8\}$
- Progressions of size 2: $\{x_1, x_3, x_5, x_7\}, \{x_2, x_4, x_6, x_8\}$.

This list of seven facets exhaust the list of all facets for $\text{vdw}(8, 3)$.

We now examine several different properties of the van der Waerden simplicial complexes.

PROPOSITION 2.55. *The van der Waerden simplicial complexes are pure for all values of n and k .*

PROOF. This is immediate by the way they are constructed, as $\dim \text{vdw}(n, k) = k$. \square

A result of Hooper and Van Tuyl [27] classifies which van der Waerden simplicial complexes are shellable based on the values of n and k . We present that statement here:

PROPOSITION 2.56. *Let $1 \leq k < n$. Then $\text{vdw}(n, k)$ is shellable if and only if:*

- $n \leq 6$, or
- $n > 6$ and $k = 1$, or

- $n > 6$ and $n/2 \leq k < n$.

A recent result of Hibi and Van Tuyl [23] states that for van der Waerden simplicial complexes, the three main classes of simplicial complexes we discussed earlier all merge into one, and can all be classified using Hooper's findings above:

PROPOSITION 2.57. *Let $1 \leq k < n$. The following are equivalent:*

$\text{vdw}(n, k)$ is shellable $\iff \text{vdw}(n, k)$ is Cohen–Macaulay $\iff \text{vdw}(n, k)$ is vertex decomposable.

Our last note on the properties of the van der Waerden simplicial complexes Δ is regarding the socle vector for the Artinian ring $R/(I_\Delta, x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n})$ with $a_i \geq 2$ for all i . From Theorem 2.33, we know the socle vector for a general Artinian ring of this form is (s_1, s_2, \dots, s_n) where s_i denotes the number of facets in the simplicial complex's facet set with i elements. But $\text{vdw}(n, k)$ is pure for all valid values of n and k , so all facets are of the same size. Furthermore, we see by construction that all the facets have $k + 1$ elements. This gives us the following result:

PROPOSITION 2.58. *The socle vector of $R/(I_{\text{vdw}(n, k)}, x_1^2, x_2^2, \dots, x_n^2)$ is $(0, 0, \dots, s_{k+1})$, where s_{k+1} is the number of facets of $\text{vdw}(n, k)$.*

5. Graph Theory Background

In this section, we recall some key definitions and theorems needed from graph theory for this thesis.

DEFINITION 2.59. A graph G is said to be **bipartite** if its set of vertices can be divided into two disjoint sets, A and B , such that each edge of G has one edge in A and the other edge in B . Alternatively, a graph is bipartite if it does not contain any cycles of odd-length.

We also have the following lemma classifying when the complete graph on n vertices (K_n , previously defined Example 2.6) are bipartite.

LEMMA 2.60. *The graph K_n is bipartite if and only if $n = 1, 2$.*

PROOF. K_1 and K_2 are both bipartite, the former trivially as it is only one vertex with no edges, and the latter as it is two vertices that we can place into separate sets, and the sole edge between them. K_n for $n \geq 3$ is not bipartite, as picking any three arbitrary vertices are connected, and so a triangle (odd-cycle) is present in all such graphs. \square

We now present some more concepts of basic graph theory used in this thesis.

DEFINITION 2.61. Given a graph, a **walk** is a sequence of edges (e_1, e_2, \dots, e_s) such that $e_i \cap e_{i+1} \neq \emptyset$ for all i . A walk where the all edges and vertices are distinct is referred to as a **path**. A path that starts and ends at the same vertex is called a **cycle**.

DEFINITION 2.62. Two vertices of a graph are **connected** if there exists an edge between them. A graph is **connected** if there exists a path between any two vertices.

DEFINITION 2.63. A **component** of a graph G is a connected sub-graph of G that is not part of any larger connected sub-graph.

REMARK 2.64. K_n is a connected graph, as by definition each vertex is connected to another by an edge, for all values of n . Thus, K_n has only one connected component, namely itself.

DEFINITION 2.65. A **tree** is an undirected, connected graph with no cycles

We now present a definition that bridges concepts between graph theory and simplicial complexes.

DEFINITION 2.66. The **dual graph** of a pure, d -dimensional simplicial complex Δ is the graph created by placing a vertex for each facet of Δ , and placing an edge between two vertices if and only if the facets represented by them share a ridge.

6. Linear Algebra Background

In this final section, we review some fundamental concepts of linear algebra that we will be using throughout this thesis. While these concepts may be elementary, they are essential for the simplification of the central problem of this thesis into a study of maximal rank. All results from this section can be found in [4] and [42]. We begin with the concept of matrix representations of linear transformations.

THEOREM 2.67. *Let N and M be two vector spaces, of dimensions n and m respectively. Let $A = \{a_1, a_2, \dots, a_n\}$ be a basis of N , and $B = \{b_1, b_2, \dots, b_m\}$ be a basis of M . If T is a linear transformation $T : N \rightarrow M$, then there is a unique $m \times n$ matrix representation of the linear transformation, denoted by $T_{B \leftarrow A}$, constructed by letting the i -th column be $T(a_i)$, written as a linear combination of the basis elements in B . This matrix has the property that for all $n^* \in N$, $T_{B \leftarrow A} \cdot [n^*]_A = [T(n^*)]_B$, where $[n^*]_A$ and $[T(n^*)]_B$ are the respective coordinate vectors.*

EXAMPLE 2.68. Consider the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(x, y) = (x + y, 2x - 3y)$. Using the basis $b_1 = (1, 0)$ and $b_2 = (0, 1)$ of \mathbb{R}^2 , we see $T(b_1) = (1, 2)$ and $T(b_2) = (1, -3)$. So the associated matrix would be

$$\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}$$

This key theorem allows us to alternate between a linear transformation and its equivalent matrix representation to see which properties, such as injectivity, surjectivity, and maximal rank, are present.

We now review some core definitions and properties of matrices:

DEFINITION 2.69. Let v_1, v_2, \dots, v_n be a set of n vectors in a vector space V over a field F . The set of all possible linear combinations of these vectors using coefficients in F is called the **span** of these vectors, that is,

$$\text{span}(v_1, v_2, \dots, v_n) = \{a_1v_1 + a_2v_2 + \dots + a_nv_n \mid a_i \in F \text{ for all } i\}.$$

DEFINITION 2.70. The **column rank** of a $m \times n$ matrix is the dimension of the span of its columns. The **row rank** of a $m \times n$ matrix is the dimension of the span of its rows.

We now present several classic theorems and corollaries about the rank of a matrix.

THEOREM 2.71. *For a $m \times n$ matrix A , its row rank is equal to its column rank.*

THEOREM 2.72. *A matrix and its transpose matrix both have the same rank.*

COROLLARY 2.73. *The rank of a $m \times n$ matrix A must be less than or equal to $\min(m, n)$*

THEOREM 2.74. *Suppose T is a linear transformation $T : N \rightarrow M$, where N and M have dimension n and m respectively. Suppose $M(T)_{m \times n}$ is the matrix representation of T . Then:*

- (1) *T is injective if and only if the rank of $M(T)_{m \times n}$ is n (the matrix has maximal column rank)*
- (2) *T is surjective if and only if the rank of $M(T)_{m \times n}$ is m (the matrix has maximal row rank).*

We end this chapter by recalling the Fundamental Theorem of Linear Maps:

THEOREM 2.75. (**The Fundamental Theorem of Linear Maps**) *If V is a finite-dimensional vector space, and $T : V \rightarrow W$ is a linear transformation, then the range of T is a finite dimensional space such that $\dim V = \dim \text{null } T + \dim \text{range } T$.*

CHAPTER 3

Families of $A(\text{vdw}(n, k))$ that have the Weak Lefschetz Property

In this chapter, we identify some cases where the Weak Lefschetz Property holds for $A(\text{vdw}(n, k))$. We show that for the smallest possible values of k , namely 1 and 2, $A(\text{vdw}(n, k))$ always has the Weak Lefschetz Property. We use ideas from linear algebra, as well as some results from Dao-Nair [11] to prove this. We then use a classic result of Stanley [43] to prove that for the largest possible value of k , namely $k = n - 1$, $A(\text{vdw}(n, n - 1))$ also always has the Weak Lefschetz Property. We conclude the chapter by showing that all $A(\text{vdw}(n, k))$ have the Weak Lefschetz Property in degree 1, the smallest possible non-trivial degree.

1. $\text{vdw}(n, 1)$ and $\text{vdw}(n, 2)$

In this section, we consider the van der Waerden simplicial complexes $\text{vdw}(n, k)$ where the value of k is one or two. We begin by re-establishing some notation and defining some key terms required to use a result from [11].

NOTATION 3.1. *Given a simplicial complex Δ , we denote by $A(\Delta)$ the Artinian ring constructed using the Stanley-Reisner ideal of Δ , as well as modding out by all the squares of the variables. That is,*

$$A(\Delta) = \frac{R}{I_{\Delta} + (x_1^2, x_2^2, \dots, x_n^2)}.$$

DEFINITION 3.2. The **1-skeleton** of a simplicial complex Δ is a graph created by using the same vertex set of Δ , but setting the edge set to be the one-dimensional faces of Δ .

We now state, without proof, a key theorem we will use multiple times in this chapter; the detailed proof can be found in [11].

THEOREM 3.3. *Let Δ be a simplicial complex, and let $G(\Delta)$ be its 1-skeleton.*

- (1) *If $\dim_k(A_2) \geq \dim_k(A_1)$, then $A(\Delta)$ has the Weak Lefschetz Property in degree 1 if and only if $G(\Delta)$ has no bipartite components.*
- (2) *If $\dim_k(A_2) < \dim_k(A_1)$, then $A(\Delta)$ has the Weak Lefschetz Property in degree 1 if and only if every bipartite component of $G(\Delta)$ is a tree (contains no cycles), and every non-bipartite component of $G(\Delta)$ has the same number of edges as vertices. If this condition holds, then $A(\Delta)$ has the Weak Lefschetz Property in all degrees.*

This theorem allows us to prove the following result:

THEOREM 3.4. $A(\text{vdw}(n, 1))$ has the Weak Lefschetz Property for all integers $n \geq 2$.

PROOF. Let us first consider the dimension of the different A_i , our graded components of $A = A(\Delta)$ when $\Delta = \text{vdw}(n, k)$. We have $\dim_k A_0 = 1$, as the degree zero elements of the Artinian ring is the set of constants. We also have $\dim_k A_1 = n$ as it is the number of vertices, the one-dimensional elements in the ring. For $n \geq 2$, the set of all facets of $\text{vdw}(n, 1)$ consists of all possible pairs of vertices $\{x_1, x_2, \dots, x_n\}$. Thus, $\dim_k A_2 = \binom{n}{2} = \frac{n(n-1)}{2}$, the number of all possible edges using n vertices. To apply Theorem 3.3, we need to see which case of the theorem holds. We compute the following:

$$\begin{aligned} \frac{\binom{n}{2}(n-1)}{2} &\geq n \\ \text{which implies } \binom{n}{2}(n-1) &\geq 2n \\ \text{which then implies } (n-1) &\geq 2 \\ \text{or equivalently, } n &\geq 3, \end{aligned}$$

with the progression from the second to third step being due to the fact that n cannot be zero. Note that for the edge case $\text{vdw}(3, 1)$, $\dim_k A_1 = n = 3$, and $\dim_k A_2 = \frac{\binom{3}{2}}{2} = 3$.

So if $n \geq 3$, we are in case 1 of Theorem 3.3, and $A(\Delta)$ has the Weak Lefschetz Property in degree 1 if and only if $G(\Delta)$ has no bipartite components. By construction, $G(\Delta)$ is K_n , as all possible edges between different vertices are valid 1-dimensional faces of Δ . From Theorem 2.60, we know K_n is not bipartite if $n \geq 3$, and from Theorem 2.64 we know K_n has no bipartite components, as it has only one component. So by condition one of Theorem 3.3, $A(\text{vdw}(n, 1))$ has the Weak Lefschetz Property in degree 1 for all values of $n \geq 3$.

For the Weak Lefschetz Property in degree 0, if we pick an arbitrary non-zero constant from A_0 , it is sufficient to check multiplication by the standard linear form

$$\ell = x_1 + x_2 + \dots + x_n$$

has maximal rank (by Theorem 2.42). However, no non-zero constant when multiplied by ℓ yields zero, ensuring that the map is injective and has maximal rank. So $A(\text{vdw}(n, 1))$ also has the Weak Lefschetz Property in degree 0 for all values of $n \geq 3$.

Since $\text{vdw}(n, 1)$ is 1-dimensional and has facets of cardinality 2, $\text{vdw}(n, 1)$ has no faces consisting of three vertices. This implies $\dim_k A_i = 0$ for all $i \geq 3$. So the image of our map $\times \ell : A_2 \rightarrow A_3$ is trivial, and as such, surjective and has maximal rank. Thus $A(\text{vdw}(n, 1))$ has the Weak Lefschetz Property in degree 2 for all values of $n \geq 3$. Similarly, since $\dim_k A_i = 0$ for all $i \geq 3$, the maps in higher degrees are all the zero maps and are surjective and injective trivially, concluding that $A(\text{vdw}(n, 1))$ has the Weak Lefschetz Property for all degree ≥ 3 . Combined with our result directly above, $A(\text{vdw}(n, 1))$ has the Weak Lefschetz Property for all $n \geq 3$.

For $\text{vdw}(2, 1)$, $\dim_k A_1 = n = 2$, and $\dim_k A_2 = \frac{(2)(2-1)}{2} = 1$. So we are in Case 2 of Theorem 3.3. But note $G(\Delta)$ is K_2 , which is just two vertices connected by an edge. This is a bipartite graph that is a tree by construction, and has no non-bipartite components. So by Case 2 of Theorem 3.3, $A(\text{vdw}(2, 1))$ not only has the Weak Lefschetz Property in degree 1, it has the Weak Lefschetz Property in all degrees.

Combining the results above, $A(\text{vdw}(n, 1))$ has the Weak Lefschetz Property in all degrees for all $n \geq 2$. \square

We now present an alternate proof for $A(\text{vdw}(n, 1))$ having the Weak Lefschetz Property in degree 1 using concepts of linear algebra presented in Chapter 1.6, instead of Theorem 3.3. The proof presented above for $A(\text{vdw}(n, 1))$ having the Weak Lefschetz Property in degrees $k = 0$, and $k \geq 2$ still hold, as those were proved by the construction of A_i and not using Theorem 3.3.

PROOF. In this alternate proof, we consider the matrix representation of the homomorphism created by multiplication by ℓ (please see examples of such matrices presented at the end of this proof). When we create this matrix representation, the columns represent the basis elements in our domain A_1 , which are our variables x_1, \dots, x_n . The rows represent the basis elements in our co-domain A_2 , which are all possible products of any two of the variables in A_1 . Our matrix will thus have n columns, and $\binom{n}{2}$ rows. Note that for our rows, any variable can be paired with any other, so each column will have $n - 1$ ones, and the remaining entries will all be zeros. Arranging the rows lexicographically, we see the following patterns:

The first $n - 1$ rows represent the products $x_1x_2, x_1x_3, x_1x_4, \dots, x_1x_n$. The first column of these $n - 1$ rows will all be 1, as x_1 is present by construction in all these products. The remaining columns, which each represent some x_i for $2 \leq i \leq n$, will have a 1 in the row corresponding to x_1x_i , and 0 in all other rows. Since the columns were ordered lexicographically, this results in a $(n - 1) \times (n - 1)$ identity matrix.

We now consider the next $n - 2$ rows. These represent the products $x_2x_3, x_2x_4, \dots, x_2x_n$ (and we note that the row representing x_1x_2 has already been dealt with). All $n - 2$ of these rows have a 0 in column 1, as none of the products involve x_1 , and likewise they all have a 1 in column 2 as all of these products involve x_2 . The remaining $n - 2$ columns, each representing some x_i for some $2 < i \leq n$ has a 1 in the row representing x_2x_i and a 0 in all other rows. Similar to above, the lexicographic ordering of both the rows and columns result in a $(n - 2) \times (n - 2)$ identity matrix block.

The pattern continues for the remaining rows representing some product x_ix_j . The final row, which represents the product $x_{n-1}x_n$ will have a 0 in the first $n - 2$ columns, and a 1 in the last two columns representing x_{n-1} and x_n . In total, we get the following pattern for the entire matrix:

- The first $n - 1$ rows have a 1 in their first column, and is then followed by a $(n - 1) \times (n - 1)$ block identity matrix.

- The next $n - 2$ rows have a 0 in their first column, a 1 in their second column, and is then followed by a $(n - 2) \times (n - 2)$ block identity matrix.
- The next $n - 3$ rows have a 0 in their first and second columns, a 1 in their third column, and is then followed by a $(n - 3) \times (n - 3)$ block identity matrix,
- ...
- The third-last and second-last columns have a 0 in the first $n - 3$ columns, all 1s in the third-last column, followed by a 2×2 identity block.
- And finally, the last column has all 0s, except for the last two columns, which are both 1s.

For a matrix of size $\binom{n}{2} \times n$ to have maximal rank, it must have n linearly independent rows or columns. Consider the first $n - 1$ rows. These rows have a 1 in the first column followed by a $n - 1 \times n - 1$ identity matrix (which has $n - 1$ linearly independent rows). The 1 in the first column does not affect the linear independence of these rows due to the placement of the 0's in the identity block, so the first $n - 1$ rows are linearly independent. We claim that the set of rows obtained by including the last row to the first $n - 1$ rows is also linearly independent. We denote the last row by R_n and the first $n - 1$ rows, by R_1, R_2, \dots, R_{n-1} . For these rows to be linearly independent, then the equation

$$c_1 R_1 + c_2 R_2 + \dots + c_{n-1} R_{n-1} + c_n R_n = 0$$

for $c_i \in K$, $1 \leq i \leq n - 1$, must all be 0. We can alternatively rewrite this as

$$c_1 R_1 + c_2 R_2 + \dots + c_{n-1} R_{n-1} = -c_n R_n.$$

Both sides of the equation are $1 \times n$ row vectors. For the statement above to be true, we can equate coordinates so that the following must hold:

$$(c_1 + c_2 + \dots + c_{n-1}, c_1, c_2, \dots, c_{n-2}, c_{n-1}) = (0, 0, \dots, 0, -c_n, -c_n).$$

This tells us:

- (1) $\sum_{i=1}^{n-1} c_i = 0$,
- (2) $c_1 = c_2 = c_3 = \dots = c_{n-3} = 0$, and
- (3) $c_{n-2} = c_{n-1} = -c_n$.

The first two facts combined imply $c_{n-2} + c_{n-1} = 0$. This fact combined with the third fact implies $-c_n + -c_n = 0 \iff -2c_n = 0 \iff -c_n = 0 \iff c_n = 0$. But if $c_n = 0$, then by fact three we have $c_{n-2} = c_{n-1} = 0$. Thus the coefficients $c_i = 0$ for all $1 \leq i \leq n$. So R_1, R_2, \dots, R_n must all be linearly independent, as wanted. This gives us n linearly independent rows, implying the matrix has maximal rank. Thus $A(\text{vdw}(n, 1))$ has the Weak Lefschetz Property in degree 1 for all $n \geq 2$, as wanted. \square

We now present the matrix representation of $\times \ell$ for some small values of n . Note that the following matrices and the map $\times \ell$ from A_1 to A_2 both have the same rank.

For $n = 3$: Note that $\{x_1, x_2, x_3\}$ is a basis for A_1 , and $\{x_1x_2, x_1x_3, x_2x_3\}$ is a basis for A_2 . The matrix for the map $\times \ell : A_1 \rightarrow A_2$ with respect to these basis is:

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \left(\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right) & \begin{array}{l} x_1x_2 \\ x_1x_3 \\ x_2x_3 \end{array} \end{array}$$

For $n = 4$: Note that $\{x_1, x_2, x_3, x_4\}$ is a basis for A_1 , and $\{x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4\}$ is a basis for A_2 . The associated matrix is:

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \left(\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right) & \begin{array}{l} x_1x_2 \\ x_1x_3 \\ x_1x_4 \\ x_2x_3 \\ x_2x_4 \\ x_3x_4 \end{array} \end{array}$$

For $n = 5$: Note that $\{x_1, x_2, x_3, x_4, x_5\}$ is a basis for A_1 , and

$$\{x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_3, x_2x_4, x_2x_5, x_3x_4, x_3x_5, x_4x_5\}$$

is a basis for A_2 . The associated matrix is:

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \left(\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right) & \begin{array}{l} x_1x_2 \\ x_1x_3 \\ x_1x_4 \\ x_1x_5 \\ x_2x_3 \\ x_2x_4 \\ x_2x_5 \\ x_3x_4 \\ x_3x_5 \\ x_4x_5 \end{array} \end{array}$$

For $n = 6$: Note that $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ is a basis for A_1 , and

$$\{x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_2x_3, x_2x_4, x_2x_5, x_2x_6, x_3x_4, x_3x_5, x_3x_6, x_4x_5, x_4x_6, x_5x_6\}$$

is a basis for A_2 . The associated matrix is:

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{matrix} x_1x_2 \\ x_1x_3 \\ x_1x_4 \\ x_1x_5 \\ x_1x_6 \\ x_2x_3 \\ x_2x_4 \\ x_2x_5 \\ x_2x_6 \\ x_3x_4 \\ x_3x_5 \\ x_3x_6 \\ x_4x_5 \\ x_4x_6 \\ x_4x_6 \end{matrix}$$

We have shown above that if $k = 1$, then for all $n \geq 2$, $A(\text{vdw}(n, 1))$ has the Weak Lefschetz Property in all degrees. We now attempt to show the same for the next natural value of k . Before we get to the proof, we state and prove a lemma.

LEMMA 3.5. *If $\{x_i, x_j, x_n\}$ is a facet of $\text{vdw}(n, 2)$ that contains the vertex x_n , then $\{x_i, x_j\}$ is not contained in any other facet containing x_n .*

PROOF. Suppose $\{x_i, x_j\}$ is contained in a facet F of $\text{vdw}(n, 2)$ also containing x_n . So $\{x_i, x_j, x_n\} \subseteq F$. But $\text{vdw}(n, 2)$ is a pure simplicial complex of dimension 2, so $\{x_i, x_j, x_n\} = F$. Thus, if x_i, x_j, x_n are all part of some facet F , then the facet F is unique. \square

We can now state the main theorem for the case $k = 2$.

THEOREM 3.6. *$A(\text{vdw}(n, 2))$ has the Weak Lefschetz Property for all integers $n \geq 3$.*

PROOF. Let $A = A(\text{vdw}(n, 2))$. We prove this theorem in cases by showing the map $\times \ell : A_i \rightarrow A_{i+1}$ has maximal rank for all integers $i \geq 0$.

Case 1: Degree 0, i.e., the map $\times \ell : A_0 \rightarrow A_1$

Similar to $\text{vdw}(n, 2)$, for checking the Weak Lefschetz Property in degree 0, Theorem 2.42 tells us it is sufficient to check multiplication of an arbitrary non-zero constant from A_0 , by the standard linear form $\ell = x_1 + x_2 + \cdots + x_n$ by Theorem 2.42. However, no

non-zero constant when multiplied by ℓ yields zero, ensuring that the map is injective and has maximal rank. So $A(\text{vdw}(n, 2))$ also has the Weak Lefschetz Property in degree 0 for all values of $n \geq 3$.

Case 2: Degree 1, i.e., the map $\times \ell : A_1 \rightarrow A_2$

To use Theorem 3.3 in our proof, we first compute the dimensions of A_1, A_2 , and A_3 for some small values of n , noting that $\dim_k A_0$ is always one.

- (1) $\text{vdw}(3, 2)$ is generated by $\{1, 2, 3\}$, so $\dim_k A_1 = n = 3$, $\dim_k A_2 =$ number of all possible edges we can generate from $\{1, 2, 3\} = 3$, and $\dim_k A_3 =$ number of all possible triples we can generate from $\{1, 2, 3\} = 1$.
- (2) $\text{vdw}(4, 2)$ is generated by $\{1, 2, 3\}$ and $\{2, 3, 4\}$ so $\dim_k A_1 = 4$, $\dim_k A_2 = 5$, and $\dim_k A_3 = 2$.
- (3) $\text{vdw}(5, 2)$ is generated by $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$, and $\{1, 3, 5\}$ so $\dim_k A_1 = 5$, $\dim_k A_2 = 8$, and $\dim_k A_3 = 4$.
- (4) $\text{vdw}(6, 2)$ is generated by $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$, $\{4, 5, 6\}$, $\{1, 3, 5\}$ and $\{2, 4, 6\}$ so $\dim_k A_1 = 6$, $\dim_k A_2 = 11$, and $\dim_k A_3 = 6$.
- (5) $\text{vdw}(7, 2)$ is generated by $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$, $\{4, 5, 6\}$, $\{5, 6, 7\}$, $\{1, 3, 5\}$, $\{2, 4, 6\}$, $\{3, 5, 7\}$ and $\{1, 4, 7\}$ so $\dim_k A_1 = 7$, $\dim_k A_2 = 17$, and $\dim_k A_3 = 9$.
- (6) $\text{vdw}(8, 2)$ is generated by $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$, $\{4, 5, 6\}$, $\{5, 6, 7\}$, $\{6, 7, 8\}$, $\{1, 3, 5\}$, $\{2, 4, 6\}$, $\{3, 5, 7\}$, $\{4, 6, 8\}$, $\{1, 4, 7\}$ and $\{2, 5, 8\}$ so $\dim_k A_1 = 8$, $\dim_k A_2 = 23$, and $\dim_k A_3 = 12$.
- (7) $\text{vdw}(9, 2)$ is generated by $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$, $\{4, 5, 6\}$, $\{5, 6, 7\}$, $\{6, 7, 8\}$, $\{7, 8, 9\}$, $\{1, 3, 5\}$, $\{2, 4, 6\}$, $\{3, 5, 7\}$, $\{4, 6, 8\}$, $\{5, 7, 9\}$, $\{1, 4, 7\}$, $\{2, 5, 8\}$, $\{3, 6, 9\}$ and $\{1, 5, 9\}$ so $\dim_k A_1 = 9$, $\dim_k A_2 = 30$, and $\dim_k A_3 = 16$.
- (8) $\text{vdw}(10, 2)$ is generated by $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$, $\{4, 5, 6\}$, $\{5, 6, 7\}$, $\{6, 7, 8\}$, $\{7, 8, 9\}$, $\{8, 9, 10\}$, $\{1, 3, 5\}$, $\{2, 4, 6\}$, $\{3, 5, 7\}$, $\{4, 6, 8\}$, $\{5, 7, 9\}$, $\{6, 8, 10\}$, $\{1, 4, 7\}$, $\{2, 5, 8\}$, $\{3, 6, 9\}$, $\{4, 7, 10\}$, $\{1, 5, 9\}$ and $\{2, 6, 10\}$ so $\dim_k A_1 = 10$, $\dim_k A_2 = 38$, and $\dim_k A_3 = 20$.

We see the recurring pattern that $\dim_k(A_2) \geq \dim_k(A_1)$; we now prove it rigorously by induction for $\text{vdw}(n, 2)$.

For the base case, we see from the computations above that $\dim_k(A_2) = 3 \geq \dim_k(A_1) = 3$ for $n = 3$, the smallest possible value n can take when $k = 2$. For the induction hypothesis, assume that $\dim_k(A'_2) \geq \dim_k(A'_1)$ for all $n^* < n$, when $A' = \text{vdw}(n^*, 2)$.

As the induction step, we want to show $\dim_k(A_2) \geq \dim_k(A_1)$ for n . From the induction hypothesis we know $\dim_k(A'_2) \geq \dim_k(A'_1)$ for $n - 1$. However, $\dim_k(A_1)$ is always the number of vertices, so for $\text{vdw}(n - 1, 2)$ it is $n - 1$, and for $\text{vdw}(n, 2)$ it is n , implying it increases only by 1. However, when compared to $\text{vdw}(n - 1, 2)$, $\text{vdw}(n, 2)$ has the additional facet $\{n - 2, n - 1, n\}$, and this facet alone contributes $\{n - 2, n\}$ and $\{n - 1, n\}$ to $\dim_k(A_2)$, implying it increases by at least 2 (as we are not counting any other facets that are present in $\text{vdw}(n, 2)$ but are not present in $\text{vdw}(n - 1, 2)$). Therefore,

$\dim_k(A_2) \geq \dim_k(A_1)$ for n as well, as wanted. Thus, we have $\dim_k(A_2) \geq \dim_k(A_1)$ for all $n \geq 3$ for $\text{vdw}(n, 2)$.

In addition, although we do not explicitly construct the 1-skeleton of $\text{vdw}(n, 2)$, we see that $\text{vdw}(n, 2)$ has facets of cardinality 3. So any vertex in the 1-skeleton is part of some arithmetic progression of size 3, and is part of the triangle formed by it. So any component of the 1-skeleton has an odd-cycle (a three-cycle) and is thus not bipartite. Thus, the conditions of the first part of Theorem 3.3 holds, and $\text{vdw}(n, 2)$ has the Weak Lefschetz Property in degree one for all values of $n \geq 3$.

Case 3: Degree 2, i.e., the map $\times \ell : A_2 \rightarrow A_3$

Since $\text{vdw}(n, 2)$ falls into Case 1 of Theorem 3.3 for all values of $n \geq 3$, we cannot use Theorem 3.3 to prove $\text{vdw}(n, 2)$ has the Weak Lefschetz Property in degree 2. To prove this, we first consider the matrix representations of these maps for smaller values of n : For $n = 3$:

$$\begin{pmatrix} x_1x_2 & x_1x_3 & x_2x_3 \\ 1 & 1 & 1 \end{pmatrix} x_1x_2x_3$$

This matrix has rank 1, which is maximal as its rank must be $\leq \min(1, 3)$ by Theorem 2.73.

For $n = 4$

$$\begin{pmatrix} x_1x_2 & x_1x_3 & x_2x_3 & x_2x_4 & x_3x_4 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} x_1x_2x_3 \\ x_2x_3x_4 \end{matrix}$$

This matrix has rank 2, which is maximal as its rank must be $\leq \min(2, 5)$ by Theorem 2.73.

For $n = 5$

$$\begin{pmatrix} x_1x_2 & x_1x_3 & x_1x_5 & x_2x_3 & x_2x_4 & x_3x_4 & x_3x_5 & x_4x_5 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} x_1x_2x_3 \\ x_2x_3x_4 \\ x_1x_3x_5 \\ x_3x_4x_5 \end{matrix}$$

This matrix has rank 4, which is maximal as its rank must be $\leq \min(4, 8)$ by Theorem 2.73.

For $n = 6$

$$\begin{pmatrix} x_1x_2 & x_1x_3 & x_1x_5 & x_2x_3 & x_2x_4 & x_2x_6 & x_3x_4 & x_3x_5 & x_4x_5 & x_4x_6 & x_5x_6 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} x_1x_2x_3 \\ x_2x_3x_4 \\ x_1x_3x_5 \\ x_3x_4x_5 \\ x_2x_4x_6 \\ x_4x_5x_6 \end{matrix}$$

This matrix has rank 6, which is maximal as its rank must be $\leq \min(6, 11)$ by Theorem 2.73.

For $n = 7$

$$\begin{pmatrix} x_1x_2 & x_1x_3 & x_1x_4 & x_1x_5 & x_1x_7 & x_2x_3 & x_2x_4 & x_2x_6 & x_3x_4 & x_3x_5 & x_3x_7 & x_4x_5 & x_4x_6 & x_4x_7 & x_5x_6 & x_5x_7 & x_6x_7 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} x_1x_2x_3 \\ x_2x_3x_4 \\ x_1x_3x_5 \\ x_3x_4x_5 \\ x_2x_4x_6 \\ x_4x_5x_6 \\ x_1x_4x_7 \\ x_3x_5x_7 \\ x_5x_6x_7 \end{matrix}$$

This matrix has rank 9, which is maximal as its rank must be $\leq \min(9, 17)$ by Theorem 2.73.

The computations above demonstrate that the matrix representation for the map in degree 2 seems to always be of maximal rank. By construction, every row in the matrix has three 1s, and it appears the number of 1s in each column must always be either 1 or 2. Furthermore, if F_i is a facet of some $\text{vdw}(n, 2)$, then it also must be a facet for all $\text{vdw}(m, 2)$ where $m > n$. So the matrix representation of $\text{vdw}(m, 2)$ is built by adding the appropriate rows and columns to the matrix representation of $\text{vdw}(n, 2)$. We formalize this with a proof by induction.

For the base case, the computations above show the matrix representation of $\times \ell$ in degree 2 for $A(\text{vdw}(3, 2))$ has maximal rank, and 3 is the smallest value n can be when k is 2. For the induction hypothesis, we assume the matrix representation of $\times \ell$ in degree 2 for $A(\text{vdw}(n-1, 2))$ has maximal rank.

For the induction step, we want to show that the matrix representation of $\times \ell$ in degree 2 for $A(\text{vdw}(n, 2))$ has maximal rank as well. This matrix is constructed using the matrix representation of $\text{vdw}(n-1, 2)$, but appending rows for the new facets introduced by x_n , and new columns for the *new* edges introduced by those new facets. Additionally, there

must be at least one new facet introduced (namely $\{x_{n-2}, x_{n-1}, x_n\}$) so the matrix is not the same as that in the induction hypothesis. Consider an arbitrary new facet and the row that it represents. This new facet must involve x_n , so it also introduces new columns representing $x_i x_n$ for some $i \leq n$, and this row has a 1 in this column. However, the matrix we are appending to, $\mathbf{vdw}(n-1, 2)$, did not initially have this column, so for the existing rows of $\mathbf{vdw}(n-1, 2)$, this column has value 0. Since no linear combinations of 0's can yield 1, this implies that the new row added must be linearly independent to the existing rows of $\mathbf{vdw}(n-1, 2)$.

However, in order to show this new matrix has maximal rank, we must also show that all the new rows introduced are also linearly independent from each other. By construction, the facets of all such rows contain x_n , and by Lemma 3.5, all such facets have a unique edge they do not share with the others. In terms of matrix representations, the column representing this unique edge will have a one in one of these new rows and zeroes elsewhere, and as above since no linear combinations of zeroes can yield one, this implies all the added rows are linearly independent. Putting it all together, the matrix representation of $\mathbf{vdw}(n-1, 2)$ has full rank, and to it we append more linearly independent rows. So every row of the matrix representation of $\mathbf{vdw}(n, 2)$ is linearly independent. Since these matrices have more columns than rows, this ensures the matrix has maximal rank, as wanted.

Case 4: Degree 3 and above, i.e. the maps $\times \ell : A_i \rightarrow A_{i+1}$ for $i \geq 3$

$\mathbf{vdw}(n, 2)$ is 2-dimensional and has facets of cardinality 3. This implies $\dim_k A_i = 0$ for all $i \geq 4$. So the image of our map $\times \ell : A_3 \rightarrow A_4$ is trivial, and as such surjective and has maximal rank. Thus $\mathbf{vdw}(n, 2)$ has the Weak Lefschetz Property in degree 3 for all values of $n \geq 3$. Similarly, since $\dim_k A_i = 0$ for all $i \geq 3$, the maps in higher degrees are all the zero maps and are surjective and injective trivially, thus having maximal rank.

Putting all four cases together, we conclude that $A(\mathbf{vdw}(n, 2))$ has the Weak Lefschetz Property for all degrees for $n \geq 3$. \square

In Chapter 5, we provide an alternative proof that $\mathbf{vdw}(n, 2)$ has the Weak Lefschetz Property in degree 2 if $n \leq 6$ using the concept of pseudo-manifolds.

2. $k = n - 1$

Next, we consider $\mathbf{vdw}(n, n-1)$, which considers the largest possible value k can take. Before we get to the main theorem, we state some essential background. The main sources used for these concepts are [29], [15], [32], and [39]

DEFINITION 3.7. A quotient ring R/I is a **complete intersection** if I is an ideal generated by a regular sequence.

EXAMPLE 3.8. In $R = k[x_1, x_2, \dots, x_n]$, where K is an infinite field of characteristic zero, the quotient ring

$$\frac{R}{(x_1^2, x_2^2, \dots, x_n^2)}$$

is a complete intersection. To prove $x_1^2, x_2^2, \dots, x_n^2$ is a regular sequence, x_1^2 must not be a zero divisor in R . This holds, as K being a field implies it is an integral domain, and polynomial rings over integral domains are also integral domains. Thus R has no zero divisors, so x_1^2 cannot be a zero divisor in R .

Furthermore, for $x_1^2, x_2^2, \dots, x_n^2$ to be a regular sequence, x_i^2 cannot be a zero divisor in $R/\langle x_1^2, x_2^2, \dots, x_{i-1}^2 \rangle$. We prove this by induction. When $i = 1$, the ideal $\langle x_1^2, x_2^2, \dots, x_{i-1}^2 \rangle$ reduces to 0, so the quotient ring remains R . Again since R is an integral domain, it has no zero divisors so the base case is proved. For the induction hypothesis, we assume that x_{i-1}^2 is not a zero divisor in $R/\langle x_1^2, x_2^2, \dots, x_{i-2}^2 \rangle$. Now for the induction step, we recall a property of monomial ideals from [22]: for any polynomial in the ideal $\langle x_1^2, x_2^2, \dots, x_{i-1}^2 \rangle$, all of its monomials is divisible by some x_j^2 for $1 \leq j \leq i - 1$.

Now assume $x_i^2 f = 0$ in $R/\langle x_1^2, x_2^2, \dots, x_{i-1}^2 \rangle$, where f is some polynomial in our ring. By definition of quotient rings, this implies $x_i^2 f \in \langle x_1^2, x_2^2, \dots, x_{i-1}^2 \rangle$. By the property of monomial ideals stated above, this implies every monomial term of $x_i^2 f$ is divisible by x_j^2 for some $1 \leq j \leq i - 1$. But if for some monomial m , we have that x_j^2 divides $x_i^2 m$, then x_j^2 divides m , since x_i^2 and x_j are separate independent variables. This implies every monomial component of f is in the ideal $\langle x_1^2, x_2^2, \dots, x_{i-1}^2 \rangle$, so f is also in the ideal $\langle x_1^2, x_2^2, \dots, x_{i-1}^2 \rangle$. Thus, $f = 0$ in $R/\langle x_1^2, x_2^2, \dots, x_{i-1}^2 \rangle$, so x_i^2 is not a zero divisor in $R/\langle x_1^2, x_2^2, \dots, x_{i-1}^2 \rangle$, as wanted.

We have deduced that $\frac{K[x_1, x_2, \dots, x_n]}{x_1^2, x_2^2, \dots, x_n^2}$ is a complete intersection. We now state a classic result from Stanley in [43] regarding Artinian rings of this form.

THEOREM 3.9. *If K is an infinite field of characteristic 0, then the quotient ring*

$$\frac{K[x_1, x_2, \dots, x_n]}{(x_1^2, x_2^2, \dots, x_n^2)}$$

is a complete intersection that has the Weak Lefschetz Property in all degrees and for all values of n . Furthermore, this Artinian ring also has the Strong Lefschetz Property.

We now have all the essential background needed to examine $\mathbf{vdw}(n, n - 1)$. We state a lemma on the construction of $\mathbf{vdw}(n, n - 1)$ before proving the main theorem.

LEMMA 3.10. *$\mathbf{vdw}(n, n - 1)$ is generated by one facet only, namely $\langle \{x_1, x_2, x_3, \dots, x_n\} \rangle$.*

PROOF. By definition of van der Waerden simplicial complexes, $\mathbf{vdw}(n, k)$ is generated by facets of size $k + 1$. Thus $\mathbf{vdw}(n, n - 1)$ must be generated by facets of size $n - 1 + 1 = n$. However, since $\mathbf{vdw}(n, n - 1)$ only has n vertices to begin with, the only possible facet is the one consisting of all the vertices: $\{x_1, x_2, x_3, \dots, x_n\}$. Hence $\mathbf{vdw}(n, n - 1) = \langle \{x_1, x_2, x_3, \dots, x_n\} \rangle$, as wanted. \square

We can now prove the main result of this subsection.

THEOREM 3.11. *Suppose K is an infinite field of characteristic 0. Then the ring $A(\mathbf{vdw}(n, n-1))$ has the Weak Lefschetz Property for all $n \geq 2$.*

PROOF. By Lemma 3.10, we know that $\mathbf{vdw}(n, n-1)$ is generated by one facet only, $\{x_1, x_2, x_3, \dots, x_n\}$. This single facet that generates $\mathbf{vdw}(n, n-1)$ has all the vertices in our ring, and thus all possible subsets of these vertices. The non-existence of non-faces means $I_\Delta = 0$, so our Artinian ring simplifies to

$$A(\Delta) = \frac{R}{I_\Delta + \langle x_1^2, x_2^2, \dots, x_n^2 \rangle} = \frac{R}{x_1^2, x_2^2, \dots, x_n^2}.$$

By Theorem 3.8, we know our Artinian ring is a complete intersection, and by Theorem 3.9, this complete intersection has the Weak Lefschetz Property in all degrees. Hence, $\mathbf{vdw}(n, n-1)$ has the Weak Lefschetz Property for all values of n , as wanted. \square

Note that our initial assumption that K is an infinite field of characteristic zero is an essential condition for this proof to hold, in order to apply Stanley's result.

3. Degree 1 of $A(\mathbf{vdw}(n, k))$

In this last section, we examine the smallest possible degree in which $A(\mathbf{vdw}(n, k))$ can have the Weak Lefschetz Property in, namely degree 1. This results also uses Theorem 3.3, a result from Dao-Nair.

We now present an instance of the Weak Lefschetz Property in the certain degree of interest.

THEOREM 3.12. *$A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property in degree 1 for all values of n and k .*

PROOF. Consider $A(\mathbf{vdw}(n, k))$, and assume $k \geq 3$. For $k = 1, 2$, we have previously proved that $A(\mathbf{vdw}(n, k))$ always has the Weak Lefschetz Property in general, which implies it must have it in degree 1.

We claim that the 1-skeleton $G(\Delta)$ has only one component. For an arbitrary $\mathbf{vdw}(n, k)$, consider the following valid facets:

$$\begin{aligned} F_1 &= \{x_1, x_2, \dots, x_{k+1}\}, \\ F_2 &= \{x_2, x_3, \dots, x_{k+2}\}, \\ &\vdots \\ F_m &= \{x_m, x_{m+2}, \dots, x_{k+m}\}, \text{ such that } k+m=n. \end{aligned}$$

Then there is an edge between x_1 and x_2 since they are both in F_1 , an edge between x_2 and x_3 since they are both in F_2 , \dots , and finally an edge between x_{n-1} and x_n since

they are both in F_m . Since every vertex is included in this chain of edges in $G(\Delta)$, the 1-skeleton of our simplicial complex, we conclude that $G(\Delta)$ has only one component.

We now prove the theorem in cases considering $\dim_k(A_2)$ and $\dim_k(A_3)$ based on Theorem 3.3 above.

If $\dim A_2 \geq \dim A_1$, then $A(\text{vdw}(n, k))$ has the Weak Lefschetz Property if and only if $G(\Delta)$ has no bipartite components. But since $k \geq 3$, each facet by construction consists of four vertices all connected to one another. So the singular component of $G(\Delta)$ must contain several three-cycles. This means that $G(\Delta)$ does not have any bipartite components (as its singular component is not bipartite), and thus by Theorem 3.3 part 1, has the Weak Lefschetz Property in degree 1.

Although we must also consider the case when $\dim A_2 < \dim A_1$, we can actually show that this case never happens. We know that $\dim A_1$ is the number of vertices of $G(\Delta)$, which is equal to n . We also know that $\dim A_2$ is the number of edges of $G(\Delta)$. Using the same facets F_1, F_2, \dots, F_m from above, we see that there must be at least $n - 1$ edges in $G(\Delta)$. This is because there is an edge between each x_i and x_{i+1} for $1 \leq i \leq n - 1$. This shows that $\dim_k(A_2)$ must be at least $n - 1$.

Similarly, since k was said to be at least 3, and from above we have the facet $F_1 = \{x_1, x_2, \dots, x_{k+1}\}$, and we know there must be an edge between x_1 and x_3 . Thus $G(\Delta)$ has at least n edges, which implies $\dim_k(A_2) \geq n$. Thus, the initial assumption of $\dim_k(A_2) < \dim_k(A_1)$ cannot occur. Hence the only possible outcome is $\dim A_2 \geq \dim A_1$, and thus by Theorem 3.3 we see that $\text{vdw}(n, k)$ always has the Weak Lefschetz Property in degree 1. \square

Classifying when $A(\text{vdw}(n, 3))$ has the Weak Lefschetz Property

In this chapter, we aim to characterize when $A(\text{vdw}(n, 3))$ has the Weak Lefschetz Property. We begin by showing that the three smallest possible values of n , namely $n = 4, 5$, and 6 , are the only instances where $A(\text{vdw}(n, 3))$ has the Weak Lefschetz Property. We then delve into the details for the failure of the Weak Lefschetz Property in $A(\text{vdw}(7, 3))$, which requires a special argument. Finally, we show the failure of the Weak Lefschetz Property for $A(\text{vdw}(n, 3))$ for $n \geq 8$.

1. $\text{vdw}(n, 3)$ for $n = 4, 5, 6$

$\text{vdw}(n, k)$ has the Weak Lefschetz Property for only three values of n , namely the smallest three possibilities $n = 4, 5$, and 6 . We prove this by examining the matrix representation of the linear transformations induced by multiplication by ℓ . We note that all $\text{vdw}(n, k)$ have the Weak Lefschetz Property in degree 1 by Theorem 3.12. We also know that $\text{vdw}(n, 3)$ is generated by facets of cardinality 4, so $\dim_k A_m = 0$ for $m \geq 5$. Thus we examine the matrix representation of the linear map $\times \ell$ in degrees 2 and 3 only, as for the higher degrees, the domain and/or co-domain are trivial, and thus the map is of maximal rank trivially.

Case $n = 4$. When $n = 4$, $\text{vdw}(4, 3)$ is generated by only one facet, $\{x_1, x_2, x_3, x_4\}$. If $A = A(\text{vdw}(4, 3))$, a basis for A_2 is

$$\{x_1x_2 + I, x_1x_3 + I, x_1x_4 + I, x_2x_3 + I, x_2x_4 + I, x_3x_4 + I\},$$

when $I = I_{\text{vdw}(4,3)} + (x_1^2, x_2^2, x_3^2, x_4^2)$. For the matrix representation for multiplication by ℓ in degree two, we index our columns by these basis elements, but write x_ix_j instead of $x_ix_j + I$. We do a similar procedure to index the rows, and continue the same procedures for the maps of greater degrees as well. The matrix representation for multiplication by ℓ in degree two thus looks like:

$$\begin{array}{cccccc} x_1x_2 & x_1x_3 & x_1x_4 & x_2x_3 & x_2x_4 & x_3x_4 \\ \left(\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right) \begin{array}{l} x_1x_2x_3 \\ x_1x_2x_4 \\ x_1x_3x_4 \\ x_2x_3x_4 \end{array} \end{array}$$

This matrix is of maximal rank since it has rank $4 = \min(4, 6)$, and so this map is of maximal rank.

The matrix representation for multiplication by ℓ in degree three looks like:

$$\begin{array}{cccc} x_1x_2x_3 & x_1x_2x_4 & x_1x_3x_4 & x_2x_3x_4 \\ \left(\begin{array}{cccc} 1 & 1 & 1 & 1 \end{array} \right) & x_1x_2x_3x_4 & & \end{array}$$

This matrix is of maximal rank since it has rank $1 = \min(1, 4)$, and so this map is of maximal rank. Combined with the previous remarks, this concludes that $A(\text{vdw}(4, 3))$ has the Weak Lefschetz Property.

Case $n = 5$. When $n = 5$, $\text{vdw}(4, 3)$ is generated by the facets $\{x_1, x_2, x_3, x_4\}$ and $\{x_2, x_3, x_4, x_5\}$. The matrix representation for multiplication by ℓ in degree two thus looks like:

$$\begin{array}{cccccccccc} x_1x_2 & x_1x_3 & x_1x_4 & x_2x_3 & x_2x_4 & x_2x_5 & x_3x_4 & x_3x_5 & x_4x_5 & \\ \left(\begin{array}{cccccccccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right) & x_1x_2x_3 \\ & & & & & & & & & x_1x_2x_4 \\ & & & & & & & & & x_1x_3x_4 \\ & & & & & & & & & x_2x_3x_4 \\ & & & & & & & & & x_2x_3x_5 \\ & & & & & & & & & x_2x_4x_5 \\ & & & & & & & & & x_3x_4x_5 \end{array}$$

This matrix has rank $7 = \min(7, 9)$, and is thus of maximal rank.

The matrix representation for multiplication by ℓ in degree three looks like:

$$\begin{array}{ccccccc} x_1x_2x_3 & x_1x_2x_4 & x_1x_3x_4 & x_2x_3x_4 & x_2x_3x_5 & x_2x_4x_5 & x_3x_4x_5 \\ \left(\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right) & x_1x_2x_3x_4 \\ & x_2x_3x_4x_5 \end{array}$$

This matrix has rank $2 = \min(2, 7)$. Thus the matrix and the map it represents are of maximal rank. Combined with the previous remarks, this concludes that $A(\text{vdw}(5, 3))$ has the Weak Lefschetz Property in all degrees.

Case $n = 6$. When $n = 6$, $\text{vdw}(4, 3)$ is generated by the three facets $\{x_1, x_2, x_3, x_4\}$, $\{x_2, x_3, x_4, x_5\}$, and $\{x_3, x_4, x_5, x_6\}$. The matrix representation for multiplication by ℓ in degree two thus looks like:

$$\begin{pmatrix} x_1x_2 & x_1x_3 & x_1x_4 & x_2x_3 & x_2x_4 & x_2x_5 & x_3x_4 & x_3x_5 & x_2x_6 & x_4x_5 & x_4x_6 & x_5x_6 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} x_1x_2x_3 \\ x_1x_2x_4 \\ x_1x_3x_4 \\ x_2x_3x_4 \\ x_2x_3x_5 \\ x_2x_4x_5 \\ x_3x_4x_5 \\ x_3x_4x_6 \\ x_3x_5x_6 \\ x_3x_5x_6 \\ x_3x_5x_6 \end{matrix}$$

This matrix has rank $10 = \min(10, 12)$, and is thus of maximal rank.

The matrix representation for multiplication by ℓ in degree three looks like:

$$\begin{pmatrix} x_1x_2x_3 & x_1x_2x_4 & x_1x_3x_4 & x_2x_3x_4 & x_2x_3x_5 & x_2x_4x_5 & x_3x_4x_5 & x_3x_4x_6 & x_3x_5x_6 & x_4x_5x_6 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} x_1x_2x_3x_4 \\ x_2x_3x_4x_5 \\ x_3x_4x_5x_6 \end{matrix}$$

This matrix has rank $3 = \min(3, 10)$. Thus the matrix and the map it represents are of maximal rank. Combined with the previous remarks, this concludes that $A(\text{vdw}(6, 3))$ has the Weak Lefschetz Property in all degrees.

All rank computations in this subsection were completed using Matlab. We have now proved that $A(\text{vdw}(n, 3))$ has the Weak Lefschetz Property for the three smallest possible values of n , namely $n = 4, 5$ and 6 .

2. $\text{vdw}(7, 3)$

In the remaining sections of this chapter, we examine for what values of n does $A(\text{vdw}(n, 3))$ have the Weak Lefschetz Property. We have showed that when $n = 4, 5$, or 6 , $A(\text{vdw}(n, 3))$ does have the Weak Lefschetz Property. We now characterize the remaining values, and prove the following result:

THEOREM 4.1. *$A(\text{vdw}(n, 3))$ has the Weak Lefschetz Property if and only if $n = 4, 5$, or 6 . For all other values of n , $A(\text{vdw}(n, 3))$ fails to have the Weak Lefschetz Property; in particular, the map $\times \ell : A_2 \rightarrow A_3$ fails to have maximal rank.*

Computations using Macaulay2 resulted in Table 7.1 showing for which values of n and k does $A(\text{vdw}(n, k))$ have the Weak Lefschetz Property. The smallest values of n and k where we have the first instance of failure is $\text{vdw}(7, 3)$.

The simplicial complex $\text{vdw}(7, 3)$ is generated by the following facets:

$$\begin{aligned} F_0 &= \{x_1, x_2, x_3, x_4\}, \\ F_1 &= \{x_2, x_3, x_4, x_5\}, \\ F_2 &= \{x_3, x_4, x_5, x_6\}, \\ F_3 &= \{x_4, x_5, x_6, x_7\}, \\ F_4 &= \{x_1, x_3, x_5, x_7\}, \end{aligned}$$

We remind the reader that the ring we are working in is $A = R/(I_\Delta, x_1^2, x_2^2, \dots, x_n^2)$. We also note that $\text{vdw}(7, 3)$ is also the smallest van der Waerden simplicial complex that has valid facets of jump factor greater than one. In this case, the facet $\{x_1, x_3, x_5, x_7\}$ has jump factor 2.

Output from Macaulay2 indicates the reason for this failure is that ‘‘Multiplication by any linear form is not surjective in degree 2’’. To see why, we examine the matrix representation of the map $\times \ell$ in degree 2 (i.e. the map $\times \ell : A_2 \rightarrow A_3$). From the generating facets above, we see that a basis to A_2 is given by

$$\begin{aligned} \{x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_7, x_2x_3, x_2x_4, x_2x_5, x_3x_4, x_3x_5, x_3x_6, x_3x_7, \\ x_4x_5, x_4x_6, x_4x_7, x_5x_6, x_5x_7, x_6x_7\}, \end{aligned}$$

so $\dim_k A_2 = 18$, and a basis to A_3 is given by

$$\begin{aligned} \{x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_1x_3x_5, x_1x_3x_7, x_1x_5x_7, x_2x_3x_4, x_2x_3x_5, x_2x_4x_5, x_3x_4x_5, \\ x_3x_4x_6, x_3x_5x_6, x_3x_5x_7, x_4x_5x_6, x_4x_5x_7, x_5x_6x_7\}, \end{aligned}$$

so $\dim_k A_3 = 17$.

The map $\times \ell$ in degree 2 must have rank $\leq \min(17, 18)$, so for it to not have maximal rank, the rank must be less than 17. Since $\dim_k A_2 = 18$, to prove $\times \ell : A_2 \rightarrow A_3$ fails to have max rank, we need to show that the dimension of the kernel of the map $\times \ell$ is greater than or equal to two. Instead, we consider the transpose map $\times \ell^T : A_3 \rightarrow A_2$. We do this by showing that the transpose map $\times \ell^T$ from A_3 to A_2 has a kernel of dimension greater than or equal to 1. This, by the rank-nullity theorem, implies that the rank of the map ℓ^T , and thus the map ℓ , must be less than 17. We use the matrix representation of both maps in order to prove this, so we now restate this idea as a lemma in terms of matrices.

LEMMA 4.2. *Let $A = A(\text{vdw}(7, 3))$ and $M_{(7,3)}$ be the matrix representation of $\times \ell : A_2 \rightarrow A_3$, and $M_{(7,3)}^T$ its transpose. Showing that the map $\times \ell$ is not of maximal rank is equivalent to showing that the kernel of $M_{(7,3)}^T$ is non-empty.*

PROOF. Consider the map $\times \ell : A_2 \rightarrow A_3$, where $\dim_k A_2 = 18$ and $\dim_k A_3 = 17$, and its matrix representation is $M_{(7,3)}^T$. Showing non-injectivity by finding an element in the kernel does not help, as it only shows the rank is less than 18. Finding two elements in the kernel ensures that the matrix does not have maximal rank. However, by Theorem

2.72, we know $M_{(7,3)}$, and its transpose $M_{(7,3)}^T$ have the same rank. Furthermore the rank of $M_{(7,3)}^T$ must also be $\leq \min(17, 18) = 17$. By finding an element in the kernel of $M_{(7,3)}^T$, we show that $M_{(7,3)}^T$ is not injective and thus $\text{null}(M_{(7,3)}^T) \geq 1$. Then by the rank-nullity theorem, which states $\text{rank}(M_{(7,3)}^T) + \text{nullity}(M_{(7,3)}^T) = 17$, we deduce that $\text{rank}(M_{(7,3)}^T) = \text{rank}(M_{(7,3)}) < 17$, showing $M_{(7,3)}$ is not of maximal rank. \square

We can even generalize this lemma as follows:

LEMMA 4.3. *Let $A = R/I$ be an Artinian ring with I a monomial ideal. Suppose $\times \ell : A_i \rightarrow A_{i+1}$ with $\dim A_i > \dim A_{i+1}$. If the map $\times \ell^T : A_{i+1} \rightarrow A_i$ has a non-empty kernel, then the Artinian ring does not have the Weak Lefschetz Property in degree i .*

PROOF. If $\times \ell$ has maximal rank, then its rank must be $\min\{\dim A_i, \dim A_{i+1}\} = \dim A_{i+1}$. Since the map $\times \ell^T$ has a non-trivial kernel, $\text{null}(\times \ell^T) \geq 1$. Thus the Rank-Nullity equation tells us that $\text{rank}(\times \ell^T) = \dim A_{i+1} - \text{null}(\times \ell^T) < \dim A_{i+1}$. Since Theorem 2.72 tells us $\text{rank}(\times \ell) = \text{rank}(\times \ell^T)$, it follows that $\text{rank}(\times \ell) < \dim A_{i+1}$, and it is not of maximal rank, and thus the ring does not have the Weak Lefschetz Property. \square

We begin by presenting the matrix $M_{(7,3)}$ for $\times \ell : A_2 \rightarrow A_3$ with respect to the basis given as above:

$$\begin{pmatrix} x_1x_2 & x_1x_3 & x_1x_4 & x_1x_5 & x_1x_7 & x_2x_3 & x_2x_4 & x_2x_5 & x_3x_4 & x_3x_5 & x_3x_6 & x_3x_7 & x_4x_5 & x_4x_6 & x_4x_7 & x_5x_6 & x_5x_7 & x_6x_7 \\ \left(\begin{array}{cccccccccccccccc} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right) \begin{array}{l} x_1x_2x_3 \\ x_1x_2x_4 \\ x_1x_3x_4 \\ x_1x_3x_5 \\ x_1x_3x_7 \\ x_1x_5x_7 \\ x_2x_3x_4 \\ x_2x_3x_5 \\ x_2x_4x_5 \\ x_3x_4x_5 \\ x_3x_4x_6 \\ x_3x_5x_6 \\ x_3x_5x_7 \\ x_4x_5x_6 \\ x_4x_5x_7 \\ x_4x_6x_7 \\ x_5x_6x_7 \end{array} \end{pmatrix}$$

And now we compute its transpose, $M_{(7,3)}^T$:

$$\begin{pmatrix}
x_1x_2x_3 & x_1x_2x_4 & x_1x_3x_4 & x_1x_3x_5 & x_1x_3x_7 & x_1x_5x_7 & x_2x_3x_4 & x_2x_3x_5 & x_2x_4x_5 & x_3x_4x_5 & x_3x_4x_6 & x_3x_5x_6 & x_3x_5x_7 & x_4x_5x_6 & x_4x_5x_7 & x_4x_6x_7 & x_5x_6x_7 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1x_2 \\
x_1x_3 \\
x_1x_4 \\
x_1x_5 \\
x_1x_7 \\
x_2x_3 \\
x_2x_4 \\
x_2x_5 \\
x_3x_4 \\
x_3x_5 \\
x_3x_6 \\
x_3x_7 \\
x_4x_5 \\
x_4x_6 \\
x_4x_7 \\
x_5x_6 \\
x_6x_7
\end{pmatrix}$$

Using the help of Matlab, the following non-trivial element in the kernel was found. We let z_i denote the element from A_3 that the i^{th} column of the matrix represents. Set z_{17} as a free variable. Then set:

$$\begin{aligned}
z_1 &= -z_{17} & z_6 &= -z_{17} & z_{11} &= z_{17} \\
z_2 &= z_{17} & z_7 &= 0 & x_{12} &= -z_{17} \\
z_3 &= -z_{17} & z_8 &= z_{17} & z_{13} &= -z_{17} \\
z_4 &= z_{17} & z_9 &= -z_{17} & z_{14} &= 0 \\
z_5 &= z_{17} & z_{10} &= 0 & z_{15} &= z_{17} \\
& & & & z_{16} &= -z_{17}
\end{aligned}$$

Since z_{17} is a free variable, we can set it to be any value of our choice. We set z_{17} to be 1, and use the solution given above to set:

$$\begin{aligned}
z_1 &= -1 & z_6 &= -1 & z_{11} &= 1 \\
z_2 &= 1 & z_7 &= 0 & z_{12} &= -1 \\
z_3 &= -1 & z_8 &= 1 & z_{13} &= -1 \\
z_4 &= 1 & z_9 &= -1 & z_{14} &= 0 \\
z_5 &= 1 & z_{10} &= 0 & z_{15} &= 1 \\
& & & & z_{16} &= -1
\end{aligned}$$

We construct a 17×1 column matrix v , where the entry in the i^{th} row is the z_i value from above. We multiply the 18×17 matrix $M_{(7,3)}^T$ with the 17×1 matrix v . We then perform the matrix multiplication to see that the resultant 18×1 vector is indeed the zero vector, as wanted.

This proves that the column vector v is in the kernel of $M_{(7,3)}^T$, and thus the map $\times \ell^T : A_3 \rightarrow A_2$ cannot be injective. This implies $\text{null}(M_{(7,3)}^T) \geq 1$, or equivalently since the rank-nullity theorem tells us $\text{rank}(M_{(7,3)}^T) + \text{nullity}(M_{(7,3)}^T) = 17$, we get that $\text{rank}(M_{(7,3)}^T)$ must be less than 17. But $\text{rank}(M_{(7,3)}^T) = \text{rank}(M_{(7,3)})$, so $\text{rank}(M_{(7,3)})$ must also be strictly less than 17. If the matrix representing a linear transformation does not have maximal rank, then neither does the map it represents (as they will have the same rank). Thus the degree two matrix representation of $\times \ell$ in $\text{vdw}(7, 3)$ does not have maximal rank, concluding our proof that $\text{vdw}(7, 3)$ does not have the Weak Lefschetz Property.

$\text{vdw}(7, 3)$ is a special case to consider for $\text{vdw}(n, 3)$, as it is the only value of n where the Weak Lefschetz Property fails and $\dim_K A_2 > \dim_K A_3$. For the remaining values of n , we can show that since $\dim_K A_3$ is always larger than or equal to $\dim_K A_2$, so our map of multiplication by ℓ is not surjective. Thus finding an element in the kernel of $\times \ell$ is enough to show that the map is also not injective, and thus does not have the Weak Lefschetz Property.

3. $\text{vdw}(n, 3)$ for $n \geq 8$

In this final subsection of this chapter, we complete the proof that for $n \geq 7$, the ring $A(\text{vdw}(n, 3))$ does not have the Weak Lefschetz Property. We examined the smallest instance, where $n = 7$, in the section above. We now generalize this result to all values of $n \geq 7$. The key idea of this subchapter is to first prove that for all values of $n \geq 8$, $\dim_K A_3 \geq \dim_K A_2$, so the map $\times \ell$ cannot be surjective. By proving that there is some element always in the kernel, we can deduce that the map is also not injective, so for all such values of n , $A(\text{vdw}(n, 3))$ does not have the Weak Lefschetz Property.

We begin with some dimensions of $\dim_K A_2$, and $\dim_K A_3$ for motivation:

$A(\text{vdw}(n, 3))$	$\dim A_2, \dim A_3$
$A(\text{vdw}(4, 3))$	6, 4
$A(\text{vdw}(5, 3))$	9, 7
$A(\text{vdw}(6, 3))$	12, 10
$A(\text{vdw}(7, 3))$	18, 17
$A(\text{vdw}(8, 3))$	24, 24
$A(\text{vdw}(9, 3))$	29, 30
$A(\text{vdw}(10, 3))$	35, 40
$A(\text{vdw}(11, 3))$	41, 50
$A(\text{vdw}(12, 3))$	47, 60
$A(\text{vdw}(13, 3))$	56, 73
$A(\text{vdw}(14, 3))$	65, 86

TABLE 4.1. Dimensions of A_2, A_3 for $A(\text{vdw}(n, 3))$ for small n

We now aim to prove that $\dim A_2$ is always less than or equal to $\dim A_3$ if $n \geq 8$. Before we do so, we consider some necessary facts about the facets of $\text{vdw}(n, 3)$ and their possible contributions to A_2 and A_3 .

PROPOSITION 4.4. *If F is a facet of $\text{vdw}(n, 3)$, then $\text{jump}(F) \leq j$, where j is the biggest integer such that $3j + 1 \leq n \leftrightarrow j \leq \lfloor \frac{n-1}{3} \rfloor$.*

Note that this can be rephrased as: For $n \leq 6$ $\text{vdw}(n, 3)$ only has facets F such that $\text{jump}(F) = 1$. Facets with jump size 2 get introduced with $n = 7$, and following this each increase of n by 3 introduces a new jump size.

PROOF. $\text{vdw}(7, 3)$ has facet $F = \{x_1, x_3, x_5, x_7\}$ of jump size 2. For an arbitrary jump size j , the first facet introduced to have that jump size would be $\{x_1, x_{1+j}, x_{1+2j}, x_{1+3j}\}$. By definition, $1 + 3j \leq n$, so the first value of n that can have a facet of jump size j is $n = 1 + 3j$. Thus, facets of new jump sizes are introduced as n increases by 3 each time. \square

LEMMA 4.5. *For $n \geq 4$, the number of facets of $\text{vdw}(n, 3)$ is $\sum_{j=1}^d (n - 3j)$, where $d = \lfloor \frac{n-1}{3} \rfloor$.*

PROOF. The largest jump size a facet can have is $d = \lfloor \frac{n-1}{3} \rfloor$, as x_1 is the smallest index we can start with, and $3d + 1$ cannot be greater than n . We consider all the possible facets F such that $\text{jump}(F) = 1$. The possibilities are $\{x_1, x_2, x_3, x_4\}$, $\{x_2, x_3, x_4, x_5\}$, and so forth until $\{x_{n-3}, x_{n-2}, x_{n-1}, x_n\}$. Based on the starting indices, there are $n - 3(1)$ of such facets. For facets of jump 2, the possibilities are $\{x_1, x_3, x_5, x_7\}$, $\{x_2, x_4, x_6, x_8\}$, and so forth until $\{x_{n-6}, x_{n-4}, x_{n-2}, x_n\}$. So there are $n - 6 = n - 3(2)$ of such facets. The pattern continues for all possible jump sizes, as a facet of jump size j can start with the smallest possible index and be $\{x_1, x_{1+j}, x_{1+2j}, x_{1+3j}\}$, and progress continuously to the largest possible index it can end with, which is $\{x_{n-3j}, x_{n-2j}, x_{n-j}, x_n\}$. So each possible jump size j contributes $(n - 3j)$ facets, and j can range from 1 to d , proving the summation formula presented above. \square

EXAMPLE 4.6. Consider $\text{vdw}(8, 3)$. The biggest jump size a valid facet can have is $\lfloor \frac{8-1}{3} \rfloor = 2$. Using the formula, the number of facets of $\text{vdw}(8, 3)$ is $\sum_{j=1}^2 (8 - 3j) = (8 - 3) + (8 - 6) = 5 + 2 = 7$. Manually computing the facets of $\text{vdw}(8, 3)$ of jump size 1, we have $\{x_1, x_2, x_3, x_4\}$, $\{x_2, x_3, x_4, x_5\}$, $\{x_3, x_4, x_5, x_6\}$, $\{x_4, x_5, x_6, x_7\}$, and $\{x_5, x_6, x_7, x_8\}$. For facets of jump size 2, we have $\{x_1, x_3, x_5, x_7\}$ and $\{x_2, x_4, x_6, x_8\}$. We see that we have 7 facets that generate $\text{vdw}(8, 3)$, just as the formula computed.

LEMMA 4.7. *If a size-three subset of the facets of $\text{vdw}(n, 3)$ is contained in more than one facet, then the facets must have the same jump factor.*

PROOF. Suppose there exists two facets,

$$\begin{aligned} F_1 &= \{x_i, x_{i+d}, x_{i+2d}, x_{i+3d}\} \\ F_2 &= \{x_j, x_{j+d'}, x_{j+2d'}, x_{j+3d'}\} \end{aligned}$$

such that some subset of size 3, $\{x_a, x_b, x_c\}$, is a subset of both of them. Without loss of generality assume $a < b < c$. If x_a, x_b, x_c were consecutive elements in the facets (i.e. they were successive elements in an arithmetic progression) then both facets have to have the same jump size by construction. If the elements were not consecutive, then considering F_1 , $b - a$ and $c - b$ is either $2d$ or d (it cannot be $3d$ as we are only considering consecutive elements in our subset). Similarly, considering F_2 , $b - a$ and $c - b$ is either $2d'$ or d' . Thus, $d = \min\{b - a, c - b\} = d'$ by uniqueness of minimality, so $d = d'$, as wanted. \square

COROLLARY 4.8. *Any subset of size three of a facet in $\text{vdw}(n, 3)$ can only be a subset of at most two facets.*

PROOF. From Lemma 4.7, if a subset of size three is a subset of two facets, then both facets must have the same jump factor. If the three terms in the subset are consecutive, then it can either be a part of the facet that adjoins an element in the first or last position based on the jump factor d . Otherwise, having three out of four of the elements and the jump factor d uniquely determines which facet it is a subset of. In either case, any subset of size three of a facet in $\text{vdw}(n, 3)$ can only be a subset of at most two facets. \square

We now state some ideas that we will use multiple times in this section; they are modifications of a key definitions we have previously seen.

DEFINITION 4.9. We previously defined the jump factor of a facet F of $\text{vdw}(n, 3)$, $F = \{x_i, x_{i+d}, x_{i+2d}, x_{i+3d}\}$, as d , and denoted it by $\mathbf{jump}(F)$. We now generalize and modify this definition to any arbitrary elements in A_2 and A_3 of our ring $A(\text{vdw}(n, 3))$. For an arbitrary basis element of degree three $a_3 = x_a x_b x_c$ in A_3 (respectively an arbitrary basis element of degree two $a_2 = x_a x_b$ in A_2), we refer to $\{|a - b|, |b - c|, |a - c|\}$ (respectively $a_2 = \{|a - b|\}$ in A_2) as the **jumps set** of that particular element.

EXAMPLE 4.10. Consider the facet $\{x_2, x_4, x_6, x_8\}$ in $\text{vdw}(8, 3)$. This facet has jump 2. A basis element for A_3 is $x_2 x_6 x_8$, and it has jump set $\{4, 2, 6\}$, whereas a basis element for A_2 is $x_2 x_6$, and it has jump set $\{4\}$.

We now examine several properties about the facets of $\text{vdw}(n, 3)$ that will be helpful in examining whether or not $A(\text{vdw}(n, 3))$ has the Weak Lefschetz Property.

PROPOSITION 4.11. *Given an arbitrary facet of $\text{vdw}(n, 3)$ for $n \geq 7$ with $\mathbf{jump}(F) = j$, then all possible subsets of F of size 3 have the form $\{x_i, x_{i+j}, x_{i+2j}\}$, $\{x_i, x_{i+j}, x_{i+3j}\}$, or $\{x_i, x_{i+2j}, x_{i+3j}\}$.*

PROOF. Let F be an arbitrary facet of $\text{vdw}(n, 3)$ for $n \geq 7$ with $\mathbf{jump}(F) = j$. Then F has the form $\{x_i, x_{i+j}, x_{i+2j}, x_{i+3j}\}$ where $0 < i < i + 3j \leq n$. The following are then all possible subsets of F of cardinality 3:

- (1) $\{x_i, x_{i+j}, x_{i+2j}\}$
- (2) $\{x_i, x_{i+j}, x_{i+3j}\}$
- (3) $\{x_i, x_{i+2j}, x_{i+3j}\}$

$$(4) \{x_{i+j}, x_{i+2j}, x_{i+3j}\}.$$

We see that the first three forms match those in the proposition exactly. For the subset of the form $\{x_{i+j}, x_{i+2j}, x_{i+3j}\}$, replacing the starting index i with $i + j$, we see it is equivalent to $\{x_{i+j}, x_{i+j+j}, x_{i+j+2j}\}$, which is also of the first form in the proposition: $\{x_i, x_{i+j}, x_{i+2j}\}$. Hence, all possible subsets of F of cardinality 3 have one of the following three forms $\{x_i, x_{i+j}, x_{i+2j}\}$, $\{x_i, x_{i+j}, x_{i+3j}\}$, or $\{x_i, x_{i+2j}, x_{i+3j}\}$. \square

COROLLARY 4.12. *Any subset of F of cardinality 3 has jumps set $\{j, j, 2j\}$ or $\{j, 2j, 3j\}$.*

PROOF. Any such subset of F with cardinality 3 must fall into one of the three categories in Proposition 4.11.

- (1) If F' has the form $\{x_i, x_{i+j}, x_{i+2j}\}$, then its jump set is $\{j, j, 2j\}$;
- (2) If F' has the form $\{x_i, x_{i+j}, x_{i+3j}\}$, then its jump set is $\{j, 2j, 3j\}$;
- (3) If F' has the form $\{x_i, x_{i+2j}, x_{i+3j}\}$, then its jump set is $\{2j, j, 3j\} = \{j, 2j, 3j\}$.

\square

COROLLARY 4.13. *Any subset of $F \in \text{vdw}(n, 3)$ with $\text{jump}(F) = j$ and cardinality 2 must have jump set $\{j\}$, $\{2j\}$, or $\{3j\}$.*

PROOF. Any subset of F with cardinality 2 must be a subset of one of the F with cardinality 3 described above. The jump set of these subsets describe the difference in indices of two arbitrary elements in them, and that is precisely the jump set of a subset of F with cardinality 2. By the corollary above, the only possibilities here are $\{j\}$, $\{2j\}$, or $\{3j\}$. \square

DEFINITION 4.14. Let a_2 be an arbitrary basis element of A_2 derived from of a facet of $\text{vdw}(n, 3)$. From above, the jump set of a_2 must be $\{j\}$, $\{2j\}$, or $\{3j\}$. If the single number in that set is even, we call a_2 an **even 2-set**. If the single number in that set is odd, we call a_2 an **odd 2-set**.

EXAMPLE 4.15. Consider $\text{vdw}(7, 3)$, generated by facets

$$F_0 = \{x_1, x_2, x_3, x_4\}, F_1 = \{x_2, x_3, x_4, x_5\}, F_2 = \{x_3, x_4, x_5, x_6\} \\ , F_3 = \{x_4, x_5, x_6, x_7\}, \text{ and } F_4 = \{x_1, x_3, x_5, x_7\}.$$

In F_0 , with $\text{jump}(F) = j = 1$, the subset of size two $\{x_1, x_4\}$ has jump set $\{3\}$, so this subset is an odd 2-set. In the same facet, the subset of size two $\{x_2, x_4\}$ has jump set $\{2\}$, and is thus an even 2-set.

The structure propositions above tell us that the odd-even parity of j can have some impact on the elements of A_3 that we work with. Let F be an arbitrary facet of $\text{vdw}(n, 3)$. Then F must have form $\{x_i, x_{i+j}, x_{i+2j}, x_{i+3j}\}$. Consider all the two-dimensional subsets one can create from the elements of F , in other words all its possible contributions to A_2 .

Suppose j is even. The two-dimensional subsets one can create from the elements of F must be a subset of one of the cardinality three subsets of F , denoted F' . By Corollary

4.11 F' has the form $\{x_i, x_{i+j}, x_{i+2j}\}$, $\{x_i, x_{i+j}, x_{i+3j}\}$, or $\{x_i, x_{i+2j}, x_{i+3j}\}$. By Corollary 4.12, the jump set of F' must be $\{j, j, 2j\}$ or $\{j, 2j, 3j\}$. Since j is even, so are all multiples of it, so $2j$ and $3j$ are also even. Thus, any two-dimensional subsets obtained from F' must be an even 2-set.

Now suppose j is odd. Similar to the argument above, we examine the cardinality three subset F' of F that our two-dimensional subsets arises from. Suppose F' has the form $\{x_i, x_{i+j}, x_{i+2j}\}$ and jump set $\{j, j, 2j\}$. Then j is odd, but $2j$ is even, so F' contributes two odd 2-sets, and one even 2-set. Now suppose F' has the form $\{x_i, x_{i+j}, x_{i+3j}\}$ or $\{x_i, x_{i+2j}, x_{i+3j}\}$. Then its jump set is $\{j, 2j, 3j\}$. If j is odd, then so is $3j$, but $2j$ is even. So in both cases, if j is odd, F' contributes one even 2-set and two odd 2-sets.

We now use these result to present a closed formula for the dimension of A_3 , the number of possible subsets of size three we can create using the facets of $\text{vdw}(n, 3)$.

THEOREM 4.16. *The dimension of A_3 is:*

- (1) $\sum_{j=1}^{\lfloor \frac{n}{4} \rfloor} (4j + 3(n - 4j)) + \sum_{j=\lceil \frac{n}{4} \rceil}^d (4(n - 3j))$ if n is not divisible by 4,
- (2) $\sum_{j=1}^{\frac{n}{4}} (4j + 3(n - 4j)) + \sum_{j=\frac{n}{4}+1}^d (4(n - 3j))$ if n is divisible by 4,

where j sums over the jump sizes of the facets and $d = \lfloor \frac{n-1}{3} \rfloor$.

PROOF. Suppose we list all the facets of an arbitrary jump size j . Then by Lemma 4.5, there are $n-3j$ such facets, namely $F_1 = \{x_1, x_{1+j}, x_{2+j}, x_{1+3j}\}$, $F_2 = \{x_2, x_{2+j}, x_{2+j}, x_{2+3j}\}$, all the way to $F_{n-3j} = \{x_{n-3j}, x_{n-2j}, x_{n-j}, x_n\}$. Note that by construction, keeping this same ordering of the F_i , all the elements in the facets F_1 up to and including F_j are unique, as the first repetition of vertices occurs in F_{j+1} , where the vertex x_{j+1} is present, when it was already present as the second vertex in F_1 .

Suppose $j \geq \lceil \frac{n}{4} \rceil$, and consider all the facets created using this jump factor j . Then $j \geq \frac{n}{4} \leftrightarrow 4j = 3j + j \geq n \leftrightarrow j \geq n - 3j \leftrightarrow 1 + j > n - 3j$. This implies that all $n - 3j$ of our facets have distinct elements, as the facets where we start to get repetitions are not even valid facets in this case, since we would get vertices with indices greater than n . So for all such facets, each vertex is distinct, so we can make 4, or equivalently $\binom{4}{3}$, sets of cardinality 3. These are precisely the elements of A_3 , so for all facets of jump size j where $j \geq \lceil \frac{n}{4} \rceil$, there are $n - 3j$ possible facets, and each of these contribute 4 to $\dim_K A_3$, so in total they contribute $\sum_{j=\lceil \frac{n}{4} \rceil}^d (4(n - 3j))$, the second component of the sum in our theorem. Our sum ends at d , which was said to be the maximal jump size allowed.

Now suppose $j \leq \lfloor \frac{n}{4} \rfloor$, and consider all the facets created using this jump factor j . This implies $j \leq \frac{n}{4} \leftrightarrow 4j = 3j + j \leq n \leftrightarrow j \leq n - 3j \leftrightarrow 1 + j < n - 3j$. Based on the same reasoning as above, this implies that now we do have different facets with common vertices x_i , that is F' and F'' can have $F' \cap F'' \neq \emptyset$. We know that for each such j , the first j facets F_1, F_2, \dots, F_j all have distinct elements. So these j facets contribute $\binom{4}{3} = 4$ elements to A_3 . Of the remaining $(n - 3j) - j = n - 4j$ facets, there exists repetition amongst elements. By Lemma 4.7 and its corollary, if a subset of size three is contained in

more than one subset, it can only be contained in at most two subsets that have the same jump factor. So for each such j , the remaining facets contribute only 3 to the dimension of A_3 . In total for each such j , the addition to $\dim_K A_3$ is $\sum_{j=1}^{\lfloor \frac{n}{4} \rfloor} (4j + 3(n - 4j))$, the first component of the sum in the theorem.

For values of n divisible by 4, we only want it included in the first component of our sum to avoid over counting, thus the separate formula provided for this case. \square

For a facet of $\text{vdw}(n + 1, 3)$ containing x_{n+1} , we now state a lemma about its contributions to $\dim A_2$, only considering subsets that include the vertex x_{n+1} .

LEMMA 4.17. *Let $\lfloor \frac{n}{3} \rfloor = 3z + r$, where $r \in \{0, 1, 2\}$. The facets of $\text{vdw}(n + 1, 3)$ that include x_{n+1} contribute at least $6z$ subsets containing x_{n+1} to $\dim A_2$.*

PROOF. Consider an arbitrary facet F of $\text{vdw}(n + 1, 3)$ that include x_{n+1} . These facets are of the form:

$$\begin{aligned} F_1 &= \{x_{n+1}, x_n, x_{n-1}, x_{n-2}\}, \\ F_2 &= \{x_{n+1}, x_{n-1}, x_{n-3}, x_{n-5}\}, \\ F_3 &= \{x_{n+1}, x_{n-2}, x_{n-5}, x_{n-8}\}, \\ &\vdots \\ F_j &= \{x_{n+1}, x_{n+1-j}, x_{n+1-2j}, x_{n+1-3j}\}. \end{aligned}$$

The number of possible subsets of size two we can make with each of these facets is $\binom{4}{2} = 6$. However, there are only 3 subsets of size two that contain the vertex x_{n+1} . Note how when considering F_1, F_2, F_3 , the subsets of size two $\{x_{n+1}, x_{n-1}\}$, $\{x_{n+1}, x_{n-2}\}$, and $\{x_{n+1}, x_{n-5}\}$ repeat amongst two F_i . So considering repetitions, there are $3 \times 3 - 3 = 6$ contributions these three facets make to $\dim A_2$.

In fact, we can generalize this for consecutive blocks of 3 as well. We let $m \geq 0$. When we considered the facets F_1, F_2, F_3 , these were precisely the facets F_{3m+1}, F_{3m+2} , and F_{3m+3} for $m = 0$, the first block. Now we consider some block of three for some arbitrary m :

$$\begin{aligned} F_{3m+1} &= \{x_{n+1}, x_{n+1-(3m+1)}, x_{n+1-2(3m+1)}, x_{n+1-3(3m+1)}\}, \\ F_{3m+2} &= \{x_{n+1}, x_{n+1-(3m+2)}, x_{n+1-2(3m+2)}, x_{n+1-3(3m+2)}\}, \\ F_{3m+3} &= \{x_{n+1}, x_{n+1-(3m+3)}, x_{n+1-2(3m+3)}, x_{n+1-3(3m+3)}\} \end{aligned}$$

We know if $m = 0$ we get the facets F_1, F_2, F_3 ; if $m = 1$ we get the facets F_4, F_5, F_6 ; if $m = 2$ we get the facets F_7, F_8, F_9 ; etc. In any set of three consecutive integers, $(3m + 1, 3m + 2, 3m + 3)$, at least one is divisible by 2, and one (potentially the same) is divisible by 3. This is because the possible remainders r_1 when dividing by 2 are $\{0, 1\}$ and the possible remainders r_2 when dividing by 3 are $\{0, 1, 2\}$. So in our set of 3 integers,

there must be 1 that when divided by 2 has remainder 0, and when divided by 3 has remainder 0.

If $3m+l$ is divisible by 2, then $F_{\frac{3m+l}{2}}$ is a valid facet that we consider (since we consider all such facets F_j where $1 \leq j \leq \lfloor \frac{n}{3} \rfloor$). So the element preceding x_{n+1} in $F_{\frac{3m+l}{2}}$ is also in F_{3m+l} , separated by one element from x_{n+1} . However, the pair shows up in both facets, so there is one repetition in F_{3m+l} .

Now suppose $3m+l'$ is divisible by 3. Then by the same logic as above, the element that precedes x_{n+1} directly in $F_{3m+l'}$ is separated by two elements from x_{n+1} in $F_{\frac{3m+l'}{2}}$. However, the pair shows up in both facets, so there is one repetition in $F_{3m+l'}$. If $3m+l$ is divisible by both 2 and 3, two pairs with x_{n+1} will be repetitions.

Furthermore, consider $F_{3(m+1)} = \{x_{n+1}, x_{n+1-(3m+3)}, x_{n+1-2(3m+3)}, x_{n+1-3(3m+3)}\}$. Since $3(m+1)$ is divisible by 3, we have that $\frac{2}{3} \times 3(m+1) = 2(m+1)$ is an integer less than $3(m+1)$. Then $F_{2(m+1)}$ is a valid facet in our construction, and it is $F_{2(m+1)} = \{x_{n+1}, x_{n+1-(2m+2)}, x_{n+1-(4m+4)}, x_{n+1-(6m+6)}\}$. Note how the pair $\{x_{n+1}, x_{n+1-(6m+6)}\}$ repeats in both facets. So when considering three consecutive integers, $3m+1, 3m+2, 3m+3$, the facet corresponding to the index divisible by three actually has 2 elements that are repeated.

So when considering contributions to $\dim A_2$ that include the vertex x_{n+1} , each block of facets $F_{3m+1}, F_{3m+2}, F_{3m+3}$ contribute 6, as of the 9 possible contributions, 3 are repeats.

Now we know we the biggest jump size our facets of $\text{vdw}(n+1, 3)$ can have is $\lfloor \frac{n}{3} \rfloor$. Since the assumption was that $\lfloor \frac{n}{3} \rfloor = 3z+r$, we can form z such groupings of three facets of consecutive jump sizes, and each such group contributes 6 to $\dim A_2$ when considering subsets that include x_{n+1} . Thus, the facets of $\text{vdw}(n+1, 3)$ that include x_{n+1} contribute at least $6z$ subsets containing x_{n+1} to $\dim A_2$, as wanted. \square

We now present the main remaining theorem:

THEOREM 4.18. *For $\text{vdw}(n, 3)$, $n \geq 8$, $\dim_K A_2 \leq \dim_K A_3$.*

PROOF. We know from Theorem 4.16 that for $\text{vdw}(n, 3)$, the dimension of A_3 is

$$\sum_{j=1}^{\lfloor \frac{n}{4} \rfloor} (4j + 3(n - 4j)) + \sum_{j=\lceil \frac{n}{4} \rceil}^d (4(n - 3j)),$$

where j sums over the jump sizes of the facets and if n is divisible by 4 we only include it in the first component of our sum. We also know $d = \lfloor \frac{n-1}{3} \rfloor$. We consider the same partition of facets. For the sake of induction, we can use the computations provided previously in Table 4.1 to deduce that for $n = 8$, our base case, $\dim_k A_2 \leq \dim_k A_3$.

Now assume that $\dim_k A_2 \leq \dim_k A_3$ for some arbitrary $\text{vdw}(n, 3)$. Now consider $\text{vdw}(n+1, 3)$. We know that the new facets being added must contain $n+1$, and there

are $\lfloor \frac{n}{3} \rfloor$ such new facets added. These are

$$\{x_{n-2}, x_{n-1}, x_n, x_{n+1}\}, \{x_{n-5}, x_{n-3}, x_{n-1}, x_{n+1}\} \cdots \{x_{n+1-3d}, x_{n+1-2d}, x_{n+1-d}, x_{n+1}\}.$$

When $j \leq \frac{n}{4}$, and we are considering the facets with elements that are repeated in other existing facets. We also want to group our facets into groups of 3 in order to use Lemma 4.17. In order to do so, we must examine the total number of facets we add when we go from $\text{vdw}(n, 3)$ to $\text{vdw}(n+1, 3)$, which is $d = \lfloor \frac{n}{3} \rfloor$. We consider cases on the divisibility of d by 3:

Case 1: $d \equiv 0 \pmod{3}$

If $d \equiv 0 \pmod{3}$, then all the facets we add when we go from $\text{vdw}(n, 3)$ to $\text{vdw}(n+1, 3)$ can be evenly divided into groups of 3. Let us consider the partition of these facets of jump size j into $j \leq \lfloor \frac{n+1}{4} \rfloor$ and $j \geq \lfloor \frac{n+1}{4} \rfloor$.

If we were to group all the facets of jump size $j \leq \lfloor \frac{n+1}{4} \rfloor$ into groups of 3, we will have at most $\frac{n+1}{4} \times \frac{1}{3} = \frac{n+1}{12}$ such groups. By Lemma 4.17, each such group contributes 6 to $\dim A_2$, namely the subsets of size two that contain x_{n+1} . Furthermore, since $j \leq \lfloor \frac{n+1}{4} \rfloor$, we know these facets all have corresponding facets in $\text{vdw}(n, 3)$ shifted by 1 unit, so all subsets of size two that do not contain x_{n+1} are repeats. Thus, in total they contribute at most $\frac{n+1}{12} \times 6 = \frac{n+1}{2}$ to $\dim A_2$.

For each facet we can still create 3 new elements of A_3 , namely the three subsets that include $n+1$.

However, each such facet contributes 3 to $\dim A_3$ by Theorem 4.16, so in total they contribute at least $\frac{n+1}{4} \times 3 = \frac{3}{4}(n+1)$ to $\dim A_3$.

For example, consider an arbitrary jump j , and the associated facet

$$\{x_{n+1-3j}, x_{n+1-2j}, x_{n+1-j}, x_{n+1}\}.$$

This facet adds at least 3 to $\dim A_3$, by adding

$$\{x_{n+1-3j}, x_{n+1-2j}, x_{n+1}\}, \{x_{n+1-3j}, x_{n+1-j}, x_{n+1}\}, \text{ and } \{x_{n+1-2j}, x_{n+1-j}, x_{n+1}\}.$$

The subset of size three $\{x_{n+1-3j}, x_{n+1-2j}, x_{n+1-j}\}$ may or may not be a new addition. When considering all the possible additions to $\dim A_2$, namely

$$\begin{aligned} &\{x_{n+1-3j}, x_{n+1-2j}\}, \{x_{n+1-3j}, x_{n+1-j}\}, \{x_{n+1-2j}, x_{n+1-j}\}, \{x_{n+1-3j}, x_{n+1}\}, \{x_{n+1-2j}, x_{n+1}\}, \\ &\text{and } \{x_{n+1-j}, x_{n+1}\}, \end{aligned}$$

the first 3 subsets of size two already appear in $\text{vdw}(n, 3)$. They are all subsets of the facet $\{x_{n-3j}, x_{n-2j}, x_{n-j}, x_n\}$ in $\text{vdw}(n, 3)$, and are not new additions to $\dim A_2$. Thus $\dim A_2$ can only increase by at most 3, since the 3 possible new subsets of size 2 may also coincide with other new facets added for $\text{vdw}(n+1, 3)$.

We now consider facets of jump size $j \geq \lfloor \frac{n+1}{4} \rfloor$. There are at least $\lfloor \frac{n}{3} \rfloor - \lfloor \frac{n+1}{4} \rfloor$ such facets. To get an accurate estimate of this number, we consider dividing by 12, the lowest

common multiple of our denominators. Let $n = 12z + r$ for some positive integer z and $r \in \{0, 1, 2, \dots, 11\}$. Then:

$$\left\lfloor \frac{n}{3} \right\rfloor = \left\lfloor \frac{12z + r}{3} \right\rfloor = 4z + \left\lfloor \frac{r}{3} \right\rfloor, \text{ where } \left\lfloor \frac{r}{3} \right\rfloor \in \{0, 1, 2, 3\}$$

We also can simplify

$$\left\lfloor \frac{n+1}{4} \right\rfloor = \left\lfloor \frac{12z + r + 1}{4} \right\rfloor = 3z + \left\lfloor \frac{r+1}{4} \right\rfloor, \text{ where } \left\lfloor \frac{r+1}{4} \right\rfloor \in \{0, 1, 2, 3\}$$

So $\left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n+1}{4} \right\rfloor = z + \left\lfloor \frac{r}{3} \right\rfloor - \left\lfloor \frac{r+1}{4} \right\rfloor$. Based on the possible values r can take, we get the following:

$$\begin{aligned} & \text{Number of facets with large jumps} = \\ L = \left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n+1}{4} \right\rfloor &= \begin{cases} z & \text{if } r \in \{0, 1, 2, 3, 4, 5, 8, 11\} \\ z + 1 & \text{if } r \in \{6, 7, 9, 10\} \end{cases} \end{aligned}$$

So in order for $\dim A_2 \leq \dim A_3$, we need

$$\frac{n+1}{2} + 6L \leq \frac{3}{4}(n+1) + 4L$$

$$\frac{n+1}{2} + 2L \leq \frac{3}{4}(n+1)$$

$$2L \leq \frac{3}{4}(n+1) - \frac{n+1}{2}$$

$$2L \leq \frac{3}{4}(n+1) - \frac{2}{4}(n+1)$$

$$2L \leq \frac{1}{4}(n+1)$$

$$L \leq \frac{1}{8}(n+1)$$

$$L \leq \frac{1}{8}(12z + r + 1) = \frac{3}{2}z + \frac{r+1}{8}.$$

When $r \in \{0, 1, 2, 3, 4, 5, 8, 11\}$, and $L = z$, the inequality becomes:

$$z \leq \frac{3}{2}z + \frac{r+1}{8}$$

$$0 \leq \frac{1}{2}z + \frac{r+1}{8}.$$

But since z is a positive integer, and $r \in \{0, 1, 2, 3, 4, 5, 8, 11\}$ is also always positive, this is true in all cases.

Now if $r \in \{6, 7, 9, 10\}$, and $L = z + 1$, the inequality becomes:

$$\begin{aligned} z + 1 &\leq \frac{3}{2}z + \frac{r + 1}{8} \\ 1 &\leq \frac{1}{2}z + \frac{r + 1}{8}. \end{aligned}$$

Since the list of the possible values r can take is small and finite, we can work with direct computations:

- (1) If $r = 6 \rightarrow \frac{r+1}{8} = \frac{7}{8}$
- (2) If $r = 7 \rightarrow \frac{r+1}{8} = \frac{8}{8} = 1$
- (3) If $r = 9 \rightarrow \frac{r+1}{8} = \frac{10}{8}$
- (4) If $r = 10 \rightarrow \frac{r+1}{8} = \frac{11}{8}$

For all values of r listed except for 6, $\frac{r+1}{8} \geq 1$, so the inequality holds easily. For $n = 6$, even though $\frac{r+1}{8} < 1$, since n is a positive integer, its value is at least 1, so the inequality still holds. Thus, in either case, the inequality holds, and so $\dim A_2 \leq \dim A_3$.

Case 2: $d \equiv 1 \pmod{3}$

If $d \equiv 1 \pmod{3}$, then we have one facet of large jump size that was not grouped into a group of 3. Since $j \geq \lceil \frac{n+1}{4} \rceil$, this facet will contribute 6 to $\dim A_2$ and 4 to $\dim A_3$. Thus our inequality

$$\frac{n+1}{2} + 6L \leq \frac{3}{4}(n+1) + 4L$$

becomes

$$\frac{n+1}{2} + 6L + 6 \leq \frac{3}{4}(n+1) + 4L + 4 \iff \frac{n+1}{2} + 6L + 2 \leq \frac{3}{4}(n+1) + 4L$$

And so our final inequality becomes

$$L \leq \frac{3}{2}z + \frac{r+1}{8} - 1.$$

When $r \in \{0, 1, 2, 3, 4, 5, 8, 11\}$, and $L = z$, the inequality becomes:

$$\begin{aligned} z &\leq \frac{3}{2}z + \frac{r+1}{8} - 1 \\ 1 &\leq \frac{1}{2}z + \frac{r+1}{8} \end{aligned}$$

But since z is a positive integer, if $z \geq 2$ this always hold. If $z = 1$ this inequality holds if $r \geq 3$. So this is true for all $n \geq 15$, but for values of n less than 15, we know the inequality holds for these values of n based on Table 4.1.

Now if $r \in \{6, 7, 9, 10\}$, and $L = z + 1$, the inequality becomes:

$$z + 1 \leq \frac{3}{2}z + \frac{r + 1}{8} - 1$$

$$2 \leq \frac{1}{2}z + \frac{r + 1}{8}$$

If $z \geq 4$, then $\frac{1}{2}z \geq 2$, so this inequality always holds since r is a positive integer.

If $z = 3$, this inequality holds if $\frac{r+1}{8} \geq \frac{1}{2} \iff r \geq 3$, which is always true in this case.

If $z = 2$, this inequality holds if $\frac{r+1}{8} \geq 1 \iff r \geq 7$. So the inequality is not true if $z = 2$ and $r = 6$, where we get $n = 30$. However, we check this directly in Macaulay2 and confirm that $\dim A_2 \leq \dim A_3$ using the data in Table 4.2.

If $z = 1$, this inequality holds if $\frac{r+1}{8} \geq \frac{3}{2} \iff r \geq 11$. So for $z = 1$ and $r = 6, 7, 9, 10$ and get $n = 18, 19, 21, 22$, we check this directly in Macaulay2 and confirm that $\dim A_2 \leq \dim A_3$ using the data in Table 4.2.

So in all sub-cases within Case 2, the inequality holds, and so $\dim A_2 \leq \dim A_3$.

Case 3: $d \equiv 2 \pmod{3}$

In this case, we have two facets with jump size $j \geq \lceil \frac{n+1}{4} \rceil$, so they contribute 12 to $\dim A_2$ and 8 to $\dim A_3$. Thus our inequality

$$\frac{n+1}{2} + 6L \leq \frac{3}{4}(n+1) + 4L$$

becomes

$$\frac{n+1}{2} + 6L + 12 \leq \frac{3}{4}(n+1) + 4L + 8 \iff \frac{n+1}{2} + 6L + 4 \leq \frac{3}{4}(n+1) + 4L,$$

and our final inequality becomes

$$L \leq \frac{3}{2}z + \frac{r+1}{8} - 2.$$

When $r \in \{0, 1, 2, 3, 4, 5, 8, 11\}$, and $L = z$, the inequality becomes:

$$z \leq \frac{3}{2}z + \frac{r+1}{8} - 2$$

$$2 \leq \frac{1}{2}z + \frac{r+1}{8}$$

$$4 \leq z + \frac{r+1}{4}$$

$$4 - \frac{r+1}{4} \leq z$$

Since the minimum value r can take here is 0, this inequality is true for all $z \geq 4$.

Manually checking, we see that the inequality holds for :

(1) $z = 3$ if $r \geq 3$

(2) $z = 2$ if $r \geq 7$

(3) $z = 1$ if $r \geq 11$.

So we must check if the inequality $\dim A_2 \leq \dim A_3$ is true directly using Macaulay2 for $n = 36, 37, 38$; $n = 24, 25, 26, 27, 28, 29$; and $n = 15, 16, 17, 20$. This was computed, and the data in Table 4.2 shows that this inequality holds in all cases for this sub-case.

Now if $r \in \{6, 7, 9, 10\}$, and $L = z + 1$, the inequality becomes:

$$z + 1 \leq \frac{3}{2}z + \frac{r+1}{8} - 2$$

$$3 \leq \frac{1}{2}z + \frac{r+1}{8}$$

$$24 \leq 4z + r + 1$$

$$23 - r \leq 4z$$

$$\frac{23 - r}{4} \leq z$$

This inequality holds if $z \geq 5$. So for $z = 1, 2, 3, 4$, we check manually based on the 4 possible values of the remainder, and see that the inequality holds if:

(1) $z = 1$ and $r \geq 19$

(2) $z = 2$ and $r \geq 15$

(3) $z = 3$ and $r \geq 11$

(4) $z = 4$, and $r \geq 7$

So we check for $n = 18, 19, 21, 22$; $n = 30, 31, 33, 34$; $n = 42, 43, 45, 46$; and $n = 54$ manually using Macaulay2, and confirm using the data in Table 4.2 that the inequality $\dim A_2 \leq \dim A_3$ holds.

Thus in all cases, the dimension of A_3 grows faster than the dimension of A_2 , thus for $\text{vdw}(n+1, 3)$, $\dim_K A_2 \leq \dim_K A_3$. \square

We now provide a table showing that the inequality holds for values of n where we had to check using computations on Macaulay2:

$\mathbf{A}(\text{vdw}(n, 3))$	$\dim A_2, \dim A_3$
$A(\text{vdw}(15, 3))$	74, 99
$A(\text{vdw}(16, 3))$	89, 116
$A(\text{vdw}(17, 3))$	103, 132
$A(\text{vdw}(18, 3))$	117, 148
$A(\text{vdw}(19, 3))$	132, 168
$A(\text{vdw}(20, 3))$	147, 188
$A(\text{vdw}(21, 3))$	159, 207
$A(\text{vdw}(22, 3))$	177, 230
$A(\text{vdw}(24, 3))$	213, 276
$A(\text{vdw}(25, 3))$	234, 302
$A(\text{vdw}(26, 3))$	255, 328
$A(\text{vdw}(27, 3))$	276, 354
$A(\text{vdw}(28, 3))$	298, 384
$A(\text{vdw}(29, 3))$	317, 413
$A(\text{vdw}(30, 3))$	336, 442
$A(\text{vdw}(31, 3))$	358, 475
$A(\text{vdw}(33, 3))$	401, 540
$A(\text{vdw}(34, 3))$	428, 576
$A(\text{vdw}(36, 3))$	482, 648
$A(\text{vdw}(37, 3))$	510, 687
$A(\text{vdw}(38, 3))$	538, 726
$A(\text{vdw}(42, 3))$	666, 892
$A(\text{vdw}(43, 3))$	702, 938
$A(\text{vdw}(45, 3))$	771, 1029
$A(\text{vdw}(46, 3))$	805, 1078
$A(\text{vdw}(54, 3))$	1107, 1498

TABLE 4.2. Dimensions of A_2, A_3 for $\mathbf{A}(\text{vdw}(n, 3))$ that were checked via Macaulay2

We have established that $\dim A_2 \leq \dim A_3$ for all $\text{vdw}(n, 3)$ for $n \geq 8$, and by extension the map $\times \ell$ cannot be surjective. Thus for the map $\times \ell$ to have maximal rank, it must be injective. We now prove that for $n \geq 8$, this is not possible by finding an element in the kernel of the matrix representation of this map.

THEOREM 4.19. $A(\text{vdw}(n, 3))$, for $n \geq 7$, does not have the Weak Lefschetz Property.

PROOF. The case $n = 7$ was proved in the preceding section, so we consider $n \geq 8$. Similar to the case of $A(\text{vdw}(n, 1))$ and $A(\text{vdw}(n, 2))$, we work with the matrix representation of the homomorphism induced by the multiplication by ℓ . We consider the map from A_2 to A_3 . Thus, in our matrix, the columns are indexed by the basis elements of A_2 , the domain of our map $\times \ell : A_2 \rightarrow A_3$, and the rows are indexed by the basis elements of A_3 . The columns represent our even and odd 2-sets from above. We arrange the columns so that all the columns representing the even 2-sets are placed first, then the columns representing the odd 2-sets are placed next. The even and odd blocks are also arranged in ascending order, by the size of $j, 2j$, or $3j$.

Suppose this matrix, which we will denote $M_{(n,3)}$ is of dimension $a \times b$. We construct a $b \times 1$ vector \vec{v} that is in the kernel of $M_{(n,3)}$. By construction, each row of $M_{(n,3)}$ has all entries as 0 but three, which take the value 1. This is since each row represents some subset of a facet that has three elements in it, and the columns representing each of the two element subsets one can create from this subset of size three gets the value 1, whereas the others all get a 0.

We construct \vec{v} as the following: for each row i in $M_{(n,3)}$, consider the three columns that have the value 1. We consider the jump set of the basis element in A_3 that our row represents. By Corollary 4.12, this jump set must be either $\{j, j, 2j\}$ or $\{j, 2j, 3j\}$, where j is the jump factor of the facet the element of A_3 was derived from. Consider column m in our matrix, and the 2-set it represents. For each of these 2-sets, let q be the jump set of this 2-set. Then $q = 2^k m$ for some odd m by the unique primary decomposition. Since each row in \vec{v} corresponds to a column in $M_{(n,3)}$, for each column in $M_{(n,3)}$ with jump set $q = 2^k m$, we construct \vec{v} so that the entry in the row corresponding to that column has the value $(-1)^{k+1}(2^k)$. Please note that a detailed example is given just below the proof.

Note that this also implies if q is odd, so $q = 2^0 m'$, then row m entry of $\vec{v} = -1$, as the decomposition of q into primes would have an exponent of 0 for the prime number 2.

We want to show that when each row of $M_{(n,3)}$ multiplied by \vec{v} , it yields zero. Thus the entire product of the matrix $M_{(n,3)}$ by \vec{v} will also yield the 0 vector in the end. We focus on the output we get when an arbitrary row of $M_{(n,3)}$ is multiplied by \vec{v} .

Suppose q is odd. If q is odd, then $q = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_n^{a_n}$, where all p_i are primes not equal to two, and all the a_i are arbitrary integers. If the row of $M_{(n,3)}$ that this two-set is derived from is of the form $\{j, j, 2j\}$, then it must be of the form $\{q, q, 2q\}$ or $\{\frac{1}{2}q, \frac{1}{2}q, q\}$, based on the order of the elements. But q is assumed to be odd, so $\frac{1}{2}q$ is not an integer, and this case cannot happen by construction. If $q = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_n^{a_n}$ is

odd, then $2q = 2p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_n^{a_n}$, so its corresponding row entry in \vec{v} is $2^1 = 2$. So when $M_{(n,3)}$ is multiplied by \vec{v} , the three non-zero terms in row m will be $1 \cdot (-1) + 1 \cdot (-1)$ (since q is odd so its corresponding row in \vec{v} is -1) $+ 2 = 0$. If the row of $M_{(n,3)}$ that this two-set is derived from is of the form $\{j, 2j, 3j\}$, then it must be of the form $\{q, 2q, 3q\}$ or $\{\frac{1}{2}q, q, \frac{3}{2}q\}$, or $\{\frac{1}{3}q, \frac{2}{3}q, q\}$. Again, the middle case is not possible since q is odd, so we are left with $\{q, 2q, 3q\}$ or $\{\frac{1}{3}q, \frac{2}{3}q, q\}$, where the latter only holds if q is divisible by 3.

In either case, since q is odd, $3q, \frac{1}{3}q$ will also be odd since the products and powers of odd numbers are odd, and $2q, \frac{2}{3}q$ are even numbers such that dividing them by 2 yields an odd number, so 2 in their primary decomposition has an exponent of 1. When $M_{(n,3)}$ is multiplied by \vec{v} , the three non-zero terms in row m will be $1 \cdot (-1) + 1 \cdot (-1) + 1 \cdot 2 = 0$. In either case, if q is odd, the corresponding entries in the product of $M_{(n,3)}$ by \vec{v} is 0.

Now suppose q is even. Then $q = 2^k m$ for some odd m . q being even also implies $2q = 2^{k+1}m$ and $3q = 2^k(3m)$ are both also even, with $3m$ being odd. We now consider sub-cases based on the odd and even parity of k .

If k is even, evidently $k + 1, k - 1$ is odd and $k + 2$ is even.

If the row of $M_{(n,3)}$ that this two-set is derived from is of the form $\{j, j, 2j\}$, then it must be of the form $\{q, q, 2q\}$ or $\{\frac{1}{2}q, \frac{1}{2}q, q\}$. In the first case, when $M_{(n,3)}$ is multiplied by \vec{v} , the three non-zero terms in the row will be $1 \cdot (-2^k) + 1 \cdot (-2^k) + 1 \cdot (2^{k+1})$, since $q = 2^k m$ and k is even so $k + 1$ is odd, and using the construction of \vec{v} above. But this simplifies to $2 \cdot (-2^k) + 2^{k+1} = -2^{k+1} + 2^{k+1} = 0$, resulting in a zero entry in the matrix. In the second case, the three non-zero terms in the multiplication will be $1 \cdot (-2^{k-1}) + 1 \cdot (-2^{k-1}) + 1 \cdot (2^k)$, since $\frac{1}{2}q$ implies that the exponent of 2 in the decomposition of q decreases by 1, and if k is even then $k - 1$ must be odd. But this simplifies to $2 \cdot (-2^{k-1}) + 2^k = -2^k + 2^k = 0$, resulting in a zero entry in the matrix yet again.

If the row of $M_{(n,3)}$ that this two-set is derived from is of the form $\{j, 2j, 3j\}$, then it must be of the form $\{q, 2q, 3q\}$ or $\{\frac{1}{2}q, q, \frac{3}{2}q\}$, or $\{\frac{1}{3}q, \frac{2}{3}q, q\}$.

In the first case, when $M_{(n,3)}$ is multiplied by \vec{v} , the three non-zero terms in the row will be $1 \cdot (-2^k) + 1 \cdot (2^{k+1}) + 1 \cdot (-2^k)$, since k is even, $k + 1$ is odd, and multiplying q by 3 does not change the exponent of 2 in its prime decomposition. This simplifies to $-2 \cdot (-2^k) + (2^{k+1}) = 0$.

In the second case, the three non-zero terms in the multiplication will be $1 \cdot (2^{k-1}) + 1 \cdot (-2^k) + 1 \cdot (-2^k)$ since again dividing by 2 is equivalent to subtracting one from the exponent of 2 in the prime decomposition of q , and if k is even then $k - 1$ is odd. Also multiplying q by $\frac{3}{2}$ is equivalent to subtracting one from the exponent of 2 and adding one to the exponent of 3 decomposition of q , with the latter having no impact on the formation of \vec{v} . This also simplifies to $-2 \cdot (-2^k) + (2^{k+1}) = 0$.

Finally, in the third case, the three non-zero terms in the multiplication will be $1 \cdot (-2^k) + 1 \cdot (2^{k+1}) + 1 \cdot (-2^k)$, using the fact that multiplying q by $\frac{1}{3}$ does not change the exponent of 2 in its prime decomposition, whereas multiplying q by $\frac{2}{3}$ increases it by 1. This also simplifies to $-2 \cdot (-2^k) + (2^{k+1}) = 0$. In all cases the resultant row reduces to 0.

If k is odd, evidently $k + 1, k - 1$ are even and $k + 2$ is odd.

If the row of $M_{(n,3)}$ that this two-set is derived from is of the form $\{j, j, 2j\}$, then it must be of the form $\{q, q, 2q\}$ or $\{\frac{1}{2}q, \frac{1}{2}q, q\}$.

In the first case, when $M_{(n,3)}$ is multiplied by \vec{v} , the three non-zero terms in the row will be $1 \cdot (2^k) + 1 \cdot (2^k) + 1 \cdot (-2^{k+1})$, since $q = 2^k m$ and k is odd so $k + 1$ is even. This simplifies to $2 \cdot (2^k) - 2^{k+1} = 2^{k+1} - 2^{k+1} = 0$, resulting in a zero entry in the matrix.

In the second case, the three non-zero terms in the multiplication will be $1 \cdot (2^{k-1}) + 1 \cdot (2^{k-1}) - 1 \cdot (2^k)$, since $\frac{1}{2}q$ implies that the exponent of 2 in the decomposition of q decreases by 1. But this simplifies to $2 \cdot (2^{k-1}) + 2^k = 2^k - 2^k = 0$, resulting in a zero entry in the matrix yet again.

If the row of $M_{(n,3)}$ that this two-set is derived from is of the form $\{j, 2j, 3j\}$, then it must be of the form $\{q, 2q, 3q\}$ or $\{\frac{1}{2}q, q, \frac{3}{2}q\}$, or $\{\frac{1}{3}q, \frac{2}{3}q, q\}$.

In the first case, when this row of $M_{(n,3)}$ is multiplied by \vec{v} , the three non-zero terms in the row will be $1 \cdot (2^k) + 1 \cdot (-2^{k+1}) + 1 \cdot (2^k)$, since multiplying q by 3 does not change the exponent of 2 in its prime decomposition. This simplifies to $2 \cdot (2^k) - (2^{k+1}) = 0$.

In the second case, the three non-zero terms in the multiplication will be $1 \cdot (-2^{k-1}) + 1 \cdot (2^k) + 1 \cdot (-2^{k-1})$ since again dividing by 2 is equivalent to subtracting one from the exponent of 2 in the prime decomposition of q . This also simplifies to $2 \cdot (-2^{k-1}) + (2^k) = 0$.

Finally, in the third case, the three non-zero terms in the multiplication will be $1 \cdot (2^k) + 1 \cdot (-2^{k+1}) + 1 \cdot (2^k)$, using the fact that multiplying q by $\frac{1}{3}$ does not change the exponent of 2 in its prime decomposition, whereas multiplying q by $\frac{2}{3}$ increases it by 1. This also simplifies to $2 \cdot (2^k) - (2^{k+1}) = 0$. In all cases the resultant row reduces to 0.

Since q was chosen as an arbitrary column in our matrix $M_{(n,3)}$, we have shown that in all possible cases, when multiplied by \vec{v} as defined above, the product of $M_{(n,3)}$ and \vec{v} yields a column vector of all zeros, the zero vector. Thus \vec{v} is in the kernel of $M_{(n,3)}$. Since $\dim_k A_2 < \dim_k A_3$, the map in question can only have maximal rank if its rank is $\dim_k A_2$.

Since the kernel is non-empty, by the Rank-Nullity theorem the rank of the map is strictly less than $\dim_k A_2$. Thus the map is not of maximal rank, and consequently $A(\text{vdw}(n, 3))$ does not have the Weak Lefschetz Property. \square

CHAPTER 5

van der Waerden simplicial complexes that are pseudo-manifolds

Certain simplicial complexes where the facets meet in a pre-described manner are also pseudo-manifolds. Deducing for which values of n and k the associated Artinian algebras of the van der Waerden simplicial complex, that is $A(\mathbf{vdw}(n, k))$, is a pseudo-manifold allows us to use known results of Dao and Nair in [11] to deduce when our algebras $A(\mathbf{vdw}(n, k))$ have the strong or weak Lefschetz properties.

In this chapter, we produce a complete characterization of when $\mathbf{vdw}(n, k)$ is a pseudo-manifold. We begin this chapter with an essential lemma, and then give a detailed classification for which values of n and k the associated van der Waerden simplicial complex is a pseudo-manifold. We then use this result to deduce some Lefschetz properties for our algebras.

First we state a useful lemma:

LEMMA 5.1. *Any facet of $\mathbf{vdw}(n, k)$ has either:*

- (1) $k + 1$ odd entries, or
- (2) $\lceil (k + 1)/2 \rceil$ odd entries and $\lfloor (k + 1)/2 \rfloor$ even entries, or
- (3) $\lfloor (k + 1)/2 \rfloor$ odd entries and $\lceil (k + 1)/2 \rceil$ even entries, or
- (4) $k + 1$ even entries.

We note that we use the term even and odd entries by examining the indices of each vertex, in accordance with Definition 2.51.

PROOF. Let F be an arbitrary facet of $\mathbf{vdw}(n, k)$ of the form

$$F = \{x_i, x_{i+a}, x_{i+2a}, x_{i+3a}, \dots, x_{i+ka}\},$$

i.e., F has jump factor a . Note that for $[n] = \{1, 2, 3, \dots, n\}$:

- (1) If n is even, there are exactly $\frac{n}{2}$ even elements and $\frac{n}{2}$ odd elements among $[n]$.
- (2) If n is odd, there are exactly $\frac{(n-1)}{2}$ even elements and $\frac{(n+1)}{2}$ odd elements among $[n]$. Equivalently, there are $\lfloor \frac{n}{2} \rfloor$ even elements and $\lceil \frac{n}{2} \rceil$ odd elements in $[n]$.

We consider two cases, depending on whether the jump factor a is odd or even.

If a is even, then adding an even number or a multiple of an even number (which is also an even number) to i does not change the parity of i . So, if a is even, and:

- (1) If i is odd, then the facet is comprised entirely of odd entries, and falls under case one.
- (2) If i is even, then the facet is comprised entirely of even entries, and falls under case four.

If a is odd, then adding it to an odd number results in an even number, and alternatively adding it to an even number results in an odd number. If a is odd and i is odd, then the first entry of the facet is odd, followed by an even entry, and the pattern continues: odd, even, odd, even, If a is odd and i is even, then the alternating pattern also exists but the order is switched, that is even, odd, even, odd,

However, the pattern goes on for $k+1$ iterations, the size of our facet. We now consider two more cases:

If $k+1$ is even, then exactly half the entries of our facet is even and half are odd. So F has $\frac{(k+1)}{2} = \lceil (k+1)/2 \rceil = \lfloor (k+1)/2 \rfloor$ odd (and analogously even) entries, and falls under both cases two and three.

If $k+1$ is odd, then the same pattern implies there will be either one more odd entry than even, or one more even entry than odd. Using the note above on $[n] = \{1, 2, 3, \dots, n\}$, we see that if we start with an odd entry, then the number of odd entries will be one more than the number of even entries. This will fall under case two. Similarly, if we start with an even entry, then the number of even entries will be one more than the number of odd entries, and thus will fall under case three, as desired.

These cases now show the statement is true. □

Using this essential lemma, we now define pseudo-manifolds, and classify for which values of n and k the simplicial complex $\text{vdw}(n, k)$ is a pseudo-manifold.

DEFINITION 5.2. A k -dimensional simplicial complex is a **pseudo-manifold** if all three of the following conditions hold:

- (1) the simplicial complex is pure (all facets are of the same dimension);
- (2) all $k-1$ dimensional faces of the simplicial complex are contained in a maximum of two facets; and
- (3) given any two facets, F_a, F_b of the simplicial complex, there exists a sequence of facets $F_a = G_0, G_1, G_2, \dots, G_n = F_b$ such that the dimension of $G_i \cap G_{i+1}$ is $k-1$ for all $0 \leq i \leq n-1$.

Criteria two of Definition 5.2 can also be restated in terms of the ridges of a pure, d -dimensional simplicial complex, which we defined earlier in Definition 2.4 as any face of dimension $d-1$. Criteria two thus requires that all ridges of $\text{vdw}(n, k)$ are contained in at most two facets.

By a result of Dao and Nair, if Δ is a pseudo-manifold, this has implications on whether $A(\Delta)$ has the Weak Lefschetz Property. Before we examine this result, we state a useful lemma about the jumps of the van der Waerden simplicial complexes.

LEMMA 5.3. *If $\frac{n}{2} \leq k \leq n$, then $\text{jump}(F) = 1$ for all facets $F \in \text{vdw}(n, k)$*

PROOF. Let $\frac{n}{2} \leq k \leq n$, and F be an arbitrary facet of $\text{vdw}(n, k)$. Then there exists $i \in \{1, 2, 3, \dots, n\}$ and $e \in \mathbb{Z}$ such that $F = \{x_i, x_{i+e}, \dots, x_{i+ke}\}$ and $i + ke \leq n$. By construction, $\text{jump}(F) = e$. Suppose for the sake of a contradiction that $e \geq 2$. Then $i + ke \geq i + 2k$ since $e \geq 2$. But this means $i + ke \geq i + (\frac{2n}{2} = n)$ since $k \geq \frac{n}{2}$. This implies $i + ke \geq i + n > n$.

This is a contradiction, as $i + ke \notin \{1, 2, 3, \dots, n\}$. Thus e cannot be greater than one, so all facets of $\text{vdw}(n, k)$ where $\frac{n}{2} \leq k \leq n$ have $\text{jump}(F) = 1$. \square

We now use this result to classify for which values of n and k the simplicial complex $\text{vdw}(n, k)$ is a pseudo-manifold.

THEOREM 5.4. *Suppose $n > k \geq 3$. Then $\text{vdw}(n, k)$ is a pseudo-manifold if and only if $\frac{n}{2} \leq k < n$.*

PROOF. We begin with the backward direction. Let $\frac{n}{2} \leq k \leq n$, and consider $\text{vdw}(n, k)$. By construction, $\text{vdw}(n, k)$ is pure. By Definition 2.49, we know that all facets F of $\text{vdw}(n, k)$ must have $\text{jump}(F) = 1$. Thus all the facets of $\text{vdw}(n, k)$ have the following natural order:

$$\begin{aligned} F_1 &= \{x_1, x_2, \dots, x_{k+1}\} \\ F_2 &= \{x_2, x_3, \dots, x_{k+2}\} \\ &\vdots \\ F_d &= \{x_d, x_{d+1}, \dots, x_{k+d} = x_n\}. \end{aligned}$$

where $d = n - k$.

For each facet F_i of dimension k (and thus cardinality $k + 1$), we construct faces of dimension $k - 1$ (cardinality k) by omitting one element from the facet. Let us denote the element that we omit using the circumflex symbol, that is, \hat{x} . Then for each facet, we can construct $k + 1$ of such faces. For example, for the first two facets, we have:

From $F_1 = \{x_1, x_2, \dots, x_{k+1}\}$, we construct:

$$\begin{aligned} G_{1,1} &= \{\hat{x}_1, x_2, \dots, x_{k+1}\} \\ G_{1,2} &= \{x_1, \hat{x}_2, \dots, x_{k+1}\} \\ &\vdots \\ G_{1,k+1} &= \{x_1, x_2, \dots, \hat{x}_{k+1}\}. \end{aligned}$$

From $F_2 = \{x_2, x_3, \dots, x_{k+2}\}$, we construct:

$$\begin{aligned} G_{2,1} &= \{\widehat{x}_2, x_3, \dots, x_{k+2}\} \\ G_{2,2} &= \{x_2, \widehat{x}_3, \dots, x_{k+2}\} \\ &\vdots \\ G_{2,k+1} &= \{x_2, x_2, \dots, \widehat{x}_{k+2}\}. \end{aligned}$$

Similarly, for the last two facets in our ordering, we have:

$F_{d-1} = \{x_{d-1}, x_{d-2}, \dots, x_{k+d-1}\}$, we construct:

$$\begin{aligned} G_{d-1,1} &= \{\widehat{x}_{d-1}, x_d, \dots, x_{k+d-1}\} \\ G_{d-1,2} &= \{x_{d-1}, \widehat{x}_{d-2}, \dots, x_{k+d-1}\} \\ &\vdots \\ G_{d-1,k+1} &= \{x_{d-1}, x_d, \dots, \widehat{x}_{k+d-1}\}. \end{aligned}$$

And finally, for $F_d = \{x_d, x_{d-2}, \dots, x_{k+d}\}$, we construct:

$$\begin{aligned} G_{d,1} &= \{\widehat{x}_d, x_{d-2}, \dots, x_{k+d}\} \\ G_{d,2} &= \{x_d, \widehat{x}_{d-2}, \dots, x_{k+d}\} \\ &\vdots \\ G_{d,k+1} &= \{x_d, x_{d-2}, \dots, \widehat{x}_{k+d}\}. \end{aligned}$$

Note that $G_{j,1} = G_{j+1,k+1}$ for all $1 \leq j \leq d-1$, as they both consist of the elements, $x_{j+1}, x_{j+2}, \dots, x_{j+k}$ by construction. So these $k-1$ dimensional faces are contained in two facets already, namely F_j and F_{j+1} . By containing x_{j+1} , this face cannot be contained in facets F_k for all $k \geq j+1$. Similarly, by containing x_{j+k} , this face cannot be contained in facet F_k for all $k \leq j-1$. Hence faces of this form are strictly contained in two facets.

Now, we consider an arbitrary $k-1$ dimensional face, $G_{j,m}$, that is not in F_d . So $x_j \in G_{j,m}$, for $1 < m \leq k+1$, so $G_{j,m} \neq G_{j+q,m}$ for all $j < j+q \leq d$. This implies $G_{j,m} \not\subset F_z$ for all $z > j$. Similarly, $x_{k+m} \in G_{j,p}$ for all $1 \leq p < k+1$, so $G_{j,m} \neq G_{j-q,p}$ for all $1 \leq j-q < j$. This implies $G_{j,m} \not\subset F_z$ for all $z < j$. This shows that all arbitrary $k-1$ dimensional faces of this form are only contained in one facet. So all subsets of size k of F_m for $1 \leq m < d$ are only contained in at most two facets.

The only case remaining is F_d , and its $k-1$ dimensional faces of the form $G_{d,l}$ for $1 \leq l < k+1$, but excluding $G_{d,k+1}$ from the first case. Note that these faces all have the element x_{k+d} by construction, so can only be a subset of F_d . Thus all $k-1$ dimensional faces of any facet are contained in at most two facets.

To prove the third requirement for pseudo-manifolds, we note that in the ordering above, $F_i \cap F_{i+1}$ has dimension $k-1$, as they only differ by one element. So for any two facets, F_a, F_b with $a < b$ without loss of generality, we can choose the sequence

$F_a, F_{a+1}, F_{a+2}, \dots, F_{a+l=b}$ to be the sequence we need to meet requirement three. Hence if $\frac{n}{2} \leq k \leq n$, then $\text{vdw}(n, k)$ is a pseudo-manifold.

Now we prove the forward direction, that is, if $\text{vdw}(n, k)$ is a pseudo-manifold, then $\frac{n}{2} \leq k \leq n$.

Suppose $3 \leq k < \frac{n}{2}$. Note that in this case, we have the following facet, constructed entirely of odd entries:

$$F_1 = \{x_1, x_3, x_5, \dots, x_{2k+1}\}$$

Since $k < \frac{n}{2}$ implies $2k + 1 < n + 1$, so $2k + 1 \leq n$, making this a valid facet.

We also always have the facet generated by jump size 1:

$$F_2 = \{x_1, x_2, x_3, \dots, x_{k+1}\}.$$

Using Lemma 5.1 on $[n] = \{1, 2, 3, \dots, n\}$, this facet has $\frac{(k+1)}{2}$ even entries if k is odd, or $\lfloor \frac{(k+1)}{2} \rfloor$ even entries if k is even.

Assume for a contradiction that $\text{vdw}(n, k)$ is a pseudo-manifold. Then by property 3 of 5.2, there must be a sequence of facets $F_1 = G_0, G_1, \dots, G_n = F_2$ such that the dimension of $G_i \cap G_{i+1}$ is $k - 1$ for all $0 \leq i \leq n - 1$, i.e., consecutive facets in the chain differ by only one element. Since F_1 has all odd entries, but F_2 does not, there must be some facet G_m in this chain that is the last to consist of all odd entries only. By property 3, we have $G_m \cap G_{m+1}$ is of dimension $k + 1 - 1 = k$. Since G_m consists of all odd entries only, this implies their intersection of k elements must contain only all odd entries, so G_{m+1} has at least k odd entries. The facet F_2 has both odd and even entries, so G_{m+1} cannot have only odd entries either, so of its $k + 1$ elements, k are odd entries and one entry is an even entry. This is a contradiction to 5.1, as we know G_{m+1} has entries of mixed parity, so it must fall into cases 2 or 3. Using the fact that $k \geq 3$, in either case, G_{m+1} must have at least $\lfloor (k + 1)/2 \rfloor \geq \lfloor (3 + 1)/2 \rfloor = 2$ even entries. Thus no such chain of facets can exist between F_1 and F_2 , contradicting the fact that $\text{vdw}(n, k)$ is a pseudo-manifold in this case. \square

The previous case required the assumption that $k \geq 3$. We now present a lemma to consider the cases where $k = 1$ or $k = 2$.

LEMMA 5.5. *$\text{vdw}(n, 1)$ is a pseudo-manifold if and only if $n \leq 3$. $\text{vdw}(n, 2)$ is a pseudo-manifold if and only if $n \leq 6$.*

PROOF. Case 1: $k = 1$

We consider the $k = 1$ case by considering the cases $n = 2, n = 3$ and $n \geq 4$ separately.

- (1) $n = 2$. In this case $\text{vdw}(2, 1) = \langle \{x_1, x_2\} \rangle$. This simplicial complex has only one facet and thus easily satisfies all three conditions to be a pseudo-manifold.

- (2) $n = 3$. In this case $\mathbf{vdw}(3, 1) = \langle \{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_3\} \rangle$. This complex is also pure, each vertex is contained in exactly two facets, and for any of the three facets, changing either of its vertices gives the other two facets, again satisfying all three conditions for it to be a pseudo-manifold.
- (3) $n \geq 4$. In this case, we claim $\mathbf{vdw}(n, 1)$ is not a pseudo-manifold. For all such $\mathbf{vdw}(n, 1)$, the following is a partial list of valid facets:

$$\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \text{ and } \{x_2, x_4\}.$$

We see that when considering the $k - 1 = 1 - 1 = 0$ dimensional faces (i.e, the vertices), x_2 is already contained three facets, violating condition 2 of the definition of pseudo-manifolds. This $\mathbf{vdw}(n, 1)$ is not a pseudo-manifold for all $n \geq 4$.

Case 2: $k = 2$

Again, we consider the $k = 2$ case by considering the cases $n = 2, 3, 4, 5, 6$ and $n \geq 7$ cases separately.

- (1) $n = 3$. In this case $\mathbf{vdw}(3, 2)$ is generated by one facet only: $\{x_1, x_2, x_3\}$. Thus it satisfies all three conditions to be a pseudo-manifold.
- (2) $n = 4$. In this case $\mathbf{vdw}(4, 2)$ has two facets: $\{x_1, x_2, x_3\}$, and $\{x_2, x_3, x_4\}$. It is pure by construction, and while the 1-dimensional face $\{x_1, x_2\}$ appears in two facets, all other 1-dimensional faces only appear in one facet, satisfying condition 2. The intersection of the two facets is $\{x_2, x_3\}$, which has dimension 1, ensuring that $\mathbf{vdw}(4, 2)$ is a pseudo-manifold.
- (3) $n = 5$. In this case $\mathbf{vdw}(5, 2)$ is generated by the facets:

$$F_0 = \{x_1, x_2, x_3\}, F_1 = \{x_2, x_3, x_4\}, F_2 = \{x_3, x_4, x_5\}, \text{ and } F_3 = \{x_1, x_3, x_5\}.$$

It is pure by construction, and all 1-dimensional faces are in one facet except for $\{x_1, x_3\}$, $\{x_2, x_3\}$, $\{x_3, x_4\}$, and $\{x_3, x_5\}$, which appear in two. Thus all 1-dimensional facets are contained in at most two facets, satisfying property two. Finally, the sequence F_0, F_1, F_2, F_3 satisfies property 3, confirming $\mathbf{vdw}(5, 2)$ is a pseudo-manifold.

- (4) $n = 6$. In this case $\mathbf{vdw}(6, 2)$ is generated by the facets:

$$F_0 = \{x_1, x_2, x_3\}, F_1 = \{x_2, x_3, x_4\}, F_2 = \{x_3, x_4, x_5\}, F_3 = \{x_4, x_5, x_6\}, \\ F_4 = \{x_1, x_3, x_5\}, \text{ and } F_5 = \{x_2, x_4, x_6\}.$$

Here, the 1-dimensional faces

$$\{x_1, x_3\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}, \{x_3, x_5\}, \{x_4, x_5\}, \text{ and } \{x_4, x_6\}$$

appear in two facets, while all other 1-dimensional faces only appear in one facet, satisfying condition 2. For condition 3, the sequence $F_4, F_0, F_1, F_2, F_3, F_5$ satisfies the third property, resulting in $\mathbf{vdw}(6, 2)$ also being a pseudo-manifold.

- (5) $n \geq 7$. In this case, the following three sets are valid facets of $\mathbf{vdw}(n, 2)$ for all values of $n \geq 7$:

$$\{x_3, x_4, x_5\}, \{x_1, x_3, x_5\}, \{x_3, x_4, x_5\}.$$

But the 1-dimensional face $\{x_3, x_5\}$ is contained within all three of these facets, violating condition two and proving $\text{vdw}(n, 2)$ is not a pseudo-manifold for $n \geq 7$.

This now finishes the proof. \square

We summarize these results in the following corollary:

COROLLARY 5.6. *$\text{vdw}(n, k)$ is a pseudo-manifold if and only if:*

- (1) $k = 1$, and $n = 2$ or 3 , or
- (2) $k = 2$, and $n = 3, 4, 5$, or $n = 6$, or
- (3) $k \geq 3$, and $\frac{n}{2} \leq k < n$.

Before we can apply a theorem of Dao and Nair that relates when Artinian rings constructed from pseudo-manifolds have the Weak Lefschetz Property in certain degrees, we must introduce the definition of a boundary.

DEFINITION 5.7. A k -dimensional pseudo-manifold has a **boundary** if it has at least one $k - 1$ dimensional face that belongs to exactly one facet.

The following corollary summarizes when the pseudo-manifolds classified above have boundaries.

COROLLARY 5.8. *$\text{vdw}(n, k)$ is a pseudo-manifold with boundary if and only if:*

- $k = 1$, and $n = 2$, or
- $k = 2$, and $n = 3, 4, 5$, or $n = 6$, or
- $k \geq 3$ and $\frac{n}{2} \leq k < n$.

PROOF. We begin with the case $k \geq 3$ and $\frac{n}{2} \leq k < n$, which we prove separately. For an arbitrary $\text{vdw}(n, k)$ for $\frac{n}{2} \leq k < n$, we stated that these are all pseudo-manifolds, and that they all have the following facets:

$$\begin{aligned} F_1 &= \{x_1, x_2, \dots, x_{k+1}\} \\ F_2 &= \{x_2, x_3, \dots, x_{k+2}\} \\ &\vdots \\ F_d &= \{x_d, x_{d+1}, \dots, x_{k+d} = x_n\}. \end{aligned}$$

The $k - 1$ dimensional face $\{x_1, x_2, \dots, x_k\}$ is only contained in F_1 , as it contains x_1 , which is not in any of the other facets. So all such $\text{vdw}(n, k)$ are pseudo-manifolds with boundaries.

Now, we consider the van der Waerden simplicial complexes with small values of n and k that are pseudo-manifolds:

- (1) $\text{vdw}(2, 1)$: this simplicial complex is generated by only one facet, so any $k - 1 = 0$ dimensional face (vertex) must be contained in one facet only.

- (2) $\mathbf{vdw}(3, 1)$: each $k - 1 = 0$ dimensional face (vertex) is contained in 2 facets by construction, so this is **not** a pseudo-manifold with a boundary.
- (3) $\mathbf{vdw}(3, 2)$: this simplicial complex is also generated by only one facet, so any $k - 1 = 1$ dimensional face (edge) must be contained in one facet only.
- (4) $\mathbf{vdw}(4, 2)$: $\{x_1, x_2\}$ is a 1-dimensional face that is contained only in the facet $\{x_1, x_2, x_3\}$, so this is a pseudo-manifold with boundary.
- (5) $\mathbf{vdw}(5, 2)$: $\{x_1, x_5\}$ is a 1-dimensional face that is contained only in the facet $\{x_1, x_3, x_5\}$, so this is a pseudo-manifold with boundary.
- (6) $\mathbf{vdw}(6, 2)$: $\{x_5, x_6\}$ is a 1-dimensional face that is contained only in the facet $\{x_4, x_5, x_6\}$, so this is a pseudo-manifold with boundary.

□

So all of the $\mathbf{vdw}(n, k)$ that we have classified as pseudo-manifolds have boundaries except $\mathbf{vdw}(3, 1)$. We can now use a result from the existing literature [11], to see when $A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property.

THEOREM 5.9. *Let Δ be a pseudo-manifold with $\dim(\Delta) = d$. Then $A(\Delta)$ has the Weak Lefschetz Property in degree d if and only if:*

- (1) *The pseudo-manifold has a boundary, or*
- (2) *The pseudo-manifold does not have a boundary, but its dual graph is not bipartite.*

By the results discussed in this chapter, we have the following corollaries regarding some values of n and k such that $A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property in degree k :

COROLLARY 5.10. *Let $k \geq 3$. Then $A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property in degree k if $\frac{n}{2} \leq k < n$.*

PROOF. By Corollary 5.6, $\mathbf{vdw}(n, k)$ is a pseudo-manifold if and only if $\frac{n}{2} \leq k < n$, and that all pseudo-manifolds of this form have a boundary. So all of these k -dimensional pseudo-manifolds have the Weak Lefschetz Property in degree k , by Theorem 5.9. □

COROLLARY 5.11. *$A(\mathbf{vdw}(n, 1))$ has the Weak Lefschetz Property in degree 1 if $n \leq 3$. $A(\mathbf{vdw}(n, 2))$ has the Weak Lefschetz Property in degree 2 if $n \leq 6$.*

PROOF. The results of this chapter show these are the values of k for which $\mathbf{vdw}(n, 1)$ and $\mathbf{vdw}(n, 2)$ are pseudo-manifolds. We have shown that all of these cases except $\mathbf{vdw}(3, 1)$ are pseudo-manifolds with boundaries, so they have the Weak Lefschetz Property in degree one or two correspondingly by the first condition of Theorem 5.9. While $\mathbf{vdw}(3, 1)$ is a pseudo-manifold without a boundary, its dual graph (presented below), is the graph of a 3-cycle and is not bipartite. Hence $\mathbf{vdw}(3, 1)$ also has the Weak Lefschetz Property in degree 1 by the second condition of Theorem 5.9. □

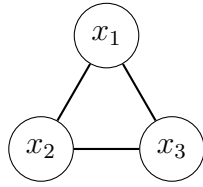


FIGURE 5.1. A 3-cycle, the dual graph of $\text{vdw}(3,1)$.

CHAPTER 6

Future Directions

This thesis mainly examines the presence or absence of the Weak Lefschetz Property of $A(\mathbf{vdw}(n, k))$, for various different values of n and k . However, some of the data computed indicated the presence or absence of the Strong Lefschetz Property of $A(\mathbf{vdw}(n, k))$ for several values of n and k too. In this chapter we examine some of these results. We then examine conjectures on the Weak Lefschetz Property formed using computations done on Macaulay2 ,[18] and state some possible areas of further research.

1. The Strong Lefschetz Property

We now examine several cases where $A(\mathbf{vdw}(n, k))$ has the Strong Lefschetz Property. We begin by recalling that a graded Artinian ring has the Strong Lefschetz Property if the map $\times \ell^d : A_i \rightarrow A_{i+d}$ has maximal rank for all positive values of i and d .

THEOREM 6.1. *$A(\mathbf{vdw}(n, 1))$ has the Strong Lefschetz Property for all $n \geq 2$.*

PROOF. In Theorem 3.4, we proved that $A(\mathbf{vdw}(n, 1))$ has the Weak Lefschetz Property for all values of n . The proof involved showing that for $\dim A_1 = n$, $\dim A_2 = \binom{n}{2}$, and $\dim A_i = 0$ for all $i \geq 3$. The Strong Lefschetz Property considers when the multiplication map $\times \ell^d : A_i \rightarrow A_{i+d}$ has maximal rank. But since $\dim A_i = 0$ for $i \geq 3$, the only non-trivial map of this form is $\times \ell : A_1 \rightarrow A_2$, which we have already deduced has maximal rank. Thus, $A(\mathbf{vdw}(n, 1))$ has the Strong Lefschetz Property for all values of $n \geq 2$, as wanted. \square

Another one of our results about the Weak Lefschetz Property can be extended to the Strong Lefschetz Property, as follows:

THEOREM 6.2. *$A(\mathbf{vdw}(n, n - 1))$ has the Strong Lefschetz Property for all $n \geq 2$*

PROOF. Using the same argument as Theorem 3.11, the Artinian Ring $A(\mathbf{vdw}(n, n - 1))$ simplifies to

$$\frac{R}{x_1^2, x_2^2, \dots, x_n^2},$$

a complete intersection. Then by the same result of Stanley, see Theorem 3.9 from [43], this Artinian ring has the Strong Lefschetz Property. This is true for all values of n , so $A(\mathbf{vdw}(n, n - 1))$ always has the Strong Lefschetz Property, as wanted. \square

Examining for what values of n and k $A(\mathbf{vdw}(n, k))$ has the Strong Lefschetz Property is a promising area for further research. Studying when $A(\mathbf{vdw}(n, k))$ has the Strong Lefschetz Property will also shed more insight on when $A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property, as the former implies the latter.

While this thesis predominantly focused on the Weak Lefschetz Property, we did run computations on Macaulay2 to obtain some data on this, as seen in Table 7.2. We now state some conjectures based on this data.

CONJECTURE 6.3. *For all values of $k \geq 2$, there exists some n such that $A(\mathbf{vdw}(n, k))$ does not have the Strong Lefschetz Property.*

This can be seen from the data in Table 7.2, and is one of main differences between the Weak and Strong Lefschetz Property for $A(\mathbf{vdw}(n, k))$. As we will examine further in the next section, there are some values of n such that the data shows $A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property for all values of k , but the same appears to be untrue for the Strong Lefschetz Property. There appears to be no k such that $A(\mathbf{vdw}(n, k))$ always has the Strong Lefschetz Property for all valid values of n , except for the ones shown previously.

CONJECTURE 6.4. *For a fixed $k \geq 3$, $A(\mathbf{vdw}(n, k))$ does not have the Strong Lefschetz Property for $n > 2k$.*

The data indicates that for a fixed value of $k \geq 3$, while the smallest possible values for n seem to always have the Strong Lefschetz Property, this is not true for any value of $n > 2k$. In fact, $2k + 1$ seems to be the exact instance for when $A(\mathbf{vdw}(n, k))$ no longer has the Strong Lefschetz Property.

Similar to the Weak Lefschetz Property, for a fixed n , and larger values of k , $A(\mathbf{vdw}(n, k))$ always seem to have the Strong Lefschetz Property:

CONJECTURE 6.5. *If $k \geq \lceil \frac{n}{2} \rceil$, then $A(\mathbf{vdw}(n, k))$ always has the Strong Lefschetz Property.*

It appears as though when k is large enough, the behavior of $A(\mathbf{vdw}(n, k))$ is well enough to always have the Strong Lefschetz Property.

Furthermore, although we consider this conjecture in detail in the next section, we observe that for all valid values of n and k , if $A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property, then so does $A(\mathbf{vdw}(m, k))$ for all values of $m < n$. However, this is not true for the Strong Lefschetz Property, and can be seen for the smallest possible value of k this can occur in, $k = 2$. $A(\mathbf{vdw}(5, 2))$, $A(\mathbf{vdw}(6, 2))$, and $A(\mathbf{vdw}(7, 2))$ all do not have the Strong Lefschetz Property, but $A(\mathbf{vdw}(n, 2))$ does have the Strong Lefschetz Property for $8 \leq n \leq 20$. While there is no specific conjecture to form about this instance, studying the reason for this phenomena would be a possible future area of research. The conditions that allow this phenomenon to happen for the Strong Lefschetz Property but not the

Weak Lefschetz Property might help shed more insight on the structure of the rings we are working with. Due to the behavior of $k = 3, 4, 5, 6$, and 7 , as well as the results we had about the Weak Lefschetz Property, we form this final conjecture about the Strong Lefschetz Property:

CONJECTURE 6.6. $A(\mathbf{vdw}(n, 2))$ has the Strong Lefschetz Property for all values of n except $n = 5, 6$, and 7 .

2. Further Conjectures on the Weak Lefschetz Property

In this final section of this thesis, we form conjectures based on the results we have proven, as well as the Macaulay2 data seen in Table 7.1. These conjectures are possible questions further research on this topic can attempt to solve. For certain conjectures that we have attempted to prove, we state results and theorems that we suppose may be helpful in proving them as well.

CONJECTURE 6.7. If $A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property, then so does $A(\mathbf{vdw}(m, k))$ for all $k < m < n$.

This was the first conjecture that was formed using the data shown in Table 7.1. Although $\mathbf{vdw}(m, k)$ is a subset of $\mathbf{vdw}(n, k)$ for all $m \leq n$, the conjecture is not trivial since the Artinian rings they form are different. The initial idea was to try an inductive argument on $m = n - 1$, and write $A(\mathbf{vdw}(m, k))$ as a quotient of $A(\mathbf{vdw}(n, k))$ by x_n^2 , but this is not correct. New elements may be included in $A(\mathbf{vdw}(n, k))$ that do not necessarily contain the variable x_n^2 itself. However, a modified approach might be the right idea, as if one is able to relate the rings $A(\mathbf{vdw}(m, k))$ and $A(\mathbf{vdw}(n, k))$ using either tensor products or quotients in a particular way, then one might be able to use a combination of the following two theorems, the first from [30] and second from [24] to obtain the desired proof.

THEOREM 6.8. Let A be a ring with an ideal I , and M be an A -module, then

$$(A/I) \otimes M \cong M/IM$$

THEOREM 6.9. Let K be a field of characteristic 0,

$$B = \frac{K[x_1, x_2, \dots, x_n]}{(x_1, x_2, \dots, x_n)^2} \otimes \frac{K[y_1, y_2, \dots, y_m]}{J}$$

where J is a monomial ideal and $A = \frac{K[y_1, y_2, \dots, y_m]}{J}$ is of socle degree D . Define

$$\ell = y_1 + y_2 + \dots + y_m,$$

$$\ell' = x_1 + x_2 + \dots + x_n + \ell.$$

Then for $1 \leq i \leq D - 1$, the map

$$\times \ell' : [B]_i \rightarrow [B]_{i+1}$$

has maximal rank if and only if the maps

$$\begin{aligned} \times \ell &: [A]_{i-1} \rightarrow [A]_i, \text{ and} \\ \times \ell^2 &: [A]_{i-1} \rightarrow [A]_{i+1} \end{aligned}$$

both also have maximal rank.

Another approach of using flat extensions was also attempted. We define the key concepts, which can be found in [7] here:

DEFINITION 6.10. For a group G , $A \subseteq G$ is a **free basis** of G if every function $\varphi : A \rightarrow B$ for some other group B can be extended uniquely to another homomorphism $\tilde{\varphi} : G \rightarrow B$ such that $\tilde{\varphi}(a) = \varphi(a)$ for all $a \in A$. Any group G with such a subset A is referred to as a **free group**.

DEFINITION 6.11. Given two graded rings A and B , a ring homomorphism $\phi : B \rightarrow A$ is a **flat extension** if B is a subring of A , such that for all i , $B_i \subset A_i$, and A has a free basis over B

We now state the theorem from [48] that we attempted to use:

THEOREM 6.12. Let $\phi : B \rightarrow A$ be a flat extension with fiber $C = A/((B_+)A)$, where $B_+ = \{b \in B_i \mid i > 0\}$. If B and C have the Strong Lefschetz Property, then so does A .

This approach was considered, but not pursued since we have evidence from the previous section on the Strong Lefschetz Property that if $A(\mathbf{vdw}(n, k))$ has the Strong Lefschetz Property, then it does not necessarily mean $A(\mathbf{vdw}(m, k))$ has the Strong Lefschetz Property for $k < m < n$.

Another approach to solving Conjecture 6.7 is using induction and considering the matrix representation of the linear map $\times \ell : A_i \rightarrow A_{i+1}$. Using the inductive approach, we need to prove that if $A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property for some n , then so does $A(\mathbf{vdw}(n-1, k))$. Then by induction, if $A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property, then so does $A(\mathbf{vdw}(m, k))$ for all $m < n$. For each degree, we can consider the matrix representation of $\times \ell$ of $A(\mathbf{vdw}(n-1, k))$ as a sub-matrix of the matrix representation of $\times \ell$ of $A(\mathbf{vdw}(n, k))$. Arranging rows and columns so that those representing elements of A_i that do contain x_n are placed at the end, we get a matrix of full rank M that is the matrix representation of the map $\times \ell$ for $A(\mathbf{vdw}(n, k))$, and it can be partitioned into 4 sub-matrices, where one partition is the matrix representation of the map $\times \ell$ for $A(\mathbf{vdw}(n-1, k))$ and another partition is a zero block. If we are able to prove that if the bigger matrix M having maximal rank implies the sub-matrix that is the matrix representation $\times \ell$ for $A(\mathbf{vdw}(n-1, k))$ also has maximal rank, then we are done.

Conjecture 6.7 examined instances of the Weak Lefschetz Property for $A(\mathbf{vdw}(n, k))$ for a fixed k when some n was known to have it. We now examine several conjectures, still for a fixed k , but when we know for certain some $A(\mathbf{vdw}(n, k))$ does not have the Weak Lefschetz Property.

CONJECTURE 6.13. *For every odd value of $k \geq 3$, there exists some \tilde{n} such that $A(\mathbf{vdw}(n, k))$ does not have the Weak Lefschetz Property for all $n \geq \tilde{n}$. Furthermore, for an arbitrary odd $k = 2p + 1$, we can take \tilde{n} to be:*

$$\tilde{n} = \begin{cases} 2k + 1 & \text{if } p \text{ is odd} \\ 2k + 2 & \text{if } p \text{ is even} \end{cases}$$

While there currently is no rationale for this formula for \tilde{n} , it fits every relevant example seen in Table 7.1. Furthermore, we know by Theorem 5.10 that if $k \geq 3$ and $\frac{n}{2} \leq k < n$, then $A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property in degree k , so the boundary being close to $n/2$ appears to be a recurring idea.

The above conjecture focused on odd values of $k \geq 3$, as for all such values there was some n such that $A(\mathbf{vdw}(n, k))$ did not have the Weak Lefschetz Property. However, the data in Table 7.1 shows no such pattern for even values of k . The even-odd parity of k seems to have a noticeable influence on whether or not $A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property. This would be an interesting area of further research, as why does the dimension of our simplicial complex k being odd create more complicated behaviors than when it is even. Does the even dimension introduce some sort of symmetric behavior that allows for $A(\mathbf{vdw}(n, k))$ to have the Weak Lefschetz Property; or is it perhaps that the odd dimension introduces some sort of problematic behavior, such as non-unimodal Hilbert Functions, which we know is an indicator of the absence of the Weak Lefschetz Property.

If $A(\mathbf{vdw}(n, k))$ does not have the Weak Lefschetz Property, the data from Macaulay2 also indicates the smallest degree at which this failure occurs. When $k = 3$ and $n \geq 7$, the first degree of failure is always 2. For $k = 5$ and $n \geq 12$, the first degree of failure is always 3. For $k = 7$ and $n \geq 15$, the first degree of failure is always 4. This recurring pattern brings us to our next conjecture:

CONJECTURE 6.14. *For an odd k , if $A(\mathbf{vdw}(n, k))$ fails to have the Weak Lefschetz Property, the smallest degree at which this occurs is $\lceil \frac{k}{2} \rceil$.*

Examining more into the even-odd parity of k brings us to our next conjecture:

CONJECTURE 6.15. *For all even values of k , $A(\mathbf{vdw}(n, k))$ has the Weak Lefschetz Property for all valid values of n .*

We proved this for the smallest possible even value of k , $k = 2$, in Chapter 3. The proof involved the fact that $\dim A_2 \geq \dim A_1$ for degree 1, and that the matrix representation of the map in degree two was always of maximal rank. If we are able to generalize these ideas to higher values of even k , with results about the dimensions of consecutive A_i , then we may be able to prove the conjecture.

Similar to Conjecture 6.5 for the Strong Lefschetz Property, we have the following for the Weak Lefschetz Property:

CONJECTURE 6.16. *If $k \geq \lceil \frac{n}{2} \rceil$, then $A(\mathbf{vdw}(n, k))$ always has the Weak Lefschetz Property.*

Similar to the case of the Strong Lefschetz Property, it would appear that for larger values of k , the behavior of $A(\mathbf{vdw}(n, k))$ is optimal enough for $A(\mathbf{vdw}(n, k))$ to have the Weak Lefschetz Property. However, the reason we state this separately because the lower bound for k is not as consistent as it was for the Strong Lefschetz Property. For example, when $n = 15$, Table 7.1 shows that $A(\mathbf{vdw}(15, k))$ has the Weak Lefschetz Property for all values of $k \geq \lceil \frac{15}{2} \rceil = 8$. But for $n = 17$, the conjecture tell us $A(\mathbf{vdw}(17, k))$ has the Weak Lefschetz Property for all values of $k \geq \lceil \frac{17}{2} \rceil = 9$. However, the data in Table 7.1 indicates we can lower k to 8. Proving this conjecture can also lead to a better lower bound for k in the case of the Weak Lefschetz Property.

3. Further Areas of Study

In this final section of this thesis, we examine some further areas of study. One major area of future research for the van der Waerden simplicial complexes $\Delta = \mathbf{vdw}(n, k)$ is to study different Artinian rings of the form $R/(I_\Delta, x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n})$, denoted by $A = A(\Delta, a_1, a_2, \dots, a_n)$, where the $a_i \neq 2$ for all i . We modded out the Stanley-Reisner ring by the squares of our variables so that the ring becomes a square-free Artinian ring. But having other values of $a_i \neq 2$ also creates Artinian rings, they just will not be square-free. Furthermore, we can even form Artinian rings using a_i such that $a_i \neq a_j$ for $i \neq j$. Working with such constructions will be more complex, as many of the tools we were able to use when $a_i = 2$ for all i will no longer apply, such as the simplified form of the socle vector formula, and simplified multiplication by ℓ , where the matrix representation of the map was simply the incidence matrix. Furthermore, if $a_i \neq a_j$ for $i \neq j$, the existing theorems we can use to find instances of the Lefschetz Properties get even smaller. However, it would be interesting to see what properties hold true for $A(\mathbf{vdw}(n, k))$ also hold true when the a_i are changed, and which ones break.

Lastly, in this thesis we mainly examined the Weak Lefschetz Property, as well as a couple cases of the Strong Lefschetz Property. However, the Macaulay2 code that was used to perform the computations (please see the code in the Appendix) also mentions the Medium Lefschetz Property. This would be an intermediate property between the Weak and Strong Lefschetz Property, where the map $\times \ell : A_i \rightarrow A_{i+1}$ always has maximal rank, and the maps $\times \ell^d : A_i \rightarrow A_{i+d}$ has maximal rank for some values of $d \geq 0$, but not for all such values of d . Studying for which values of n and k does $A(\mathbf{vdw}(n, k))$ have the Medium Lefschetz Property, and for what degrees d are the maps $\times \ell^d$ of maximal rank, is also a possible future area of study.

CHAPTER 7

Appendix

In this chapter, we provide some tables that summarize computations done on Macaulay2 that indicate for which values of n and k does the Artinian ring $A(\text{vdw}(n, k))$ have the Weak or Strong Lefschetz Properties, and when is $\text{vdw}(n, k)$ a pseudo-manifold. We also provide a Macaulay2 package that can be used to perform computations regarding $\text{vdw}(n, k)$.

1. Summary of Computations

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	✓															
3	✓	✓														
4	✓	✓	✓													
5	✓	✓	✓	✓												
6	✓	✓	✓	✓	✓											
7	✓	✓	✗	✓	✓	✓										
8	✓	✓	✗	✓	✓	✓	✓									
9	✓	✓	✗	✓	✓	✓	✓	✓								
10	✓	✓	✗	✓	✓	✓	✓	✓	✓							
11	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓						
12	✓	✓	✗	✓	✗	✓	✓	✓	✓	✓	✓					
13	✓	✓	✗	✓	✗	✓	✓	✓	✓	✓	✓	✓				
14	✓	✓	✗	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓			
15	✓	✓	✗	✓	✗	✓	✗	✓	✓	✓	✓	✓	✓	✓		
16	✓	✓	✗	✓	✗	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	–
17	✓	✓	✗	✓	✗	✓	✗	✓	✓	✓	✓	✓	✓	–	–	–

TABLE 7.1. A table showing which of the van der Waerden complexes have the Weak Lefschetz Property as verified by Macaulay2. Note entries with – are those whose computations timed out in Macaulay2. Though not shown on the table, $A(\text{vdw}(20, 9))$ does not have the Weak Lefschetz Property.

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	✓															
3	✓	✓														
4	✓	✓	✓													
5	✓	✗	✓	✓												
6	✓	✗	✓	✓	✓											
7	✓	✗	✗	✓	✓	✓										
8	✓	✓	✗	✓	✓	✓	✓									
9	✓	✓	✗	✗	✓	✓	✓	✓								
10	✓	✓	✗	✗	✓	✓	✓	✓	✓							
11	✓	✓	✗	✗	✗	✓	✓	✓	✓	✓						
12	✓	✓	✗	✗	✗	✓	✓	✓	✓	✓	✓					
13	✓	✗	✗	✗	✗	✗	✓	✓	✓	✓	✓	✓				
14	✓	✓	✗	✗	✗	✗	✓	✓	✓	✓	✓	✓	✓			
15	✓	✓	✗	✗	✗	✗	✗	✓	✓	✓	✓	✓	✓	✓		
16	✓	✓	✗	✗	✗	✗	✗	✓	✓	✓	✓	-	-	-	-	
17	✓	✓	✗	✗	✗	✗	✗	✗	✓	✓	-	-	-	-	-	-

TABLE 7.2. A table showing which of the van der Waerden complexes have the Strong Lefschetz Property as verified by Macaulay2. Again note entries with $-$ are those whose computations timed out in Macaulay2. Though not shown on the table, $A(\text{vdw}(20, 9))$ does not have the Strong Lefschetz Property

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	✓															
3	✓	✓														
4	✗	✓	✓													
5	✗	✓	✓	✓												
6	✗	✓	✓	✓	✓											
7	✗	✗	✗	✓	✓	✓										
8	✗	✗	✗	✓	✓	✓	✓									
9	✗	✗	✗	✗	✓	✓	✓	✓								
10	✗	✗	✗	✗	✓	✓	✓	✓	✓							
11	✗	✗	✗	✗	✗	✓	✓	✓	✓	✓						
12	✗	✗	✗	✗	✗	✓	✓	✓	✓	✓	✓					
13	✗	✗	✗	✗	✗	✗	✓	✓	✓	✓	✓	✓				
14	✗	✗	✗	✗	✗	✗	✓	✓	✓	✓	✓	✓	✓			
15	✗	✗	✗	✗	✗	✗	✗	✓	✓	✓	✓	✓	✓	✓		
16	✗	✗	✗	✗	✗	✗	✗	✓	✓	✓	✓	✓	✓	✓	✓	
17	✗	✗	✗	✗	✗	✗	✗	✗	✓	✓	✓	✓	✓	✓	✓	✓

TABLE 7.3. A table showing which of the van der Waerden complexes are pseudo-manifolds

2. Macaulay2 Package Code

The following is a Macaulay2 [18] package written for the Computational Commutative Algebra and Computational Algebraic Geometry course at the Fields Institute with Professor Mike Stillman. It was written to easily compute several of the defining characteristics of the van der Waerden simplicial complexes, and also examine whether or not they have the Weak or Strong Lefschetz Properties. Several functions in this code were written by Dr. Adam Van Tuyl and Dr. Lisa Nicklasson, and are indicated as so.

```

newPackage(
  "VanDerWaerdenSimplicialComplexes",
  Version => "1.0",
  Date => "April 20, 2025",
  Authors => {
    {Name => "Naveena Ragunathan", Email => "ragunatn@mcmaster.ca"}},
  Headline => "Macaulay2 package for the Van Der Waerden Simplicial Complexes ",
  Keywords => {"Documentation"},
  AuxiliaryFiles => false,
  PackageExports => {"SimplicialComplexes"},
  DebuggingMode => false
)

export {"VanDerWaerden", "ArtinianRing", "SocleVector", "isShellable",
"isCohenMacaulay", "VDW", "hvec", "trunc", "compareHilb", "toZZT", "isArtinian",
"hasMaxRank", "checkMRP", "checkWRP", "checkSRP" }
-----

VanDerWaerden = method(TypicalValue => List)
VanDerWaerden(Ring, ZZ, ZZ) := SimplicialComplex => (R,n,k) -> (
  if (k>=n) then error "n must be greater than k";
  l = {};--We make the ring also an input so that the ArtinianRing function also
  refers to the same ring
  for i from 1 to n do (
    for d from 1 to floor((n-1)/k) do ( --looking at the possible jump factors,
      and i is the starting point
      if (i+d*k <=n) then (
        sublist = {};
        for j from 0 to k do (
          sublist = append(sublist,x_(i+d*j)) );
        l = append(l, product sublist));
      );
    );
  return l; --this returns a list of all the valid arithmetic progressions;
  this will be the generators of our simplicial complex
)
-----

ArtinianRing = method(TypicalValue => QuotientRing)
ArtinianRing(ZZ, ZZ) := QuotientRing => (n,k) -> (
  if (k>=n) then error "n must be greater than k";
  R = QQ[x_1..x_n];
  gen = gens(R); --getting all the variables in our ring

```

```

SquareIdeal = ideal apply(gen, x -> x^2); --squaring all of our variables and
      creating an ideal out of them
vdwIdeal = monomialIdeal(simplicialComplex(VanDerWaerden(R,n,k))); --creating
return R / (SquareIdeal + vdwIdeal); --creating our Artinian ring where a_i=2
)

```

an ideal from t

```

-----
SocleVector = method(TypicalValue => Sequence)
SocleVector (Ring, ZZ,ZZ) := Sequence => (R, n, k) -> (
  Y = facets(simplicialComplex(VanDerWaerden(R,n,k)));
  j = length(toList(factor(Y#0)));
  L := {};
  for i from 0 to j - 1 do (
    L = if i == j - 1 then append(L, j) else append(L, 0)
  );
  return toSequence(L)
)

```

```

-----
isShellable = method(TypicalValue => Boolean)
isShellable (ZZ,ZZ) := Boolean => (n, k) -> (
  if (k>=n) then error "n must be greater than k"
  else if (n <= 6) then return "true"
  else if (n > 6 and k == 1) then return "true"
  else if (n > 6 and (n/2 <= k and k < n)) then return "true"
  else return "false";
)

```

```

-----
isCohenMacaulay = method(TypicalValue => Boolean)
isCohenMacaulay (ZZ,ZZ) := Boolean => (n, k) -> (
  if (k>=n) then error "n must be greater than k"
  else if (n <= 6) then return "true"
  else if (n > 6 and k == 1) then return "true"
  else if (n > 6 and (n/2 <= k and k < n)) then return "true"
  else return "false";
)

```

```

-----
isVertexDecomposable = method(TypicalValue => Boolean)
isVertexDecomposable (ZZ,ZZ) := Boolean => (n, k) -> (
  if (k>=n) then error "n must be greater than k"
  else if (n <= 6) then return "true"
  else if (n > 6 and k == 1) then return "true"
  else if (n > 6 and (n/2 <= k and k < n)) then return "true"
  else return "false";
)

```

```

-----
PseudoManifold =
method(TypicalValue => String)
PseudoManifold (ZZ,ZZ) := String => (n, k) -> (
  if (k>=n) then error "n must be greater than k"

```

```

else if (n==2 and k==1) then return "Pseudomanifold with Boundary"
else if (n==3 and k==1) then return "Pseudomanifold without Boundary"
else if ((n==2 or n==4 or n==5 or n==6) and (k==2)) then return "Pseudomanifold
with Boundary"
else if (n==3 and k==2) then return "Pseudomanifold without Boundary"
else if ((n/2 <= k and k < n) and k>=3) then return "Pseudomanifold
with Boundary"
else return "Not a Pseudomanifold";
)
)
-----
--The following ten are functions from Dr Van Tuyl and Lisa Nicklason. Please see
below on how to use these functions to determine whether the Artinian Ring created
from vdw(n,k) and the squares of variables have the Weak or Strong Lefschetz
Properties.

--(1)Creating the Van Der Waerden simplicial complex; this is what our first function
is based on
VDW = (n,k) -> (
  R=QQ[x_1..x_n];
  l = {};
  for i from 1 to n do (
for d from 1 to floor((n-1)/k) do (
  if (i+d*k <=n) then (
sublist = {};
for j from 0 to k do (
  sublist = append(sublist,x_(i+d*j))
);
l = append(l, product sublist));
);
);
  return simplicialComplex(l);
);
)
-----
--(2)hvec: Gives the coefficients of a polynomial as a list
hvec=(p)->(
  d:=(degree p)#0;
  T:= first gens ring p;
  return for i from 0 to d list coefficient(T^i,p))
)
-----
--(3)trunc: Truncate a list at the first non-negative element
trunc=(v)->(
  a:=v#0;
  i:=0;
  l:={};
  while a>0 do(
l=l||{a};
i=i+1;
if i> length v-1 then return l;

```

```

a=v#i;
);
    return l;
)
-----
--(4)compareHilb: Compares Hilbert Series with one of max rank; if unequal this
tells you where is the first place they differ, does it fail to be injective or
surjective, and the dim (which is the difference of the dim of the kernel if not
inj, codim of the image if not surj)
compareHilb=(hilbs,exps)->(
    lh:=length hilbs;
    le:=length exps;
    for i to le-1 do if hilbs#i!=exps#i then return {i, hilbs#i-exps#i,"inj"};
    if lh>le then return {le, hilbs#le, "sur"};
    return true;
)
-----
--(5)toZZT: takes an element in a DegreesRing, and sends it to ZZ[T]
toZZT=(p)->(
    DR:=ring p;
    T:=first gens DR;
    F:=map(ZZ[T],DR);
    return F(p);
)
-----
--(6)isArtinian: Checks if a ring is Artinian
isArtinian=(R)->( dim(R)==0)
-----
--(7) hasMaxRank: Takes an element f, and checks if multiplication by it has max
rank in all degrees
hasMaxRank=method()
hasMaxRank(RingElement):=(f)->(
    S:=ring f;
    if not isQuotientOf(PolynomialRing,S) then error "The argument must be an
element of a quotient of a polynomial ring.";
    if not isArtinian(S) then error "The argument must be an element of an Artinian
ring.";
    if ((not isHomogeneous(f)) or isConstant(f)) then error "The argument must be a
non-constant homogeneous element.";
    d:=(degree f)#0;
    p:=numerator reduceHilbert hilbertSeries S;
    p=toZZT(p);
    T:=first gens ring p;
    h:=trunc hvec ((1-T^d)*p);
    v:=hvec numerator reduceHilbert hilbertSeries (S/ideal{f});
    result:=compareHilb(v,h);
    if result===true then return true;
    faildeg:=(result#0)-d;
)

```

```

if result#2=="inj" then return ("Multiplication by " | toString(f) | " fails
    to be injective in degree " | toString(faildeg) | ". The kernel has dimension"
    | result#1 | ".");
if result#2=="sur" then return ("Multiplication by " | toString(f) | " fails
    to be surjective in degree " | toString(faildeg) | ". The image has
    codimension" | result#1 | ".");
-----
--(8)checkMLP: Checks if an Artinian Ring has Medium Lefschetz Property (MLP)
checkMRP=method()
checkMRP(QuotientRing):=(R)->(
    if not isQuotientOf(PolynomialRing,R) then error "The argument must be a quotient
    of a polynomial ring.";
    if not isArtinian(R) then error "The argument must be an Artinian ring.";
    p:=numerator reduceHilbert hilbertSeries R;
    p=toZZT(p);
    T:=first gens ring p;
    t:=(degree p)#0;
    H:=for j from 1 to t list trunc hvec((1-T^j)*p);
    canFindMREl:=(d)->(
faildeg=-1;
failtype="";
faildim=0;
for i to 4 do(
    f:=random(d,R);
        v:=hvec numerator reduceHilbert hilbertSeries (R/ideal{f});
        result:=compareHilb(v,H#(d-1));
        if result===true then return true;
        if (result#0)-d>faildeg then (
faildeg=(result#0)-d;
faildim=result#1;
failtype=result#2;
);
);
return {faildeg,faildim, failtype};
);
    for d from 1 to t do(
result:=canFindMREl(d);
if not result===true then (
    if result#2=="inj" then return ("Could not find an element of degree " | d | "
        such that multiplication is injective in degree " | result#0 | ". The kernel
        has dimension " | result#1 | ".");
    if result#2=="sur" then return ("Could not find an element of degree " | d | "
        such that multiplication is surjective in degree " | result#0 | ". The image
        has codimension " | result#1 | ".");
);
);
return true;
)

```

```

-----
-- (9) checkWLP: Checks if an Artinian Ring has Weak Lefschetz Property (WLP)
checkWLP=method()
checkWLP(QuotientRing):=(S)->(
  I:=trim ideal(S);
  if not isQuotientOf(PolynomialRing,S) then error "The argument must be a quotient
    of a polynomial ring.";
  if not isHomogeneous(I) then error "The defining ideal must be homogeneous";
  if not isArtinian(S) then error "The argument must be an Artinian ring.";
  f:=sum gens S;
  faildeg:=-1;
  faildim:=0;
  failtype:="";
  p:=numerator reduceHilbert hilbertSeries S;
  p=toZZT(p);
  T:=first gens ring p;
  h:=trunc hvec((1-T)*p);
  isWLEl:=(f)->(
v:=hvec numerator reduceHilbert hilbertSeries(S/ideal(f));
return compareHilb(v,h);
);
    if isMonomialIdeal I then (
result:=isWLEl(f);
if result===true then return true;
if result#2=="inj" then return ("Multiplication by any linear form is not injective
  in degree " | (result#0)-1 | ". The dimension of the kernel is " | result#1 | ".");
if result#2=="sur" then return ("Multiplication by any linear form is not surjective
  in degree " |(result#0)-1 | ". The codimension of the image is" | result#1 | ".");
);
    for i to 4 do (
result:=isWLEl(f);
    if result===true then return true;
if (result#0)-1>faildeg then (
  faildeg=(result#0)-1;
  faildim=result#1;
  failtype=result#2;
);
f=random(1,S);
);
    if failtype=="inj" then return ("Could not find a linear form such that
  multiplication is injective in degree " | faildeg | ". The kernel has dimension
  " | faildim | ".");
    if failtype=="sur" then return ("Could not find a linear form such that
  multiplication is surjective in degree " | faildeg | ". The image has
  codimension " | faildim | ".");
)
-----

```

```

--(10) checkSLP: Checks if an Artinian Ring has Strong Lefschetz Property (SLP)

```

```

checkSLP=method()
checkSLP(QuotientRing):=(S)->(
  I:=trim ideal(S);
  if not isQuotientOf(PolynomialRing,S) then error "The argument must be a
    quotient of a polynomial ring.";
  if not isHomogeneous(I) then error "The defining ideal must be homogeneous.";
  if not isArtinian(S) then error "The argument must be an Artinian ring.";
  p:=numerator reduceHilbert hilbertSeries S;
  p=toZZT(p);
  T:= first gens ring p;
  t:=(degree p)#0;
  H:=for j from 1 to t list trunc hvec((1-T^j)*p);
  failpow:=-1;
  faildeg:=-1;
  faildim:=0;
  failtype:="";
  isSLEl:=(f)->(
    for j from 1 to t do (
      v:=hvec numerator reduceHilbert hilbertSeries (S/ideal{f^j});
      h:=H#(j-1);
      result:=compareHilb(v,h);
      if not result===true then return ({j}| result);
    );
    return true;);
  f:=sum gens S;
  if isMonomialIdeal(I) then (
result:=isSLEl(f);
if result===true then return true;
if result#3=="inj" then return ("Multiplication by the " | result#0 | "'th power of
  any linear form is not injective in degree " | (result#1)-(result#0) | ".
  The dimension of the kernel is " | result#2 | ".");
if result#3=="sur" then return ("Multiplication by the " | result#0 | "'th power of
  any linear form is not surjective in degree " | (result#1)-(result#0) | ".
  The codimension of the image is " | result#2 | "."););
  for i to 4 do (
result:=isSLEl(f);
if result===true then return true;
if result#1>failpow+faildeg then (
  failpow=result#0;
  faildeg=result#1-failpow;
  faildim=result#2;
  failtype=result#3;
  );
  f=random(1,S););
  if failtype=="inj" then return ("Could not find a linear form such that
    multiplication by its " | failpow | "'th power is injective in degree " |
    faildeg | ". The kernel has dimension " | faildim | ".");
  if failtype=="sur" then return ("Could not find a linear form such that

```

```

multiplication by its " | failpow | "'th power is surjective in degree " |
faildeg | ". The image has codimension " | faildim | ".");
)

```

```

-----
--To check whether VanDerWaerden(R,n,k) has any of the Lefschetz Properties, we
--1. Run all code starting from function (1) VDS
--2. We define our polynomial ring R
--3. We create i = monomialIdeal(VDW(n,k))
--4. We create j = the ideal generated by the squares of R's generators
--5. We define S = R / (i+j)
--6. We can now checkWLP(S), checkSLP(S), checkMRP(S)
--See example for more details
-----

```

DOCUMENTATION

```

-----
beginDocumentation()
doc ///
Node
Key
  VanDerWaerdenSimplicialComplexes
Headline
  A package on Van Der Waerden simplicial complexes
Description
  A Macaulay2 package to compute specific properties of the Van Der Waerden
  simplicial complexes
Description
Caveat
  The last 10 functions are from Dr. Van Tuyl and Lisa Nicklason
Subnodes
  VanDerWaerden
  ArtinianRing
  SocleVector
  isShellable
  isCohenMacaulay
  VDW
  hvec
  trunc
  compareHilb
  toZZT
  isArtinian
  hasMaxRank
  checkMRP
  checkWRP
  checkSRP
///
-----

```

```

doc ///
Node
Key   VanDerWaerden
      (VanDerWaerden, Ring, Integer, Integer)
Headline
Computes a list of generators for the Van Der Waerden simplicial complex
associated with the integers n,k:(n,k).
Usage
VanDerWaerden(R,n,k)
Inputs
R: A polynomial ring
n: A positive integer
k: A positive integer less than n
Outputs
: A list of elements (arithmetic progressions) that generate the Van Der Waerden
simplicial complex associated with the integers n,k
Example
R = QQ[x_1..x_10]
VanDerWaerden(R,7,2) =
{{x_1*x_2*x_3}, {x_2*x_3*x_4}, {x_3*x_4*x_5}, {x_4*x_5*x_6}, {x_5*x_6*x_7},
{x_1*x_3*x_5}, {x_2*x_4*x_6}, {x_3*x_5*x_7},
{x_1*x_4*x_7}}
VanDerWaerden(R,8,3) =
{{x_1*x_2*x_3*x_4}, {x_2*x_3*x_4*x_5}, {x_3*x_4*x_5*x_6}, {x_4*x_5*x_6*x_7},
{x_5*x_6*x_7*x_8},
{x_1*x_3*x_5*x_7}, {x_2*x_4*x_6*x_8}}
VanDerWaerden(R,10,20) = {}
///
-----
doc ///
Node
Key   ArtinianRing
      (ArtinianRing, Integer, Integer)
Headline
Creates an Artinian Ring associated to the Van Der Waerden simplicial complex,
by computing the Stanley Reisner Ring and additionally modding out the squares
of all generators
Usage
ArtinianRing(n,k)
Inputs
n: A positive integer
k: A positive integer less than n
Outputs
: an Artinian Ring associated to the Van Der Waerden simplicial complex, by
computing the Stanley Reisner Ring and additionally modding out the squares of
all generators
Example

```

```

R = QQ[x_1..x_10]
ArtinianRing(7,2) = R/(x_1^2,x_2^2,x_3^2,x_4^2,x_5^2,x_6^2,x_7^2,x_1*x_2*x_4,
  x_1*x_3*x_4,x_2*x_5,x_1*x_4*x_5,x_1*x_6,x_3*x_6,x_2*x_7, x_1*x_3*x_7,x_3*x_4*x_7,
  x_1*x_5*x_7,x_4*x_5*x_7,x_4*x_6*x_7)
ArtinianRing(8,3) = R/(x_1^2,x_2^2,x_3^2,x_4^2,x_5^2,x_6^2,x_7^2,x_8^2,
  x_1*x_2*x_5,x_1*x_4*x_5,x_1*x_6,x_2*x_3*x_6,x_2*x_5*x_6,x_2*x_7,x_1*x_4*x_7,
  x_3*x_4*x_7,x_3*x_6*x_7,x_1*x_8,x_3*x_8,x_2*x_5*x_8,x_4*x_5*x_8,x_4*x_7*x_8)
--The non-square components for the last 2 examples are from built-in functions,
monomialIdeal and simplicialComplex
ArtinianRing(10,20) = -- expected at least one facet
///
-----
doc ///
Node
Key
SocleVector
(SocleVector, Ring, Integer, Integer)
Headline
Computes the socle vector of the Artinian Ring found above in a simplified manner
using a theorem from Van Tuyl and Zanello
Usage
SocleVector(R,n,k)
Inputs
R: A polynomial ring
n: A positive integer
k: A positive integer less than n
Outputs
: The socle vector of the associated Artinian Ring
Example
R = QQ[x_1..x_10]
facets(simplicialComplex(VanDerWaerden(R,7,2))) = {x_5*x_6*x_7,x_3*x_5*x_7,
  x_1*x_4*x_7,x_4*x_5*x_6,x_2*x_4*x_6,x_3*x_4*x_5,x_1*x_3*x_5,x_2*x_3*x_4,
  x_1*x_2*x_3} --this is a pre-defined Macaulay2 function, and we see all facets
are of size 3 (which makes sense as the simplicial complex is pure)
SocleVector(R,7,2) = (0,0,3)
facets(simplicialComplex(VanDerWaerden(R,8,3))) = {x_5*x_6*x_7*x_8,x_2*x_4*x_6*x_8,
  x_4*x_5*x_6*x_7,x_1*x_3*x_5*x_7,x_3*x_4*x_5*x_6,x_2*x_3*x_4*x_5,x_1*x_2*x_3*x_4}
SocleVector(R,8,3) = (0,0,0,4)
SocleVector (R, 10, 20)-- expected at least one facet
///
-----
doc ///
Node
Key
isShellable
(isShellable, Integer, Integer)
Headline
Determines whether the Van Der Waerden simplicial complex generated by the given

```

n and k value is shellable, using a result from Becky Hopper.

Usage

isShellable(n,k)

Inputs

n: A positive integer

k: A positive integer less than n

Outputs

: A string stating whether or not the Van Der Waerden simplicial complex generated by the given n and k value is shellable

Example

isShellable(5,1) = true

isShellable(9,3) = false

isShellable(9,10) = ERROR: n must be greater than k

///

doc ///

Node

Key

isCohenMacaulay

(isCohenMacaulay, Integer, Integer)

Headline

Determines whether the Van Der Waerden simplicial complex generated by the given n and k value is CohenMacaulay, using a result from Becky Hopper.

Usage

isCohenMacaulay(n,k)

Inputs

n: A positive integer

k: A positive integer less than n

Outputs

: A string stating whether or not the Van Der Waerden simplicial complex generated by the given n and k value is isCohenMacaulay

Example

isCohenMacaulay(5,1) = true

isCohenMacaulay(9,3) = false

isCohenMacaulay(9,10) = ERROR: n must be greater than k

///

doc ///

Node

Key

isVertexDecomposable

(isVertexDecomposable, Integer, Integer)

Headline

Determines whether the Van Der Waerden simplicial complex generated by the given n and k value is VertexDecomposable, using a result from Becky Hopper, Adam Van Tuyl and Takayuki Hibi.

```

Usage
  isVertexDecomposable(n,k)
Inputs
  n: A positive integer
  k: A positive integer less than n
Outputs
  : A string stating whether or not the Van Der Waerden simplicial complex generated
  by the given n and k value is is VertexDecomposable
Example
  isVertexDecomposable(5,1) = true
  isVertexDecomposable = false
  isCohenMacaulay(9,10) = ERROR: n must be greater than k
///
-----
doc ///
Node
  Key
    PseudoManifold
    (PseudoManifold, Integer, Integer)
  Headline
    Determines whether the Van Der Waerden simplicial complex generated by the given
    n and k value is a pseudo-manifold; and states whether or not it has a boundary.
Usage
  PseudoManifold(n,k)
Inputs
  n: A positive integer
  k: A positive integer less than n
Outputs
  : A string stating whether or not the Van Der Waerden simplicial complex
  generated by the given n and k value is a pseudomanifold with or without boundary.
Example
  PseudoManifold (5,1) = Not a Pseudomanifold
  PseudoManifold (5,4) = Pseudomanifold with Boundary
  PseudoManifold (3,1) = Pseudomanifold without Boundary
  Pseudomanifold (9,10) = ERROR: n must be greater than k
///
-----

TESTS

-----
TEST1///
R = QQ[x_1..x_10]
assert(VanDerWaerden(R,7,2) == {{x_1*x_2*x_3}, {x_2*x_3*x_4}, {x_3*x_4*x_5},
  {x_4*x_5*x_6}, {x_5*x_6*x_7}, {x_1*x_3*x_5}, {x_2*x_4*x_6}, {x_3*x_5*x_7},
  {x_1*x_4*x_7}})
///
-----

```

```
TEST2///
R = QQ[x_1..x_10]
assert(ArtinianRing(7,2) == R/(x_1^2,x_2^2,x_3^2,x_4^2,x_5^2,x_6^2,x_7^2,
  x_1*x_2*x_4,x_1*x_3*x_4,x_2*x_5,x_1*x_4*x_5,x_1*x_6,x_3*x_6,x_2*x_7,
  x_1*x_3*x_7,x_3*x_4*x_7,x_1*x_5*x_7,x_4*x_5*x_7,x_4*x_6*x_7))
///
-----
TEST3///
R = QQ[x_1..x_10]
assert(SocleVector(R,7,2) == (0,0,3))
///
-----
TEST4///
assert(isShellable(9,3) == false)
///
-----
TEST5///
assert(isCohenMacaulay(5,1) = true)
///
-----
TEST6///
assert(isVertexDecomposable(7,3) = false)
///
-----
TEST7///
assert(PseudoManifold(10,8) = Pseudomanifold with Boundary)
///
-----
```

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