IMPACT OF A MEAN-REVERTING SDE

ON OPTIMAL PORTFOLIOS

THE IMPACT OF A MEAN-REVERTING STOCHASTIC DIFFERENTIAL EQUATION ON OPTIMAL PORTFOLIO ALLOCATION

By

JOHN MCNAIR

A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfillment of the Requirements

for the Degree

Master of Science

McMaster University

MASTER OF SCIENCE (2005) McMaster University

(Mathematics and Statistics) Hamilton, Ontario

Title: The Impact of a Mean-Reverting Stochastic Differential Equation on Optimal Portfolio Allocation

AUTHOR: John McNair

SUPERVISOR: Dr. T. R. Hurd

NUMBER OF PAGES: V, 50

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Abstract

We consider the dilemma an investor contemplates when faced with the decision to allocate proportions of initial wealth within a multi-"risky" asset framework in order to maximize terminal wealth. It is assumed that Geometric Brownian Motion generates the "risky" asset price paths for this problem formulation. We consider the particular setting where the "risky" assets exhibit dependence on an Arithmetic Ornstein-Uhlenbeck stochastic differential equation (AOU) in the form of correlation and an embedded adjustment to the stochastic drift. We exemplify the motivation for this problem formulation by highlighting the empirical dependence that has occurred between the daily returns[1] of the TSX Composite Index (s_t), the daily returns of the Scotia Capital Overall Bond Index (b_t) and the Yield Ratio[2](R_t). We then derive the optimal portfolio control for this investor, using the Hamilton-Jacobi-Bellman equation method. We then construct optimal portfolios using Mean-Variance-Optimization (MVO) and compare terminal wealth for two investors using Monte-Carlo Simulation. Investor A is incognizant of the above dependence whereas Investor B is cognizant. We vary the above dependence parameters and assess the overall impact on the probability distribution of terminal wealth.

[1] Returns here are assumed to be log returns i.e. $Ln(X_{t+1})- Ln(X_t)$

[2] Ratio of TSX dividend yield over SCU yield

Acknowledgement

This thesis is the result of 3 years of work, on a part-time basis, where the support and accompaniment of various people was essential towards its completion. I am grateful for the opportunity to express a sincere thanks to them all.

The first person I would like to thank is my thesis advisor Dr. Tom Hurd. I first brought my thesis topic to him early in September of 2001. I recall engaging in a very colourful discussion on the concepts of my prospective topic where he insisted on playing the role of "Devil's Advocate" challenging every concept I put forward. Since my enrollment into the graduate studies program at McMaster, I have been doing my best to survive in two very different worlds, that of an investment practitioner and that of a mathematician. During my frequent visits with Dr. Hurd, my investment practitioner brain would, every now and then, make a cameo appearance and Dr. Hurd would quickly respond to this by saying "think as a mathematician". Dr. Hurd's constant guidance and expertise on the topic of optimal portfolio allocation and financial mathematics helped to transform my prospective topic from a concept into a formal thesis towards my M.Sc. degree in mathematics. I would also like to thank Bogdan Traicu, who made it quite clear that the Hamilton-Jacobi-Bellman optimal portfolio selection problem is not "rocket science its mathematics". I also want to thank my parents who, from a very young age, instilled a catch phrase deep within brain "it's not how smart you are it's how bad you want it". Last but definitely not least, I want to thank my wife whose love, patience and constant encouragement proved to be the essential ingredients needed for completion of this thesis.

Table of Contents

1.0 Introduction. 1

1.1 Background. 2

2.0 Motivation – Multivariate Time Series Analysis. 5

3.0 Geometric Brownian Motion model for "risky" asset price dynamics. 8

4.0 Empirical Yield Ratio – Time Series Analysis. 9

4.1 Arithmetic Ornstein-Uhlenbeck Model for Yield Ratio dynamics. 15

4.2 Monte Carlo Simulation. 16

5.0 Trivariate Arithmetic Ornstein-Uhlenbeck Model. 17

5.1 Monte Carlo Simulation. 23

6.0 Optimal portfolio selection – Hamilton-Jacobi-Bellman Framework. 25

7.0 Optimal portfolio selection – Mean Variance Framework. 29

8.0 Monte Carlo Simulation of Terminal Wealth. 32

8.1 Monte Carlo Simulation Comparison of Terminal Wealth. 33

9.0 Conclusions. 38

Appendices. 41

1.0 Introduction

In this paper we examine the problem that an investor contemplates when deciding to apportion proportions of his/her initial wealth to n "risky" assets so that terminal wealth is maximized. It is assumed that the "risky" asset price dynamics are governed by the standard Geometric Brownian Motion stochastic differential equation. We consider the particular scenario where the price processes of the "risky" assets display dependence on an Arithmetic Ornstein-Uhlenbeck stochastic differential equation in the form of correlation and an embedded adjustment to their stochastic drift. We demonstrate the motivation for this problem formulation by conducting time series analysis using the minimum AICC Yule-Walker estimation on the trivariate set (s_t, b_t, R_t) and highlighting the autoregressive relationship that these "risky" assets have with R_t. We then exemplify the autoregressive nature of the R_t and how it can be modeled using an AOU process by discretizing the AOU stochastic differential equation and making use of Monte-Carlo simulation. We then derive the optimal portfolio control for this investor by considering the particular case where $n = 2$, using the Hamilton-Jacobi-Bellman equation framework, allowing for correlation between the "risky" assets. We then compare the terminal wealth, using Monte-Carlo simulation, of two investors who each construct three portfolios via the Mean-Variance Optimization method. The initial capital for all six portfolios is \$1000. Three of these portfolios are constructed by an Investor A, who is incognizant of the above dependence between the "risky" assets and the AOU. Investor B, who is aware of the above dependence, constructs the other three portfolios. Investor A optimizes once at the beginning of the year and holds the optimal weights constant throughout 250 trading days whereas Investor B optimizes daily. A minimum variance, mid-variance and maximum expected value portfolio will be constructed for Investor A and B and pertinent profit/loss characteristics will be compared. We vary the level of dependence between the "risky" assets and the AOU and assess the overall impact on the probability distribution of terminal wealth.

1.1 Background

Portfolio optimization, a relatively mature field of study in mathematical finance, owes its roots to the likes of Merton [7] (and the "Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case") and Markowitz (and the Mean-Variance-Optimization method) [1,3] to name a few. Both of these approaches result in a time independent set of optimal portfolios on a reward to risk basis. The first is a continuous time framework and the second a single period optimization algorithm. Both Merton and Markowitz derived optimal portfolios by considering the reward to risk relationship of the "risky" assets themselves. In this thesis, the formulation differs from that of Merton and Markowitz in that I consider the existence of an external factor (an asset to which no wealth will be assigned) that may help in the decision to allocate wealth to one asset or the other. In fact, plan sponsors, trustees appointed to effectively manage pension plan assets for a specific company, and investment management firms often make use of external factors such as the Fed Model[3] to dictate how they should apportion wealth to stocks and bonds. For example, some strategists argue that when the Fed Model is above its long-term average then one should hold an overweight position to stocks. If the Fed Model is below its long-term average then hold an overweight position in bonds. Considering empirical data and using the Yield Ratio as the governing metric (instead of the Fed Model) there exists substantive evidence that this investment philosophy is sound. Consider the following graph in which investor B begins with $100 and decides to hold 100% of his/her portfolio in stocks (TSX composite total return index) or bonds (Scotia Capital Overall Bond total return index), depending on whether or not the Yield Ratio is above or below its 4-year rolling average respectively, and investor A who also begins with $100 but decides to allocate his/her wealth evenly between stocks and bonds over all periods.

[3] A model thought to be used by the Federal Reserve that hypothesizes a relationship between long-term treasury notes and the market return of equities. It was coined the "Fed Model" by Prudential Securities strategist Ed Yardeni.

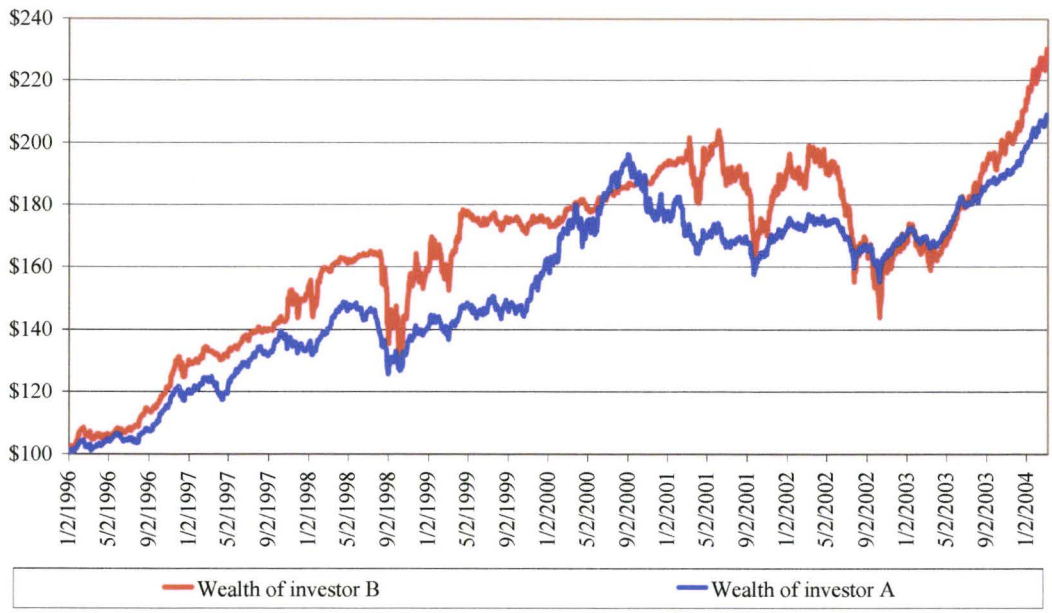


Figure 1.1.1 - Cumulative wealth of Investor A and B

In this example, not only does investor B have more wealth at the end of the period but he/she also has more wealth over most of the time horizon.

Often, however, practitioners do not make use of such governing metrics when constructing optimal portfolio allocation and therefore there exists the potential of exposing their wealth to performance shortfall *vis à vis* a more optimal approach. This thesis addresses this concern by incorporating such a governing metric into an optimal portfolio allocation context.

The following assumptions are to be taken for granted for the remainder of this document:

Assumptions

1. Transaction costs or any other cost associated with apportioning wealth are non-existent.
2. All securities can be purchased in any amount (including fractions)
3. Apportioning wealth does not affect the probability distribution of the securities available for investment.
4. There exists a risk-less asset (or "sure" investment) whose performance over a specific period is exactly known and all risk-less assets have the same performance over this period.
5. Short selling is an allowable strategy and the borrowing rate equals the lending rate.

2.0 Motivation – Multivariate Time Series Analysis

In this section we conduct multivariate time series analysis on the empirical trivariate set (s_t, b_t, R_t) to illustrate the serial interdependence that exists between these data. The first step in any time series analysis is to consider the classical decomposition of the empirical data to be analyzed:

$$\vec{X}_t = \vec{m}_t + \vec{c}_t + \vec{Y}_t \tag{2.1}$$

Where $\vec{X}_t$ is the observation at time t, $\vec{m}_t$ is a trend component for the empirical data, $\vec{c}_t$ is a seasonal component for the empirical data and $\vec{Y}_t$ is a random noise component, which is weakly stationary.

Definition 2.1: The multivariate time series $\{\vec{X}_t\}$ is weakly stationary if:

$$\begin{aligned} &i)\ \mu_{\vec{X}}(t) \text{ is independent of time } (t), \text{ and} \\ &ii)\ \gamma_{\vec{X}}(t+h,t) \text{ is independent of time } (t) \text{ for each } h \end{aligned} \tag{2.2}$$

where $\mu_{\vec{X}}(t)$ is the mean function and $\gamma_{\vec{X}}(t+h,t)$ the auto-covariance function of the time series. The aim is to estimate the deterministic components $\vec{m}_t$ and $\vec{c}_t$ and to subtract them from the original data to obtain a series of stationary residuals $\vec{Y}_t$. We can then use time series analysis theory [6] to find satisfactory probabilistic models for these residuals and ultimately a probabilistic model for the original time series.

For the purpose of this time series analysis, we use the multivariate minimum AICC Yule-Walker [6] methodology to find the multivariate autoregressive model that best fits the 250 trading days for every year since 1999.

Recall that a trivariate AR(1) process has the following form:

$$\hat{\vec{X}}_t = \Phi_1 + \Phi_{13}\hat{\vec{X}}_{t-1} + \vec{Z}_{13}, \vec{Z}_{13} \sim WN(\vec{0}, \Sigma) \tag{2.3}$$

The following table lists the results of this methodology.

1999 – Multivariate Minimum AICC Yule-Walker estimates

$$\hat{\vec{X}}_t = \begin{bmatrix} s_t \\ b_t \\ R_t \end{bmatrix} = \begin{bmatrix} 0.005970 \\ 0.000631 \\ 0.005353 \end{bmatrix} + \begin{bmatrix} 0.123 & 0.295 & -0.019 \\ 0.002 & 0.139 & -0.002 \\ -0.038 & -0.133 & 0.980 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ b_{t-1} \\ R_{t-1} \end{bmatrix} + \begin{bmatrix} Z_{t1} \\ Z_{t2} \\ Z_{t3} \end{bmatrix},$$

$$\begin{bmatrix} Z_{t1} \\ Z_{t2} \\ Z_{t3} \end{bmatrix} \sim WN\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.000081 & 0.000007 & -0.000017 \\ 0.000007 & 0.000007 & 0.000005 \\ -0.000017 & 0.000005 & 0.000032 \end{bmatrix} \right)$$

2000 – Multivariate Minimum AICC Yule-Walker estimates

$$\hat{\vec{X}}_t = \begin{bmatrix} s_t \\ b_t \\ R_t \end{bmatrix} = \begin{bmatrix} -0.017622 \\ -0.000157 \\ 0.007621 \end{bmatrix} + \begin{bmatrix} 0.133 & 1.071 & 0.096 \\ 0.006 & -0.020 & 0.003 \\ -0.027 & -0.180 & 0.959 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ b_{t-1} \\ R_{t-1} \end{bmatrix} + \begin{bmatrix} Z_{t1} \\ Z_{t2} \\ Z_{t3} \end{bmatrix},$$

$$\begin{bmatrix} Z_{t1} \\ Z_{t2} \\ Z_{t3} \end{bmatrix} \sim WN\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.000267 & -0.000003 & -0.000053 \\ -0.000003 & 0.000005 & 0.000003 \\ -0.000053 & 0.000003 & 0.000021 \end{bmatrix} \right)$$

2001 – Multivariate Minimum AICC Yule-Walker estimates

$$\hat{\vec{X}}_t = \begin{bmatrix} s_t \\ b_t \\ R_t \end{bmatrix} = \begin{bmatrix} -0.007810 \\ 0.000880 \\ 0.003850 \end{bmatrix} + \begin{bmatrix} -0.002 & 0.283 & 0.025 \\ -0.002 & 0.194 & -0.002 \\ -0.009 & 0.082 & 0.986 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ b_{t-1} \\ R_{t-1} \end{bmatrix} + \begin{bmatrix} Z_{t1} \\ Z_{t2} \\ Z_{t3} \end{bmatrix},$$

$$\begin{bmatrix} Z_{t1} \\ Z_{t2} \\ Z_{t3} \end{bmatrix} \sim WN\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.000139 & -0.000007 & -0.000045 \\ -0.000007 & 0.000008 & 0.000010 \\ -0.000045 & 0.000010 & 0.000045 \end{bmatrix} \right)$$

2002 – Multivariate Minimum AICC Yule-Walker estimates

$$\hat{\vec{X}}_t = \begin{bmatrix} s_t \\ b_t \\ R_t \end{bmatrix} = \begin{bmatrix} -0.007102 \\ 0.001185 \\ 0.005730 \end{bmatrix} + \begin{bmatrix} -0.004 & -0.049 & 0.018 \\ 0.025 & 0.086 & -0.002 \\ 0.043 & 0.201 & 0.984 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ b_{t-1} \\ R_{t-1} \end{bmatrix} + \begin{bmatrix} Z_{t1} \\ Z_{t2} \\ Z_{t3} \end{bmatrix},$$

$$\begin{bmatrix} Z_{t1} \\ Z_{t2} \\ Z_{t3} \end{bmatrix} \sim WN\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.000107 & -0.000014 & -0.000061 \\ -0.000014 & 0.000007 & 0.000016 \\ -0.000061 & 0.000016 & 0.000075 \end{bmatrix} \right)$$

2003 – Multivariate Minimum AICC Yule-Walker estimates

$$\hat{\vec{X}}_t = \begin{bmatrix} s_t \\ b_t \\ R_t \end{bmatrix} = \begin{bmatrix} -0.008495 \\ 0.005725 \\ 0.015601 \end{bmatrix} + \begin{bmatrix} 0.072 & 0.284 & 0.023 \\ 0.014 & 0.014 & -0.014 \\ 0.040 & -0.095 & 0.961 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ b_{t-1} \\ R_{t-1} \end{bmatrix} + \begin{bmatrix} Z_{t1} \\ Z_{t2} \\ Z_{t3} \end{bmatrix},$$

$$\begin{bmatrix} Z_{t1} \\ Z_{t2} \\ Z_{t3} \end{bmatrix} \sim WN\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.000036 & -0.000005 & -0.000023 \\ -0.000005 & 0.000007 & 0.000013 \\ -0.000023 & 0.000013 & 0.000035 \end{bmatrix} \right)$$

A goodness of fit test can then be implemented by calculating the sample autocorrelations of the residuals [6]. If fewer than 2 of the 20 autocorrelations lie beyond the bounds $\pm 1.96/\sqrt{n}$, where 1.96 is the .975 quantile of the standard normal distribution, then we accept the independently identically distributed (iid) hypothesis of the residuals and conclude that the model $\hat{\vec{X}}_t$ is a good fit for the data $\vec{X}_t$. Appendix 1 exhibits the sample autocorrelations of the residuals for the 250 trading days for every year since 1999. These charts visually exhibit the goodness of fit of the above models for the empirical data.

3.0 Geometric Brownian Motion model for "risky" asset price dynamics

One model for future asset price dynamics generally accepted by mathematicians is the Geometric Brownian Motion stochastic differential equation for asset price S_t, with stochastic drift μ_S (constant) and volatility σ_S (constant) defined by

$$dS_t = \mu_S S_t dt + \sigma_S S_t dW_t^S \tag{3.1}$$

Where W_t^S is the standard Wiener process. The explicit solution for S_t can be found in the following manner:

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dW_t^S \Leftrightarrow \int_0^t \frac{dS_s}{S_s} = \mu_S t + \sigma_S W_t^S \tag{3.2}$$

Using Ito's formula [4] we can solve (3.2) using the following steps:

$$\text{Let } F(t,S_t) = F(0,S_0) + \int_0^t \frac{\partial F}{\partial s} ds + \int_0^t \frac{\partial F}{\partial S} dS_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial S^2} dVarS_s \tag{3.3}$$

now :

$$dVarS_t = dS_t dS_t = \mu_S^2 S_t^2 dtdt + 2\mu_S \sigma_S dtdW_t^S + \sigma_S^2 S_t^2 dW_t^S dW_t^S = \sigma_S^2 S_t^2 dt \tag{3.4}$$

$$\text{choose } F = \ln S \Rightarrow dF = \frac{1}{S} = d(\ln S) \Rightarrow d^2F = \frac{-1}{S^2} \tag{3.5}$$

$$\therefore \ln S_t - \ln S_0 = \int_0^t \frac{dS_s}{S_s} + \frac{1}{2}\int_0^t \frac{-\sigma_S^2 S_s^2}{S_s^2} ds \Rightarrow \ln \frac{S_t}{S_0} = \int_0^t \frac{dS_s}{S_s} - \frac{\sigma_S^2}{2}\int_0^t ds \tag{3.6}$$

$$\text{but : } \int_0^t \frac{dS_s}{S_s} = \mu_S t + \sigma W_t^S \tag{3.7}$$

$$\therefore \ln \frac{S_t}{S_0} + \frac{\sigma_S^2 t}{2} = \mu_S t + \sigma W_t^S \Rightarrow S_t = S_0 e^{\sigma_S W_t^S + t(\mu_S - \frac{\sigma_S^2}{2})} \tag{3.8}$$

Monte Carlo simulation can be used to generate the asset price path of this model but it will be necessary to discretize the time [2] in (3.8) by introducing a time step δ_t and recalling the underlying properties of Brownian [4] motion to yield:

$$S_{t+\delta_t} = S_t e^{((\mu_S - \frac{\sigma_S^2}{2})\delta_t + \sigma_S \sqrt{\delta_t}\varepsilon)}, \varepsilon \sim N(0,1) \tag{3.9}$$

4.0 Empirical Yield Ratio – Time Series Analysis

As depicted below, the actual historical daily values of R_t have an inherent tendency to revert to a long-term average value. Some practitioners take for granted that this long-term average value represents the equilibrium point of this metric.

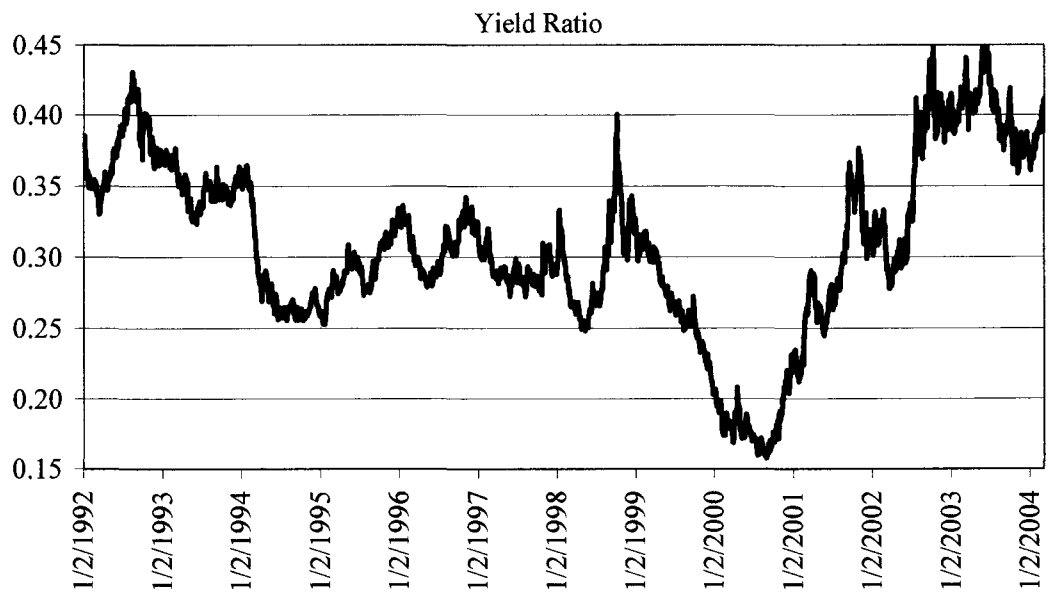


Figure 4.1 - Empirical daily values of the Yield Ratio

To determine the viability of the Arithmetic Ornstein-Uhlenbeck process (AOU) as a model for R_t, first recall that the mean adjusted discretized AOU stochastic differential equation is simply a first-order auto-regression process or AR(1) process satisfying the relationship

$$\begin{aligned} &R_t = \phi R_{t-1} + Z_t,\ t = 0, \pm 1, \pm 2... \\ &\text{where } \{Z_t\} \sim WN(0, \sigma^2) \end{aligned} \tag{4.1}$$

Let us now examine the main features of the empirical R_t by first examining the scatter plot of the $(R_{t_{i-1}}, R_{t_i})$ pairs for the trading days since the beginning of 1999 until the end of 2003.

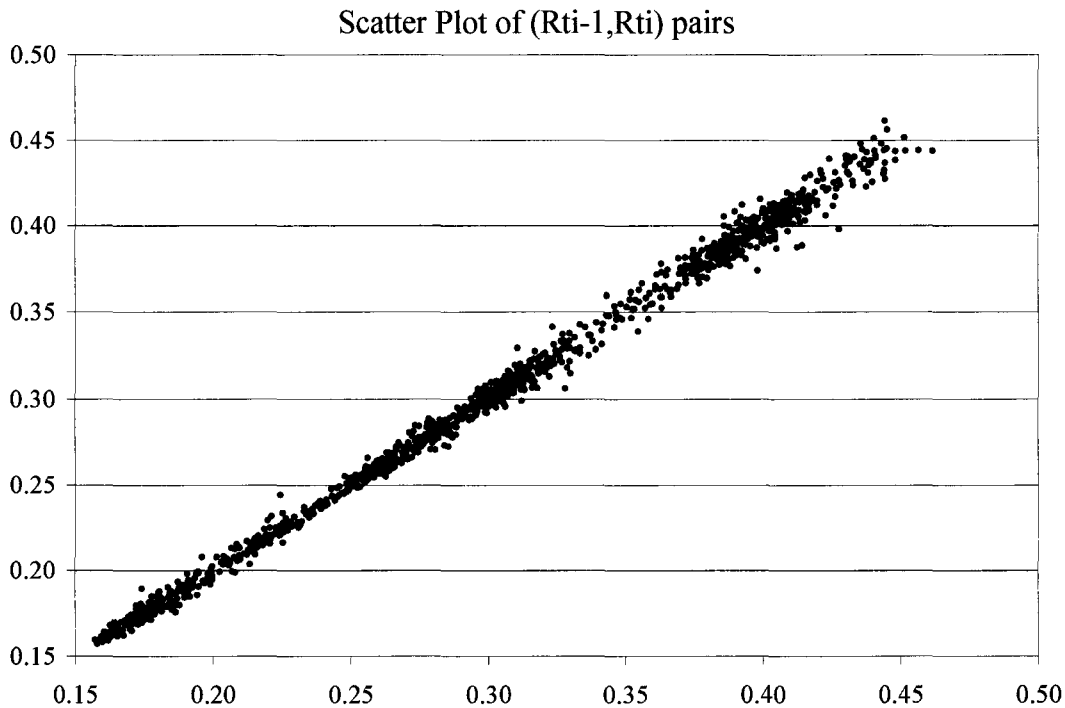


Figure 4.2 - Scatter plot of the Yield Ratio at time t versus the Yield Ratio a time t-1

Figure 4.2 does indeed irrevocably suggest a linear relationship between R_{t_i} and $R_{t_{i-1}}$ over this period. Using least squares to fit a straight line through the above pairs, we obtain the following model:

$$R_t = 0.99589R_{t-1} + Z_t,\ t = 0, \pm 1, \pm 2...$$
$$\text{where } \{Z_t\} \text{ is iid noise with variance}$$
$$\sigma^2 = \frac{\sum_{i=2}^{1254}(R_t - 0.99589R_{t-1})^2}{1253} \tag{4.2}$$
$$= 2.77778 \times 10^{-5}$$

Also note that each individual trading year (consisting of 250 trading days) also shares the above linear relationship.

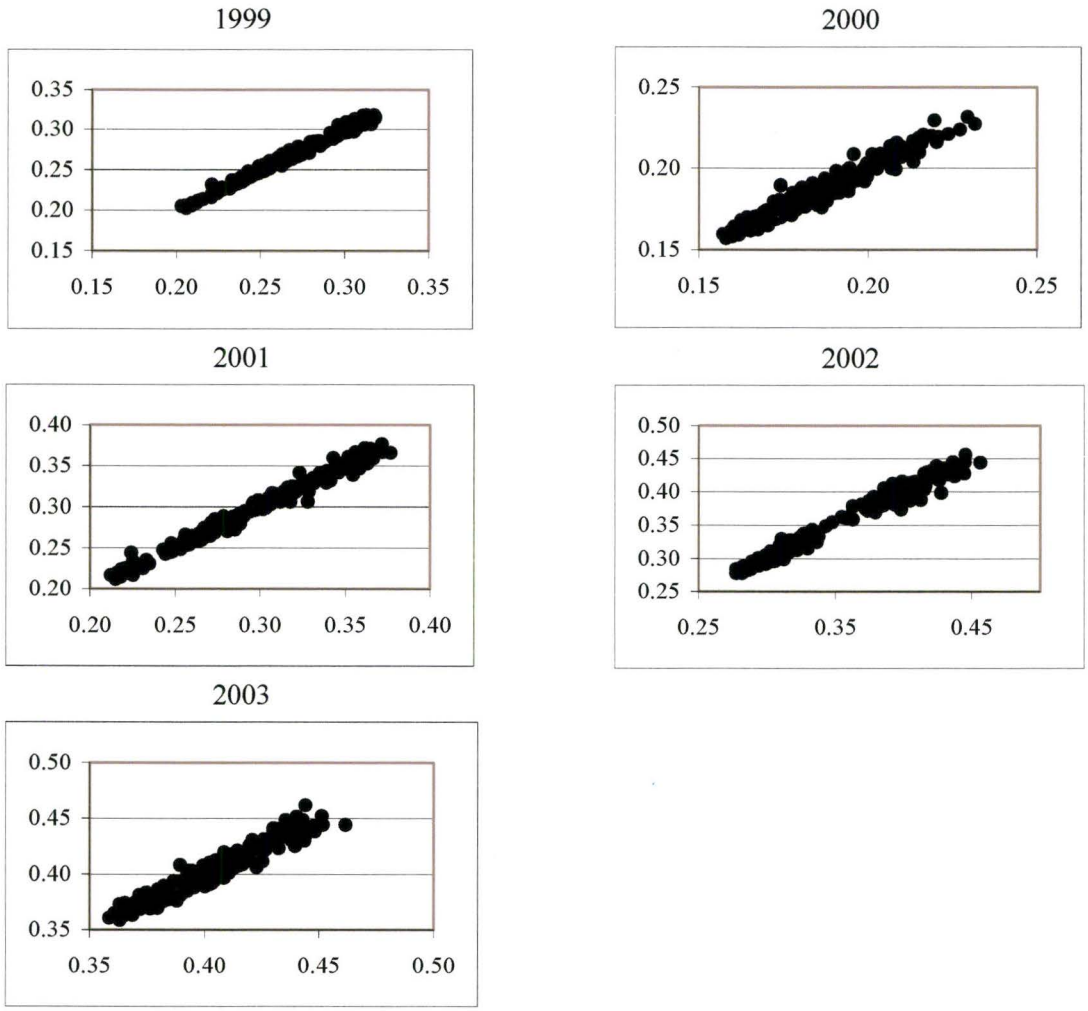


Figure 4.3 - Scatter plot of the Yield Ratio at time t versus the Yield Ratio a time t-1 for each calendar year since 1999

Examination of the geometric decay of the autocorrelations at higher lags shows that the linear operator ϕ does not change significantly over time and that the AR(1) autoregressive model is more appropriate. Consider the following graph, which depicts the rolling 250 trading day autocorrelation functions at lag-1, lag-5, lag-10, lag-15 and lag-20 for all trading days since 1999.

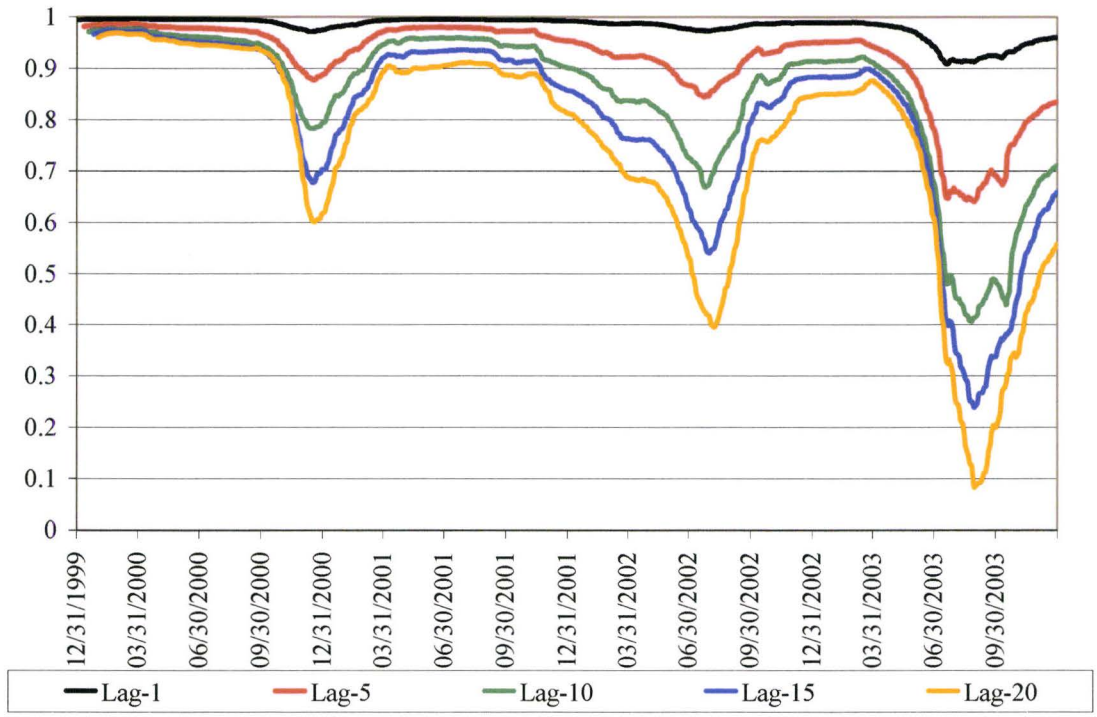


Figure 4.4 - Rolling 250 trading day autocorrelation functions at lag-1, lag-5, lag-10, lag-15 and lag-20 for all trading days since 1999

Notice that at lag-1 the autocorrelation remains in the range 0.9 to 1.0 whereas at higher lags the autocorrelations vary significantly.

For a more rigorous approach, one can use the minimum AICC Yule-Walker estimation to find the appropriate model for the actual R_t. The following tables list the results of this method over the 250 trading days for every year since 1999.

1999 – Minimum AICC Yule-Walker estimates

$$\hat{R}_t = 0.9814R_{t-1} + Z_t, \text{ where } Z_t \sim WN(0, 0.000011) \tag{4.3}$$

2000 – Minimum AICC Yule-Walker estimates

$$\hat{R}_t = 0.9603R_{t-1} + Z_t, \text{ where } Z_t \sim WN(0, 0.000014) \tag{4.4}$$

2001 – Minimum AICC Yule-Walker estimates

$$\hat{R}_t = 0.9863R_{t-1} + Z_t, \text{ where } Z_t \sim WN(0, 0.000028) \tag{4.5}$$

2002 – Minimum AICC Yule-Walker estimates

$$\hat{R}_t = 0.9839R_{t-1} + Z_t, \text{ where } Z_t \sim WN(0, 0.000052) \tag{4.6}$$

2003 – Minimum AICC Yule-Walker estimates

$$\hat{R}_t = 0.9581R_{t-1} + Z_t, \text{ where } Z_t \sim WN(0, 0.000032) \tag{4.7}$$

For large n we can compute confidence intervals for the true value of the parameter ϕ for the above AR(1) processes. Let $\Phi_{1-\alpha}$ be the $(1-\alpha)$ quantile of the standard normal distribution and let $\upsilon_{11} = [1 - \hat{\rho}^2(1)]$ (where $\hat{\rho}(1)$ is the autocorrelation function at lag-1) then:

$$\hat{\phi} \pm \frac{\sqrt{\upsilon_{11}}\Phi_{1-\alpha/2}}{\sqrt{n}} \tag{4.8}$$

will encapsulate ϕ with probability close to $(1-\alpha)$.

1999 – Confidence interval for the AR(1) linear operator

$$\phi = 0.9814 \pm 0.013 \tag{4.9}$$

2000 – Confidence interval for the AR(1) linear operator

$$\phi = 0.9603 \pm 0.028 \tag{4.10}$$

2001 – Confidence interval for the AR(1) linear operator

$$\phi = 0.9863 \pm 0.016 \tag{4.11}$$

2002 – Confidence interval for the AR(1) linear operator

$$\phi = 0.9839 \pm 0.018 \tag{4.12}$$

2003 – Confidence interval for the AR(1) linear operator

$$\phi = 0.9581 \pm 0.034 \tag{4.13}$$

The minimum AICC Yule-Walker estimation methodology yields an AR(1) process for every trading year since 1999 and therefore justifies the use of the Arithmetic Ornstein-Uhlenbeck process to model R_t.

4.1 Arithmetic Ornstein-Uhlenbeck Model for Yield Ratio dynamics

One generally accepted model that has been proved successful in modeling such mean-reverting behaviour is the Arithmetic Ornstein-Uhlenbeck process (AOU) defined by

$$dR_t = -\eta(R_t - \bar{R})dt + \sigma_R dW_t^R \tag{4.1.1}$$

where $\bar{R}$ is the long-term average or the value to which (4.1.1) mean-reverts and η is the rate or frequency at which this model reverts. An explicit solution for R_t can be found in the following manner:

$$dR_t = \eta\bar{R}dt - \eta R_t dt + \sigma_R dW_t^R \tag{4.1.2}$$

$$dR_t + \eta R_t dt = \eta\bar{R}dt + \sigma_R dW_t^R \tag{4.1.3}$$

$$e^{\eta t}(dR_t + \eta R_t dt) = e^{\eta t}(\eta\bar{R}dt + \sigma_R dW_t^R) \Rightarrow d(e^{\eta t}R_t) = \eta\bar{R}e^{\eta t}dt + \sigma_R e^{\eta t}dW_t^R \tag{4.1.4}$$

$$e^{\eta t}R_t - R_0 = \eta\bar{R}\int_0^t e^{\eta s}ds + \sigma_R\int_0^t e^{\eta s}dW_s^R \tag{4.1.5}$$

$$e^{\eta t}R_t - R_0 = \eta\bar{R}(\frac{e^{\eta t}-1}{\eta}) + \sigma_R\int_0^t e^{\eta s}dW_s^R \tag{4.1.6}$$

$$R_t = R_0 e^{-\eta t} + \bar{R}(1-e^{-\eta t}) + \sigma_R e^{-\eta t}\int_0^t e^{\eta s}dW_s^R \tag{4.1.7}$$

Monte Carlo simulation can be used to generate the path of (4.1.7) but it will be necessary to discretize [2] the time by introducing a time step δ_t and recalling the basic properties of Brownian [4] motion to yield:

$$R_{t+\delta_t} = R_t e^{-\eta\delta_t} + \bar{R}(1-e^{-\eta\delta_t}) + \sigma_R(\frac{1-e^{-2\eta\delta_t}}{2\eta})^{1/2}\varepsilon, \varepsilon \sim N(0,1) \tag{4.1.8}$$

Notice how equation (4.1.8) resembles (4.1) where $\phi = e^{-\eta\delta_t}$. Let us make use of Monte-Carlo simulation with the results of the 2003 Minimum AICC Yule-Walker estimates.

4.2 Monte Carlo Simulation

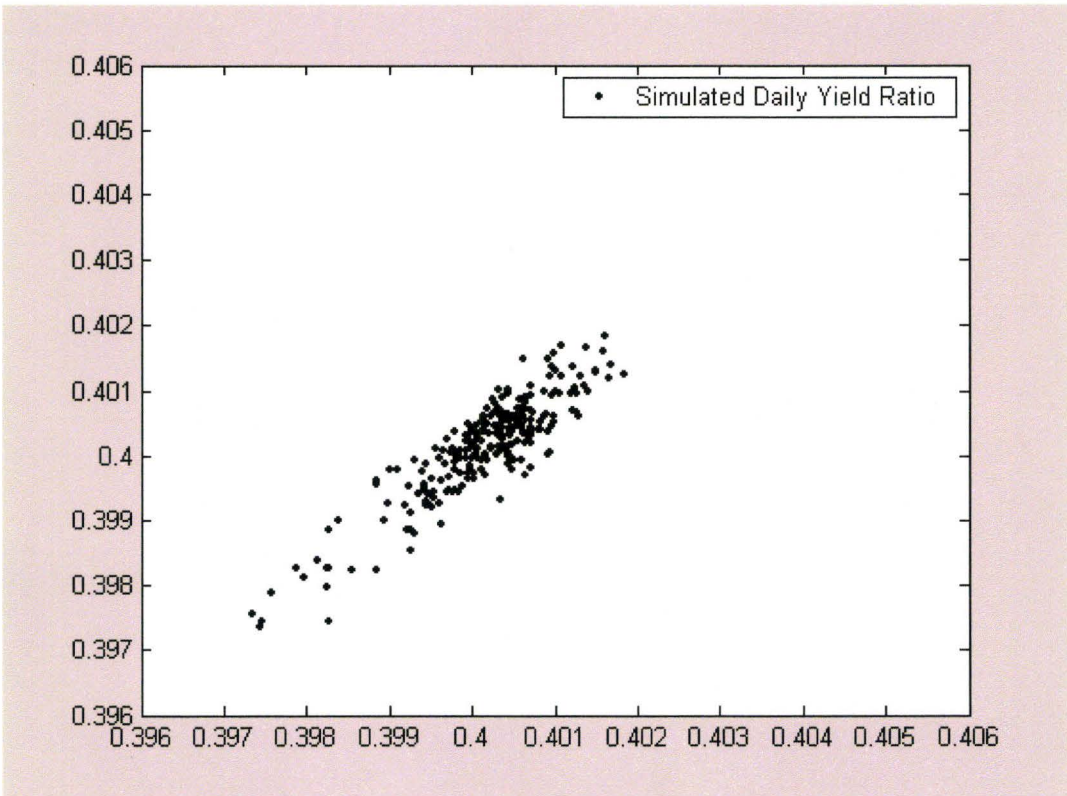


Figure 4.2.1 - Scatter plot of the pairs $(R_{t_{i-1}}, R_{t_i})$ using the average of 500 simulations

The above scatter plot also demonstrates the linear dependency at lag-1 of the AOU process.

5.0 Trivariate Arithmetic Ornstein-Uhlenbeck Model

Let the following Geometric Brownian motion and AOU models represent the price path dynamics of the TSX(S_t), the SCU(B_t) and R_t respectively:

$$\begin{aligned} dS_t &= (\kappa + \alpha(R_t - \bar{R}))S_t dt + \sigma_S S_t dW_t^S \\ dB_t &= (\lambda + \gamma(R_t - \bar{R}))B_t dt + \sigma_B B_t dW_t^B \\ dR_t &= -\eta(R_t - \bar{R})dt + \sigma_R dW_t^R \end{aligned} \tag{5.1}$$

Notice here that the stochastic drifts for the process dS_t and dB_t are $(\kappa + \alpha(R_t - \bar{R}))$ and $(\lambda + \gamma(R_t - \bar{R}))$ respectively.

Now if we let $s_t = \log S_t$ *and* $b_t = \log B_t$ (5.1) becomes:

$$\begin{aligned} ds_t &= \left(\kappa + \alpha(R_t - \bar{R}) - \frac{\sigma_S^2}{2}\right)dt + \sigma_S dW_t^S \\ db_t &= \left(\lambda + \gamma(R_t - \bar{R}) - \frac{\sigma_B^2}{2}\right)dt + \sigma_B dW_t^B \\ dR_t &= -\eta(R_t - \bar{R})dt + \sigma_R dW_t^R \end{aligned} \tag{5.2}$$

Rewriting (5.2) as a multivariate model allowing for correlation between the processes yields:

$$d\vec{R}_t = \begin{bmatrix} \kappa + \alpha(R_t - \bar{R}) - 0.5\sigma_S^2 \\ \lambda + \gamma(R_t - \bar{R}) - 0.5\sigma_B^2 \\ -\eta(R_t - \bar{R}) \end{bmatrix} dt + \begin{bmatrix} \sigma_{SS} & \sigma_{SB} & \sigma_{SR} \\ \sigma_{SB} & \sigma_{BB} & \sigma_{BR} \\ \sigma_{SR} & \sigma_{BR} & \sigma_{RR} \end{bmatrix}^{1/2} d\vec{W}_t \tag{5.3}$$

or, more appropriately, (5.3) can be written as a multivariate Arithmetic Ornstein-Uhlenbeck stochastic differential equation:

$$d\vec{R}_t = (\vec{\mu} - A\vec{R}_t)dt + \Sigma^{1/2} d\vec{W}_t \tag{5.4}$$

where:

$$\vec{\mu} = \begin{bmatrix} \kappa - \alpha\bar{R} - 0.5\sigma_S^2 \\ \lambda - \gamma\bar{R} - 0.5\sigma_B^2 \\ \eta\bar{R} \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & -\alpha \\ 0 & 0 & -\gamma \\ 0 & 0 & \eta \end{bmatrix}, \vec{R}_t = \begin{bmatrix} s_t \\ b_t \\ R_t \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} \sigma_{SS} & \sigma_{SB} & \sigma_{SR} \\ \sigma_{SB} & \sigma_{BB} & \sigma_{BR} \\ \sigma_{SR} & \sigma_{BR} & \sigma_{RR} \end{bmatrix} \tag{5.5}$$

An explicit solution for $\vec{R}_t$ can be found as a generalization of the solution for the univariate Arithmetic Ornstein-Uhlenbeck model:

$$e^{At}(d\vec{R}_t + A\vec{R}_t dt) = e^{At}\vec{\mu}dt + e^{At}\Sigma^{1/2}d\vec{W}_t \tag{5.6}$$

where e^{At} is the matrix exponential.

Now

$$(de^{At}\vec{R}_t) = e^{At}\vec{\mu}dt + e^{At}\Sigma^{1/2}d\vec{W}_t \tag{5.7}$$

$$e^{At}\vec{R}_t - e^{A0}\vec{R}_0 = \int_0^t e^{As}\vec{\mu}ds + \int_0^t e^{As}\Sigma^{1/2}d\vec{W}_s \tag{5.8}$$

Now note that:

$$At = \begin{bmatrix} 0 & 0 & -\alpha t \\ 0 & 0 & -\gamma t \\ 0 & 0 & \eta t \end{bmatrix} = P_1 D_1 P_1^{-1} = \begin{bmatrix} 1 & 0 & -\alpha/\eta \\ 0 & 1 & -\gamma/\eta \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \eta t \end{bmatrix}\begin{bmatrix} 1 & 0 & \alpha/\eta \\ 0 & 1 & \gamma/\eta \\ 0 & 0 & 1 \end{bmatrix} \tag{5.9}$$

$$-At = \begin{bmatrix} 0 & 0 & \alpha t \\ 0 & 0 & \gamma t \\ 0 & 0 & -\eta t \end{bmatrix} = P_1 D_2 P_1^{-1} = \begin{bmatrix} 1 & 0 & -\alpha/\eta \\ 0 & 1 & -\gamma/\eta \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\eta t \end{bmatrix}\begin{bmatrix} 1 & 0 & \alpha/\eta \\ 0 & 1 & \gamma/\eta \\ 0 & 0 & 1 \end{bmatrix} \tag{5.10}$$

and therefore:

$$e^{At} = P_1 e^{D_1} P_1^{-1} = \begin{bmatrix} 1 & 0 & -\alpha/\eta \\ 0 & 1 & -\gamma/\eta \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\eta t} \end{bmatrix}\begin{bmatrix} 1 & 0 & \alpha/\eta \\ 0 & 1 & \gamma/\eta \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dfrac{\alpha(1-e^{\eta t})}{\eta} \\ 0 & 1 & \dfrac{\gamma(1-e^{\eta t})}{\eta} \\ 0 & 0 & e^{\eta t} \end{bmatrix} \tag{5.11}$$

$$e^{-At} = P_1 e^{D_2} P_1^{-1} = \begin{bmatrix} 1 & 0 & -\alpha/\eta \\ 0 & 1 & -\gamma/\eta \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\eta t} \end{bmatrix}\begin{bmatrix} 1 & 0 & \alpha/\eta \\ 0 & 1 & \gamma/\eta \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dfrac{\alpha(1-e^{-\eta t})}{\eta} \\ 0 & 1 & \dfrac{\gamma(1-e^{-\eta t})}{\eta} \\ 0 & 0 & e^{-\eta t} \end{bmatrix} \tag{5.12}$$

Which implies that:

$$e^{-At}e^{At} = I \tag{5.13}$$

Also note that

$$e^{A0} = I \tag{5.14}$$

The matrix A is not invertible so it will be necessary to make use of the following identity to solve (5.8):

$$e^{At} = I + \frac{(e^{\eta t} - 1)}{\eta} A \tag{5.15}$$

Now

$$\vec{R}_t = e^{-At}\vec{R}_0 + e^{-At}\int_0 e^{As}\vec{\mu} ds + e^{-At}\int_0 e^{As}\Sigma^{1/2} d\vec{W}_s \tag{5.16}$$

$$\vec{R}_t = e^{-At}\vec{R}_0 + \int_0 e^{A(s-t)}\vec{\mu} ds + \int_0 e^{A(s-t)}\Sigma^{1/2} d\vec{W}_s \tag{5.17}$$

$$\vec{R}_t = e^{-At}\vec{R}_0 + \int_0 (I + \frac{(e^{\eta(s-t)} - 1)}{\eta} A)\vec{\mu} ds + \int_0 (I + \frac{(e^{\eta(s-t)} - 1)}{\eta} A)\Sigma^{1/2} d\vec{W}_s \tag{5.18}$$

$$\vec{R}_t = e^{-At}\vec{R}_0 + \left(tI + \left(\frac{(1 - e^{-\eta t})}{\eta^2} - \frac{t}{\eta} \right) A \right)\vec{\mu} + \int_0 (I + \frac{(e^{\eta(s-t)} - 1)}{\eta} A)\Sigma^{1/2} d\vec{W}_s \tag{5.19}$$

The equation (5.19) is the explicit solution to the trivariate Arithmetic Ornstein-Uhlenbeck stochastic differential equation. Monte Carlo simulation can be used to generate the path of (5.19) but it will be necessary to discretize [2] the time by introducing a time step δ_t and recalling the basic properties of Brownian [4] motion. To achieve this, we first must calculate $E[\vec{R}_t]$ and $Var[\vec{R}_t]$:

For the expected value of (5.19), we do the following:

$$E[\vec{R}_t] = E[e^{-At}\vec{R}_0] + E\left[\left(tI + \left(\frac{(1-e^{-\eta t})}{\eta^2} - \frac{t}{\eta}\right)A\right)\bar{\mu}\right] + E\left[\int_0^t (I + \frac{(e^{\eta(s-t)}-1)}{\eta}A)\Sigma^{1/2}d\vec{W}_s\right]$$

and by the basic properties of Brownian motion we have that:

$$E[\vec{R}_t] = e^{-At}\vec{R}_0 + \left(tI + \left(\frac{(1-e^{-\eta t})}{\eta^2} - \frac{t}{\eta}\right)A\right)\bar{\mu}$$

$$E[\vec{R}_t] = \begin{bmatrix} s_0 + \frac{\alpha R_0(1-e^{-\eta t})}{\eta} + (\kappa - 0.5\sigma_S^2)t - \alpha\bar{R}\left(\frac{1-e^{-\eta t}}{\eta}\right) \\ b_0 + \frac{\gamma R_0(1-e^{-\eta t})}{\eta} + (\lambda - 0.5\sigma_B^2)t - \gamma\bar{R}\left(\frac{1-e^{-\eta t}}{\eta}\right) \\ e^{-\eta t}R_0 + \bar{R}(1-e^{-\eta t}) \end{bmatrix} \quad (5.20)$$

Now for the variance of (5.19) recall that:

$$Var[\vec{R}_t] = E[\vec{R}_t\vec{R}_t^T] - E[\vec{R}_t]E[\vec{R}_t^T] \quad (5.21)$$

which reduces to:

$$Var[\vec{R}_t] = E\left[\left(\int_0^t (I + \frac{(e^{\eta(s-t)}-1)}{\eta}A)\Sigma^{1/2}d\vec{W}_s\right)\left(\int_0^t (I + \frac{(e^{\eta(s-t)}-1)}{\eta}A)\Sigma^{1/2}d\vec{W}_s\right)^T\right] \quad (5.22)$$

We must make use of the Itô isometry to solve (5.22).

Theorem 5.1 (The Itô isometry): If $\phi(t,\omega)$ is bounded and elementary then:

$$E\left[\left(\int_s^T \phi dW_t\right)^2\right] = E\left[\int_s^T \phi^2 dt\right] \quad (5.23)$$

The multivariate extension to theorem 5.1 is derived as follows. Consider the following expression:

$$E\left[\left(\int_s^T M d\vec{W}_t\right)^2\right] \quad (5.24)$$

Since the ij^{th} component of matrix M is bounded and elementary the following is true

$$\int_S^T M d\vec{W}_t = \sum_{j\geq 0} M_j(\vec{W}_{t_{j+1}} - \vec{W}_{t_j}) = \sum_j \begin{bmatrix} e_{11_j} & e_{12_j} \ldots e_{1n_j} \\ e_{21_j} & e_{22_j} \ldots e_{2n_j} \\ . \\ e_{n1_j} & e_{n2_j} \ldots e_{nn_j} \end{bmatrix} \begin{bmatrix} \Delta W_{t_j}^1 \\ \Delta W_{t_j}^2 \\ . \\ \Delta W_{t_j}^n \end{bmatrix} \tag{5.25}$$

Then

$$E\left[e_{mk_j} e_{pl_i} \Delta W_{t_j}^k \Delta W_{t_i}^l\right] = \begin{cases} 0 & k \neq l, i \neq j \\ E\left[e_{mk_j} e_{pl_i}\right]\left(t_{j+1} - t_j\right) & k = l, i = j \\ & m, p = 1 \text{ to } n \end{cases} \tag{5.26}$$

Thus

$$E\left[\left(\int_s^T M d\vec{W}_t\right)^2\right] = E\left[\int_s^T MM^T dt\right] \tag{5.27}$$

Using the above result, let us now solve (5.22):

$$Var[\vec{R}_t] = E\left[\int_0^t ((I + \frac{(e^{\eta(s-t)} - 1)}{\eta} A)\Sigma^{1/2})((I + \frac{(e^{\eta(s-t)} - 1)}{\eta} A)\Sigma^{1/2})^T ds\right]$$

$$Var[\vec{R}_t] = E\left[\int_0^t ((I + \frac{(e^{\eta(s-t)} - 1)}{\eta} A)\Sigma^{1/2})((\Sigma^{1/2})^T (I + \frac{(e^{\eta(s-t)} - 1)}{\eta} A^T))ds\right]$$

$$Var[\vec{R}_t] = E\left[\int_0^t (\Sigma^{1/2} + \frac{(e^{\eta(s-t)} - 1)}{\eta} A\Sigma^{1/2})((\Sigma^{1/2})^T + \frac{(e^{\eta(s-t)} - 1)}{\eta}(\Sigma^{1/2})^T A^T)ds\right]$$

$$Var[\vec{R}_t] = E\left[\int_0^t \begin{array}{l} \Sigma^{1/2}(\Sigma^{1/2})^T + \frac{(e^{\eta(s-t)} - 1)}{\eta}\Sigma^{1/2}(\Sigma^{1/2})^T A^T + \frac{(e^{\eta(s-t)} - 1)}{\eta} A\Sigma^{1/2}(\Sigma^{1/2})^T \\ + \frac{(e^{\eta(s-t)} - 1)^2}{\eta^2} A\Sigma^{1/2}(\Sigma^{1/2})^T A^T ds \end{array}\right]$$

$$Var[\vec{R}_t] = E\left[\int_0^t \begin{array}{l} \Sigma^{1/2}(\Sigma^{1/2})^T + \frac{(e^{\eta(s-t)} - 1)}{\eta}(\Sigma^{1/2}(\Sigma^{1/2})^T A^T + A\Sigma^{1/2}(\Sigma^{1/2})^T) \\ + \frac{(e^{2\eta(s-t)} - 2e^{\eta(s-t)} + 1)}{\eta^2} A\Sigma^{1/2}(\Sigma^{1/2})^T A^T ds \end{array}\right]$$

and therefore we have that:

$$Var[\vec{R}_t] = t\Sigma^{1/2}(\Sigma^{1/2})^T + \left(\frac{1-\eta t - e^{-\eta t}}{\eta^2}\right)(\Sigma^{1/2}(\Sigma^{1/2})^T A^T + A\Sigma^{1/2}(\Sigma^{1/2})^T) + \left(\frac{-3+2\eta t - e^{-2\eta t} + 4e^{-\eta t}}{2\eta^3}\right) A\Sigma^{1/2}(\Sigma^{1/2})^T A^T \tag{5.28}$$

Now let

$$\begin{bmatrix} s_0 + (\kappa - 0.5\sigma_S^2)\delta_t - \alpha\bar{R}\left(\frac{1-e^{-\eta\delta_t}}{\eta}\right) \\ b_0 + (\lambda - 0.5\sigma_B^2)\delta_t - \gamma\bar{R}\left(\frac{1-e^{-\eta\delta_t}}{\eta}\right) \\ \bar{R}(1-e^{-\eta\delta_t}) \end{bmatrix} = \Pi, \begin{bmatrix} 0 & 0 & \frac{\alpha(1-e^{-\eta\delta_t})}{\eta} \\ 0 & 0 & \frac{\gamma(1-e^{-\eta\delta_t})}{\eta} \\ 0 & 0 & e^{-\eta\delta_t} \end{bmatrix} = \Delta \tag{5.29}$$

and let

$$\delta_t\Sigma^{1/2}(\Sigma^{1/2})^T + \left(\frac{1-\eta\delta_t - e^{-\eta\delta_t}}{\eta^2}\right)(\Sigma^{1/2}(\Sigma^{1/2})^T A^T + A\Sigma^{1/2}(\Sigma^{1/2})^T) + \left(\frac{-3+2\eta\delta_t - e^{-2\eta\delta_t} + 4e^{-\eta\delta_t}}{2\eta^3}\right) A\Sigma^{1/2}(\Sigma^{1/2})^T A^T = \Omega \tag{5.30}$$

or rather:

$$\delta_t\Sigma + \left(\frac{1-\eta\delta_t - e^{-\eta\delta_t}}{\eta^2}\right)(\Sigma A^T + A\Sigma) + \left(\frac{-3+2\eta\delta_t - e^{-2\eta\delta_t} + 4e^{-\eta\delta_t}}{2\eta^3}\right) A\Sigma A^T = \Omega \tag{5.31}$$

Therefore the discretized trivariate Arithmetic Ornstein-Uhlenbeck process is:

$$\vec{R}_{t+\delta_t} = \Pi + \Delta\vec{R}_t + \Omega^{1/2}\vec{\varepsilon}, \vec{\varepsilon} \sim N(0,1) \tag{5.31}$$

5.1 Monte Carlo Simulation

For the purpose of Monte-Carlo simulation within this section we consider the case where the trivariate set (s_t, b_t, R_t) exhibited the following relationship:

$$\hat{\vec{X}}_{t+\delta_t} = \begin{bmatrix} s_{t+\delta_t} \\ b_{t+\delta_t} \\ R_{t+\delta_t} \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \phi_{13} \\ 0 & 0 & \phi_{23} \\ 0 & 0 & \phi_{33} \end{bmatrix} \begin{bmatrix} s_t \\ b_t \\ R_t \end{bmatrix} + \vec{Z}_t,\ \vec{Z}_t \sim WN(\vec{0}, \Sigma) \tag{5.1.1}$$

Here the $(s_{t+\delta_t}, b_{t+\delta_t}, R_{t+\delta_t})$ trivariate set exhibits a linear relationship with R_t. Considering the empirical data of section 2.0 and assuming the appropriate parameters to be zero, we can derive the inputs necessary for Monte-Carlo simulation by equating the appropriate matrices of (5.31) and (2.3):

$$\begin{bmatrix} 0 & 0 & \alpha(1-e^{-\eta\delta_t})/\eta \\ 0 & 0 & \gamma(1-e^{-\eta\delta_t})/\eta \\ 0 & 0 & e^{-\eta\delta_t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \phi_{13} \\ 0 & 0 & \phi_{23} \\ 0 & 0 & \phi_{33} \end{bmatrix} \tag{5.1.2}$$

which implies that:

$$\begin{aligned} & e^{-\eta\delta_t} = \phi_{33} \Rightarrow \eta = -\ln(\phi_{33})/\delta_t \\ & \alpha = \eta\phi_{13}/(1-\phi_{33}) \\ & \gamma = \eta\phi_{23}/(1-\phi_{33}) \end{aligned} \tag{5.1.3}$$

Now using the results of (5.1.3) and equating (with appropriate values for s_0 and b_0):

$$\begin{bmatrix} s_0 + (\kappa - 0.5\sigma_S^2)\delta_t - \alpha\bar{R}(1-e^{-\eta\delta_t})/\eta \\ b_0 + (\lambda - 0.5\sigma_B^2)\delta_t - \gamma\bar{R}(1-e^{-\eta\delta_t})/\eta \\ \bar{R}(1-e^{-\eta\delta_t}) \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \tag{5.1.4}$$

results in the following relationships:

$$\begin{aligned} & \bar{R} = \phi_3/(1-\phi_{33}) \\ & \kappa = \phi_1 + \phi_{13}\phi_3/(1-\phi_{33}) + 0.5\sigma_S^2 - s_0 \\ & \lambda = \phi_2 + \phi_{23}\phi_3/(1-\phi_{33}) + 0.5\sigma_B^2 - b_0 \end{aligned} \tag{5.1.5}$$

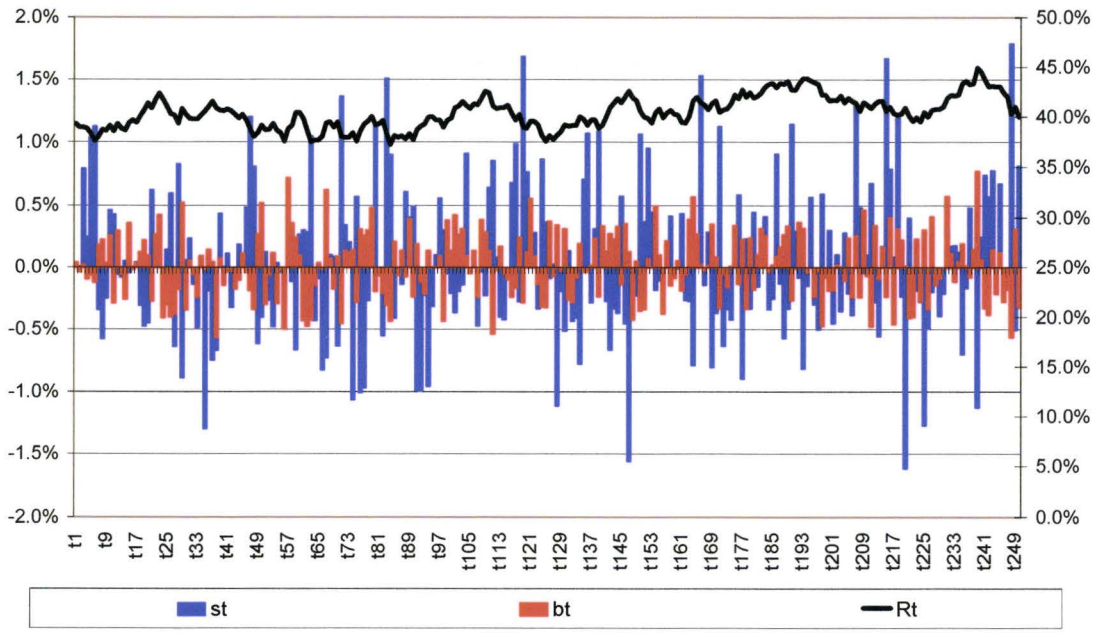


Figure 5.1 - Monte-Carlo Simulation of stochastic differential equation (5.4)

The simulation in Figure 5.1, was derived by using the 2003 Multivariate Minimum AICC Yule-Walker estimates:

$$\Pi = \begin{bmatrix} -0.008495 \\ 0.005725 \\ 0.015601 \end{bmatrix}, \Delta = \begin{bmatrix} 0 & 0 & 0.023 \\ 0 & 0 & -0.014 \\ 0 & 0 & 0.961 \end{bmatrix}, \Omega^{1/2} = \begin{bmatrix} 0.005956 & 0.000000 & 0.000000 \\ -0.000788 & 0.002490 & 0.000000 \\ -0.003720 & 0.003846 & 0.002240 \end{bmatrix}$$

6.0 Optimal portfolio selection – Hamilton-Jacobi-Bellman Framework

Let $X(t)$ denote an investor's wealth at time t. The investor, at each time instant, can allocate a portion of his/her wealth $(\omega_i, i = 1,...,n)$ to the "risky" assets (6.1) whose drift components are dependent on the AOU process (6.3) and therefore $(1-\sum_{i=1}^{n}\omega_i)$ to his/her bank account or "safe asset" (6.2):

$$dS_t^i = (\mu_i + \alpha_i R_t)S_t^i dt + \sigma_i S_t^i dW_t^i \quad for\ i = 1\ to\ n \tag{6.1}$$

$$dC_t = mC_t dt \tag{6.2}$$

$$dR_t = -\eta R_t dt + \sigma_R dW_t^R \tag{6.3}$$

Notice that (6.1) and (6.3) differs from (5.1) in that this AOU process has been mean adjusted so that $\overline{R} = 0$.

This leads to the following stochastic differential equation for wealth:

$$dX_t = [(\sum_{i=1}^{n}(\mu_i + \alpha_i R_t)\omega_i) + m(1-\sum_{i=1}^{n}\omega_i)]X_t dt + (\sum_{i=1}^{n}\sigma_i\omega_i dW_t^i)X_t \tag{6.4}$$

Suppose the investor's initial wealth is $X_t = x > 0$. Now the investor desires to maximize the expected utility of his wealth at some future time $t_0 > t$. If the investor is not permitted to borrow any further assets and we are given a utility $N:[0,\infty) \to [0,\infty), N(0) = 0$ (usually assumed to be increasing and concave), the problem is then to find $\Phi(s,x,r)$ and a Markov control $\omega^* = \omega^*(t, X_t, R_t), 0 \le \omega^* \le 1$ such that:

$$\Phi(s,x,r) = \sup\{J^{\omega}(s,x,r) : 0 \le \omega^* \le 1\} = J^{\omega^*}(s,x,r) \tag{6.5}$$

where:

$$J^{\omega}(s,x,r) = E^{s,x,r}[N(X_T^{\omega})] \tag{6.6}$$

The Hamilton-Jacobi-Bellman operator $(L^{\omega}\Phi)$ for this problem is therefore:

$$L^{\omega}\Phi = \Phi_t + x\Phi_x[(\sum_{i=1}^{n}(\mu_i + \alpha_i r)\omega_i) + m(1-\sum_{i=1}^{n}\omega_i)] - \Phi_r \eta r$$
$$+ \tfrac{1}{2}x^2\Phi_{xx}[\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}\omega_i\omega_j] + \tfrac{1}{2}\sigma_R^2\Phi_{rr} + x\Phi_{xr}[\sum_{i=1}^{n}\sigma_{iR}\omega_i] \tag{6.7}$$

The Hamilton-Jacobi-Bellman optimal portfolio problem is to find (Φ, ω^*) which solves

$$\left\{\begin{array}{l} \max_{\tilde{\omega}} \Phi_t + x\Phi_x[(\sum_{i=1}^{n}(\mu_i + \alpha_i r)\omega_i) + m(1-\sum_{i=1}^{n}\omega_i)] - \Phi_r \eta r \\ + \tfrac{1}{2}x^2\Phi_{xx}[\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}\omega_i\omega_j] + \tfrac{1}{2}\sigma_R^2\Phi_{rr} + x\Phi_{xr}[\sum_{i=1}^{n}\sigma_{iR}\omega_i] = 0 \\ \Phi(T,x,r) = N(x) \end{array}\right\} \tag{6.8}$$

Now if we let $\Phi(t,x,r) = x^p\Gamma(t,r)$, with $\Gamma(T,r) = 1$, where Γ is a function of t and r, then $L^{\omega}\Phi$ becomes

$$L^{\omega}\Phi = (x^p\Gamma(t,r))_t + x(x^p\Gamma(t,r))_x[(\sum_{i=1}^{n}(\mu_i + \alpha_i r)\omega_i) + m(1-\sum_{i=1}^{n}\omega_i)]$$
$$-(x^p\Gamma(t,r))_r \eta r + \tfrac{1}{2}x^2(x^p\Gamma(t,r))_{xx}[\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}\omega_i\omega_j] \tag{6.9}$$
$$+ \tfrac{1}{2}\sigma_R^2(x^p\Gamma(t,r))_{rr} + x(x^p\Gamma(t,r))_{xr}[\sum_{i=1}^{n}\sigma_{iR}\omega_i]$$

Now taking the partial derivatives and dividing by x^p yields the following:

$$x^{-p}L^{\omega}\Phi = \Gamma_t(t,r) + p\Gamma(t,r)[(\sum_{i=1}^{n}(\mu_i + \alpha_i r)\omega_i) + m(1-\sum_{i=1}^{n}\omega_i)] - \Gamma_r(t,r)\eta r$$
$$+ \tfrac{1}{2}p(p-1)\Gamma(t,r)[\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}\omega_i\omega_j] + \tfrac{1}{2}\sigma_R^2\Gamma_{rr}(t,r) + p\Gamma_r(t,r)[\sum_{i=1}^{n}\sigma_{iR}\omega_i] \tag{6.10}$$

Notice that (6.10) is now a function of t and r alone and that optimal wealth is therefore independent of current wealth. Consider now the case where $n = 2$

$$x^{-p}L^{\omega}\Phi = \Gamma_t + p\Gamma[(\mu_1 + \alpha_1 r)\omega_1 + (\mu_2 + \alpha_2 r)\omega_2 + m(1-\omega_1-\omega_2)] - \Gamma_r \eta r$$
$$+ \tfrac{1}{2}p(p-1)\Gamma[\sigma_1^2\omega_1^2 + \sigma_2^2\omega_2^2 + 2\sigma_{12}\omega_1\omega_2] + \tfrac{1}{2}\sigma_R^2\Gamma_{rr} + p\Gamma_r[\sigma_{1R}\omega_1 + \sigma_{2R}\omega_2] \tag{6.11}$$

To derive the optimal control ω^* we must compute the partial derivatives $\frac{\partial L^{\omega}\Phi}{\partial \omega_i}$ for $i = 1, 2$ and equate them to zero and then isolate the components of the optimal control:

$$\omega_1 = \frac{\Gamma(m - \mu_1 - \alpha_1 r) - (p-1)\Gamma\sigma_{12}\omega_2 - \Gamma_r\sigma_{1R}}{(p-1)\Gamma\sigma_1^2} \tag{6.12}$$

and similarly

$$\omega_2 = \frac{\Gamma(m - \mu_2 - \alpha_2 r) - (p-1)\Gamma\sigma_{12}\omega_1 - \Gamma_r\sigma_{2R}}{(p-1)\Gamma\sigma_2^2} \tag{6.13}$$

Substituting (6.13) into (6.12) yields the following expressions for the optimal control with two risky assets:

$$\omega_1 = \frac{\Gamma(m - \mu_1 - \alpha_1 r)}{(p-1)\Gamma\sigma_1^2} - \frac{\sigma_{12}}{\sigma_1^2}\left(\frac{\Gamma(m - \mu_2 - \alpha_2 r) - (p-1)\Gamma\sigma_{12}\omega_1 - \Gamma_r\sigma_{2R}}{(p-1)\Gamma\sigma_2^2}\right) - \frac{\Gamma_r\sigma_{1R}}{(p-1)\Gamma\sigma_1^2} \tag{6.14}$$

$$\omega_1 = \frac{\sigma_2^2\Gamma(m - \mu_1 - \alpha_1 r) - \sigma_{12}[\Gamma(m - \mu_2 - \alpha_2 r) - (p-1)\Gamma\sigma_{12}\omega_1 - \Gamma_r\sigma_{2R}] - \sigma_2^2\Gamma_r\sigma_{1R}}{(p-1)\Gamma\sigma_1^2\sigma_2^2} \tag{6.15}$$

and therefore

$$\omega_1 = \frac{\sigma_2^2\Gamma(m - \mu_1 - \alpha_1 r) - \sigma_{12}\Gamma(m - \mu_2 - \alpha_2 r) + \sigma_{12}\Gamma_r\sigma_{2R} - \sigma_2^2\Gamma_r\sigma_{1R}}{(p-1)\Gamma\sigma_1^2\sigma_2^2(1-\rho_{12}^2)} \tag{6.16}$$

or rather

$$\omega_1 = \frac{[\sigma_2^2(m - \mu_1) - \sigma_{12}(m - \mu_2)]\Gamma + (\alpha_2\sigma_{12} - \alpha_1\sigma_2^2)r\Gamma + (\sigma_{12}\sigma_{2R} - \sigma_2^2\sigma_{1R})\Gamma_r}{(p-1)\Gamma\sigma_1^2\sigma_2^2(1-\rho_{12}^2)} \tag{6.17}$$

similarly

$$\omega_2 = \frac{[\sigma_1^2(m - \mu_2) - \sigma_{12}(m - \mu_1)]\Gamma + (\alpha_1\sigma_{12} - \alpha_2\sigma_1^2)r\Gamma + (\sigma_{12}\sigma_{1R} - \sigma_1^2\sigma_{2R})\Gamma_r}{(p-1)\Gamma\sigma_1^2\sigma_2^2(1-\rho_{12}^2)} \tag{6.18}$$

Let:

$$\begin{aligned} d_1 &= \frac{\sigma_2^2(m-\mu_1)-\sigma_{12}(m-\mu_2)}{(p-1)\sigma_1^2\sigma_2^2(1-\rho_{12}^2)}; d_2 = \frac{(\alpha_2\sigma_{12}-\alpha_1\sigma_2^2)}{(p-1)\sigma_1^2\sigma_2^2(1-\rho_{12}^2)}; \\ d_3 &= \frac{(\sigma_{12}\sigma_{2R}-\sigma_2^2\sigma_{1R})}{(p-1)\sigma_1^2\sigma_2^2(1-\rho_{12}^2)}; d_4 = \frac{\sigma_1^2(m-\mu_2)-\sigma_{12}(m-\mu_1)}{(p-1)\sigma_1^2\sigma_2^2(1-\rho_{12}^2)} \\ d_5 &= \frac{(\alpha_1\sigma_{12}-\alpha_2\sigma_1^2)}{(p-1)\sigma_1^2\sigma_2^2(1-\rho_{12}^2)}; d_6 = \frac{(\sigma_{12}\sigma_{1R}-\sigma_1^2\sigma_{2R})}{(p-1)\sigma_1^2\sigma_2^2(1-\rho_{12}^2)} \end{aligned} \tag{6.19}$$

be known constants then:

$$\begin{aligned} \omega_1 &= d_1 + d_2 r + d_3 \Gamma_r / \Gamma; \\ \omega_2 &= d_4 + d_5 r + d_6 \Gamma_r / \Gamma \end{aligned} \tag{6.20}$$

Notice that the optimal control still depends on Γ. The first step in determining the nature of Γ is to substitute (6.20) back into the Hamilton-Jacobi-Bellman equation (6.11). This yields the following partial differential equation:

$$\left\{ \begin{aligned} &L^{\omega}\Phi = \lambda_1\Gamma^2 + \lambda_2 r\Gamma^2 + \lambda_3\Gamma\Gamma_r + \lambda_4 r^2\Gamma^2 + \lambda_5 r\Gamma\Gamma_r + \lambda_6\Gamma_r^2 + \Gamma\Gamma_t + \lambda_7\Gamma\Gamma_{rr} = 0 \\ &\Gamma(T,r) = 1 \end{aligned} \right\} \tag{6.21}$$

Where λ_i for $i = 1 \, to \, 7$ are known constants. Solving this partial differential equation for $\Gamma(t,r)$ and substituting back into (6.20) would yield the explicit solution for the optimal control for this investor.

In practice it is very difficult to find explicit solutions for non-linear PDE's of second order such as (6.21) or any PDE for that matter. There exist many methods that aim to find approximate solutions to these by way of perturbation or the finite elements method (FEM). We will not discuss these methods here but refer the reader to [8].

7.0 Optimal portfolio selection – Mean Variance Framework

The Standard Mean-Variance Optimization (MVO) method, as developed by Nobel Laureate Harry Markowitz, aims to find efficient portfolios. MVO Efficient portfolios, are those which have the highest expected portfolio return for a given level of expected portfolio variance or, conversely, the lowest level of expected portfolio variance for a given level of expected portfolio return. Plotted on a return to variance graph, these efficient portfolios generate the "efficient frontier". The inputs for MVO are the expected returns μ_i $(i = 1,...,n)$, the variances σ_{ii} $(i = 1,...,n)$ and the covariances σ_{ij} $(i, j = 1,...,n, i \neq j)$ of the assets being considered for optimization. For the case where n assets are being considered for optimization the MVO problem is as follows. Minimize portfolio variance $\sigma_{PP} = \vec{\omega}^T \Sigma \vec{\omega}$ as

$$\min_{\vec{\omega}} \vec{\omega}^T \Sigma \vec{\omega} \tag{7.1}$$

subject to the following constraints:

$$\begin{aligned} &\vec{\omega}^T \vec{\mu} = \mu_P, \\ &\vec{\omega}^T I = 1, \\ &\text{and perhaps other constraints.} \end{aligned} \tag{7.2}$$

where $\vec{\omega}$ and $\vec{\mu}$ are the vector of weights and returns assigned to the assets, Σ is the covariance matrix of the assets and μ_P is the portfolio return. For the purpose of the problem formulation within this section we will consider the following additional constraint:

$$\omega_i \geq 0, i = 1,...,n \tag{7.3}$$

This additional constraint disallows the investor to short sell his/her investments. In order to develop efficient portfolios within the MVO framework it is necessary to understand the feasible interval for efficient portfolio variance and efficient portfolio return. To demonstrate how this is accomplished let us consider the case where $n = 2$.

The absolute minimum variance portfolio within the MVO framework is calculated by taking the partial derivative of (7.1) with respect to ω_S and equating it to zero. Note here that $\omega_B = 1 - \omega_S$. This portfolio has the following weights:

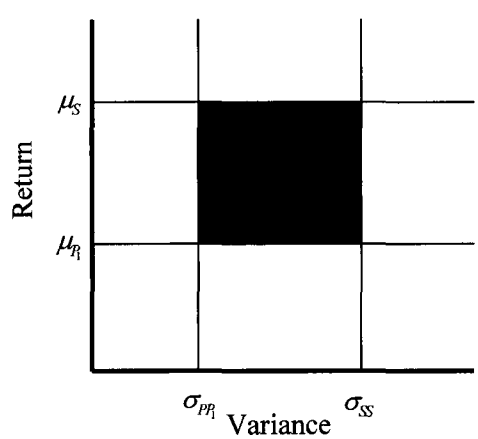
$$\begin{bmatrix} \omega_S \\ \omega_B \end{bmatrix} = \begin{bmatrix} \dfrac{\sigma_{BB} - \sigma_{SB}}{\sigma_{SS} + \sigma_{BB} - 2\sigma_{SB}} \\ 1 - \dfrac{\sigma_{BB} - \sigma_{SB}}{\sigma_{SS} + \sigma_{BB} - 2\sigma_{SB}} \end{bmatrix} \tag{7.4}$$

and is essentially the portfolio on the efficient frontier with the lowest portfolio variance. The variance and return associated with this efficient portfolio are respectively:

$$\sigma_{P_{\mathrm{I}}} = \begin{bmatrix} \dfrac{\sigma_{BB} - \sigma_{SB}}{\sigma_{SS} + \sigma_{BB} - 2\sigma_{SB}} & 1 - \dfrac{\sigma_{BB} - \sigma_{SB}}{\sigma_{SS} + \sigma_{BB} - 2\sigma_{SB}} \end{bmatrix} \begin{bmatrix} \sigma_{SS} & \sigma_{SB} \\ \sigma_{SB} & \sigma_{BB} \end{bmatrix} \begin{bmatrix} \dfrac{\sigma_{BB} - \sigma_{SB}}{\sigma_{SS} + \sigma_{BB} - 2\sigma_{SB}} \\ 1 - \dfrac{\sigma_{BB} - \sigma_{SB}}{\sigma_{SS} + \sigma_{BB} - 2\sigma_{SB}} \end{bmatrix} \tag{7.5}$$

$$\mu_{P_{\mathrm{I}}} = \begin{bmatrix} \dfrac{\sigma_{BB} - \sigma_{SB}}{\sigma_{SS} + \sigma_{BB} - 2\sigma_{SB}} & 1 - \dfrac{\sigma_{BB} - \sigma_{SB}}{\sigma_{SS} + \sigma_{BB} - 2\sigma_{SB}} \end{bmatrix} \begin{bmatrix} \kappa + \alpha(R_t - \bar{R}) \\ \lambda + \gamma(R_t - \bar{R}) \end{bmatrix} \tag{7.6}$$

Let $\max(\mu_S, \mu_B)$, be the maximum return portfolio on the efficient frontier. If $\mu_S > \mu_B$ then the maximum return portfolio is $[\omega_s \;\; \omega_B] = [1 \;\; 0]$. The feasible range of portfolio variance and portfolio return, in this case, within the MVO framework when $n = 2$ can be depicted as follows:



The shaded region above is where the efficient frontier will reside. Now to find any one efficient portfolio reduces to solving the following quadratic equation for ω_s for a given level of feasible portfolio variance using the quadratic formula

$$\omega_S^2(\sigma_{SS}+\sigma_{BB}-2\sigma_{SB})+2\omega_S(\sigma_{SB}-\sigma_{BB})+\sigma_{BB}=\text{Feasible Portfolio Variance} \quad (7.7)$$

8.0 Monte Carlo Simulation of Terminal Wealth

In this section we compare the terminal wealth of two investors, who each construct three portfolios via the Mean-Variance Optimization method, using Monte-Carlo simulation. The log returns (s_t, b_t) for the underlying assets of their optimal portfolios behave as in (5.31). The initial capital for all optimal portfolios is \$1000. Investor A, however, is incognizant of R_t and therefore optimizes once at the beginning of the year and holds the optimal weights constant throughout the 250 trading days whereas Investor B optimizes daily because he/she is aware of the relationship between (s_t, b_t) and R_t. Let A1/B1 be investor A/B's minimum variance portfolio under the MVO method, let A2/B2 be investor A/B's portfolio with variance equal to the mid point of the feasible variance discussed in section 7.0 and let A3/B3 be investor A/B's maximum expected value portfolio. Both investors have the same 250-day expected value. In order for Investor B to construct his/her maximum expected value portfolio he/she must compare

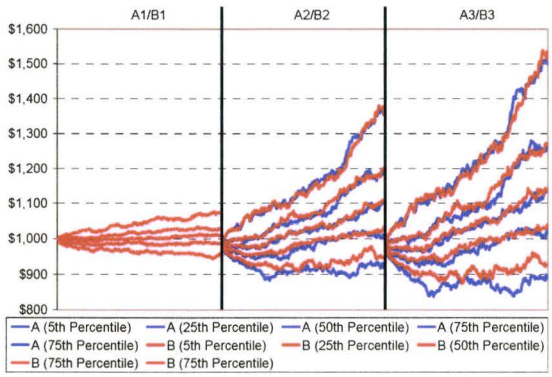
$$s_0 + (\kappa - 0.5\sigma_S^2)\delta_t - \alpha\bar{R}\left(\frac{1-e^{-\eta\delta_t}}{\eta}\right) + \frac{\alpha R_t(1-e^{-\eta\delta_t})}{\eta} \tag{8.1}$$

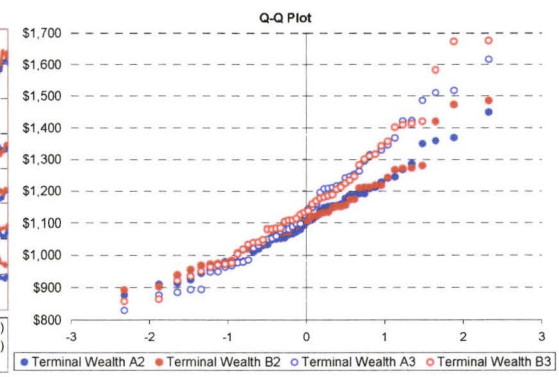
and

$$b_0 + (\lambda - 0.5\sigma_B^2)\delta_t - \gamma\bar{R}\left(\frac{1-e^{-\eta\delta_t}}{\eta}\right) + \frac{\gamma R_t(1-e^{-\eta\delta_t})}{\eta} \tag{8.2}$$

on a daily basis. Whichever is greater dictates which asset receives 100% of the wealth for the next day.

8.1 Monte Carlo Simulation Comparison of Terminal Wealth





	A2	B2	A3	B3
Average Percent Profit	11.0%	11.9%	15.4%	16.9%
Standard Deviation of Profit	13.1%	13.2%	18.7%	18.9%
Avg. Profit/STDEV	83.7%	90.5%	82.6%	89.5%
Value at Risk (1 in 20)	-8.1%	-5.3%	-11.0%	-7.2%
Correlation Test of Normality [9] (α=5%)	Accept	Reject	Accept	Reject
Kurtosis	-0.16	0.94	-0.36	0.68
Skewness	0.46	0.82	0.45	0.83

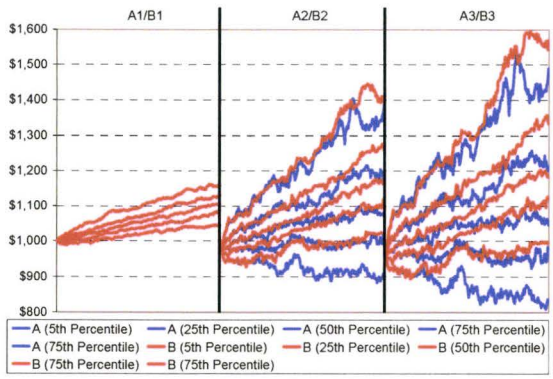
Figure 8.1.1 - Percentile ranking and terminal wealth characteristics of 50 simulations for investor A's and B's wealth over 250 trading days

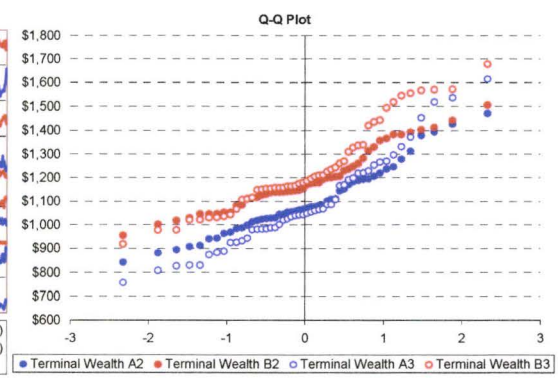
The 1999 minimum AICC Yule-Walker estimates were used as inputs:

$$\Pi = \begin{bmatrix} 0.005970 \\ 0.000631 \\ 0.005353 \end{bmatrix}, \Delta = \begin{bmatrix} 0 & 0 & -0.019 \\ 0 & 0 & -0.002 \\ 0 & 0 & 0.980 \end{bmatrix}, \Omega^{1/2} = \begin{bmatrix} 0.009018 & 0.000000 & 0.000000 \\ 0.000773 & 0.002528 & 0.000000 \\ -0.001900 & 0.002526 & 0.004623 \end{bmatrix}$$

$$\alpha = -0.019193, \gamma = -0.002020$$

Investor B yields a higher percentage profit over the 250 days than Investor A with comparable standard deviation of profit leading to a more favorable risk-adjusted percentage profit. The 1 in 20 downside event, as depicted by Value at Risk for Investor B, is less than that of Investor A. Investor B, however, fails the correlation test for normality. The realized distribution of profit for Investor B is more concentrated around its mean with more right asymmetry as compared to Investor A's strategy.





	A2	B2	A3	B3
Average Percent Profit	9.7%	19.2%	8.8%	23.5%
Standard Deviation of Profit	14.3%	12.8%	19.6%	18.5%
Avg. Profit/STDEV	68.2%	149.9%	45.1%	127.6%
Value at Risk (1 in 20)	-10.1%	2.3%	-17.2%	-0.2%
Correlation Test of Normality [9] (α=5%)	Accept	Accept	Reject	Reject
Kurtosis	0.32	-0.31	0.37	-0.33
Skewness	0.71	0.57	0.78	0.67

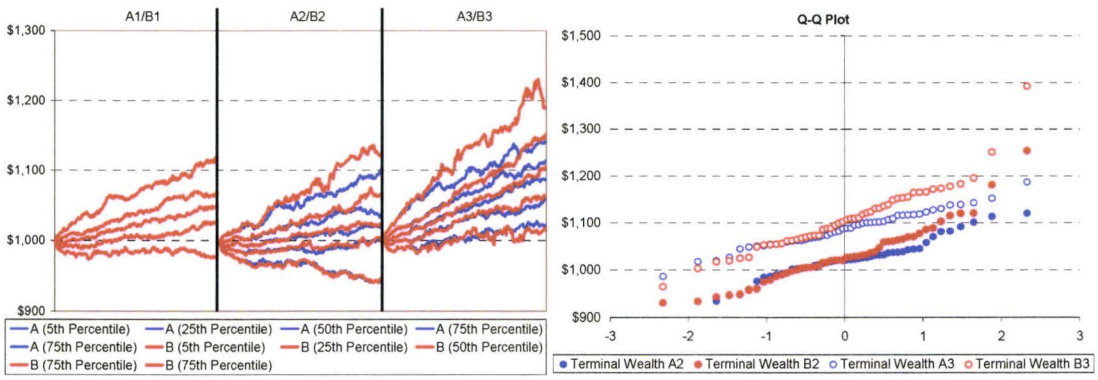
Figure 8.1.2 - Percentile ranking and terminal wealth characteristics of 50 simulations for investor A's and B's wealth over 250 trading days

The 2000 minimum AICC Yule-Walker estimates were used as inputs:

$$\Pi=\begin{bmatrix}-0.017622\\-0.000157\\0.007621\end{bmatrix},\Delta=\begin{bmatrix}0&0&0.096\\0&0&0.003\\0&0&0.959\end{bmatrix},\Omega^{1/2}=\begin{bmatrix}0.016185&0.000000&0.000000\\-0.000181&0.002231&0.000000\\-0.003146&0.001075&0.003015\end{bmatrix}$$

$$\alpha = 0.098024, \gamma = 0.003063$$

For this simulation, Investor B yields a higher percentage profit over the 250 days than Investor A with less standard deviation of profit leading to a more favorable risk-adjusted percentage profit. The 1 in 20 downside event, as depicted by Value at Risk for Investor B, is less than that of Investor A. Investor B's mid variance strategy passes the correlation test for normality at level of significance $\alpha = 5\%$ whereas his/her maximum expected value strategy fails. The realized distribution of profit for Investor B is less concentrated around its mean with less right asymmetry as compared to Investor A's strategy.



	A2	B2	A3	B3
Average Percent Profit	2.0%	3.2%	8.7%	10.8%
Standard Deviation of Profit	4.3%	6.2%	3.8%	7.0%
Avg. Profit/STDEV	45.8%	51.4%	226.9%	154.9%
Value at Risk (1 in 20)	-6.0%	-5.5%	2.3%	1.8%
Correlation Test of Normality [9] (α=5%)	Accept	Reject	Accept	Reject
Kurtosis	0.40	2.59	0.42	4.48
Skewness	0.09	1.10	-0.05	1.28

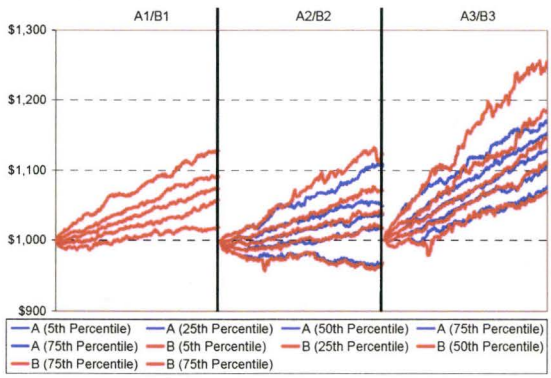
Figure 8.1.3: Percentile ranking and terminal wealth characteristics of 50 simulations for investor A's and B's wealth over 250 trading days

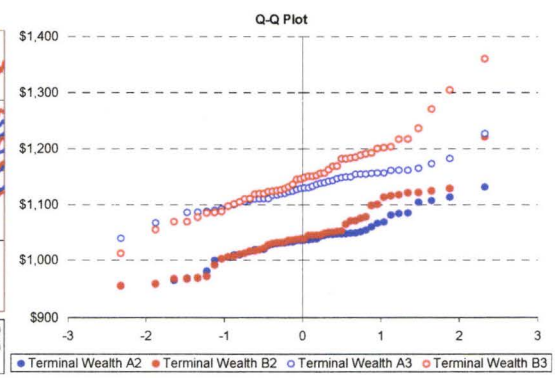
The 2001 minimum AICC Yule-Walker estimates were used as inputs:

$$\Pi = \begin{bmatrix} -0.007810 \\ 0.000880 \\ 0.003850 \end{bmatrix}, \Delta = \begin{bmatrix} 0 & 0 & 0.025 \\ 0 & 0 & -0.002 \\ 0 & 0 & 0.986 \end{bmatrix}, \Omega^{1/2} = \begin{bmatrix} 0.011742 & 0.000000 & 0.000000 \\ -0.000582 & 0.002764 & 0.000000 \\ -0.003758 & 0.002785 & 0.004743 \end{bmatrix}$$

$$\alpha = 0.025177, \gamma = -0.002014$$

Once again, Investor B yields a higher percentage profit over the 250 days than Investor A but with greater standard deviation of profit which does not necessarily lead to a more favorable risk-adjusted percentage profit. The 1 in 20 downside event, as depicted by Value at Risk for both Investors are comparable. Investor B fails the correlation test for normality at level of significance $\alpha = 5\%$. The realized distribution of profit for Investor B is more concentrated around its mean with more right asymmetry as compared to Investor A's strategy.





	A2	B2	A3	B3
Average Percent Profit	3.6%	4.6%	12.8%	15.1%
Standard Deviation of Profit	3.8%	5.2%	3.3%	6.3%
Avg. Profit/STDEV	93.3%	88.6%	384.9%	241.1%
Value at Risk (1 in 20)	-3.3%	-3.2%	7.7%	7.0%
Correlation Test of Normality [9] (α=5%)	Accept	Reject	Accept	Reject
Kurtosis	0.46	1.37	0.96	1.95
Skewness	0.09	0.74	0.00	0.85

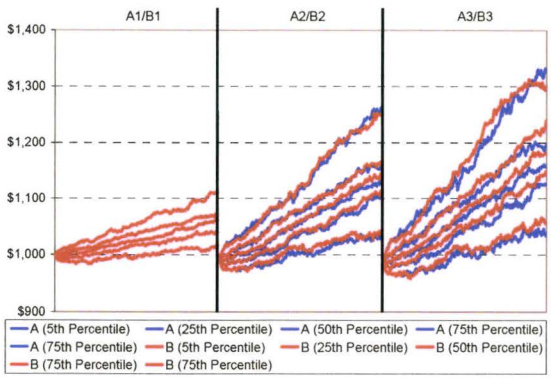
Figure 8.1.4: Percentile ranking and terminal wealth characteristics of 50 simulations for investor A's and B's wealth over 250 trading days

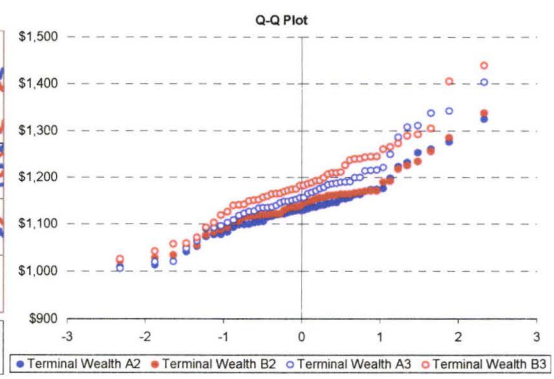
The 2002 minimum AICC Yule-Walker estimates were used as inputs:

$$\Pi=\begin{bmatrix}-0.007102\\0.001185\\0.005730\end{bmatrix},\Delta=\begin{bmatrix}0&0&0.018\\0&0&-0.002\\0&0&0.984\end{bmatrix},\Omega^{1/2}=\begin{bmatrix}0.010291&0.000000&0.000000\\-0.001341&0.002274&0.000000\\-0.005815&0.003519&0.005254\end{bmatrix}$$

$$\alpha=0.018146,\gamma=-0.002016$$

Investor B here also yields a higher percentage profit over the 250 days than Investor A but with greater standard deviation of profit which does not lead to a more favorable risk-adjusted percentage profit. The 1 in 20 downside event, as depicted by Value at Risk for both Investors are comparable. Investor B fails the correlation test for normality at level of significance $\alpha = 5\%$. The realized distribution of profit for Investor B is more concentrated around its mean with more right asymmetry as compared to Investor A's strategy.





	A2	B2	A3	B3
Average Percent Profit	13.5%	14.3%	16.7%	18.9%
Standard Deviation of Profit	6.4%	6.1%	8.2%	8.1%
Avg. Profit/STDEV	211.4%	234.7%	205.5%	233.9%
Value at Risk (1 in 20)	3.0%	4.1%	3.3%	5.8%
Correlation Test of Normality [9] (α=5%)	Reject	Reject	Reject	Accept
Kurtosis	1.19	1.66	1.03	1.56
Skewness	0.63	0.63	0.65	0.62

Figure 8.1.5: Percentile ranking and terminal wealth characteristics of 50 simulations for investor A's and B's wealth over 250 trading days

The 2003 minimum AICC Yule-Walker estimates were used as inputs:

$$\Pi=\begin{bmatrix}-0.008495\\0.005725\\0.015601\end{bmatrix},\Delta=\begin{bmatrix}0&0&0.023\\0&0&-0.014\\0&0&0.961\end{bmatrix},\Omega^{1/2}=\begin{bmatrix}0.005956&0.000000&0.000000\\-0.000788&0.002490&0.000000\\-0.003720&0.003846&0.002240\end{bmatrix}$$

$$\alpha = 0.023461, \gamma = -0.014280$$

Investor B here also yields a higher percentage profit over the 250 days than Investor A but with comparable standard deviation of profit which leads to a more favorable risk-adjusted percentage profit. The 1 in 20 downside event, as depicted by Value at Risk for Investor B, is more optimal as compared to that of Investor A. Only Investor B's maximum expected value strategy passes the correlation test for normality at level of significance $\alpha = 5\%$. The realized distribution of profit for Investor B is more concentrated around its mean as compared to Investor A's strategy. The distribution of profit for both investors depicts the same level of asymmetry towards the right.

9.0 Conclusions

Considering the 250 trading days of every year since 1999, we have shown that the actual daily log returns of the S&P/TSX Composite Total Return Index and Scotia Capital Overall Bond Total Return Index, observed within the financial market, have exhibited serial dependence at lag-1 with the Yield Ratio using trivariate minimum AICC Yule-Walker estimation. We have also shown that if the trivariate stochastic differential equation (5.1) governs the path dynamics of the trivariate set (s_t, b_t, R_t) then the optimal portfolio strategy under the Hamilton-Jacobi-Bellman framework is independent of current wealth but does depend on the path dynamics of R_t. Under the MVO framework the set of optimal portfolios also depends on R_t. We have shown, using Monte-Carlo simulation, that an investor who is cognizant of the relationship depicted in section 5.0 has the potential to harness a greater average profit by re-optimizing on a daily basis than an investor who is incognizant and consequently optimizes once every 250 trading days. In Appendix 2 we have depicted the wealth process of A2 and B2 with actual daily returns of the S&P/TSX Composite Total Return Index and the Scotia Capital Overall Bond Total Return Index. The results in Appendix 2 are not as convincing as in section 8.0. Investor B's mid-variance strategy does not always yield greater terminal wealth when considering actual market data. This can be due to many reasons, one of which is that actual daily returns observed within financial markets are not normally distributed and assign more probability to extreme events compared to simulated returns.

Within this thesis we focused on the Yield Ratio as a governing metric. It turns out that there are many such mean-reverting governing metrics that are used by investment practitioners to harness the potential for superior terminal wealth such as the earnings yield ratio or even exchange rates to name a few. Portfolio managers who must decide between corporate bonds or government bonds often consider the ratio of the underlying interest rates (yields) for these investments to govern which will receive a more significant allocation of wealth. Consider the following multivariate minimum AICC Yule-Walker [6] methodology to find the multivariate autoregressive model that

best fits the 250 trading days of 2003 for the trivariate set: Scotia Capital Government Bond Index ($SCGov_t$), Scotia Capital Corporate Bond Index ($SCCorp_t$) and the ratio of their Yields (R_t):

$$\hat{\vec{X}}_t = \begin{bmatrix} \text{SCGov}_t \\ \text{SCCorp}_t \\ R_t \end{bmatrix} = \begin{bmatrix} -0.002960 \\ -0.002875 \\ 0.013977 \end{bmatrix} + \begin{bmatrix} -0.168 & 0.188 & 0.004 \\ -0.357 & 0.404 & 0.004 \\ -0.079 & 0.213 & 0.984 \end{bmatrix} \begin{bmatrix} \text{SCGov}_{t-1} \\ \text{SCCorp}_{t-1} \\ R_{t-1} \end{bmatrix} + \begin{bmatrix} Z_{t1} \\ Z_{t2} \\ Z_{t3} \end{bmatrix},$$

$$\begin{bmatrix} Z_{t1} \\ Z_{t2} \\ Z_{t3} \end{bmatrix} \sim WN\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.000008 & 0.000007 & -0.000003 \\ 0.000007 & 0.000006 & -0.000002 \\ -0.000003 & -0.000002 & 0.000020 \end{bmatrix} \right)$$

Assumption number 1 within this document implied that there are no costs associated with the transaction of apportioning wealth from one investment to the other. Let us assume that Investor B made use of E*TRADE Canada[4] to trade in and out of s_t and into b_t. Trades on E*TRADE can cost as little as $10. Now considering the simulation for the year of 2003, Investor B would quickly deplete his assets. Investor B's mid-variance strategy executed 183 trades during the 250 trading days which would ultimately result in $1830 of transaction costs.

There are other more cost effective ways that Investor B can make use of to harness his/her potential profit. For example one can make use of derivatives such as Forward[5] contracts to trade in and out of s_t and into b_t. Forward contracts are inexpensive and their price can often be negotiated to a more favourable level. Typically, however the price associated with these Forward contracts is represented as a percentage of the notional amount of the investment. For example, if Investor B wants $1000 exposure to s_t then a typical transaction cost in the Forward market would be $1000(0.01%) = $0.10. Even if Investor B had executed 250 trades in which $1000 was shifted to another

[4] An indirect wholly-owned subsidiary of U.S. based E*TRADE Financial Corp., offers Canadians an on-line mechanism to trade Canadian and U.S. stocks, options, Canadian fixed income securities, and Canadian mutual funds.

[5] A notional market transaction in which a seller agrees to deliver a specific amount of cash to a buyer at some point in the future.

asset, he/she would have only incurred $25 in transaction costs. The point here is that Investor B's strategy is definitely not optimal when transacting real currency on a market value basis but remains optimal in the world of derivative instruments.

Investors who wish to make use of Investor B's strategy must first find a mean-reverting process within the financial markets. The underlying assets to which the investor wants to allocate wealth, must also exhibit dependence on this mean-reverting process as described in section 2. Once these preliminaries have been established the investor must be mindful that actual market returns associate more probability to extreme events and that simulated outcomes may be significantly different from actual outcomes.

Appendix 1

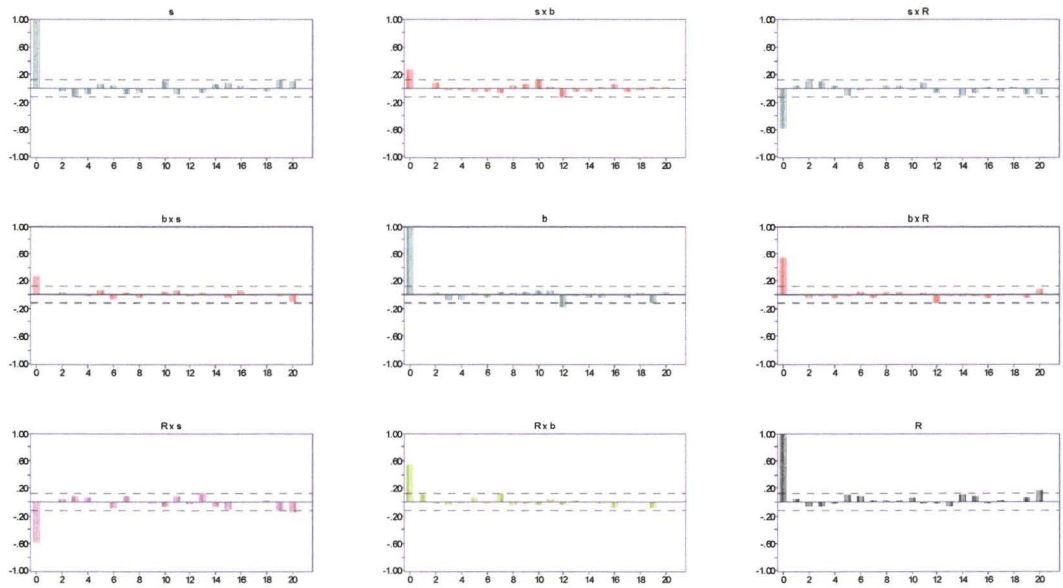


Figure 1.1 - Sample cross-correlations of the residuals after fitting the trivariate AR(1) model for the 250 trading days of 1999

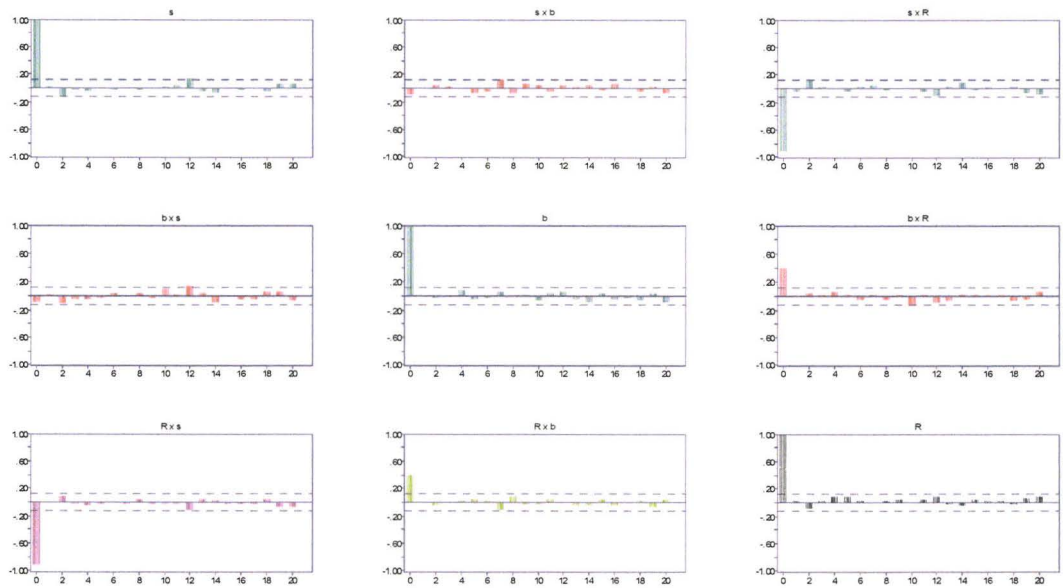


Figure 1.2 - Sample cross-correlations of the residuals after fitting the trivariate AR(1) model for the 250 trading days of 2000

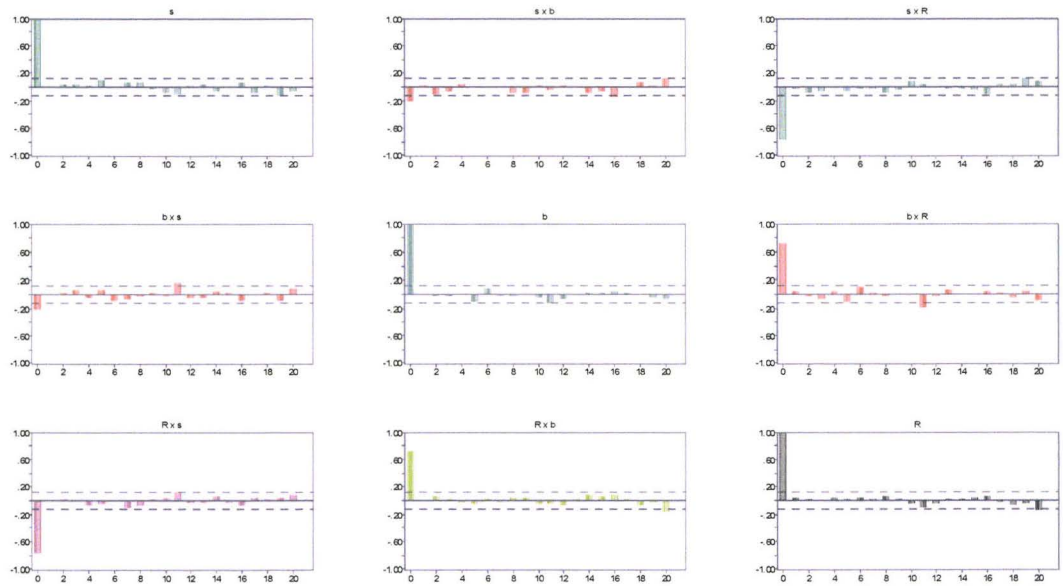


Figure 1.3 - Sample cross-correlations of the residuals after fitting the trivariate AR(1) model for the 250 trading days of 2001

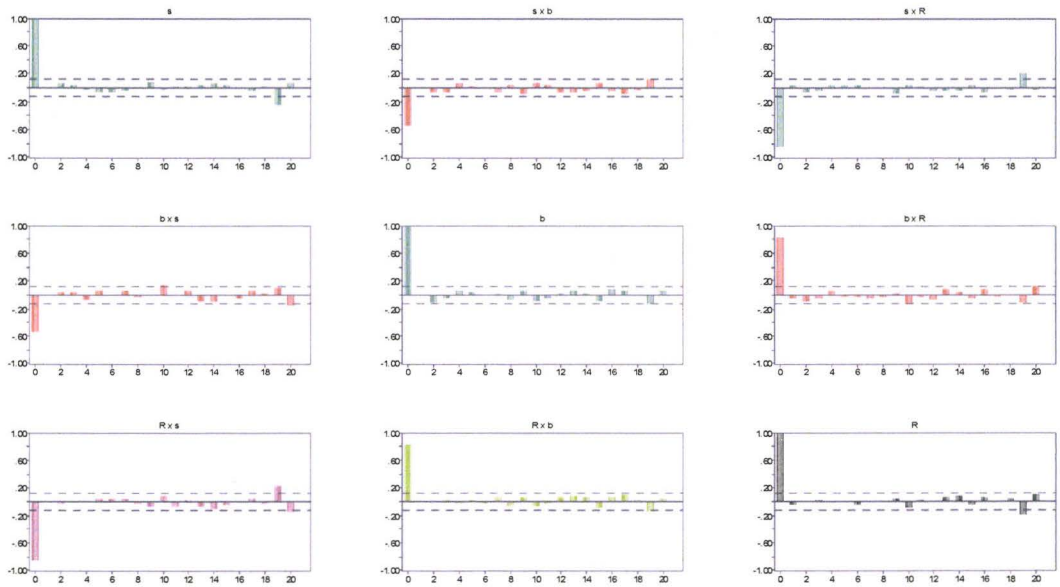


Figure 1.4 - Sample cross-correlations of the residuals after fitting the trivariate AR(1) model for the 250 trading days of 2002

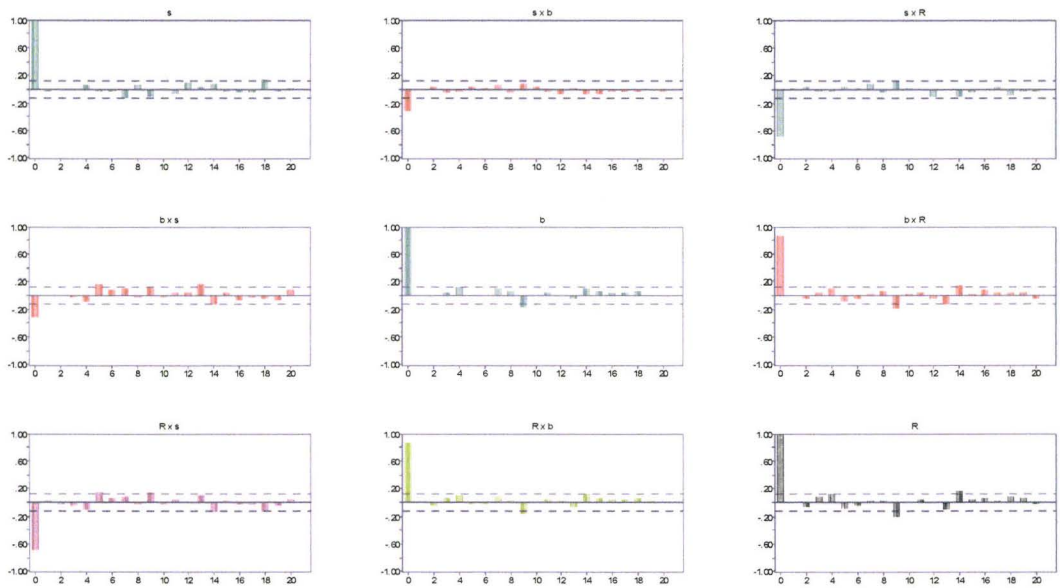


Figure 1.5 - Sample cross-correlations of the residuals after fitting the trivariate AR(1) model for the 250 trading days of 2003

Appendix 2

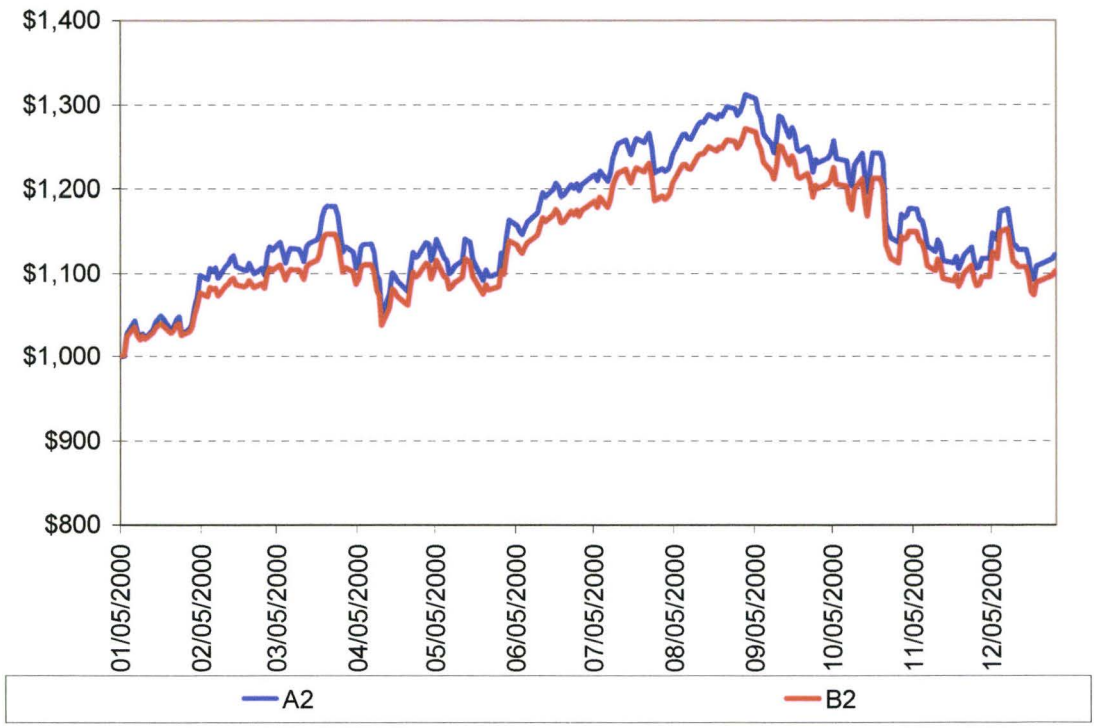


Figure 2.1 - Actual wealth for Investor A and B over the 250 trading days of 2000 using simulated portfolio weights A2 and B2 resulting from the simulation of 8.1

Figure 2.2 - Actual wealth for Investor A and B over the 250 trading days of 2001 using simulated portfolio weights A2 and B2 resulting from the simulation of 8.2.

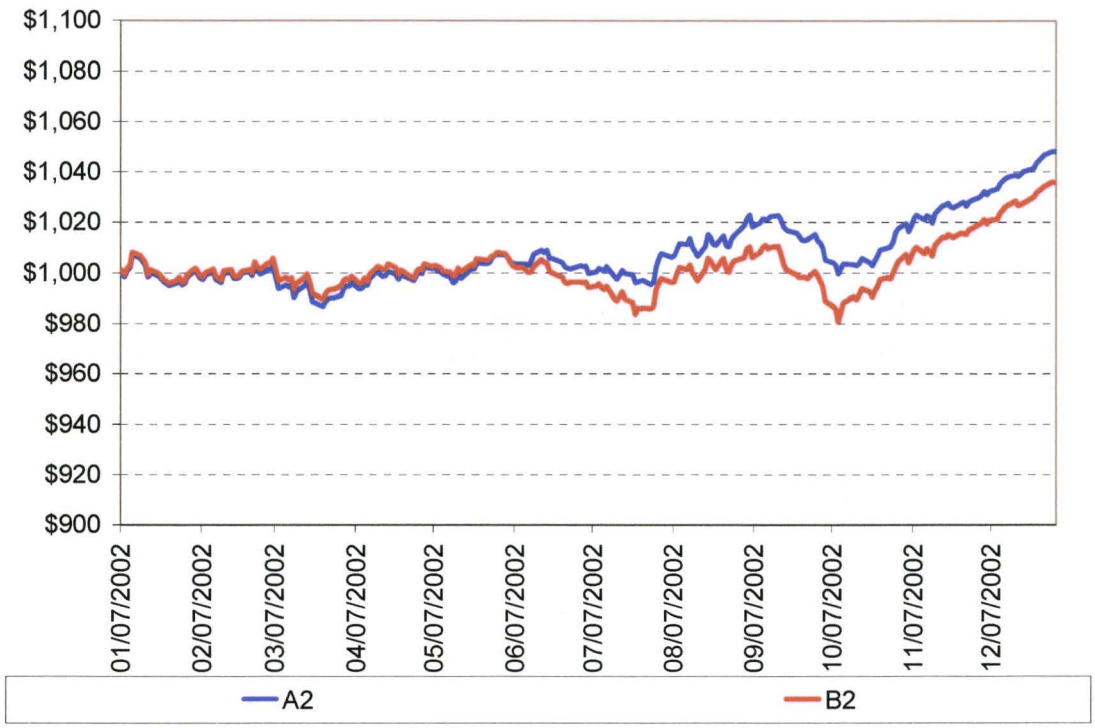


Figure 2.3 - Actual wealth for Investor A and B over the 250 trading days of 2002 using simulated portfolio weights A2 and B2 resulting from the simulation of 8.3

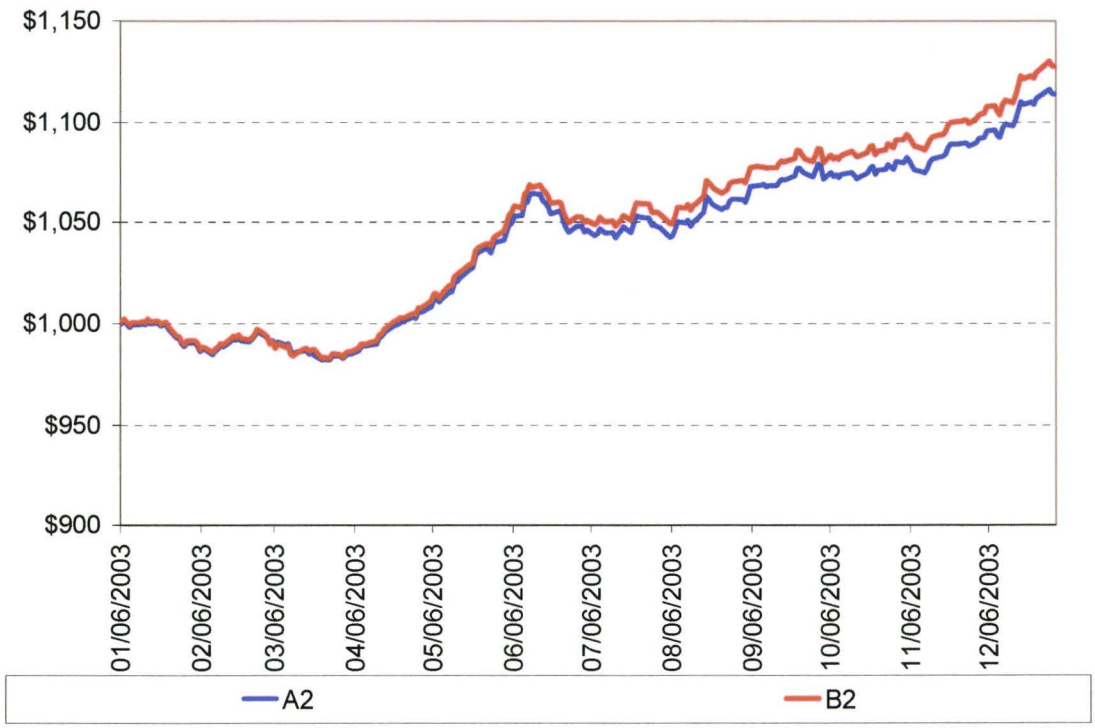


Figure 2.4 - Actual wealth for Investor A and B over the 250 trading days of 2003 using simulated portfolio weights A2 and B2 resulting from the simulation of 8.4

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